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Growth of solutions of a class of linear differential equations with entire coefficients

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Abstract. In this paper we will investigate the growth of solutions of the linear differential equation

$$f^{(n)} + P_{n-1}(z)e^z f^{(n-1)} + \cdots + P_0(z)e^z f = 0$$

where $P_0, \ldots, P_{n-1}$ are polynomials.

Contents

Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [5], [6]).

For $n \geq 2$, we consider the linear differential equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_0(z) f = 0 \tag{0.1}$$

where $A_0(z), \ldots, A_{n-1}$ are entire functions with $A_0(z) \not\equiv 0$. It is well known that all solutions of (0.1) are entire functions. A classical result, due to Wittich [7], says that all solutions of (0.1) are of finite order of growth if and only if all coefficients are polynomials. For a complete analysis of possible orders in the polynomial case, see [3]. If some (or all) of the coefficients of (0.1) are transcendental, a natural question is to ask when and how many

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linearly independent solutions of finite order may appear? Partial results have been available since the paper of Frei [1], which says that if p is the largest integer such that $A_p(z)$ is transcendental, then there can exist at most p linearly independent finite order solutions of the differential equation (0.1). Recently, in a paper to appear [4], I investigated the growth of solutions of the differential equation

$$f^{(n)} + A_{n-1}(z)\, e^z f^{(n-1)} + \cdots + A_0(z)\, e^z f = 0; \tag{0.2}$$

I proved that if $A_0(z)$ is transcendental entire function of order $\sigma(f) = 0$ and $A_1(z), \ldots, A_{n-1}(z)$ are polynomials, then every non trivial solution of (0.2) has infinite order. So, a natural question is to consider the case when all $A_j(z)$, $j = 0, \ldots, n-1$, are polynomials. This question is our investigation in this paper.

We will see that there are similarities and differences between the following differential equations

$$f^{(n)} + P_{n-1}(z)\, e^z f^{(n-1)} + \cdots + P_0(z)\, e^z f = 0, \tag{0.3}$$

$$f^{(n)} + P_{n-1}(z)\, f^{(n-1)} + \cdots + P_0(z)\, f = 0, \tag{0.4}$$

where $P_0(z) \not\equiv 0, \ldots, P_{n-1}(z)$ are polynomials.

From [1], (0.3) has at least one solution of infinite order. While, (0.3) may have a polynomial solution: for example $f(z) = z$ is a solution of the differential equation $f'' + ze^z f' - e^z f = 0$.

As in [3], we define a strictly decreasing finite sequence of nonnegative integers

$$s_1 > s_2 > \cdots > s_p \geq 0 \tag{0.5}$$

in the following manner. We choose s_1 to be the unique integer satisfying

$$\frac{d_{s_1}}{n - s_1} = \max_{0 \leq k \leq n-1} \frac{d_k}{n-k} \quad \text{and} \quad \frac{d_{s_1}}{n - s_1} > \frac{d_k}{n-k} \quad \text{for all } 0 \leq k < s_1; \tag{0.6}$$

where $d_j = \deg P_j$ if $P_j \not\equiv 0$ and for convenience $d_j = -\infty$ if $P_j \equiv 0$, $0 \leq j \leq n-1$.

Then given s_j, $j \geq 1$, we define s_{j+1} to be the unique integer satisfying

$$\begin{aligned} &\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} = \max_{0 \leq k < s_j} \frac{d_k - d_{s_j}}{s_j - k} > -1 \quad \text{and} \\ &\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} > \frac{d_k - d_{s_j}}{s_j - k} \quad \text{for all } 0 \leq k < s_{j+1}. \end{aligned} \tag{0.7}$$

For a certain p, the integer s_p will exist, but the integer s_{p+1} will not exist, and then the sequence $s_1, s_2, \ldots, s_p$ terminates with s_p. Obviously, $p \leq n$, and we also see that (0.5) holds.

Correspondingly, define for $j = 1, 2, \ldots, p$,

$$\alpha_j = 1 + \frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j}, \tag{0.8}$$

where we set

$$s_0 = n \quad \text{and} \quad d_{s_0} = d_n = 0. \tag{0.9}$$

From (0.7) and (0.8), we observe that $\alpha_j > 0$ for each j, $1 \leq j \leq p$.

