

# Noncommutative Cartan subalgebras of $C^*$ -algebras

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ABSTRACT. J. Renault has recently found a generalization of the characterization of  $C^*$ -diagonals obtained by A. Kumjian in the eighties, which in turn is a  $C^*$ -algebraic version of J. Feldman and C. Moore's well known theorem on Cartan subalgebras of von Neumann algebras. Here we propose to give a version of Renault's result in which the Cartan subalgebra is not necessarily commutative [sic]. Instead of describing a Cartan pair as a twisted groupoid  $C^*$ -algebra we use N. Sieben's notion of Fell bundles over inverse semigroups which we believe should be thought of as *twisted étale groupoids with noncommutative unit space*. En passant we prove a theorem on uniqueness of conditional expectations.

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## 1. Introduction

Building on work by Feldman–Moore [6, 7] and Kumjian [14], Renault has recently introduced a natural notion of Cartan subalgebras of  $C^*$ -algebras [17]. If  $B$  is a Cartan subalgebra of the  $C^*$ -algebra  $A$ , Renault proved the existence of an essentially principal étale groupoid  $G(B)$  and a 2-cocycle  $\Sigma(B)$ , respectively called the *Weyl groupoid* and the *Weyl twist*, such that  $A$  is isomorphic to the reduced twisted groupoid  $C^*$ -algebra  $C_r^*(G(B), \Sigma(B))$ , in such a way that  $B$  is carried onto the algebra of continuous functions vanishing at infinity on the unit space of  $G(B)$ . In other words, this gives a characterization of Cartan subalgebras in the context of  $C^*$ -algebras paralleling Feldman and Moore’s characterization of Cartan subalgebras of von Neumann algebras.

Among the important consequences of Renault’s work is the fact that the conditional expectation from a  $C^*$ -algebra to a Cartan subalgebra is unique [17, 5.7]. Uniqueness of conditional expectations is a common and useful phenomenon in the theory of von Neumann algebras and it holds whenever the von Neumann subalgebra  $B$  of the von Neumann algebra  $A$  satisfies  $B' \cap A \subseteq B$ . See [19, IX.4.3].

Renault’s result on uniqueness of conditional expectations may thus be considered as a version of this well known result, but the requirement that  $B$  be a maximal abelian subalgebra (one of the conditions for  $B$  to be a Cartan subalgebra) is way stronger than  $B' \cap A \subseteq B$ . Incidentally notice that  $B$  is a maximal abelian subalgebra of  $A$  if and only if the following two conditions hold:

- (MAX)  $B' \cap A \subseteq B$ ,
- (AB)  $B$  is abelian.

Thus, if one is hoping for a direct generalization of the von Neumann uniqueness of conditional expectations mentioned above to the context of  $C^*$ -algebras, Renault’s result should be strengthened by removing condition (AB) above from the hypotheses.

Unfortunately this is impossible. If  $A = C([0, 1]) \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the algebra of compact operators on an infinite dimensional Hilbert space, and  $B = 1 \otimes \mathcal{K}$ , then the pair  $(A, B)$  satisfies Renault’s axioms of Cartan pairs except that, in place of maximal abelianness, only axiom (MAX) holds. Nevertheless conditional expectations abound as each probability measure

on  $[0, 1]$  gives, by means of integration, a different conditional expectation from  $C([0, 1]) \otimes \mathcal{K}$  to  $1 \otimes \mathcal{K}$ .

However the validity of (MAX) in this example sounds fishy:  $B' \cap A = \{0\}$ , so it is contained in  $B$  alright but, not far outside  $A$  one does find *virtual commutants* of  $B$  not belonging to  $B$ , such as any element of the form  $f \otimes 1$ . Technically speaking, what we mean by a virtual commutant (Definition (9.2)) for an inclusion “ $B \subseteq A$ ” of  $C^*$ -algebras is a  $B$ -bimodule map

$$\varphi : J \rightarrow A,$$

where  $J$  is a closed two-sided ideal of  $B$ . If  $a \in B' \cap A$  and  $J$  is any ideal in  $B$ , then  $\varphi(x) = ax$ , gives an example of a virtual commutant defined on  $J$ . Hence these should be thought of as *generalized elements* of  $B' \cap A$ .

