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The co-universal C^*-algebra of a row-finite graph

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Abstract. Let E be a row-finite directed graph. We prove that there exists a C^*-algebra $C^*_{\min}(E)$ with the following co-universal property: given any C^*-algebra B generated by a Toeplitz–Cuntz–Krieger E-family in which all the vertex projections are nonzero, there is a canonical homomorphism from B onto $C^*_{\min}(E)$. We also identify when a homomorphism from B to $C^*_{\min}(E)$ obtained from the co-universal property is injective. When every loop in E has an entrance, $C^*_{\min}(E)$ coincides with the graph C^*-algebra $C^*(E)$, but in general, $C^*_{\min}(E)$ is a quotient of $C^*(E)$. We investigate the properties of $C^*_{\min}(E)$ with emphasis on the utility of co-universality as the defining property of the algebra.

Contents

1. Introduction

The aim of this paper is to initiate a study of C^*-algebras defined by what we refer to as co-universal properties, and to demonstrate the utility of such a property in investigating the structure of the resulting C^*-algebra. We do this by considering the specific example of co-universal C^*-algebras associated to row-finite directed graphs.

A directed graph E consists of a countable set E^0 of vertices, and a countable set E^1 of directed edges. The edge-directions are encoded by maps $r, s : E^1 \to E^0$: an edge e points from the vertex $s(e)$ to the vertex $r(e)$. In this paper, we follow the edge-direction conventions of [9]; that is, a path in E is a finite sequence $e_1e_2...e_n$ of edges such that $s(e_i) = r(e_{i+1})$ for $1 \leq i < n$.

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Let E be a directed graph. A *Toeplitz–Cuntz–Krieger E-family* in a C^*-algebra B consists of sets $\{p_v : v \in E^0\}$ and $\{s_e : s \in E^1\}$ of elements of B such that:

(T1) the p_v are mutually orthogonal projections;
(T2) $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$; and
(T3) $p_v \geq \sum_{e\in F} s_e s_e^*$ for all $v \in E^0$ and all finite $F \subset r^{-1}(v)$.

A Toeplitz–Cuntz–Krieger E-family $\{p_v : v \in E^0\}$, $\{s_e : s \in E^1\}$ is called a *Cuntz–Krieger E-family* if it satisfies

(CK) $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $0 < |r^{-1}(v)| < \infty$.

The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by a Cuntz–Krieger E-family.

To see where (T1)–(T3) come from, let E^* denote the path category of E. That is, E^* consists of all directed paths $\alpha = \alpha_1\alpha_2 \dots \alpha_m$ endowed with the partially defined associative multiplication given by concatenation. There is a natural notion of a "left-regular" representation λ of E^* on $\ell^2(E^*)$: for a path $\alpha \in E^*$, $\lambda(\alpha)$ is the operator on $\ell^2(E^*)$ such that

$$\lambda(\alpha)\xi_\beta = \begin{cases} \xi_{\alpha\beta} & \text{if } s(\alpha) = r(\beta) \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

It is not hard to verify that the elements $P_v := \lambda(v)$ and $S_e := \lambda(e)$ satisfy (T1)–(T3). Indeed, it turns out that the C^*-algebra generated by these P_v and S_e is universal for Toeplitz–Cuntz–Krieger E-families.

The final relation (CK) arises if we replace the space E^* of paths in E with its boundary $E^{\leq\infty}$ (this boundary consists of all the infinite paths in E together with those finite paths that originate at a vertex which receives no edges). A formula more or less identical to (1.1) defines a Cuntz–Krieger E-family $\{P_v^\infty : v \in E^0\}$, $\{S_e^\infty : e \in E^1\}$ in $\mathcal{B}(\ell^2(E^{\leq\infty}))$. The Cuntz–Krieger uniqueness theorem [2, Theorem 3.1] implies that when every loop in E has an entrance, the C^*-algebra generated by this Cuntz–Krieger family is universal for Cuntz–Krieger E-families. When E contains loops without entrances however, universality fails. For example, if E has just one vertex and one edge, then a Cuntz–Krieger E-family consists of a pair P, S where P is a projection and S satisfies $S^*S = P = SS^*$. Thus the universal C^*-algebra $C^*(E)$ is isomorphic to $C^*(\mathbb{Z}) = C(\mathbb{T})$. However, $E^{\leq\infty}$ consists of a single point, so $C^*(\{P_v^\infty, S_e^\infty\}) \cong \mathbb{C}$.

The definition of $C^*(E)$ is justified, when E contains loops with no entrance, by the gauge-invariant uniqueness theorem (originally due to an Huef and Raeburn; see [6, Theorem 2.3]), which says that $C^*(E)$ is the unique C^*-algebra generated by a Cuntz–Krieger E-family in which each p_v is nonzero and such that there is a *gauge action* γ of $\mathbb{T}$ on $C^*(E)$ satisfying $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$ for all $v \in E^0$, $e \in E^1$ and $z \in \mathbb{T}$.

Recently, Katsura developed a very natural description of this gauge-invariant uniqueness property in terms of what we call here a *co-universal* *property*. In the context of graph C^*-algebras, Proposition 7.14 of [7] says that $C^*(E)$ is co-universal for gauge-equivariant Toeplitz–Cuntz–Krieger E-families in which each vertex projection is nonzero. That is, $C^*(E)$ is the unique C^*-algebra such that:

- $C^*(E)$ is generated by a Toeplitz–Cuntz–Krieger E-family $\{p_v, s_e\}$ such that each p_v is nonzero, and $C^*(E)$ carries a gauge action; and
- for every Toeplitz–Cuntz–Krieger E-family $\{q_v, t_e\}$ such that each q_v is nonzero and such that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(\{q_v, t_e\})$ satisfying $\beta_z(q_v) = q_v$ and $\beta_z(t_e) = zt_e$ for all $v \in E^0$ and $e \in E^1$, there is a homomorphism $\psi_{q,t} : C^*(\{q_v, t_e\}) \to C^*(E)$ satisfying $\psi_{q,t}(q_v) = p_v$ and $\psi_{q,t}(t_e) = s_e$ for all $v \in E^0$ and $e \in E^1$.

The question which we address in this paper is whether there exists a co-universal C^*-algebra for (not necessarily gauge-equivariant) Toeplitz–Cuntz–Krieger E-families in which each vertex projection is nonzero. Our first main theorem, Theorem 3.1 shows that there does indeed exist such a C^*-algebra $C^*_{\min}(E)$, and identifies exactly when a homomorphism $B \to C^*_{\min}(E)$ obtained from the co-universal property of the latter is injective. The bulk of Section 3 is devoted to proving this theorem. Our key tool is Hong and Szymański's powerful description of the primitive ideal space of the C^*-algebra of a directed graph. We realise $\mathcal{T}C^*(E)$ as the universal C^*-algebra of a modified graph $\widetilde{E}$ to apply Hong and Szymański's results to the Toeplitz algebra.

Our second main theorem, Theorem 4.1 is a uniqueness theorem for the co-universal C^*-algebra. We then proceed in the remainder of Section 4 to demonstrate the power and utility of the defining co-universal property of $C^*_{\min}(E)$ and of our uniqueness theorem by obtaining the following as fairly straightforward corollaries:

- a characterisation of simplicity of $C^*_{\min}(E)$;
- a characterisation of injectivity of representations of $C^*_{\min}(E)$;
- a description of $C^*_{\min}(E)$ in terms of a universal property, and a uniqueness theorem of Cuntz–Krieger type;
- a realisation of $C^*_{\min}(E)$ as the Cuntz–Krieger algebra $C^*(F)$ of a modified graph F;
- an isomorphism of $C^*_{\min}(E)$ with the C^*-subalgebra of $\mathcal{B}(\ell^2(E^{\leq\infty}))$ generated by the Cuntz–Krieger E-family $\{P^{\leq\infty}_v, S^{\leq\infty}_e\}$ described earlier; and
- a faithful representation of $C^*_{\min}(E)$ on a Hilbert space $\mathcal{H}$ such that the canonical faithful conditional expectation of $\mathcal{B}(\mathcal{H})$ onto its diagonal subalgebra implements an expectation from $C^*_{\min}(E)$ onto the commutative C^*-subalgebra generated by the range projections $\{s^m_\alpha (s^m_\alpha)^* : \alpha \in E^*\}$.

Our results deal only with row-finite graphs to simplify the exposition. However, it seems likely that a similar analysis applies to arbitrary graphs.

