The A-polynomial of the $(-2, 3, 3 + 2n)$ pretzel knots

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ABSTRACT. We show that the A-polynomial A_n of the 1-parameter family of pretzel knots $K_n = (-2, 3, 3 + 2n)$ satisfies a linear recursion relation of order 4 with explicit constant coefficients and initial conditions. Our proof combines results of Tamura–Yokota and the second author. As a corollary, we show that the A-polynomial of K_n and the mirror of K_{-n} are related by an explicit $\mathrm{GL}(2, \mathbb{Z})$ action. We leave open the question of whether or not this action lifts to the quantum level.

CONTENTS

1. Introduction

1.1. The behavior of the A-polynomial under filling. In [CCGLS94], the authors introduced the A-polynomial A_W of a hyperbolic 3-manifold W with one cusp. It is a 2-variable polynomial which describes the dependence

of the eigenvalues of a meridian and longitude under any representation of $\pi_1(W)$ into $\mathrm{SL}(2,\mathbb{C})$. The A-polynomial plays a key role in two problems:

- the deformation of the hyperbolic structure of W,
- the problem of exceptional (i.e., nonhyperbolic) fillings of W.

Knowledge of the A-polynomial (and often, of its Newton polygon) is translated directly into information about the above problems, and vice-versa. In particular, as demonstrated by Boyer and Zhang [BZ01], the Newton polygon is dual to the fundamental polygon of the Culler–Shalen seminorm [CGLS87] and, therefore, can be used to classify cyclic and finite exceptional surgeries.

In [Gar10], the first author observed a pattern in the behavior of the A-polynomial (and its Newton polygon) of a 1-parameter family of 3-manifolds obtained by fillings of a 2-cusped manifold. To state the pattern, we need to introduce some notation. Let $K = \mathbb{Q}(x_1,\ldots,x_r)$ denote the field of rational functions in r variables $x_1,\ldots,x_r$.

Definition 1.1. We say that a sequence of rational functions $R_n \in K$ (defined for all integers n) is *holonomic* if it satisfies a linear recursion with constant coefficients. In other words, there exists a natural number d and $c_k \in K$ for $k = 0,\ldots,d$ with $c_d c_0 \neq 0$ such that for all integers n we have:

$$(1) \qquad \sum_{k=0}^{d} c_k R_{n+k} = 0.$$

Depending on the circumstances, one can restrict attention to sequences indexed by the natural numbers (rather than the integers).

Consider a hyperbolic manifold W with two cusps C_1 and C_2. Let (μ_i, λ_i) for $i = 1, 2$ be pairs of meridian-longitude curves, and let W_n denote the result of $-1/n$ filling on C_2. Let $A_n(M_1, L_1)$ denote the A-polynomial of W_n with the meridian-longitude pair inherited from W.

Theorem 1.1 ([Gar10]). *With the above conventions, there exists a holonomic sequence $R_n(M_1, L_1) \in \mathbb{Q}(M_1, L_1)$ such that for all but finitely many integers n, $A_n(M_1, L_1)$ divides the numerator of $R_n(M_1, L_1)$. In addition, a recursion for R_n can be computed explicitly via elimination, from an ideal triangulation of W.*

1.2. The Newton polytope of a holonomic sequence. Theorem 1.1 motivates us to study the Newton polytope of a holonomic sequence of Laurent polynomials. To state our result, we need some definitions. Recall that the *Newton polytope* of a Laurent polynomial in n variables $x_1,\ldots,x_n$ is the convex hull of the points whose coordinates are the exponents of its monomials. Recall that a *quasi-polynomial* is a function $p : \mathbb{N} \longrightarrow \mathbb{Q}$ of the form $p(n) = \sum_{k=0}^{d} c_k(n) n^k$ where $c_k : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. When $c_d \neq 0$, we call d the *degree* of $p(n)$. We will call quasi-polynomials of degree at most one (resp. two) *quasi-linear* (resp. *quasi-quadratic*). Quasi-polynomials appear in lattice point counting problems (see [Ehr62, CW10]),

in the Slope Conjecture in quantum topology (see [Gar11b]), in enumerative combinatorics (see [Gar11a]) and also in the A-polynomial of filling families of 3-manifolds (see [Gar10]).