We mention that the integers $s_1, s_2, \ldots, s_p$ can also expressed in the following manner:

$$s_1 = \min\left\{ j : \frac{d_j}{n-j} = \max_{0\leq k\leq n-1} \frac{d_k}{n-k} \right\};$$

and given s_j, $j \geq 1$, we have

$$s_{j+1} = \min\left\{ i : \frac{d_i - d_{s_j}}{s_j - i} = \max_{0\leq k< s_j} \frac{d_k - d_{s_j}}{s_j - k} > -1 \right\}.$$

We denote by $n' \leq n-1$ the largest integer such that $P_{n'}(z) \not\equiv 0$ in (0.3) in all this paper. If $n' \geq 1$ we define, as above, a strictly decreasing finite sequence of nonnegative integers

$$s'_1 > s'_2 > \cdots > s'_q \geq 0,$$

as follows:

$$\frac{d_{s'_1} - d_{n'}}{n' - s'_1} = \max_{0\leq k\leq n'-1} \frac{d_k - d_{n'}}{n'-k} > -1 \quad \text{and}$$
$$\frac{d_{s'_1} - d_{n'}}{n' - s_1} > \frac{d_k - d_{n'}}{n'-k} \quad \text{for all} \quad 0 \leq k < s'_1.$$

Then given s'_j, $j \geq 1$, we define s'_{j+1} to satisfy

$$\frac{d_{s'_{j+1}} - d_{s'_j}}{s'_j - s'_{j+1}} = \max_{0\leq k< s'_j} \frac{d_k - d_{s'_j}}{s'_j - k} > -1 \quad \text{and}$$
$$\frac{d_{s'_{j+1}} - d_{s'_j}}{s'_j - s'_{j+1}} > \frac{d_k - d_{s'_j}}{s'_j - k} \quad \text{for all } 0 \leq k < s'_{j+1}.$$

As above, this sequence terminates with s'_q, and obviously we have $q \leq n'$.

Correspondingly, define for $j = 1, \ldots, q$

$$\alpha'_j = 1 + \frac{d_{s'_j} - d_{s'_{j-1}}}{s'_{j-1} - s'_j}, \tag{0.10}$$

where we set $s'_0 = n'$, and we have also

$$\alpha'_1 > \alpha'_2 > \cdots > \alpha'_q > 0.$$

In [3], G. Gundersen, E. Steinbart and S. Wang proved the following:

Theorem 0.1 ([3])**.** *For equation* (0.4)*, the following conclusions hold:*

(i) *If f is a transcendental solution of* (0.4)*, then $\sigma(f) = \alpha_j$ for some j, $1 \leq j \leq p$.*

(ii) *If $s_1 \geq 1$ and $p \geq 2$, then the following inequalities hold:*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p \geq \frac{1}{s_{p-1} - s_p} \geq \frac{1}{s_1 - s_p} \geq \frac{1}{s_1}.$$

(iii) *If $s_1 = 0$, then any nontrivial solution f of (0.4) satisfies $\sigma(f) = 1 + \frac{d_0}{n}$.*

In this paper, we will give the possible orders of solutions of (0.3). We also give related results.

1. Statement of results

Theorem 1.1. *If (0.3) admits a transcendental solution of finite order α then*

$$\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \cup \{\alpha'_1, \alpha'_2, \ldots, \alpha'_q\},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha'_1, \alpha'_2, \ldots, \alpha'_q$ are defined in (0.8) and (0.10).

Remark 1.2. It may happen that some values of $\{\alpha'_1, \alpha'_2, \ldots, \alpha'_q\}$ are equal to some values of $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, and globally we have at most n distinct values. More precisely, if $s_1 = n'$, then $\alpha_{1+k} = \alpha'_k$ for every integer k, $1 \leq k \leq q$; and if $s_1 \neq n'$ and $s_i = s'_j$ for some integers i, j then $\alpha_{i+k} = \alpha'_{j+k}$ for every integer k, $1 \leq k \leq q - j$.

For example: if $n = 3$, $d_0 = 4$, $d_1 = 3$, $d_2 = 1$, then $p = 2$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = 2$ and $q = 2$, $\alpha'_1 = 3$, $\alpha'_2 = 2$.

Theorem 1.3. *If $s_1 = 0$, then every nontrivial solution of (0.3) satifies*

$$\sigma(f) \geq 1 + \frac{d_0}{n}.$$

A natural question is: are there cases of (0.3) when every nontrivial solution has infinite order. The answer is positive as indicated by the following example.