Certainly Hong and Szymański's characterisation of the primitive ideal space of a graph C^*-algebra is available for arbitrary graphs. In principle one can argue along exactly the same lines as we do in Section 3 to obtain a co-universal C^*-algebra for an arbitrary directed graph. Alternatively, the results of this paper could be bootstrapped to the non-row-finite situation using Drinen and Tomforde's desingularisation process [3].

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2. Preliminaries

We use the conventions and notation for directed graphs established in [9]; in particular our edge-direction convention is consistent with [9] rather than with, for example, [1, 2, 5].

A *path* in a directed graph E is a concatenation $\lambda = \lambda_1\lambda_2 \dots \lambda_n$ of edges $\lambda_i \in E^1$ such that $s(\lambda_i) = r(\lambda_{i+1})$ for $i < n$; we write $r(\lambda)$ for $r(\lambda_1)$ and $s(\lambda)$ for $s(\lambda_n)$. We denote by E^* the collection of all paths in E. For $v \in E^0$ we write vE^1 for $\{e \in E^1 : r(e) = v\}$; similarly $E^1v = \{e \in E^1 : s(e) = v\}$.

A *cycle* in E is a path $\mu = \mu_1 \dots \mu_{|\mu|}$ such that $r(\mu) = s(\mu)$ and such that $s(\mu_i) \neq s(\mu_j)$ for $1 \leq i < j \leq |\mu|$. Given a cycle μ in E, we write $[\mu]$ for the set

$$[\mu] = \{\mu,\ \mu_2\mu_3\cdots\mu_{|\mu|}\mu_1,\ \dots,\ \mu_{|\mu|}\mu_1\cdots\mu_{n-1}\}$$

of cyclic permutations of μ. We write $[\mu]^0$ for the set $\{s(\mu_i) : 1 \leq i \leq |\mu|\} \subset E^0$, and $[\mu]^1$ for the set $\{\mu_i : 1 \leq i \leq |\mu|\} \subset E^1$. Given a cycle μ in E and a subset M of E^0 containing $[\mu]^0$, we say that μ has *no entrance in* M if $r(e) = r(\mu_i)$ and $s(e) \in M$ implies $e = \mu_i$ for all $1 \leq i \leq |\mu|$. We denote by $C(E)$ the set $\{[\mu] : \mu$ is a cycle with no entrance in $E^0\}$. By $C(E)^1$ we mean $\bigcup_{C\in C(E)} C^1$, and by $C(E)^0$ we mean $\bigcup_{C\in C(E)} C^0$.

A *cutting set* for a directed graph E is a subset X of $C(E)^1$ such that for each $C \in C(E)$, $X \cap C^1$ is a singleton. Given a cutting set X for E, for each $x \in X$, we write $\mu(x)$ for the unique cycle in E such that $r(\mu) = r(x)$, and let $\lambda(x) = \mu(x)_2\mu(x)_3 \dots \mu(x)_{|\mu(x)|}$; so $\mu(x) = x\lambda(x)$ for all $x \in X$, and $C(E) = \{[\mu(x)] : x \in X\}$.

3. Existence of the co-universal C^*-algebra

Our main theorem asserts that every row-finite directed graph admits a co-universal C^*-algebra and identifies when a homomorphism obtained from the co-universal property is injective.

Theorem 3.1. *Let E be a row-finite directed graph.*

(1) *There exists a C^*-algebra $C^*_{\min}(E)$ which is co-universal for Toeplitz–Cuntz–Krieger E-families of nonzero partial isometries in the sense that $C^*_{\min}(E)$ is generated by a Toeplitz–Cuntz–Krieger E-family*

$$\{P_v : v \in E^0\}, \{S_e : e \in E^1\}$$

with the following two properties.

(a) *The vertex projections $\{P_v : v \in E^0\}$ are all nonzero.*

(b) *Given any Toeplitz–Cuntz–Krieger E-family*

$$\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$$

such that each $q_v \neq 0$ and given any cutting set X for E, there is a function $\kappa : X \to \mathbb{T}$ and a homomorphism

$$\psi_{q,t} : C^*(\{q_v, t_e : v \in E^0, e \in E^1\}) \to C^*_{\min}(E)$$

satisfying $\psi_{q,t}(q_v) = P_v$ for all $v \in E^0$, $\psi_{q,t}(t_e) = S_e$ for all $e \in E^1 \setminus X$, and $\psi_{q,t}(t_x) = \kappa(x)S_x$ for all $x \in X$.

(2) *Given a Toeplitz–Cuntz–Krieger E-family*

$$\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$$

with each q_v nonzero, the homomorphism $\psi_{q,t} : B \to C^*_{\min}(E)$ obtained from (1b) *is an isomorphism if and only if for each cycle μ with no entrance in E, the partial isometry t_μ is a scalar multiple of $q_{r(\mu)}$.*

Remarks 3.2. It is convenient in practise to work with cutting sets X and functions from X to $\mathbb{T}$ as in Theorem 3.1(1b). However, property (1b) can also be reformulated without respect to cutting sets. Indeed:

(1) The asymmetry arising from the choice of a cutting set X in Theorem 3.1(1b) can be avoided. The property could be reformulated equivalently as follows: *given a Toeplitz–Cuntz–Krieger E-family $\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$ such that each $q_v \neq 0$, there is a function $\rho : C(E)^1 \to \mathbb{T}$ and a homomorphism*

$$\psi_{q,t} : C^*(\{q_v, t_e : v \in E^0, e \in E^1\}) \to C^*_{\min}(E)$$

satisfying $\psi_{q,t}(q_v) = P_v$ for all $v \in E^0$, $\psi_{q,t}(t_e) = S_e$ for all $e \in E^1 \setminus C(E)^1$, and $\psi_{q,t}(t_e) = \rho(e)S_e$ for all $e \in C(E)^1$. One can prove that an algebra satisfying this modified condition (1b) exists using exactly the same argument as for the current theorem after making the appropriate modification to Lemma 3.7. That the resulting algebra coincides with $C^*_{\min}(E)$ follows from applications of the co-universal properties of the two algebras.

(2) Fix a row-finite graph E with no sources and a function $\kappa : C(E) \to \mathbb{T}$. Let $\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$ be a Toeplitz–Cuntz–Krieger E-family such that each $q_v \neq 0$. Then there is a homomorphism as in Theorem 3.1(1b) with respect to the fixed function κ for some cutting set X if and only if there is such a homomorphism for every cutting

set X. One can see this by following the argument of Lemma 3.11 below to see that κ does not depend on X.

(3) Given a Toeplitz–Cuntz–Krieger E-family $\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$ such that each $q_v \neq 0$ and a cutting set X, the functions $\kappa : X \to \mathbb{T}$ which can arise in Theorem 3.1(1b) are precisely those for which $\kappa([\mu])$ belongs to the spectrum $\mathrm{sp}_{q_v C^*(\{q_v, t_e : v \in E^0, e \in E^1\}) q_v}(t_\mu)$ for each cycle μ without an entrance in E. To see this, one uses Hong and Szymański's theorems to show that in the first paragraph of the proof of Lemma 3.11, the complex numbers z which can arise are precisely the elements of the spectrum of the unitary $s_{\alpha(\mu)} + I$ in the corner $(p_{\alpha(r(\mu))} + I)\big(C^*(\widetilde{E})/I\big)(p_{\alpha(r(\mu))} + I)$.

Corollary 3.3. *Let E be a row-finite directed graph in which every cycle has an entrance. Then $C^*(E) \cong C^*_{\min}(E)$. In particular, if $\{q_v : v \in E^0\}$, $\{t_e : e \in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family in a C^*-algebra B such that each q_v is nonzero, then there is a homomorphism*

$$\psi_{q,t} : C^*(\{q_v, t_e : v \in E^0, e \in E^1\}) \to C^*(E)$$

such that $\psi_{q,t}(q_v) = p_v$ for all $v \in E^0$ and $\psi_{q,t}(t_e) = s_e$ for all $e \in E^1$.

Proof. For the first statement, observe that the co-universal property of $C^*_{\min}(E)$ induces a surjective homomorphism $\psi_{p,s} : C^*(E) \to C^*_{\min}(E)$. Since every cycle in E has an entrance, the condition in Theorem 3.1(2) is trivially satisfied, so $\psi_{p,s}$ is an isomorphism.