However not every virtual commutant is of this form. In the above example in which  $B = 1 \otimes \mathcal{K}$  and  $A = C([0, 1]) \otimes \mathcal{K}$ , let  $f$  be a nonconstant function and put

$$\varphi(1 \otimes x) = f \otimes x \quad \forall x \in \mathcal{K}.$$

Then  $\varphi$  is a virtual commutant defined on  $J = B$ , but it is not of the elementary form above. One may then replace (MAX) above by the following stronger axiom:

(MAX') the range of every virtual commutant is contained in  $B$ ,

in which case our badly behaved example will be knocked out. This replacement is not at all a drastic departure from tradition since (MAX'+AB) is equivalent to (MAX+AB), and hence to maximal abelianness. See (9.9) for a proof of this statement.

We thus propose to reformulate Renault’s definition of Cartan subalgebras [17, Definition 5.1] by replacing maximal abelianness with (MAX') above, without requiring (AB).

I am well aware that Cartan subalgebras, wherever they have been considered, have always been assumed to be abelian algebras. So our use of this term to refer to noncommutative subalgebras might sound slightly heretic. But since we shall provide generalizations of well known classical results for Cartan subalgebras, we hope our heresy will be forgiven.

Assuming that  $(A, B)$  is such a generalized Cartan pair, and that  $A$  is separable, we prove in Theorem (12.3) that the conditional expectation onto  $B$  is unique, thus extending Renault’s result [17, 5.7].

Our second main contribution is to generalize Renault’s version of Feldman and Moore’s theorem [17, 5.6] but, since  $B$  may be noncommutative, one would not expect to get a groupoid model of it. Instead we use Sieben’s (unpublished) notion of Fell bundles over inverse semigroups, which we believe should be thought of as *twisted étale groupoids with noncommutative unit space*.

Recall that if  $\mathcal{G}$  is an étale groupoid, then the set of open bisections forms an inverse semigroup  $\mathfrak{S}$ . If for every  $U$  in  $\mathfrak{S}$  we let  $A_U$  be the set of elements in  $C^*(\mathcal{G})$  supported in  $U$ , then  $\mathcal{A} = \{A_U\}_{U \in \mathfrak{S}}$  is a Fell bundle over  $\mathfrak{S}$  whose

cross-sectional  $C^*$ -algebra is isomorphic to  $C^*(\mathcal{G})$ : although Theorem (8.9) in [5] is stated in a slightly different language, this is precisely what it means. Even if this result does not involve twists, in all likelihood it may be proved for twisted groupoids as well.

Our second main result, Theorem (14.7), shows that, given a separable generalized Cartan pair  $(A, B)$ , there exists a Fell bundle over a countable inverse semigroup  $\mathfrak{S}$  whose reduced cross-sectional  $C^*$ -algebra is isomorphic to  $A$  in such a way that  $B$  corresponds to the restriction of the bundle to the idempotent semilattice of  $\mathfrak{S}$ .

In a sense our result is not as complete as Renault's and its predecessors since we go only as far as identifying the appropriate Fell bundle. In the commutative case (i.e., when  $B$  is commutative) these results could perhaps be interpreted as further digging into the structure of the Fell bundle and describing it by means of a twisted groupoid. Of course this task is not feasible when  $B$  is noncommutative, but perhaps there is more to say about our Fell bundle than presently meets our eyes.

Let us now briefly discuss the methods used to prove our main results. While Renault's strategy is to first find the groupoid model and then use it to prove the uniqueness of conditional expectations, we do things in the opposite order.

Starting with a  $C^*$ -algebra inclusion " $B \subseteq A$ ", we say that a *slice* (Definition (10.1)) is a closed linear subspace of the normalizer of  $B$  in  $A$  which is invariant under left and right multiplication by elements of  $B$ .