Definition 1.2. We say that a sequence N_n of polytopes is linear (resp. quasi-linear) if the coordinates of the vertices of N_n are polynomials (resp. quasi-polynomials) in n of degree at most one. Likewise, we say that a sequence N_n of polytopes is quadratic (resp. quasi-quadratic) if the coordinates of the vertices of N_n are polynomials (resp. quasi-polynomials) of degree at most two.

Theorem 1.2 ([Gar10])**.** *Let N_n be the Newton polytope of a holonomic sequence $R_n \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. Then, for all but finitely many integers n, N_n is quasi-linear.*

1.3. Do favorable links exist? Theorems 1.1 and 1.2 are general, but in favorable circumstances more is true. Namely, consider a family of knot complements K_n, obtained by $-1/n$ filling on a cusp of a 2-component hyperbolic link J. Let f denote the linking number of the two components of J, and let A_n denote the A-polynomial of K_n with respect to its canonical meridian and longitude (M, L). By definition, A_n contains all components of irreducible representations, but *not* the component $L - 1$ of abelian representations.

Definition 1.3. We say that J, a 2-component link in 3-space, with linking number f is *favorable* if $A_n(M, LM^{-f^2 n}) \in \mathbb{Q}[M^{\pm 1}, L^{\pm 1}]$ is holonomic.

The shift of coordinates, $LM^{-f^2 n}$, above is due to the canonical meridian-longitude pair of K_n differing from the corresponding pair for the unfilled component of J as a result of the nonzero linking number. Theorem 1.2 combined with the above shift implies that, for a favorable link, the Newton polygon of K_n is quasi-quadratic.

Hoste–Shanahan studied the first examples of a favorable link, the *Whitehead link* and its *half-twisted version* (see Figure 1), and consequently gave an explicit recursion relation for the 1-parameter families of A-polynomials of twist knots $K_{2,n}$ and $K_{3,n}$ respectively; see [HS04].

The goal of our paper is to give another example of a favorable link J (see Figure 1), whose 1-parameter filling gives rise to the family of $(-2, 3, 3 + 2n)$ *pretzel knots*. Our paper is a concrete illustration of the general Theorems 1.1 and 1.2 above. Aside from this, the 1-parameter family of knots K_n, where K_n is the $(-2, 3, 3 + 2n)$ pretzel knot, is well-studied in hyperbolic geometry (where K_n and the mirror of K_{-n} are pairs of geometrically similar knots; see [BH96, MM08]), in exceptional Dehn surgery (where for instance $K_2 = (-2, 3, 7)$ has three Lens space fillings $1/0$, $18/1$ and $19/1$; see [CGLS87]) and in Quantum Topology (where K_n and the mirror of K_{-n} have different Kashaev invariant, equal volume, and different subleading corrections to the volume, see [GZ]).
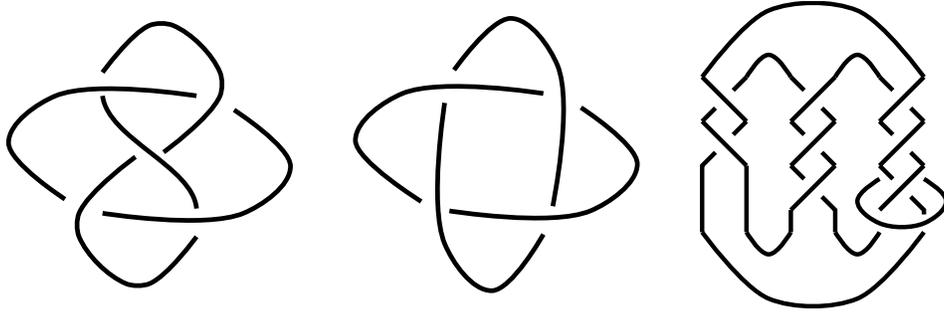
FIGURE 1. The Whitehead link on the left, the half-twisted Whitehead link in the middle and our seed link J at right.

The success of Theorems 1.3 and 1.4 below hinges on two independent results of Tamura–Yokota and the second author [TY04, Mat02], and an additional lucky coincidence. Tamura–Yokota compute an explicit recursion relation, as in Theorem 1.3, by elimination, using the gluing equations of the decomposition of the complement of J into six ideal tetrahedra; see [TY04]. The second author computes the Newton polygon N_n of the A-polynomial of the family K_n of pretzel knots; see [Mat02]. This part is considerably more difficult, and requires:

(a) The set of boundary slopes of K_n, which are available by applying the Hatcher–Oertel algorithm [HO89, Dun01] to the 1-parameter family K_n of Montesinos knots. The four slopes given by the algorithm are candidates for the slopes of the sides of N_n. Similarly, the fundamental polygon of the Culler–Shalen seminorm of K_n has vertices in rays which are the multiples of the slopes of N_n. Taking advantage of the duality of the fundamental polygon and Newton polygon, in order to describe N_n it is enough to determine the vertices of the Culler–Shalen polygon.