Example 1.4. Every nontrivial solution of the differential equation

$$f^{(n)} + P_0(z) e^z f = 0,$$

has infinite order, where $P_0(z) \not\equiv 0$ is a polynomial. In fact, if we suppose that $f \not\equiv 0$ is of finite order then by looking at $e^z = -\frac{1}{P_0(z)} \frac{f^{(n)}}{f}$, we conclude that $T(r, e^z) = O(\log r)$, a contradiction.

Theorem 1.5. *If $s_p = 0$, then there is no polynomial solutions $f \not\equiv 0$ of (0.3).*

2. Preliminary lemmas

Lemma 2.1 ([3]). *For any fixed $j = 0, 1, \ldots, p-1$, let α be any real number satisfying $\alpha > \alpha_{j+1}$, and let k be any integer satisfying $0 \leq k < s_j$. Then*

$$n - k + d_k + k\alpha < n - s_j + d_{s_j} + s_j\alpha.$$

Lemma 2.2 ([3]). *For any fixed $j = 1, \ldots, p$, let α be any real number satisfying $\alpha < \alpha_j$, and let k be any integer satisfying $s_j < k \leq n$. Then*

$$n - k + d_k + k\alpha < n - s_j + d_{s_j} + s_j\alpha.$$

Lemma 2.3 ([3]). *Let $\alpha > 0$. Then for any integer k satisfying $0 \leq k < s_p$, we have*

$$n - k + d_k + k\alpha < n - s_p + d_{s_p} + s_p\alpha.$$

Lemma 2.4. *Let α be any real number satisfying $\alpha > \alpha_1$, and k be any integer satisfying $0 \leq k < s_1$. Then*

$$n - k + d_k + k\alpha < n - s_1 + d_{s_1} + s_1\alpha.$$

Proof. We have

$$n - k + d_k + k\alpha = n - s_1 + d_{s_1} + s_1\alpha + \alpha\,(k - s_1) + d_k - d_{s_1} + s_1 - k,$$

and since $\alpha > \alpha_1$ and $0 \leq k < s_1$ we obtain

$$n - k + d_k + k\alpha < n - s_1 + d_{s_1} + s_1\alpha + \alpha_1\,(k - s_1) + d_k - d_{s_1} + s_1 - k.$$

And we have

$$\begin{aligned}
&\alpha_1\,(k - s_1) + d_k - d_{s_1} + s_1 - k \\
&= \left(1 + \frac{d_{s_1}}{n - s_1}\right)(k - s_1) - \frac{d_k}{n - k}\,(k - n) \\
&\quad - (k - s_1) - d_{s_1} \\
&= \left(\frac{d_{s_1}}{n - s_1}\right)(k - s_1) - \frac{d_k}{n - k}\,(k - s_1 + s_1 - n) - d_{s_1} \\
&= \left(\frac{d_{s_1}}{n - s_1} - \frac{d_k}{n - k}\right)(k - s_1) + \left(\frac{d_{s_1}}{n - s_1} - \frac{d_k}{n - k}\right)(s_1 - n) \\
&= \left(\frac{d_{s_1}}{n - s_1} - \frac{d_k}{n - k}\right)(k - n)\,.
\end{aligned}$$

From the definition of s_1 in (0.6), we obtain

$$\left(\frac{d_{s_1}}{n - s_1} - \frac{d_k}{n - k}\right)(k - n) < 0,$$

for $0 \leq k < s_1$. Thus, we deduce that

$$n - k + d_k + k\alpha < n - s_1 + d_{s_1} + s_1\alpha,$$

for $0 \leq k < s_1$. □

Lemma 2.5. *Let α be any real number satisfying $\alpha > \alpha_1$, and k be any integer satisfying $0 \le k < n'$ (n' is the largest integer such that $P_{n'}(z) \not\equiv 0$ in (0.3)). If $s_1 < n'$ and $\frac{d_{s_1}}{n-s_1} = \frac{d_{n'}}{n-n'}$, then we have*