Under suitable hypotheses every slice  $M$  satisfies  $M^*M \subseteq B$  so it may be thought of as Hilbert  $B$ -module, with inner product  $\langle m, n \rangle = m^*n$ . In particular we may use Kasparov's stabilization theorem to prove the existence of a *frame*, namely a sequence  $\{u_i\}_{i \in \mathbf{N}}$  of elements in  $M$  such that

$$m = \sum_{i \in \mathbf{N}} u_i \langle u_i, m \rangle \quad \forall m \in M.$$

Experimenting with variants of the notion of Jones–Watatani index for conditional expectations [12, 20], we thought of analyzing the series

$$\tau = \sum_{i=1}^{\infty} P(u_i)P(u_i^*),$$

where  $P$  is a conditional expectation onto  $B$ . Surprisingly it converges in the strict topology of the multiplier algebra of the ideal  $R(M) := \overline{MM^*} \subseteq B$ , as proved in Lemma (11.5). Since the Jones–Watatani index is central [20, 1.2.8] it is perhaps not surprising that  $\tau$  lies in the center of the multiplier algebra of  $R(M)$ . We moreover prove the following curious and highly meaningful identity:

$$\tau mn^* = P(m)P(n^*) \quad \forall m, n \in M.$$

One of its consequences is that  $\|P(m)\| = \|\tau^{1/2}m\|$ , for every  $m$  in  $M$ , and hence the correspondence

$$P(m) \mapsto \tau^{1/2}m$$

extends to an isometric map which turns out to be a virtual commutant defined in the ideal  $\overline{P(M)}$ . Assuming (MAX') we therefore deduce that  $\tau^{1/2}m \in B$ , for every  $m$ , which puts us just a few steps away from the proof of the uniqueness of conditional expectations.

The machinery developed to prove this result may then be used to achieve the decomposition of a generalized Cartan pair  $(A, B)$  via a Fell bundle. One proves without much difficulty that the set  $\mathfrak{S}_{A,B}$  of all slices forms an inverse semigroup (Proposition (13.3)), and then the Fell bundle comes naturally by setting  $A_M = M$ , for every  $M \in \mathfrak{S}_{A,B}$ .

This paper is divided in two parts, the first one, comprising Sections (2–8), is designed not only to explain Sieben’s theory of Fell bundles over inverse semigroups (Sections (2–3)), but also to define the reduced cross-sectional  $C^*$ -algebra and to prove that the fibers of the bundle are faithfully represented. The work in doing so is considerable because, unlike the theory of Fell bundles over *groups*, there is no readily available conditional expectation, a useful tool for constructing *regular representations* in other contexts.

In order to overcome this difficulty we begin in Section (4) to study the much simpler case of Fell bundles over *semilattices*, i.e., inverse semigroups consisting of idempotent elements. In Sections (5–8) we then develop a machinery designed to extend states from the cross-sectional algebra of the bundle restricted to the idempotent semilattice up to the whole cross-sectional algebra. These extended states are then used to get nontrivial representations.

Part two, comprising Sections (9–14), is where we prove our main results. We begin with Section (9) by introducing and studying our notion of virtual commutants. Theorem (9.5), one of the main results of this section, gives a concrete model for these gadgets, stating that every virtual commutant is in fact represented by an element which commutes with the subalgebra  $B$ , although it lives outside of  $A$ . Not all of the results proved in this section are used towards our main results but they are included in the hope that the notion of virtual commutant might find applications elsewhere.

In Section (10) we study slices and in Section (11) we prove the strict convergence of the series describing  $\tau$  above and develop its consequences. Section (12) starts with our proposed generalization of the notion of Cartan subalgebras and is where our Theorem (12.3) on uniqueness of conditional expectations is to be found.

In Section (13) we construct the inverse semigroup and the Fell bundle from a generalized Cartan pair which will be used in the next and final section to provide our generalization (Theorem (14.7)) of Renault’s version of Feldman–Moore’s theorem.

After this paper circulated as a preprint, some new results on Fell bundles over inverse semigroups, relevant to the present work, appeared in [2] and [3]. In particular it is proven in [2] that a Fell bundle whose fibers over idempotent elements are commutative algebras must necessarily come from a twisted étale groupoid.