Since every cycle in E has an entrance, a cutting set for E has no elements. Hence the second statement is just a re-statement of Theorem 3.1(1b). $\square$

The remainder of this section will be devoted to proving Theorem 3.1. Our key technical tool in proving Theorem 3.1 will be Hong and Szymański's description of the primitive ideal space of a graph C^*-algebra. To do this, we first realise the Toeplitz algebra $\mathcal{T}C^*(E)$ as a graph algebra in its own right. This construction is known, but we have found it difficult to pin down in the literature. The idea is to augment E with an additional copy of each vertex which receives a nonzero finite number of edges.

Notation 3.4. Let E be a directed graph. Define a directed graph $\widetilde{E}$ as follows (in the equations below, the symbols $\alpha(v)$, $\alpha(e)$, $\beta(v)$ and $\beta(e)$ are formal symbols used to indicate new copies of vertices v and edges e of E):

$$\widetilde{E}^0 = \{\alpha(v) : v \in E^0\} \sqcup \{\beta(v) : v \in E^0, 0 < |vE^1| < \infty\}$$

$$\widetilde{E}^1 = \{\alpha(e) : e \in E^1\} \sqcup \{\beta(e) : e \in E^1, 0 < |s(e)E^1| < \infty\}$$

$$r(\alpha(e)) = \alpha(r(e)) \text{ and } s(\alpha(e)) = \alpha(s(e)) \text{ for all } e \in E^1, \text{ and}$$

$$r(\beta(e)) = \alpha(r(e)), \text{ and } s(\beta(e)) = \beta(s(e)) \text{ whenever } 0 < |s(e)E^1| < \infty.$$

For $\lambda \in E^*$ with $|\lambda| \geq 2$, we define $\alpha(\lambda) := \alpha(\lambda_1) \dots \alpha(\lambda_{|\lambda|})$. If E is row-finite, $\widetilde{E}$ is also row-finite.

Lemma 3.5. *Let E be a directed graph and let $\widetilde{E}$ be as in Notation* 3.4. *For $v \in E^0$ and $e \in E^1$, let*

$$q_v := \begin{cases} p_{\alpha(v)} + p_{\beta(v)} & \text{if } 0 < |vE^1| < \infty \\ p_{\alpha(v)} & \text{otherwise,} \end{cases}$$

and

$$t_e := \begin{cases} s_{\alpha(e)} + s_{\beta(e)} & \text{if } 0 < |s(e)E^1| < \infty \\ s_{\alpha(e)} & \text{otherwise.} \end{cases}$$

Then there is an isomorphism $\phi : \mathcal{T}C^*(E) \to C^*(\widetilde{E})$ satisfying $\phi(p_v^{\mathcal{T}}) = q_v$ and $\phi(s_e^{\mathcal{T}}) = t_e$ for all $v \in E^0$ and $e \in E^1$.

Proof. Routine calculations show that $\{q_v : v \in E^0\}$, $\{t_e : e \in E^1\}$ is a Toeplitz–Cuntz–Krieger E-family in $C^*(\widetilde{E})$. The universal property of $\mathcal{T}C^*(E)$ therefore implies that there is a homomorphism $\phi : \mathcal{T}C^*(E) \to C^*(\widetilde{E})$ satisfying $\phi(p_v^{\mathcal{T}}) = q_v$ for all $v \in E^0$ and $\phi(s_e^{\mathcal{T}}) = t_e$ for all $e \in E^1$.

To see that ϕ is surjective, fix $v \in \widetilde{E}^0$. To see that $p_v \in \operatorname{range}(\phi)$, we consider three cases: (a) $v = \alpha(w)$ for some w with wE^1 either empty or infinite; (b) $v = \alpha(w)$ for some w with $0 < |wE^1| < \infty$; or (c) $v = \beta(w)$ for some w with $0 < |wE^1| < \infty$. In case (a), we have $p_v = p_{\alpha(w)} = \phi(p_w^{\mathcal{T}})$ by definition. In case (b), the set $v\widetilde{E}^1 = \{\alpha(e), \beta(e) : e \in wE^1\}$ is nonempty and finite. Hence the Cuntz–Krieger relation in $C^*(\widetilde{E})$ ensures that

$$p_v = p_{\alpha(w)} = \sum_{e \in vE^1} s_{\alpha(e)} s_{\alpha(e)}^* + s_{\beta(e)} s_{\beta(e)}^* = \sum_{e \in vE^1} t_e t_e^* \in \operatorname{range}(\phi). \tag{3.1}$$

In case (c), we have $p_v = q_v - p_{\alpha(w)} \in \operatorname{range}(\phi)$ by case (b). Now fix $e \in \widetilde{E}^1$. To see that $s_e \in \operatorname{range}(\phi)$, observe that if $e = \alpha(f)$ for some $f \in E^1$, then $s_e = s_{\alpha(f)} = \phi(s_f^{\mathcal{T}}) p_{\alpha(s(f))} \in \operatorname{range}(\phi)$, and if $e = \beta(f)$, then $s_e = s_{\beta(f)} = \phi(s_f^{\mathcal{T}}) p_{\beta(s(f))} \in \operatorname{range}(\phi)$ also.

To finish the proof, observe that if $0 < |vE^1| < \infty$, then $q_v - \sum_{r(e)=v} t_e t_e^* = p_{\beta(v)} \neq 0$. Since the t_e are all nonzero and have mutually orthogonal ranges, it follows that for each $v \in E^0$ and each finite subset F of vE^1, we have $q_v - \sum_{e \in F} t_e t_e^* \neq 0$. Thus the uniqueness theorem [4, Theorem 4.1] for $\mathcal{T}C^*(E)$ implies that ϕ is injective. □

Notation 3.6. Let E be a directed graph.

(1) For $v \in E^0$ such that $0 < |vE^1| < \infty$, we define

$$\Delta_v := p_v^{\mathcal{T}} - \sum_{r(e)=v} s_e^{\mathcal{T}} (s_e^{\mathcal{T}})^* \in \mathcal{T}C^*(E).$$

(2) Given a function $\kappa : C(E) \to \mathbb{T}$, we denote by I^κ the ideal of $\mathcal{T}C^*(E)$ generated by $\{\Delta_v : v \in E^0\} \cup \{\kappa(C) p_{r(\mu)}^{\mathcal{T}} - s_{\alpha(\mu)}^{\mathcal{T}} : C \in C(E), \mu \in C\}$.

Lemma 3.7. *Let E be a directed graph. Let κ be a function from $C(E) \to \mathbb{T}$, and let $1 : C(E) \to \mathbb{T}$ denote the constant function $1(C) = 1$ for all $C \in C(E)$. Fix a cutting set X for E, and for each $x \in X$, let $C(x)$ be the unique element of $C(E)$ such that $x \in C(x)^1$. Then there is an isomorphism $\widetilde{\tau_{\overline{\kappa}}} : \mathcal{T}C^*(E)/I^1 \to \mathcal{T}C^*(E)/I^\kappa$ satisfying*

$$\begin{aligned} \widetilde{\tau_{\overline{\kappa}}}(p_v^{\mathcal{T}} + I^1) &= p_v + I^\kappa && \text{for all } v \in E^0 \\ \widetilde{\tau_{\overline{\kappa}}}(s_e^{\mathcal{T}} + I^1) &= s_e + I^\kappa && \text{for all } e \in E^1 \setminus X, \text{ and} \\ \widetilde{\tau_{\overline{\kappa}}}(s_x^{\mathcal{T}} + I^1) &= \overline{\kappa(C(x))} s_x + I^\kappa && \text{for all } x \in X. \end{aligned}$$

Proof. By the universal property of $\mathcal{T}C^*(E)$, there is an action τ of $\mathbb{T}^{C(E)}$ on $\mathcal{T}C^*(E)$ such that for $\rho \in \mathbb{T}^{C(E)}$, we have

$$\begin{aligned} \tau_\rho(p_v^{\mathcal{T}}) &= p_v^{\mathcal{T}} && \text{for all } v \in E^0 \\ \tau_\rho(s_e^{\mathcal{T}}) &= s_e^{\mathcal{T}} && \text{for all } e \in E^1 \setminus X, \text{ and} \\ \tau_\rho(s_x^{\mathcal{T}}) &= \rho(C(x)) s_x^{\mathcal{T}} && \text{for all } x \in X. \end{aligned}$$

By definition of I^1 and I^κ and of the action τ, we have $\tau_{\overline{\kappa}}(I^1) = I^\kappa$. Hence $\tau_{\overline{\kappa}}$ determines an isomorphism

$$\widetilde{\tau_{\overline{\kappa}}} : \mathcal{T}C^*(E)/I^1 \to \tau_{\overline{\kappa}}(\mathcal{T}C^*(E))/I^\kappa = \mathcal{T}C^*(E)/I^\kappa$$

satisfying $\widetilde{\tau_{\overline{\kappa}}}(a + I^1) = \tau_{\overline{\kappa}}(a) + I^\kappa$ for all $a \in \mathcal{T}C^*(E)$. □

Lemma 3.8. *Let E be a directed graph. Fix a function $\kappa : C(E) \to \mathbb{T}$. Then $s_v^{\mathcal{T}} \notin I^\kappa$ for all $v \in E^0$.*

To prove this lemma, we need a little notation.