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha.$$

Proof. We have

$$n - k + d_k + k\alpha = n - n' + d_{n'} + n'\alpha + \alpha\left(k - n'\right) + d_k - d_{n'} + n' - k,$$

and since $\alpha > \alpha_1$ and $0 \le k < n'$ we obtain

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha + \alpha_1\left(k - n'\right) + d_k - d_{n'} + n' - k;$$

and we have

$$\begin{aligned}
&\alpha_1\left(k - n'\right) + d_k - d_{n'} + n' - k \\
&= \frac{d_{s_1}}{n-s_1}\left(k - n'\right) + d_k - d_{n'} \\
&= \frac{d_{s_1}}{n-s_1}(k-n) + \frac{d_{s_1}}{n-s_1}\left(n - n'\right) + d_k - d_{n'} \\
&= \frac{d_{s_1}}{n-s_1}(k-n) + \frac{d_{n'}}{n-n'}\left(n - n'\right) + d_k - d_{n'} \\
&= \frac{d_{s_1}}{n-s_1}(k-n) + d_k \\
&= \left(\frac{d_{s_1}}{n-s_1} - \frac{d_k}{n-k}\right)(k-n) \le 0,
\end{aligned}$$

for any k satisfying $0 \le k < n'$. Thus, we deduce that

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha. \qquad \square$$

Lemma 2.6. *If $s_1 < n'$ and $\frac{d_{s_1}}{n-s_1} > \frac{d_{n'}}{n-n'}$, then $\alpha_1' > \alpha_1$.*

Proof. We have the following equivalences:

$$\begin{aligned}
\frac{d_{s_1}}{n-s_1} > \frac{d_{n'}}{n-n'} \quad &\Leftrightarrow \quad d_{s_1} - \frac{n'-s_1}{n-s_1}d_{s_1} > d_{n'} \\
&\Leftrightarrow \quad d_{s_1} - d_{n'} > \frac{n'-s_1}{n-s_1}d_{s_1} \\
&\Leftrightarrow \quad \frac{d_{s_1} - d_{n'}}{n'-s_1} > \frac{d_{s_1}}{n-s_1}.
\end{aligned}$$

Which implies that

$$\max_{0 \le k < n'} \frac{d_k - d_{n'}}{n'-k} > \frac{d_{s_1}}{n-s_1},$$

and so

$$\alpha_1' > \alpha_1. \qquad \square$$

Lemma 2.7. *If $\alpha_1' > \alpha_1$, then for $\alpha > \alpha_1'$ and $0 \leq k < n'$, we have*

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha.$$

Proof. We have

$$n - k + d_k + k\alpha = n - n' + d_{n'} + n'\alpha + \alpha\left(k - n'\right) + d_k - d_{n'} + n' - k,$$

and since $\alpha > \alpha_1'$ and $0 \leq k < n'$ we obtain

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha + \alpha_1'\left(k - n'\right) + d_k - d_{n'} + n' - k;$$

and we have

$$\begin{aligned}\alpha_1'\left(k - n'\right) + d_k - d_{n'} + n' - k &= \left(\frac{d_{s_1'} - d_{n'}}{n' - s_1'}\right)\left(k - n'\right) + d_k - d_{n'} \\ &= \left(\frac{d_{s_1'} - d_{n'}}{n' - s_1'} - \frac{d_k - d_{n'}}{n' - k}\right)\left(k - n'\right) \leq 0,\end{aligned}$$

for any k satisfying $0 \leq k < n'$. Thus, we deduce that

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha. \qquad \square$$

By using the same proofs as for Lemma 2.1 and Lemma 2.2, we can obtain the two following lemmas:

Lemma 2.8. *For any fixed $j = 0, 1, \ldots, q - 1$, let α be any real number satisfying $\alpha > \alpha_{j+1}'$, and let k be any integer satisfying $0 \leq k < s_j'$. Then*

$$n - k + d_k + k\alpha < n - s_j' + d_{s_j'} + s_j'\alpha.$$

Lemma 2.9. *For any fixed $j = 1, \ldots, q$, let α be any real number satisfying $\alpha < \alpha_j'$, and let k be any integer satisfying $s_j' < k \leq n'$. Then*

$$n - k + d_k + k\alpha < n - s_j' + d_{s_j'} + s_j'\alpha.$$

Lemma 2.10 ([3])**.** *Suppose that $s_m + 1 \leq k < n$ for two positive integers m and k. Then*

$$d_k \leq d_{s_{m-1}} + \left(s_{m-1} - k\right)\left(\alpha_m - 1\right).$$

Lemma 2.11 ([2])**.** *Let $f \not\equiv 0$ be a meromorphic function of finite order β, and let $k \geq 1$ be an integer. Then for any given $\varepsilon > 0$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{k(\beta-1)+\varepsilon},$$

where $|z| \notin [0,1] \cup E$, E is a set in $(1, \infty)$ that has finite logarithmic measure.

3. Proof of Theorem 1.1

Suppose that (0.3) admits a transcendental entire solution f of finite order $\sigma(f) = \alpha$.