(b) Use of the exceptional 1/0 filling and two fortunate exceptional Seifert fillings of K_n with slopes $4n + 10$ and $4n + 11$ to determine exactly the vertices of the Culler–Shalen polygon and consequently N_n. In particular, the boundary slope 0 is not a side of N_n (unless $n = -3$) and the Newton polygon is a hexagon for all hyperbolic K_n.

Given the work of [TY04] and [Mat02], if one is lucky enough to match N_n of [Mat02] with the Newton polygon of the solution of the recursion relation of [TY04] (and also match a leading coefficient), then Theorem 1.3 below follows; i.e., J is a favorable link.

1.4. Our results for the pretzel knots K_n. Let $A_n(M, L)$ denote the A-polynomial of the pretzel knot K_n, using the canonical meridian-longitude coordinates. Consider the sequences of Laurent polynomials $P_n(M, L)$ and
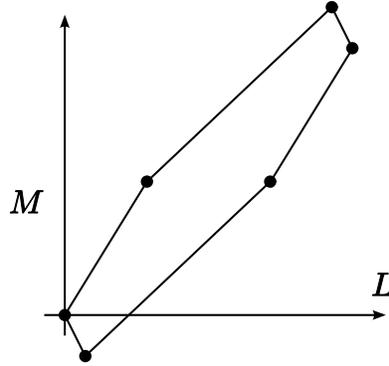
FIGURE 2. The Newton polygon NP_n.

$Q_n(M, L)$ defined by:

$$(2) \qquad\qquad P_n(M, L) = A_n(M, LM^{-4n})$$

for $n > 1$ and

$$(3) \qquad\qquad Q_n(M, L) = A_n(M, LM^{-4n})M^{-4(3n^2+11n+4)}$$

for $n < -2$ and $Q_{-2}(M, L) = A_{-2}(M, LM^{-8})M^{-20}$. In the remaining cases $n = -1, 0, 1$, the knot K_n is not hyperbolic (it is the torus knot 5_1, 8_{19} and 10_{124} respectively), and one expects exceptional behavior. This is reflected in the fact that P_n for $n = 0, 1$ and Q_n for $n = -1, 0$ can be defined to be suitable rational functions (rather than polynomials) of M, L. Let NP_n and NQ_n denote the Newton polygons of P_n and Q_n respectively.

Theorem 1.3.

(a) P_n and Q_n satisfy linear recursion relations

$$(4) \qquad\qquad \sum_{k=0}^{4} c_k P_{n+k} = 0, \qquad n \geq 0$$

and

$$(5) \qquad\qquad \sum_{k=0}^{4} c_k Q_{n-k} = 0, \qquad n \leq 0$$

where the coefficients c_k and the initial conditions P_n for $n = 0, \ldots, 3$ and Q_n for $n = -3, \ldots, 0$ are given in Appendix A.

(b) In (L, M) coordinates, NP_n and NQ_n are hexagons with vertices

$$(6) \quad \{\{0, 0\}, \{1, -4n + 16\}, \{n - 1, 12n - 12\}, \{2n + 1, 16n + 18\},$$
$$\{3n - 1, 32n - 10\}, \{3n, 28n + 6\}\}$$

for P_n with $n > 1$ and

(7) $\{\{0, 4n + 28\}, \{1, 38\}, \{-n, -12n + 26\}, \{-2n - 3, -16n - 4\},$
$$\{-3n - 4, -28n - 16\}, \{-3n - 3, -32n - 6\}\}$$

for Q_n with $n < -1$.