Notation 3.9. Given a directed graph E, we denote by $E^{\leq\infty}$ the collection $E^\infty \cup \{\alpha \in E^* : s(\alpha)E^1 = \emptyset\}$. There is a Cuntz–Krieger E-family in $\mathcal{B}(\ell^2(E^{\leq\infty}))$ determined by

$$P_v^\infty \xi_x = \begin{cases} \xi_x & \text{if } r(x) = v \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_e^\infty \xi_x = \begin{cases} \xi_{ex} & \text{if } r(x) = s(e) \\ 0 & \text{otherwise.} \end{cases}$$

If μ is a cycle with no entrance in E, then $r(\mu)E^{\leq\infty} = \{\mu^\infty\}$, so $S_\mu^\infty = P_{r(\mu)}^\infty$.

Proof of Lemma 3.8. By Lemma 3.7, it suffices to show that I^1 contains no vertex projections. Let $\pi_{P^\infty,S^\infty} : \mathcal{T}C^*(E) \to \mathcal{B}(\ell^2(E^{\leq\infty}))$ be the representation obtained from the universal property of $\mathcal{T}C^*(E)$ applied to the Cuntz–Krieger family of Notation 3.9. Then $\ker(\pi_{P^\infty,S^\infty})$ contains all the generators of I^1, so $I^1 \subset \ker(\pi_{P^\infty,S^\infty})$. Moreover, $\ker(\pi_{P^\infty,S^\infty})$ contains no vertex projections because each vertex of E is the range of at least one $x \in E^{\leq\infty}$. $\square$

From this point onward we make the simplifying assumption that our graphs are row-finite. Though there is no obvious obstruction to our analysis without this restriction, the added generality would complicate the details of our arguments. In any case, if the added generality should prove useful, it should not be difficult to bootstrap our results to the non-row-finite setting by means of the Drinen–Tomforde desingularisation process applied to the graph $\widetilde{E}$ of Notation 3.4.

Proposition 3.10. *Let I be an ideal of $\mathcal{T}C^*(E)$ such that $p_v^{\mathcal{T}} \notin I$ for all $v \in E^0$. There is a function $\kappa : C(E) \to \mathbb{T}$ such that $I \subset I^\kappa$.*

To prove the proposition, we first establish our key technical lemma. This lemma is implicit in Hong and Szymański's description [5] of the primitive ideal space of $C^*(\widetilde{E})$, but it takes a little work to tease a proof of the statement out of their two main theorems.

Lemma 3.11. *Let $\widetilde{E}$ be the directed graph of Notation* 3.4*. Let I be an ideal of $C^*(\widetilde{E})$ such that $p_{\alpha(v)} \notin I$ for all $v \in E^0$. There is a function $\kappa : C(E) \to \mathbb{T}$ such that I is contained in the ideal J^κ of $C^*(\widetilde{E})$ generated by $\{p_{\beta(v)} : v \in E^0\}$ and $\{\kappa(C)p_{\alpha(r(\mu))} - s_{\alpha(\mu)} : C \in C(E), \mu \in C\}$.*

Before proving the lemma, we summarise some notation and results of [5] as they apply to the row-finite directed graph $\widetilde{E}$ in the situation of Lemma 3.11. A *maximal tail* of $\widetilde{E}$ is a subset M of $\widetilde{E}^0$ such that:

(MT1) $w \in M$ and $v\widetilde{E}^*w \neq \emptyset$ imply $v \in M$;

(MT2) if $v \in M$ and $v\widetilde{E}^1 \neq \emptyset$, then there exists $e \in v\widetilde{E}^1$ such that $s(e) \in M$; and

(MT3) if $u, v \in M$, then there exists $w \in M$ such that $u\widetilde{E}^*w \neq \emptyset$ and $v\widetilde{E}^*w \neq \emptyset$.

We denote by $\mathcal{M}_\gamma(\widetilde{E})$ the collection of maximal tails M of $\widetilde{E}$ such that every cycle μ satisfying $[\mu]^0 \subset M$ has an entrance in M. We denote by $\mathcal{M}_\tau(\widetilde{E})$ the collection of maximal tails M of $\widetilde{E}$ such that there is a cycle μ in $\widetilde{E}$ for which $[\mu]^0 \subset M$ but μ has no entrance in M. Since each $\beta(v)$ is a source in $\widetilde{E}$, the cycles in $\widetilde{E}$ are the paths of the form $\alpha(\mu)$ where μ is a cycle in E. Moreover $\alpha(\mu) \in C(\widetilde{E})$ if and only if $\mu \in C(E)$. Thus if $M \in M_\tau(\widetilde{E})$, then there is a unique $C_M \in C(E)$ such that $\alpha(C_M^0) \subset M$ and $\alpha(\mu)$ has no entrance in M for each $\mu \in C_M$. We may recover M from C_M:

$$M = \{\alpha(v) : v \in E^0, vE^*C_M^0 \neq \emptyset\}.$$

The gauge-invariant primitive ideals of $C^*(\widetilde{E})$ are indexed by $\mathcal{M}_\gamma(\widetilde{E})$; specifically, $M \in \mathcal{M}_\gamma(\widetilde{E})$ corresponds to the primitive ideal PI_M^γ generated by $\{p_w : w \in \widetilde{E}^0 \setminus M\}$. The non-gauge-invariant primitive ideals of $C^*(\widetilde{E})$

are indexed by $\mathcal{M}_\tau(\widetilde{E}) \times \mathbb{T}$; specifically, the pair (M, z) corresponds to the primitive ideal $\mathrm{PI}^\tau_{M,z}$ generated by $\{p_w : w \in E^0 \setminus M\} \cup \{zp_{r(\mu)} - s_\mu\}$ for any $\mu \in C_M$ (the ideal does not depend on the choice of $\mu \in C_M$). Corollary 3.5 of [5] describes the closed subsets of the primitive ideal space of $C^*(\widetilde{E})$ in terms of subsets of $\mathcal{M}_\gamma(\widetilde{E}) \sqcup \mathcal{M}_\tau(\widetilde{E}) \times \mathbb{T}$.

Proof of Lemma 3.11. We begin by constructing the function κ. Fix $C \in C(E)$. Since $p_{\alpha(v)} \notin I$ for each $v \in C^0$, we may fix a primitive ideal J^C of $C^*(\widetilde{E})$ such that $I \subset J^C$ and $p_{\alpha(v)} \notin J^C$ for $v \in C^0$. By [5, Corollary 2.12], we have either $J^C = \mathrm{PI}^\gamma_M$ for some $M \in \mathcal{M}_\gamma(\widetilde{E})$, or $J^C = \mathrm{PI}^\tau_{M,z}$ for some $M \in \mathcal{M}_\tau(\widetilde{E})$ and $z \in \mathbb{T}$. Since $p_{\alpha(v)} \notin J^C$ for $v \in C^0$, we must have $\alpha(C^0) \subset M$, and then the maximal tail condition forces

$$M = M_C := \{\alpha(v) : v \in E^0, vE^*C^0 \neq \emptyset\} \in M_\tau(\widetilde{E}).$$

Hence $J^C = \mathrm{PI}^\tau_{M,z}$ for some $z \in \mathbb{T}$; we set $\kappa(C) := z$.

For $C \in C(E)$ and $v \in C^0$, let $J^v := \mathrm{PI}^\tau_{M_C,\kappa(C)}$. Since $\beta(v) \notin M_C$ for all $v \in E^0$, [5, Lemma 2.8] implies that $p_{\beta(v)} \in J^v$ for all $v \in E^0$, and our definition of J^v ensures that $p_{\alpha(v)} \notin J^v$.

We claim that the assignment $v \mapsto J^v$ of the preceding paragraph extends to a function $v \mapsto J^v$ from E^0 to the primitive ideal space of $C^*(\widetilde{E})$ such that for every $v \in E^0$:

(1) $I \subset J^v$;
(2) $p_{\alpha(v)} \notin J^v$;
(3) $p_{\beta(w)} \in J^v$ for all $w \in E^0$; and
(4) either $J^v = \mathrm{PI}^\tau_{M_C,\kappa(C)}$ for some $C \in C(E)$, or J^v is gauge-invariant.