From (0.3), we can write

$$e^{-z}\frac{f^{(n)}}{f} + P_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + P_1(z)\frac{f'}{f} + P_0(z) = 0. \tag{3.1}$$

If $V(r)$ denotes the central index of f, then

$$V(r) = (1 + o(1))\, Cr^{\alpha}, \tag{3.2}$$

as $r \to \infty$, where C is a positive constant. In addition, from the Wiman-Valiron theory it follows that there exists a set $E \subset (1, \infty)$ that has finite logarithmic measure, such that for all $j = 1, 2, \dots, n$ we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V(r)}{z_r}\right)^j \tag{3.3}$$

as $r \to \infty$, $r \notin E$, where z_r is a point on the circle $|z| = r$ that satisfies $|f(z_r)| = \max_{|z|=r} |f(z)|, \ \ 0 < r < \infty$.

Let b_k denote the leading coefficient of the polynomial $P_k(z)$, and set $a_k = C^k |b_k|$, where $C > 0$ is the constant in (3.2) and set $a_n = C^n$.

Substituting (3.2) and (3.3) in (3.1) and multiplying the both side by z_r^n, we get an equation whose the left side consists of a sum of $n+1$ terms whose moduli are asymptotic (as $r \to \infty$, $r \notin E$) to the following $n+1$ terms:

$$e^{-r\cos\theta_r} a_n r^{n\alpha},\ a_{n-1} r^{1+d_{n-1}+(n-1)\alpha}, \dots, a_k r^{n-k+d_k+k\alpha}, \dots, a_0 r^{n+d_0}. \tag{3.4}$$

We discuss three cases according to the limit of $e^{-r\cos\theta_r}$.

Case 1. $\overline{\lim_{r\to\infty}} e^{-r\cos\theta_r} = \infty$. In this case, $e^{-r\cos\theta_r} a_n r^{n\alpha}$ is the unique dominant term (as $r \to \infty$, $r \notin E$) in (3.4). This is impossible.

Case 2. $\overline{\lim_{r\to\infty}} e^{-r\cos\theta_r} = c$ where $0 < c < \infty$. If $\alpha_{j+1} < \alpha < \alpha_j$ for some $j = 1, \dots, p-1$, Then from Lemma 2.1 and Lemma 2.2, we have

$$n - k + d_k + k\alpha < n - s_j + d_{s_j} + s_j\alpha,$$

for any $k \neq s_j$, then $a_{s_j} r^{n-s_j+d_{s_j}+s_j\alpha}$ is the unique dominant term (as $r \to \infty$, $r \notin E$) in (3.4). This is impossible also. Now if $0 < \alpha < \alpha_p$ then from Lemma 2.2 and Lemma 2.3, we have

$$n - k + d_k + k\alpha < n - s_p + d_{s_p} + s_p\alpha,$$

for any $k \neq s_p$, then $a_{s_p} r^{n-s_p+d_{s_p}+s_p\alpha}$ is the unique dominant term (as $r \to \infty$, $r \notin E$) in (3.4). This gives a contradiction in (3.1). Finaly if $\alpha > \alpha_1$ then from Lemma 2.1, we have

$$n - k + d_k + k\alpha < n\alpha, \tag{3.5}$$

for any $0 \le k < n$; so $e^{-r\cos\theta_r} a_n r^{n\alpha}$ is the unique dominant term (as $r \to \infty$, $r \notin E$) in (3.4). Also, this leads to a contradiction in (3.1).

Case 3. $\overline{\lim_{r\to\infty}} e^{-r\cos\theta_r} = 0$. If $0 < \alpha < \alpha_p$ or $\alpha_{j+1} < \alpha < \alpha_j$ for some $j = 1, \ldots, p-1$, we find the same contradiction as in Case 2. Now if $\alpha > \alpha_1$, although we have (3.5), $e^{-r\cos\theta_r} a_n r^{n\alpha}$ is not the dominant term because

$$\lim_{r\to\infty} e^{-r\cos\theta_r} a_n r^{n\alpha} = 0.$$

If $s_1 = n'$, then from Lemma 2.4, we have

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha,$$

for any $0 \le k < n'$. So, there exists only one dominant term in (3.4) (as $r \to \infty$, $r \notin E$). A contradiction.