## Part I. Fell bundles over inverse semigroups and their cross-sectional $C^*$ -algebras

### 2. Fell bundles over inverse semigroups

In this section we introduce the notion of Fell bundles over inverse semigroups, which is the main object of interest in this work. This is essentially the same as Sieben's homonymous notion introduced in a talk given in the Groupoid Fest, held at Arizona State University in November 1998, with the title "Fell bundles over inverse semigroups and  $r$ -discrete groupoids". We thank Sieben for giving us access to his unpublished work which we describe here with some small modifications.

Throughout this section we let  $S$  be an inverse semigroup, whose idempotent semilattice will be denoted by  $E(S)$ . We refer the reader to [15] for a comprehensive account of the theory of inverse semigroups.

**2.1. Definition.** A *Fell bundle over  $S$*  is a quadruple

$$\mathcal{A} = \left( \{A_s\}_{s \in S}, \{\mu_{s,t}\}_{s,t \in S}, \{\text{star}_s\}_{s \in S}, \{j_{t,s}\}_{s,t \in S, s \leq t} \right)$$

where, for each  $s, t \in S$ :

- (a)  $A_s$  is a complex Banach space;
- (b)  $\mu_{s,t} : A_s \otimes A_t \rightarrow A_{st}$  is a linear map;
- (c)  $\text{star}_s : A_s \rightarrow A_{s^*}$  is a conjugate-linear isometric map;
- (d)  $j_{t,s} : A_s \hookrightarrow A_t$  is a linear isometric map for every  $s \leq t$ .

It is moreover required that for every  $r, s, t \in S$ , and every  $a \in A_r$ ,  $b \in A_s$ , and  $c \in A_t$ :

- (i)  $\mu_{rs,t}(\mu_{r,s}(a \otimes b) \otimes c) = \mu_{r,st}(a \otimes \mu_{s,t}(b \otimes c))$ ;
- (ii)  $\text{star}_{rs}(\mu_{r,s}(a \otimes b)) = \mu_{s^*,r^*}(\text{star}_s(b) \otimes \text{star}_r(a))$ ;
- (iii)  $\text{star}_{s^*}(\text{star}_s(a)) = a$ ;
- (iv)  $\|\mu_{r,s}(a \otimes b)\| \leq \|a\| \|b\|$ ;
- (v)  $\|\mu_{r^*,r}(\text{star}_r(a) \otimes a)\| = \|a\|^2$ ;
- (vi)  $\mu_{r^*,r}(\text{star}_r(a) \otimes a) \geq 0$ , in  $B_{r^*r}$ ;
- (vii) if  $r \leq s \leq t$ , then  $j_{t,r} = j_{t,s} \circ j_{s,r}$ ;

(viii) if  $r \leq r'$ , and  $s \leq s'$ , then the diagrams

$$\begin{array}{ccc}
 A_r \otimes A_s & \xrightarrow{\mu_{r,s}} & A_{rs} \\
 \downarrow j_{r',r} \otimes j_{s',s} & & \downarrow j_{r's',rs} \\
 A_{r'} \otimes A_{s'} & \xrightarrow{\mu_{r',s'}} & A_{r's'}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A_s & \xrightarrow{\text{star}_s} & A_{s^*} \\
 \downarrow j_{s',s} & & \downarrow j_{s',s} \\
 A_{s'} & \xrightarrow{\text{star}_{s'}} & A_{s'^*}
 \end{array}$$

commute.

If no confusion is likely to arise we shall use the simplified notations

$$(2.2) \quad ab := \mu_{r,s}(a \otimes b) \quad \text{and} \quad a^* := \text{star}_r(a),$$

whenever  $a \in A_r$  and  $b \in A_s$ . Axioms (2.1.i-vi) then take on the more familiar aspect:

- (i')  $(ab)c = a(bc)$ ;
- (ii')  $(ab)^* = b^*a^*$ ;
- (iii')  $a^{**} = a$ ;
- (iv')  $\|ab\| \leq \|a\|\|b\|$ ;
- (v')  $\|aa^*\| = \|a\|^2$ ;
- (vi')  $aa^* \geq 0$ , in  $B_{rr^*}$ .

If  $s \leq t$ , we shall often use the map  $j_{t,s}$  to identify  $A_s$  as a subspace of  $A_t$ . Axiom (2.1.viii) then says that the multiplication and star operations are compatible with such an identification.