To prove the claim, fix $v \in E^0$. If $vE^*C^0 \neq \emptyset$ for some $C \in C(E)$, then $J^v := \mathrm{PI}^\tau_{M_C,\kappa(C)}$ has the desired properties. So suppose that $vE^*C^0 = \emptyset$ for all $C \in C(E)$. Then there exists $x^v \in vE^{\leq\infty}$ such that x^v does not have the form $\lambda\mu^\infty$ for any $\lambda \in E^*$ and cycle $\mu \in E$. The set

$$M^v := \{\alpha(w) : w \in E^0, wE^*x^v(n) \neq \emptyset \text{ for some } n \in \mathbb{N}\}$$

is a maximal tail of $\widetilde{E}$ which contains $\alpha(v)$ and does not contain $\beta(w)$ for any $w \in E^0$. By construction of x^v, every cycle in M^v has an entrance in M^v, and so $J^v := \mathrm{PI}^\gamma_{M^v}$ satisfies (2)–(4). It therefore suffices to show that $I \subset \mathrm{PI}^\gamma_{M^v}$. For each $n \in \mathbb{N}$, we have $p_{\alpha(x^v(n))} \notin I$, so there is a primitive ideal J of $C^*(\widetilde{E})$ containing I and not containing $p_{\alpha(x^v(n))}$. The set $M_n = \{w \in \widetilde{E}^0 : p_w \notin J\}$ is a maximal tail of $\widetilde{E}$, so either $M_n \in \mathcal{M}_\gamma(\widetilde{E})$ and $J = \mathrm{PI}^\gamma_{M_n}$, or $M_n \in \mathcal{M}_\tau(\widetilde{E})$ and $J = \mathrm{PI}^\tau_{M_n,z}$ for some $z \in \mathbb{T}$. By definition, $x^v(n) \in M_n$, and then (MT1) forces $\alpha(w) \in M_n$ for all $w \in E^0$ such that $wE^*x^v(m) \neq \emptyset$ for some $m \leq n$. Hence $M^v \subset \cup_{n\in\mathbb{N}} M_n$. Hence parts (1) and (3) of [5, Corollary 3.5] imply that J^v belongs to the closure of the set of primitive ideals of $C^*(\widetilde{E})$ which contain I, and hence contains I itself. This proves the claim.

Observe that $I \subset \bigcap_{v\in E^0} J^v$. To prove the result, it therefore suffices to show that $\bigcap_{v\in E^0} J^v$ is generated by

$$\{p_{\beta(v)} : v \in E^0\} \quad \text{and} \quad \{\kappa(C)p_{\alpha(r(\mu))} - s_{\alpha(\mu)} : C \in C(E), \mu \in C\}.$$

For this, first observe that for $v \in E^0$, we have $p_{\beta(v)} \in \bigcap_{v\in E^0} J^v$ because $p_{\beta(v)}$ belongs to each J^v. Fix $C \in C(E)$ and $\mu \in C$. Since the cycle without an entrance belonging to a given maximal tail of E is unique, for each $v \in E^0$, we have either $J^v = \mathrm{PI}^\tau_{M_C,\kappa(C)}$, or $p_{r(\mu)} \in J^v$. In particular, $\kappa(C)p_{\alpha(r(\mu))} - s_{\alpha(\mu)} \in J^v$ for each $v \in E^0$, so $\kappa(C)p_{\alpha(r(\mu))} - s_{\alpha(\mu)} \in \bigcap_{v\in E^0} J^v$. Hence $J^\kappa \subset \bigcap_{v\in E^0} J^v$, and it remains to establish the reverse inclusion.

Fix a primitive ideal J of $C^*(\widetilde{E})$ which contains all the generators of J^κ. It suffices to show that $\bigcap_{v\in E^0} J^v \subset J$. Under the bijection between primitive ideals of $C^*(\widetilde{E})$ and elements of $\mathcal{M}_\gamma(\widetilde{E}) \sqcup \mathcal{M}_\tau(\widetilde{E}) \times \mathbb{T}$, the collection $\{J_v : v \in E^0\}$ is sent to

$$\{M^v : vE^*C^0 = \emptyset \text{ for all } C \in C(E)\} \sqcup \{(M_C, \kappa(C)) : C \in C(E)\}.$$

Since each J^v trivially contains $\bigcap_{v\in E^0} J^v$, it therefore suffices to show that the element N_J of $\mathcal{M}_\gamma(\widetilde{E}) \sqcup \mathcal{M}_\tau(\widetilde{E}) \times \mathbb{T}$ corresponding to J satisfies

$$N_J \in \overline{\{M^v : vE^*C^0 = \emptyset \text{ for all } C \in C(E)\}} \sqcup \overline{\{(M_C, \kappa(C)) : C \in C(E)\}}. \tag{3.2}$$

Let $M_J := \{v \in \widetilde{E}^0 : p_v \notin J\}$. Then either J is gauge-invariant and $N_J = M_J$, or J is not gauge-invariant, and $N_J = (M_J, z)$ for some $z \in \mathbb{T}$. Observe that

$$M_J \subset \{\alpha(v) : v \in E^0\} = \Big(\bigcup \{M^v : vE^*C^0 = \emptyset \text{ for all } C \in C(E)\}\Big) \cup \Big(\bigcup \{(M_C, \kappa(C)) : C \in C(E)\}\Big). \tag{3.3}$$

We now consider three cases.

Case 1: J is gauge-invariant. Then $M_J \in M_\gamma(\widetilde{E})$, and $N_J = M_J$. In this case, (3.3) together with parts (1) and (3) of [5, Corollary 3.5] give (3.2).

Case 2: J is not gauge-invariant, and $M_J \neq M_C$ for all $C \in C(E)$. We have $N_J = (M_J, z)$ for some $z \in \mathbb{T}$, and since $M_J \neq M_C$ for all $C \in C(E)$, it follows that N_J does not belong to the subset

$$\{(M_C, \kappa(C)) : C \in C(E)\}_{\min} \subset \{(M_C, \kappa(C)) : C \in C(E)\}$$

defined on page 58 of [5]. Hence parts (2) and (4ii) of [5, Corollary 3.5] imply (3.2).

Case 3: $M_J = M_C$ for some $C \in C(E)$. Fix $\mu \in C$. Since J contains $\kappa(C)p_{\alpha(r(\mu))} - s_{\alpha(\mu)}$, we have $N_J = (M_C, \kappa(C))$ and then part (4iii) of [5, Corollary 3.5] implies (3.2). This completes the proof. □

Proof of Proposition 3.10. Let $\phi : \mathcal{T}C^*(E) \to C^*(\widetilde{E})$ be the isomorphism of Lemma 3.5. Observe that by (3.1), we have $\phi(\Delta_v) = p_{\beta(v)}$ for all $v \in E^0$ such that $vE^1 \neq \emptyset$. We claim that $p_{\alpha(v)} \notin \phi(I)$ for all $v \in E^0$. To see this, first suppose that $vE^1 = \emptyset$. Then $p_{\alpha(v)} = \phi(p_v^{\mathcal{T}}) \notin \phi(I)$ by assumption. Now suppose that $vE^1 \neq \emptyset$, say $r(e) = v$. Then

$$s_{\alpha(e)}^* p_{\alpha(v)} s_{\alpha(e)} + s_{\beta(e)}^* p_{\alpha(v)} s_{\beta(e)} = p_{\alpha(s(e))} + p_{\beta(s(e))} = \phi(p_{s(e)}^{\mathcal{T}}) \notin \phi(I)$$

by assumption. This forces $p_{\alpha(v)} \notin \phi(I)$.

Lemma 3.11 therefore applies to the ideal $\phi(I)$ of $C^*(\widetilde{E})$. Let $\kappa : C(E) \to \mathbb{T}$ and $J^\kappa \lhd C^*(\widetilde{E})$ be the resulting function and ideal. Then $I^\kappa := \phi^{-1}(J^\kappa)$ is generated by $\{\Delta_v : v \in E^0\}$ and $\{\kappa(C)p_{r(\mu)}^{\mathcal{T}} - s_{\alpha(\mu)}^{\mathcal{T}} : C \in C(E), \mu \in C\}$ by definition of ϕ, and contains I by construction. $\square$

We are now ready to prove our main theorem.