If $s_1 < n'$ and $\dfrac{d_{s_1}}{n - s_1} = \dfrac{d_{n'}}{n - n'}$, then from Lemma 2.5, we have

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha,$$

for any $0 \le k < n'$. As above, there exists one dominant term in (3.4) (as $r \to \infty$, $r \notin E$), which leads to a contradiction.

If $s_1 < n'$ and $\dfrac{d_{s_1}}{n - s_1} > \dfrac{d_{n'}}{n - n'}$, then from Lemma 2.6, we have $\alpha_1' > \alpha_1$. Now we will use the sequence $\alpha_1', \alpha_2', \ldots, \alpha_q'$. If $0 < \alpha < \alpha_q'$ or $\alpha_{j+1}' < \alpha < \alpha_j'$ for some $j = 1, \ldots, q-1$, by using Lemma 2.8 and Lemma 2.9, we find the same previous contradiction. Now if $\alpha > \alpha_1'$, from the Lemma 2.7, we have

$$n - k + d_k + k\alpha < n - n' + d_{n'} + n'\alpha,$$

for any $0 \le k < n'$. As above, this gives a contradiction.

Thus, the possible values of α are $\{\alpha_1, \alpha_2, \ldots, \alpha_p\} \cup \left\{\alpha_1', \alpha_2', \ldots, \alpha_q'\right\}$.

4. Proof of Theorem 1.3

Suppose to the contrary that there exists a non trivial solution f of (0.3) which satisfies $\sigma(f) = \beta < 1 + \frac{d_0}{n}$. Then, we can write

$$\beta = 1 + \frac{d_0}{n} - \tau, \tag{4.1}$$

where τ is a positive constant.

Since $s_1 = 0$, from (0.6) we have

$$d_k \le \frac{n-k}{n} d_0, \quad k = 1, 2, \ldots, n-1. \tag{4.2}$$

From (0.3) we can write

$$-P_0(z) = e^{-z}\frac{f^{(n)}}{f} + P_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + P_1(z)\frac{f'}{f}. \tag{4.3}$$

By taking $\arg z \in \left(0, \frac{\pi}{2}\right)$, we can get $|e^{-z}| < 1$; and from Lemma 2.11 and (4.3), we obtain

$$|P_0(z)| < \sum_{k=1}^{n} |z|^{d_k + k(\beta-1)+\varepsilon}, \tag{4.4}$$

where $|z|$ large enough ($|z| \notin E$) and $d_n = 0$.

From (4.4), (4.2) and (4.1), we get

$$|P_0(z)| < \sum_{k=1}^{n} |z|^{d_0 - k\tau + \varepsilon} < n\,|z|^{d_0 - \tau + \varepsilon},$$

where $|z|$ is sufficiently large $(|z| \notin E)$. This is not possible if we choose $0 < \varepsilon < \tau$. Thus $\sigma(f) \geq 1 + \frac{d_0}{n}$.

5. Proof of Theorem 1.5

Suppose that $f \not\equiv 0$ is a polynomial solution of (0.3).

From (0.3), we can write

$$-P_0(z) = e^{-z}\frac{f^{(n)}}{f} + P_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + P_1(z)\frac{f'}{f},$$

which implies that f must be of degree at most $n-1$. So we get

$$-P_0(z) = P_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + P_1(z)\frac{f'}{f}. \tag{5.1}$$

It follows from (5.1) that

$$d_0 \leq \max_{1 \leq k \leq n-1}\{d_k - k\}. \tag{5.2}$$

By Lemma 2.10, we have

$$d_k \leq d_{s_{p-1}} + (s_{p-1} - k)(\alpha_p - 1), \tag{5.3}$$

for all $k = 1, \ldots, n-1$, since $s_p = 0$. Therefore, from (5.3), the definition of α_p in (0.8), and the fact that $s_p = 0$, we obtain for any $1 \leq k \leq n-1$,

$$\begin{aligned} d_k - k &\leq d_{s_{p-1}} - k + (s_{p-1} - k)(\alpha_p - 1) \\ &\leq d_0 + \frac{k}{s_{p-1}}\left(d_{s_{p-1}} - d_0 - s_{p-1}\right). \end{aligned} \tag{5.4}$$

Since $\alpha_p > 0$ and $s_p = 0$, it follows from the definition of α_p in (0.8) that $d_{s_{p-1}} < s_{p-1} + d_0$. Hence from (5.4), we obtain $d_k - k < d_0$ for all $1 \leq k \leq n-1$. But this contradicts (5.2). This completes the proof of Theorem 1.5.

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