Let us now discuss a few immediate consequences of the definition.

**2.3. Proposition.**

- (a) If  $e \in E(S)$ , then  $A_e$  is a  $C^*$ -algebra. Incidentally this is the  $C^*$ -algebra structure with respect to which (2.1.vi) refers to positivity.
- (b) For every  $s \in S$ , one has that  $j_{s,s}$  is the identity map on  $A_s$ .
- (c) If  $e, f \in E(S)$ , and  $e \leq f$ , then  $j_{f,e}(A_e)$  is a closed two-sided ideal in  $A_f$ .

**Proof.** We skip (a), since it is obvious. Given  $s \in S$ , notice that  $j_{s,s}$  is an isometric linear map from  $A_s$  to itself which is idempotent by (2.1.vii). Therefore  $j_{s,s}$  must be the identity map on  $A_s$ , proving (b).

With respect to (c), let  $a \in A_e$  and  $b \in A_f$ . Then

$$j_{f,e}(a)b = \mu_{f,f}(j_{f,e}(a) \otimes j_{f,f}(b)) \stackrel{(2.1.viii)}{=} j_{f,e}(\mu_{e,f}(a \otimes b)) \in j_{f,e}(A_e),$$

and similarly  $b j_{f,e}(a) \in j_{f,e}(A_e)$ . This shows that  $j_{f,e}(A_e)$  is a two-sided ideal in  $A_f$ . It is closed because  $A_e$  is a Banach space and  $j_{f,e}$  is an isometric map. □

If  $r, s, t \in S$  are such that  $r, s \leq t$ , let

$$v = ts^*sr^*r \quad (= sr^*r = rs^*s).$$

Then clearly  $v \leq r, s$ , and it is easy to see that  $v$  is the maximum among the elements of  $S$  which are both smaller than  $r$  and  $s$ . Therefore  $v$  is effectively the *infimum* of  $r$  and  $s$  with respect to the natural order structure of  $S$ , so we shall denote  $v$  by  $r \wedge s$ . This is despite the fact that  $S$  is *not* necessarily a semilattice with respect to its order: the existence of  $r \wedge s$  is only guaranteed once the set  $\{r, s\}$  is known to admit an upper bound, namely  $t$ .

**2.4. Proposition.** *Let  $r, s, t \in S$  be such that  $r, s \leq t$ , and set*

$$v = r \wedge s = ts^*sr^*r.$$

*Then*

$$\begin{array}{ccc} & A_r & \\ j_{r,v} \nearrow & & \searrow j_{t,r} \\ A_v & & A_t \\ j_{s,v} \searrow & & \nearrow j_{t,s} \\ & A_s & \end{array}$$

*is a pull-back diagram.*

**Proof.** Initially notice that the diagram commutes since

$$j_{t,r}j_{r,v} = j_{t,v} = j_{t,s}j_{s,v}$$

by (2.1.vii). To prove the statement we must show that whenever  $a_r \in A_r$ , and  $a_s \in A_s$  are such that

$$j_{t,r}(a_r) = j_{t,s}(a_s),$$

there exists a unique  $a_v \in A_v$  such that

$$j_{r,v}(a_v) = a_r \quad \text{and} \quad j_{s,v}(a_v) = a_s.$$

Since all maps involved are injective, this is equivalent to saying that

$$j_{t,v}(A_v) = j_{t,r}(A_r) \cap j_{t,s}(A_s).$$

From the commutativity of our diagram it easily follows that  $j_{t,v}(A_v) \subseteq j_{t,r}(A_r) \cap j_{t,s}(A_s)$ , so it suffices to prove the reverse inclusion, which we will do shortly.

As already seen  $A_{s^*s}$  is a  $C^*$ -algebra, and the reader will have no difficulty in proving that  $A_s$  is a right Hilbert module over  $A_{s^*s}$  relative to the operations

$$xa := \mu_{s,s^*s}(x \otimes a) \quad \text{and} \quad \langle x, y \rangle := \mu_{s^*,s}(\text{star}_s(x) \otimes y),$$

defined for all  $a \in A_{s^*s}$ , and  $x, y \in A_s$ . Given an approximate unit  $\{u_i\}_i$  for  $A_{s^*s}$ , we then have by [11, 1.1.4] that  $x = \lim_i xu_i$ , for every  $x \in A_s$ .