Proof of Theorem 3.1. With $I^1 \lhd \mathcal{T}C^*(E)$ defined as in Notation 3.6 and Lemma 3.7, define $C^*_{\min}(E) := \mathcal{T}C^*(E)/I^1$. For $v \in E^0$ and $e \in E^1$, let $P_v := p_v^{\mathcal{T}} + I^1$ and $S_e := s_v^{\mathcal{T}} + I^1$. Then $\{P_v : v \in E^0\}$, $\{S_e : e \in E^1\}$ is a Cuntz–Krieger E-family which generates $C^*_{\min}(E)$. The P_v are all nonzero by Lemma 3.8.

Now let $\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$ be a Toeplitz–Cuntz–Krieger E-family such that $q_v \neq 0$ for all v, and let $B := C^*(\{q_v, t_e : v \in E^0, e \in E^1\})$. The universal property of $\mathcal{T}C^*(E)$ implies that there is a homomorphism $\pi_{q,t} : \mathcal{T}C^*(E) \to B$ satisfying $\pi_{q,t}(p_v^{\mathcal{T}}) = q_v$ for all $v \in E^0$ and $\pi_{q,t}(s_e^{\mathcal{T}}) = t_e$ for all $e \in E^1$. Since each q_v is nonzero, $I = \ker(\pi_{q,t})$ is an ideal of $\mathcal{T}C^*(E)$ such that $p_v^{\mathcal{T}} \notin I$ for all $v \in E^0$. Let $\kappa : C(E) \to \mathbb{T}$ and I^κ be as in Corollary 3.10. Since $I \subset I^\kappa$, there is a well-defined homomorphism $\psi_0 : B \to \mathcal{T}C^*(E)/I^\kappa$ satisfying $\psi_0(q_v) = p_v^{\mathcal{T}} + I^\kappa$ for all $v \in E^0$ and $\psi_0(t_e) = s_e^{\mathcal{T}} + I^\kappa$ for all $e \in E^1$. Let $\widetilde{\tau_{\overline{\kappa}}} : \mathcal{T}C^*(E)/I^1 \to \mathcal{T}C^*(E)/I^\kappa$ be as in Lemma 3.7. Then $\psi_{q,t} := \widetilde{\tau_{\overline{\kappa}}}^{-1} \circ \psi_0$ has the desired property. This proves statement (1).

For statement (2), suppose first that $\psi_{q,t}$ is injective. For each $\mu \in C(E)$, we have $S_\mu = P_{r(\mu)}$ by definition of I^1. Let $x(\mu)$ be the unique element of the cutting set X which belongs to $[\mu]^1$. With $\psi_{q,t}$ and κ as in (1), we have $\psi_{q,t}(t_\mu) = \kappa(x(\mu))\psi_{q,t}(q_{r(\mu)})$. Since $\psi_{q,t}$ is injective, we must have $t_\mu = \kappa(x(\mu))q_{r(\mu)}$ for all $\mu \in C(E)$. Now suppose that there is a function $\kappa : C(E) \to \mathbb{T}$ such that $t_\mu = \kappa([\mu])q_{r(\mu)}$ for every cycle μ with no entrance in E. Then the kernel $I_{q,t}$ of the canonical homomorphism $\pi_{q,t} : \mathcal{T}C^*(E) \to B$ contains the generators of I^κ, and hence contains I^κ. Since the q_v are all nonzero, Corollary 3.10 implies that we also have $I_{q,t} \subset I^\lambda$ for some $\lambda : C(E) \to \mathbb{T}$. We claim that $\kappa = \lambda$; for if not, then there exists $C \in C(E)$ such that $k := \kappa(C)$ is distinct from $l = \lambda(C)$. For $\mu \in C$, we then have $kp_v^{\mathcal{T}} - s_\mu^{\mathcal{T}}, lp_v^{\mathcal{T}} - s_\mu^{\mathcal{T}} \in I^\lambda$. But then $(k-l)p_v^{\mathcal{T}} \in I^\lambda$, which is impossible by Lemma 3.8. Hence $I^\lambda = I_{q,t} = I^\kappa$, and $\psi_{q,t}$ is injective. $\square$

4. Properties of the co-universal C^*-algebra

In this section we prove a uniqueness theorem for $C^*_{\min}(E)$ in terms of its co-universal property. We go on to explore the structure and properties of the co-universal algebra. Throughout this section we have preferred proofs which emphasise the utility of the co-universal property over other techniques.

Let E be a row-finite directed graph. We say that a Cuntz–Krieger E-family $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ is a *reduced Cuntz–Krieger E-family* if

(R) for every cycle μ without an entrance in E^0, there is a scalar $\kappa(\mu) \in \mathbb{T}$ such that $s_\mu = \kappa(\mu) p_{r(\mu)}$.

We say that $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ is a *normalised reduced Cuntz–Krieger E-family* if $s_\mu = p_{r(\mu)}$ for each cycle μ without an entrance in E^0.

Theorem 4.1. *Let E be a row-finite directed graph.*

(1) *There is a normalised reduced Cuntz–Krieger E-family*
$$\{p^m_v : v \in E^0\} \cup \{s^m_e : E \in E^1\}$$
that generates $C^*_{\min}(E)$ and satisfies Theorem 3.1(1a) and (1b). *In particular, given any cutting set X for E, $C^*_{\min}(E)$ is generated by $\{p^m_v : v \in E^0\} \cup \{s^m_e : E \in E^1 \setminus X\}$.*

(2) *Any other C^*-algebra generated by a Toeplitz–Cuntz–Krieger E-family satisfying Theorem* 3.1(1a) and (1b) *is isomorphic to $C^*_{\min}(E)$.*

Proof. For (1) let $\{P^\infty_v : v \in E^0\}$, $\{S^\infty_e : e \in E^1\}$ be the Cuntz–Krieger E-family of Notation 3.9. Theorem 3.1(1b) ensures that there is a function $\kappa : X \to \mathbb{T}$ and a homomorphism ψ_{P^∞,S^∞} from $C^*(\{P^\infty_v, S^\infty_e : v \in E^0, e \in E^1\})$ onto $C^*_{\min}(E)$ such that $\psi_{P^\infty,S^\infty}(P^\infty_v) = P_v$ for all $v \in E^0$, $\psi_{P^\infty,S^\infty}(S^\infty_e) = S_e$ for all $e \in E^1 \setminus X$, and $\psi_{P^\infty,S^\infty}(S^\infty_x) = \kappa(x) S_x$ for all $x \in X$. Hence $p^m_v := \psi_{P^\infty,S^\infty}(P^\infty_v)$ and $s^m_e := \psi_{P^\infty,S^\infty}(S^\infty_e)$ generate $C^*_{\min}(E)$ and satisfy Theorem 3.1(1a) and (1b). To prove the last assertion of (1), fix $x \in X$ and calculate:
$$\begin{aligned} S^\infty_x &= S^\infty_x S^\infty_{\lambda(x)} (S^\infty_{\lambda(x)})^* \\ &= S^\infty_{\mu(x)} (S^\infty_{\lambda(x)})^* \\ &= (S^\infty_{\lambda(x)})^* \\ &\in C^*(\{P^\infty_v, S^\infty_e : v \in E^0, E \in E^1 \setminus X\}. \end{aligned}$$

For (2), let A be another C^*-algebra generated by a Toeplitz–Cuntz–Krieger family $\{p^A_v : v \in E^0\}$, $\{s^A_e : e \in E^1\}$ with each p^A_v nonzero, and suppose that A has the same two properties as $C^*_{\min}(E)$. Applying the co-universal properties, we see that there are surjective homomorphisms $\phi : C^*_{\min}(E) \to A$ and $\psi : A \to C^*_{\min}(E)$ such that $\phi(p^m_v) = p^A_v$, $\phi(s^m_e) = s^A_e$, $\psi(p^A_v) = p^m_v$, and $\psi(s^A_e) = s^m_e$ for all $v \in E^0$ and $e \in E^1 \setminus X$. In particular, ϕ and ψ are inverse to each other, and hence are isomorphisms. $\square$

Remark 4.2. Of course statement (1) of Theorem 4.1 follows from the definition of $C^*_{\min}(E)$ (embedded in the proof of Theorem 3.1). However the argument given highlights how it follows from the co-universal property.

Corollary 4.3. *Let E be a row-finite directed graph. Let $\phi : C^*_{\min}(E) \to B$ be a homomorphism. Then ϕ is injective if and only if $\phi(p^m_v) \neq 0$ for all $v \in E^0$.*

Proof. Suppose that ϕ is injective. Then that each $p^m_v \neq 0$ implies that each $\phi(p^m_v) \neq 0$ also.