Remark 1.4. We can give a single recursion relation valid for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ as follows. Define

(8) $R_n(M, L) = A_n(M, LM^{-4n}) b^{|n|} \epsilon_n(M),$

where

(9) $b = \dfrac{1}{LM^8(1 - M^2)(1 + LM^{10})},$

$$c = \frac{L^3 M^{12} (1 - M^2)^3}{(1 + LM^{10})^3},$$

$$\epsilon_n(M) = \begin{cases} 1 & \text{if } n > 1 \\ cM^{-4(3+n)(2+3n)} & \text{if } n < -2 \\ cM^{-28} & \text{if } n = -2. \end{cases}$$

Then, R_n satisfies the palindromic fourth order linear recursion

(10) $\displaystyle\sum_{k=0}^{4} \gamma_k R_{n+k} = 0$

where the coefficients γ_k and the initial conditions R_n for $n = 0, \dots, 3$ are given in Appendix B. Moreover, R_n is related to P_n and Q_n by:

(11) $R_n = \begin{cases} P_n b^{|n|} & \text{if } n \geq 0 \\ Q_n b^{|n|} c M^{-8} & \text{if } n \leq 0. \end{cases}$

Remark 1.5. The computation of the Culler–Shalen seminorm of the pretzel knots K_n has an additional application, namely it determines the number of components (containing the character of an irreducible representation) of the $\mathrm{SL}(2, \mathbb{C})$ character variety of the knot, and consequently the number of factors of its A-polynomial. In the case of K_n, (after translating the results of [Mat02] for the pretzel knots $(-2, 3, n)$ to the pretzel knots $(-2, 3, 3+2n)$) it was shown by the second author [Mat02, Theorem 1.6] that the character variety of K_n has one (resp. two) components when 3 does not divide n (resp. divides n). The nongeometric factor of A_n is given by

$$\begin{cases} 1 - LM^{4(n+3)} & n \geq 3 \\ L - M^{-4(n+3)} & n \leq -3 \end{cases}$$

for $n \neq 0$ a multiple of 3.

Since the A-polynomial has even powers of M, we can define the B-polynomial by
$$B(M^2, L) = A(M, L).$$

Our next result relates the A-polynomials of the geometrically similar pair $(K_n, -K_{-n})$ by an explicit $GL(2, \mathbb{Z})$ transformation.

Theorem 1.4. *For $n > 1$ we have:*

$$(12) \quad B_{-n}(M, LM^{2n-5}) = (-L)^n M^{3(2n^2 - 7n + 7)} B_n(-L^{-1}, L^{2n+5}M^{-1})\eta_n$$

where $\eta_n = 1$ (resp. M^{22}) when $n > 2$ (resp. $n = 2$).

2. Proofs

2.1. The equivalence of Theorem 1.3 and Remark 1.4. In this subsection we will show the equivalence of Theorem 1.3 and Remark 1.4. Let $\gamma_k = c_k/b^k$ for $k = 0, \ldots, 4$ where b is given by (9). It is easy to see that the γ_k are given explicitly by Appendix B, and moreover, they are palindromic. Since $R_n = P_n b^n$ for $n = 0, \ldots, 3$ it follows that R_n and $P_n b^n$ satisfy the same recursion relation (10) for $n \geq 0$ with the same initial conditions. It follows that $R_n = P_n b^n$ for $n \geq 0$.

Solving (10) backwards, we can check by an explicit calculation that $R_n = Q_n b^{|n|} c M^{-8}$ for $n = -3, \ldots, 0$ where b and c are given by (9). Moreover, R_n and $Q_n b^{|n|} c M^{-8}$ satisfy the same recursion relation (10) for $n < 0$. It follows that $R_n = Q_n b^{|n|} c M^{-8}$ for $n < 0$. This concludes the proof of Equations (10) and (11).

2.2. Proof of Theorem 1.3. Let us consider first the case of $n \geq 0$, and denote by P'_n for $n \geq 0$ the unique solution to the linear recursion relation (4) with the initial conditions as in Theorem 1.3. Let $R'_n = P'_n b^n$ be defined according to Equation (11) for $n \geq 0$.

Remark 1.4 implies that R'_n satisfies the recursion relation of [TY04, Thm. 1]. By [TY04, Thm. 1], $A_n(M, LM^{-4n})$ divides $P'_n(M, L)$ when $n > 1$.

Next, we claim that the Newton polygon NP'_n of $P'_n(M, L)$ is given by (6). This can be verified easily by induction on n.