Given  $y \in j_{t,r}(A_r) \cap j_{t,s}(A_s)$ , write

$$y = j_{t,r}(a_r) = j_{t,s}(a_s),$$



where  $a_r \in A_r$  and  $a_s \in A_s$ . For every member  $u_i$  of the approximate unit above we have

$$\begin{aligned} j_{t,s}(a_s u_i) &= j_{t,s}(\mu_{s,s^*s}(a_s \otimes u_i)) \\ &= \mu_{t,t^*t}(j_{t,s}(a_s) \otimes j_{t^*t,s^*s}(u_i)) && (2.1.viii) \\ &= \mu_{t,t^*t}(j_{t,r}(a_r) \otimes j_{t^*t,s^*s}(u_i)) \\ &= j_{t,r s^*s}(\mu_{r,s^*s}(a_r \otimes u_i)) && (2.1.viii) \\ &\in j_{t,v}(A_v). \end{aligned}$$

So,

$$y = j_{t,s}(a_s) = \lim_i j_{t,s}(a_s u_i) \in j_{t,v}(A_v). \quad \square$$

### 3. Cross sectional $C^*$ -algebra of a Fell bundle

Throughout this section we fix a Fell bundle  $\mathcal{A}$  over the inverse semigroup  $S$ .

**3.1. Definition.** Let  $B$  be a complex  $*$ -algebra. A *pre-representation* of  $\mathcal{A}$  in  $B$  is a family  $\Pi = \{\pi_s\}_{s \in S}$ , where for each  $s \in S$

$$\pi_s : A_s \rightarrow B$$

is a linear map such that for all  $s, t \in S$ , all  $a \in A_s$ , and all  $b \in A_t$ , one has

- (i)  $\pi_{st}(\mu_{s,t}(a \otimes b)) = \pi_s(a)\pi_t(b)$ ,
- (ii)  $\pi_{s^*}(\text{star}_s(a)) = \pi_s(a)^*$ .

If moreover  $\Pi$  satisfies

- (iii)  $\pi_t \circ j_{t,s} = \pi_s$ , whenever  $s \leq t$ ,

we will say that  $\Pi$  is a *representation*.

With the simplified notations given in (2.2), axioms (3.1.i–ii) take on the simpler form:

- (i')  $\pi_{st}(ab) = \pi_s(a)\pi_t(b)$ ,
- (ii')  $\pi_{s^*}(a^*) = \pi_s(a)^*$ .

If  $e \in E(S)$  we have already seen that  $A_e$  is a  $C^*$ -algebra. Given a pre-representation  $\Pi$  of  $\mathcal{A}$  in a  $C^*$ -algebra  $B$ , it is then immediate that  $\pi_e$  is a  $*$ -homomorphism from  $A_e$  to  $B$ . In particular  $\pi_e$  is necessarily contractive<sup>1</sup>. Given any  $s \in S$ , and  $a \in A_s$ , we then have that

$$\begin{aligned} \|\pi_s(a)\|^2 &= \|\pi_s(a)^* \pi_s(a)\| \\ &= \|\pi_{s^*}(\text{star}_s(a)) \pi_s(a)\| \\ &= \|\pi_{s^*s}(\mu_{s^*,s}(\text{star}_s(a) \otimes a))\| \\ &\leq \|\mu_{s^*,s}(\text{star}_s(a) \otimes a)\| \\ &= \|a\|^2 && (2.1.v). \end{aligned}$$

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<sup>1</sup>A linear map  $T$  from a normed space  $X$  to a normed space  $Y$  is *contractive* if  $\|T(x)\| \leq \|x\|$ , for every  $x \in X$ .