Now suppose that $\phi(p^m_v) \neq 0$ for all $v \in E^0$. Then the co-universal property of $C^*_{\min}(E)$ ensures that for any cutting set X for E, there is a homomorphism $\psi : \phi(C^*_{\min}(E)) \to C^*_{\min}(E)$ satisfying $\psi(\phi(p^m_v)) = p^m_v$ for all $v \in E^0$ and $\psi(\phi(s^m_e)) = s^m_e$ for all $e \in E^1 \setminus X$. Theorem 4.1(1) implies that ψ is surjective and an inverse for ϕ. □

Corollary 4.4. *Let E be a row-finite directed graph. Then $C^*_{\min}(E)$ is simple if and only if E is cofinal.*

Proof. First suppose that E is cofinal. Fix a homomorphism $\phi : C^*_{\min}(E) \to B$. We must show that ϕ is either trivial or injective. The argument of [9, Proposition 4.2] shows that $\phi(p^m_v) = 0$ for any $v \in E^0$, then $\phi(p^m_w) = 0$ for all $v \in E^0$, which forces $\phi = 0$. On the other hand, if $\phi(p^m_w) \neq 0$ for all $w \in E^0$, then Corollary 4.3 implies that ϕ is injective.

Now suppose that E is not cofinal. Fix $v \in E^0$ and $x \in E^{\leq\infty}$ such that $vE^*x(n) = \emptyset$ for all $n \in \mathbb{N}$. Standard calculations show that

$$I_x := \overline{\operatorname{span}}\{s^m_\alpha (s^m_\beta)^* : s(\alpha) = s(\beta) = x(n) \text{ for some } n \leq |x|\}$$

is an ideal of $C^*_{\min}(E)$ which is nontrivial because it contains $p^m_{x(0)}$. To see that $p^m_v \notin I_x$, fix $n \leq |x|$ and $\alpha, \beta \in E^*x(n)$. It suffices to show that $p^m_v s^m_\alpha (s^m_\beta)^* = 0$. Let $l := |\alpha|$. By Theorem 3.1(2), $\{p^m_v : v \in E^0\}$, $\{s^m_e, e \in E^1\}$ is a reduced Cuntz–Krieger E-family, and in particular a standard inductive argument based on relation (CK) shows that

$$p^m_v = \sum_{\lambda \in vE^{\leq l}} s^m_\lambda (s^m_\lambda)^*.$$

Fix $\lambda \in vE^{\leq l}$. Since $vE^*x(n) = \emptyset$ for all $n \in \mathbb{N}$, we have $\alpha \neq \lambda\lambda'$ for all $\lambda' \in E^*$. Since $|\alpha| = l \geq |\lambda|$, it follows that $(s^m_\lambda)^* s^m_\alpha = 0$. Hence $p^m_v s^m_\alpha (s^m_\beta)^* = 0$ as claimed. □

Corollary 4.5. *Let E be a row-finite directed graph. Then $C^*_{\min}(E)$ is the universal C^*-algebra generated by a normalised reduced Cuntz–Krieger E-family. That is, if $\{q_v : v \in E^0\}$, $\{t_e : e \in E^1\}$ is another normalised reduced Cuntz–Krieger family in a C^*-algebra B, then there is a homomorphism $\pi_{q,t} : C^*_{\min}(E) \to B$ such that $\pi_{q,t}(p^m_v) = q_v$ for all $v \in E^0$ and $\pi_{q,t}(s^m_e) = t_e$ for all $e \in E^1$.*

Proof. The universal property of $\mathcal{T}C^*(E)$ implies that there is a homomorphism $\pi^{\mathcal{T}}_{q,t} : \mathcal{T}C^*(E) \to B$ such that $\pi^{\mathcal{T}}_{q,t}(p^{\mathcal{T}}_v) = q_v$ and $\pi^{\mathcal{T}}_{q,t}(s^{\mathcal{T}}_e) = t_e$ for all $v \in E^0$ and $e \in E^1$. Let $I_{q,t} := \ker(\pi^{\mathcal{T}}_{q,t})$, and let I_{p^m,s^m} be the kernel of the canonical homomorphism $\pi^{\mathcal{T}}_{p^m,s^m} : \mathcal{T}C^*(E) \to C^*_{\min}(E)$. Let $K := I_{q,t} \cap I_{p^m,s^m}$. Define $p^K_v := p^{\mathcal{T}}_v + K$ and $s^K_e := s^{\mathcal{T}}_e + K$ for all $v \in E^0$ and $e \in E^1$. Since both $\{p^m_v, s^m_e\}$ and $\{q_v, t_e\}$ are normalised reduced Cuntz–Krieger families, $\{p^K_v, s^K_e\}$ is also. Since no $p^{\mathcal{T}}_v$ belongs to I_{p^m,s^m}, each p^K_v is nonzero. Hence Theorem 3.1(1b) and (2) imply that there is an isomorphism $\psi_{p^K,s^K} : \mathcal{T}C^*(E)/K \to C^*_{\min}(E)$ such that $\psi_{p^K,s^K}(p^K_v) = p^m_v$ and $\psi_{p^K,s^K}(s^K_e) = s^m_e$ for all v, e. By definition of K, the homomorphism $\pi^{\mathcal{T}}_{q,t} : \mathcal{T}C^*(E) \to B$ descends to a homomorphism $\widetilde{\pi^{\mathcal{T}}_{q,t}} : \mathcal{T}C^*(E)/K \to B$, and then $\pi_{q,t} := \widetilde{\pi^{\mathcal{T}}_{q,t}} \circ (\psi_{p^K,s^K})^{-1}$ is the desired homomorphism. □

Lemma 4.6. *Let E be a row-finite directed graph. Fix a cutting set X for E. Define a directed graph F as follows:*

$$F^0 = \{\zeta(v) : v \in E^0\}$$
$$F^1 = \{\zeta(e) : e \in E^1 \setminus X\}$$
$$s(\zeta(e)) = \zeta(s(e)) \quad \textit{and} \quad r(\zeta(e)) = \zeta(r(e)).$$

There is an isomorphism ϕ from $C^*(F)$ to $C^*_{\min}(E)$ such that $\phi(p_{\zeta(v)}) = p^m_v$ for all $v \in E^0$ and $\phi(s_{\zeta(e)}) = s^m_e$ for all $e \in F^1$.

Proof. Let $\{p_{\zeta(v)} : v \in E^0\}$, $\{s_{\zeta(e)} : e \in E^1 \setminus X\}$ denote the universal generating Cuntz–Krieger F-family in $C^*(F)$. Recall that for $x \in X$, we write $\mu(x)$ for the unique cycle with no entrance in E such that $\mu(x)_1 = x$, and we define $\lambda(x)$ to be the path such that $\mu(x) = x\lambda(x)$. For $\nu \in E^*$ with $|\nu| \geq 2$ and $\nu_i \notin X$ for all i, we write $\zeta(\nu)$ for the path $\zeta(\nu_1) \cdots \zeta(\nu_{|\nu|}) \in F$. Define

$$\begin{aligned} q_v &:= p_{\zeta(v)} \text{ for all } v \in E^0, \\ t_e &:= s_{\zeta(e)} \text{ for all } e \in E^1 \setminus X, \text{ and} \\ t_x &:= s^*_{\zeta(\lambda(x))} \text{ for all } x \in X. \end{aligned}$$

It suffices to show that the q_v and t_e form a normalised reduced Cuntz–Krieger E-family; the result will then follow from Theorem 3.1(1b) and (2).

The q_v are mutually orthogonal projections because the $p_{\zeta(v)}$ are. This establishes (T1).

For $e \in F^1$ we have $t^*_e t_e = s^*_{\zeta(e)} s_{\zeta(e)} = p_{s(\zeta(e))} = q_{s(e)}$. For each $x \in X$, since $\mu(x)$ has no entrance in E, we have $r(x)(F)^{|\lambda(x)|} = \{\zeta(\lambda(x))\}$, so the Cuntz–Krieger relation forces $s_{\zeta(\lambda(x))} s^*_{\zeta(\lambda(x))} = p_{\zeta(s(x))}$. Hence

$$t^*_x t_x = s_{\zeta(\lambda(x))} s^*_{\zeta(\lambda(x))} = p_{\zeta(s(x))} = q_{s(x)}.$$

This establishes (T2).