Next, in [Mat02, p. 1286], the second author computes the Newton polygon N_n of the $A_n(M, L)$. It is a hexagon given in (L, M) coordinates by

$$\{\{0, 0\}, \{1, 16\}, \{n - 1, 4(n^2 + 2n - 3)\}, \{2n + 1, 2(4n^2 + 10n + 9)\},$$
$$\{3n - 1, 2(6n^2 + 14n - 5)\}, \{3n, 2(6n^2 + 14n + 3)\}\}$$

when $n > 1$,

$$\{\{-3n - 4, 0\}, \{-3(1 + n), 10\}, \{-3 - 2n, 4(3 + 4n + n^2)\},$$
$$\{-n, 2(4n^2 + 16n + 21)\}, \{0, 4(3n^2 + 12n + 11)\}, \{1, 6(2n^2 + 8n + 9)\}\}$$

when $n < -2$ and

$$\{\{0, 0\}, \{1, 0\}, \{2, 4\}, \{1, 10\}, \{2, 14\}, \{3, 14\}\}$$

when $n = -2$. Notice that the above 1-parameter families of Newton polygons are quadratic. It follows by explicit calculation that the Newton polygon of $A_n(M, LM^{-4n})$ is quadratic and exactly agrees with NP'_n for all $n > 1$.

By the above discussion, $P_n(M, L)$ is a rational mutiple of $A_n(M, LM^{-4n})$. Since their leading coefficients (with respect to L) agree, they are equal. This proves Theorem 1.3 for $n > 1$. The case of $n < -1$ is similar. $\qquad\square$

2.3. Proof of Theorem 1.4. Using Equations (2) and (3), convert Equation (12) into

$$(13) \qquad Q_{-n}(\sqrt{M}, L/M^5) = (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).$$

Note that, under the substitution $(M, L) \mapsto (i/\sqrt{L}, L^{2n+5}/M)$, LM^{4n} becomes L^5/M. Similarly, LM^{-4n} becomes L/M^5 under the substitution $(M, L) \mapsto (\sqrt{M}, LM^{2n-5})$.

It is straightforward to verify equation (13) for $n = 2, 3, 4, 5$. For $n \geq 6$, we use induction. Let c_k^- denote the result of applying the substitutions $(M, L) \mapsto (\sqrt{M}, L/M^5)$ to the c_k coefficients in the recursions (4) and (5). For example,

$$c_0^- = \frac{L^4(1+L)^4(1-M)^4}{M^2}.$$

Similarly, define c_k^+ to be the result of the substitution

$$(M, L) \mapsto (i/\sqrt{L}, L^5/M)$$

to c_k. It is easy to verify that for $k = 0, 1, 2, 3$,

$$\frac{c_k^-}{c_4^-}(-LM)^{k-4} = \frac{c_k^+}{c_4^+}.$$

Then,

$$Q_{-n}(\sqrt{M}, L/M^5) = -\frac{1}{c_4^-} \sum_{k=0}^{3} c_k^- Q_{-n+4-k}(\sqrt{M}, L/M^5)$$

$$= -\frac{1}{c_4^-} \sum_{k=0}^{3} c_k^- (-L)^{n-4+k} M^{n-4+k+13} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^-}{c_4^-}(-LM)^{k-4} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= -(-L)^n M^{n+13} \sum_{k=0}^{3} \frac{c_k^+}{c_4^+} P_{n-4+k}(i\sqrt{L}, L^5/M)$$

$$= (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).$$

By induction, Equation (13) holds for all $n > 1$ proving Theorem 1.4. $\quad\square$

Appendix A. The coefficients c_k and the initial conditions for P_n and Q_n

$c_4 = M^4$

$c_3 = 1 + M^4 + 2LM^{12} + LM^{14} - LM^{16} + L^2M^{20} - L^2M^{22} - 2L^2M^{24}$
$\quad - L^3M^{32} - L^3M^{36}$

$c_2 = (-1 + LM^{12})(-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2M^{16} + 2L^2M^{18}$
$\quad - 4L^2M^{20} - 2L^2M^{22} + 3L^2M^{24} - 3L^3M^{28} + 2L^3M^{30} + 4L^3M^{32}$
$\quad - 2L^3M^{34} + L^3M^{36} - 2L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} + L^5M^{52})$

$c_1 = -L^2(-1 + M)^2M^{16}(1 + M)^2(1 + LM^{10})^2(-1 - M^4 - 2LM^{12} - LM^{14}$
$\quad + LM^{16} - L^2M^{20} + L^2M^{22} + 2L^2M^{24} + L^3M^{32} + L^3M^{36})$

$c_0 = L^4(-1 + M)^4M^{36}(1 + M)^4(1 + LM^{10})^4$

$$P_0 = \frac{\left(-1 + LM^{12}\right)\left(1 + LM^{12}\right)^2}{(1 + LM^{10})^3}$$

$$P_1 = \frac{\left(-1 + LM^{11}\right)^2\left(1 + LM^{11}\right)^2}{1 + LM^{10}}$$

$P_2 = -1 + LM^8 - 2LM^{10} + LM^{12} + 2L^2M^{20} + L^2M^{22} - L^4M^{40}$
$\quad - 2L^4M^{42} - L^5M^{50} + 2L^5M^{52} - L^5M^{54} + L^6M^{62}$

$P_3 = (-1 + LM^{12})(-1 + LM^4 - LM^6 + 2LM^8 - 5LM^{10} + LM^{12}$
$\quad + 5L^2M^{16} - 4L^2M^{18} + L^2M^{22} + L^3M^{26} + 3L^3M^{30} + 2L^3M^{32}$
$\quad - 2L^4M^{36} - 3L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} - 2L^5M^{46} - 3L^5M^{48}$
$\quad - L^5M^{52} - L^6M^{56} + 4L^6M^{60} - 5L^6M^{62} - L^7M^{66} + 5L^7M^{68}$
$\quad - 2L^7M^{70} + L^7M^{72} - L^7M^{74} + L^8M^{78})$

$$Q_0 = -\frac{\left(-1 + LM^{12}\right)\left(1 + LM^{12}\right)^2}{L^3(-1 + M)^3M^4(1 + M)^3}$$

$$Q_{-1} = -\frac{M^{12}\left(1 + LM^{14}\right)^2}{L(-1 + M)(1 + M)}$$

$Q_{-2} = M^{20}(1 - LM^8 + 2LM^{10} + 2LM^{12} - LM^{16} + LM^{18} + L^2M^{20}$
$\quad - L^2M^{22} + 2L^2M^{26} + 2L^2M^{28} - L^2M^{30} + L^3M^{38})$

$Q_{-3} = M^{16}(-1 + LM^{12})(1 + LM^{10} + 5LM^{12} - LM^{14} - 2LM^{16} + 2LM^{18}$
$\quad - LM^{20} + 2L^2M^{20} + LM^{22} + 4L^2M^{22} + 3L^2M^{26} - 3L^2M^{28}$

$$- L^3 M^{28} + 5L^3 M^{30} + 5L^2 M^{32} - L^2 M^{34} - 3L^3 M^{34} + 3L^3 M^{36}$$
$$+ 4L^3 M^{40} + L^4 M^{40} + 2L^3 M^{42} - L^4 M^{42} + 2L^4 M^{44} - 2L^4 M^{46}$$
$$- L^4 M^{48} + 5L^4 M^{50} + L^4 M^{52} + L^5 M^{62})$$

Appendix B. The coefficients γ_k and the initial conditions for R_n

$$\gamma_4 = L^4(-1 + M)^4 M^{36}(1 + M)^4 \left(1 + LM^{10}\right)^4$$

$$\gamma_3 = L^3(-1 + M)^3 M^{24}(1 + M)^3 \left(1 + LM^{10}\right)^3 \left(-1 - M^4 - 2LM^{12} - LM^{14}\right.$$
$$+ LM^{16} - L^2 M^{20} + L^2 M^{22} + 2L^2 M^{24} + L^3 M^{32} + L^3 M^{36})$$

$$\gamma_2 = L^2(-1 + M)^2 M^{16}(1 + M)^2 \left(1 + LM^{10}\right)^2 \left(-1 + LM^{12}\right) \left(-1 - 2LM^{10}\right.$$
$$- 3LM^{12} + 2LM^{14} - L^2 M^{16} + 2L^2 M^{18} - 4L^2 M^{20} - 2L^2 M^{22}$$
$$+ 3L^2 M^{24} - 3L^3 M^{28} + 2L^3 M^{30} + 4L^3 M^{32} - 2L^3 M^{34} + L^3 M^{36}$$
$$- 2L^4 M^{38} + 3L^4 M^{40} + 2L^4 M^{42} + L^5 M^{52})$$

$$\gamma_1 = \gamma_3$$

$$\gamma_0 = \gamma_4$$

Let P_n for $n = 0, \ldots, 3$ be as in Appendix A. Then,

$$(14) \qquad\qquad\qquad R_n = P_n b^n$$

for $n = 0, \ldots, 3$ where b is given by Equation (9).

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