Fix $v \in F^0$ such that $vE^1 \neq \emptyset$. If $v = r(x)$ for some $x \in X$, then $r_E^{-1}(v) = \{x\}$, and we have

$$q_v = p_{\zeta(v)} = s^*_{\zeta(\lambda(x))} s_{\zeta(\lambda(x))} = t_x t_x^* = \sum_{e \in r_E^{-1}(v)} t_e t_e^*.$$

If $v \neq r(x)$ for all $x \in X$, then $vF^1 = \{\zeta(e) : e \in vE^1\}$, and so

$$q_v = p_{\zeta(v)} = \sum_{f \in vF^1} s_f s_f^* = \sum_{e \in vE^1} t_e t_e^*.$$

This establishes both (T3) and (CK). □

Corollary 4.7. *Let E be a row-finite directed graph. There is an isomorphism*

$$\psi_{P^\infty, S^\infty} : C^*_{\min}(E) \to C^*(\{P_v^\infty, S_e^\infty : v \in E^0, e \in E^1\})$$

satisfying $\psi_{P^\infty,S^\infty}(p_v^m) = P_v^\infty$ for all $v \in E^0$ and $\psi_{P^\infty,S^\infty}(s_e^m) = S_e^\infty$ for all $e \in E^1$.

Proof. As observed above, $\{P_v^\infty : v \in E^0\}$, $\{S_e^\infty : e \in E^1\}$ is a normalised reduced Cuntz–Krieger E-family with each P_v^∞ nonzero. The result therefore follows from Corollaries 4.3 and 4.5. □

We now identify a subspace of $\ell^2(E^{\leq\infty})$ which is invariant under the Cuntz–Krieger family of Notation 3.9. We use the resulting Cuntz–Krieger family to construct a faithful conditional expectation from $C^*_{\min}(E)$ onto its diagonal subalgebra.

Let Ω denote the collection

$$\begin{aligned}\Omega = &\{\alpha \in E^* : s(\alpha)E^1 = \emptyset\} \\ &\cup \{\alpha\mu^\infty : \alpha \in E^*, \mu \text{ is a cycle with no entrance }\} \\ &\cup \{x \in E^\infty : x \neq \alpha\rho^\infty \text{ for any } \alpha, \rho \in E^* \text{ such that } s(\alpha) = r(\rho) = s(\rho)\}.\end{aligned}$$

So $x \in E^{\leq\infty}$ belongs to Ω if and only if either x is aperiodic, or x has the form $\alpha\mu^\infty$ for some cycle μ with no entrance in E. Observe that

(4.1)
$$\text{if } x \in \Omega \text{ and if } y \in E^{\leq\infty} \text{ and } m, n \in \mathbb{N} \text{ satisfy } \sigma^m(x) = \sigma^n(y), \text{ then } y \in \Omega.$$

We regard $\ell^2(\Omega)$ as a subspace of $\ell^2(E^{\leq\infty})$. The condition (4.1) implies that $\ell^2(E^{\leq\infty})$ is invariant for the Cuntz–Krieger E-family of Notation 3.9. We may therefore define a Cuntz–Krieger E-family $\{P_v^\Omega : v \in E^0\}$, $\{S_e^\Omega : e \in E^1\}$ in $\mathcal{B}(\ell^2(\Omega))$ by

$$P_v^\Omega = P_v^\infty|_{\ell^2(\Omega)} \qquad \text{and} \qquad S_e^\Omega = S_e^\infty|_{\ell^2(\Omega)}$$

for all $v \in E^0$ and $e \in E^1$. Since every vertex of E is the range of at least one element of Ω, we have $P_v^\Omega \neq 0$ for all $v \in E^0$.

Lemma 4.8. *Let E be a row-finite directed graph. There is an isomorphism $\psi_{P^\Omega,S^\Omega} : C^*_{\min}(E) \to C^*(\{P^\Omega_v, S^\Omega_e : v \in E^0, e \in E^1\})$ satisfying $\psi_{P^\Omega,S^\Omega}(p^m_v) = P^\Omega_v$ for all $v \in E^0$ and $\psi_{P^\Omega,S^\Omega}(s^m_e) = S^\Omega_e$ for all $e \in E^1$.*

Proof. The proof is identical to that of Corollary 4.7. □

For the next proposition, let W denote the collection of paths $\alpha \in E^*$ such that $\alpha \neq \beta\mu$ for any $\beta \in E^*$ and any cycle μ with no entrance in E.

Proposition 4.9. *Let E be a row-finite directed graph.*

(1) *The C^*-algebra $C^*_{\min}(E)$ satisfies*
$$C^*_{\min}(E) = \overline{\operatorname{span}}\{s^m_\alpha (s^m_\beta)^* : \alpha, \beta \in W, s(\alpha) = s(\beta)\}.$$

(2) *Let $D := \overline{\operatorname{span}}\{s^m_\alpha (s^m_\alpha)^* : \alpha \in E^*\}$. There is a faithful conditional expectation $\Psi : C^*_{\min}(E) \to D$ such that*
$$\Psi(s^m_\alpha (s^m_\beta)^*) = \begin{cases} s^m_\alpha (s^m_\alpha)^* & \textit{if } \alpha = \beta \\ 0 & \textit{otherwise} \end{cases}$$
for all $\alpha, \beta \in W$ with $s(\alpha) = s(\beta)$.

Proof. By Lemma 4.8 it suffices to prove the corresponding statements for the C^*-algebra $B := C^*(\{P^\Omega_v, S^\Omega_e : v \in E^0, e \in E^1\}$.

(1) We have $B = \overline{\operatorname{span}}\{S^\Omega_\alpha (S^\Omega_\beta)^* : \alpha, \beta \in E^*\}$ because the same is true of $\mathcal{T}C^*(E)$. If $\alpha \in E^* \setminus W$, then $\alpha = \alpha'\mu^n$ for some $\alpha' \in W$, some cycle μ with no entrance in E and some $n \in \mathbb{N}$. Since $\{P^\Omega_v : v \in E^0\}$, $\{S^\Omega_e : e \in E^1\}$ is a normalised reduced Cuntz–Krieger E-family, $(S^\Omega_\mu)^n = P^\Omega_{r(\mu)}$, so $S^\Omega_\alpha = S^\Omega_{\alpha'}$.

(2) Let $\{\xi_x : x \in \Omega\}$ denote the standard orthonormal basis for $\ell^2(\Omega)$. For each $x \in \Omega$, let $\theta_{x,x} \in \mathcal{B}(\ell^2(\Omega))$ denote the rank-one projection onto $\mathbb{C}\xi_x$. Let Ψ denote the faithful conditional expectation on $\mathcal{B}(\ell^2(\Omega))$ determined by $\Psi(T) = \sum_{x \in \Omega} \theta_{x,x} T \theta_{x,x}$, where the convergence is in the strong operator topology. It suffices to show that

$$\Psi(S^\Omega_\alpha (S^\Omega_\beta)^*) = \begin{cases} S^\Omega_\alpha (S^\Omega_\alpha)^* & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

for all $\alpha, \beta \in W$ with $s(\alpha) = s(\beta)$.

Fix $\alpha, \beta \in W$ with $s(\alpha) = s(\beta)$. If $\alpha = \beta$, then
$$S^\Omega_\alpha (S^\Omega_\alpha)^* = \sum_{y \in s(\alpha)\Omega} p_{\alpha x},$$
and (4.2) is immediate. So suppose that $\alpha \neq \beta$. For $x \in \Omega$, we have
$$\theta_{x,x} S^\Omega_\alpha (S^\Omega_\beta)^* \theta_{x,x} = \begin{cases} \theta_{x,x} & \text{if } x = \alpha y = \beta y \\ 0 & \text{otherwise.} \end{cases}$$
Hence we must show that $\alpha y \neq \beta y$ for all $y \in s(\alpha)\Omega$. Fix $y \in s(\alpha)\Omega$. First observe that if $|\alpha| = |\beta| = l$, then $(\alpha y)(0, l) = \alpha \neq \beta = (\beta y)(0, l)$.

Now suppose that $|\alpha| \neq |\beta|$; we may assume without loss of generality that $|\alpha| < |\beta|$. We suppose that $\alpha y = \beta y$ and seek a contradiction. That $\alpha y = \beta y$ implies that $\beta = \alpha\beta'$ and $y = \beta' y$. Hence $r(\beta') = s(\beta')$ and $y = (\beta')^\infty$. Since $y \in \Omega$, it follows that $\beta' = \mu^n$ for some cycle μ with no entrance and some $n \in \mathbb{N}$, contradicting $\beta \in W$. $\square$

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