With this we have shown:

**3.2. Proposition.** *Given a pre-representation  $\Pi$  of the Fell bundle  $\mathcal{A}$  in a  $C^*$ -algebra  $B$ , one has that*

$$\|\pi_s(a)\| \leq \|a\| \quad \forall s \in S \quad \forall a \in A_s.$$

**3.3. Proposition.** *Let  $\mathcal{A}$  be a Fell bundle over the inverse semigroup  $S$  and let  $\Pi$  be a representation of  $\mathcal{A}$  in the  $C^*$  algebra  $B$ . Then for every idempotents  $e, f \in E(S)$  one has that*

$$\pi_e(A_e) \cdot \pi_f(A_f) = \pi_{ef}(A_{ef}) = \pi_e(A_e) \cap \pi_f(A_f).$$

**Proof.** Let  $a \in A_e$  and  $b \in A_f$ . Then, by (3.1.i) we have that

$$\pi_e(a)\pi_f(b) = \pi_{ef}(\mu_{e,f}(a \otimes b)) \in \pi_{ef}(A_{ef}),$$

proving that  $\pi_e(A_e) \cdot \pi_f(A_f) \subseteq \pi_{ef}(A_{ef})$ . For any  $c \in A_{ef}$  we have by (3.1.iii) that

$$\pi_{ef}(c) = \pi_e(j_{e,ef}(c)) \in \pi_e(A_e),$$

so  $\pi_{ef}(A_{ef}) \subseteq \pi_e(A_e)$ , and similarly  $\pi_{ef}(A_{ef}) \subseteq \pi_f(A_f)$ , so

$$\pi_{ef}(A_{ef}) \subseteq \pi_e(A_e) \cap \pi_f(A_f).$$

To conclude we will prove that

$$\pi_e(A_e) \cap \pi_f(A_f) \subseteq \pi_e(A_e) \cdot \pi_f(A_f).$$

It is evident that  $\pi_e$  is a  $*$ -homomorphism from  $A_e$  into  $B$ . Therefore  $\pi_e(A_e)$  is a closed  $*$ -subalgebra of  $B$ , the same applying to  $\pi_f(A_f)$ . It follows that  $\pi_e(A_e) \cap \pi_f(A_f)$  is likewise a closed  $*$ -subalgebra of  $B$ .

Given any  $b \in \pi_e(A_e) \cap \pi_f(A_f)$  we may use Cohen–Hewitt’s factorization theorem [10, 32.22] to write  $b = b_1 b_2$ , where  $b_1, b_2 \in \pi_e(A_e) \cap \pi_f(A_f)$ , so we see that  $b \in \pi_e(A_e) \cdot \pi_f(A_f)$ .  $\square$

**3.4. Definition.** The *cross-sectional  $C^*$ -algebra of  $\mathcal{A}$* , denoted  $C^*(\mathcal{A})$ , is the universal  $C^*$ -algebra generated by the disjoint union

$$\dot{\bigcup}_{s \in S} A_s,$$

subject to the relations stating that the natural maps

$$\pi_s^u : A_s \rightarrow C^*(\mathcal{A})$$

form a representation of  $\mathcal{A}$  in  $C^*(\mathcal{A})$ .

From (3.2) and [1, 1.2] one deduces the existence of  $C^*(\mathcal{A})$  as well as its uniqueness up to isomorphism. For further reference let us spell out the universal property of  $C^*(\mathcal{A})$  in detail:

**3.5. Proposition.**  *$C^*(\mathcal{A})$  is a  $C^*$ -algebra and  $\Pi^u = \{\pi_s^u\}_{s \in S}$  is a representation of  $\mathcal{A}$  in  $C^*(\mathcal{A})$ . Moreover, given any representation  $\Pi = \{\pi_s\}_{s \in S}$  of the Fell bundle  $\mathcal{A}$  in a  $C^*$ -algebra  $B$ , there exists a unique  $*$ -homomorphism  $\Phi : C^*(\mathcal{A}) \rightarrow B$  such that  $\Phi \circ \pi_s^u = \pi_s$ , for all  $s \in S$ .*

One of our main goals is to show the existence of certain nontrivial representations of  $\mathcal{A}$ , and also to show that each  $\pi_s^u$  is injective. In order to achieve this goal it is useful to have a more concrete description of  $C^*(\mathcal{A})$ , as follows. Let

$$\mathcal{L}(\mathcal{A}) = \bigoplus_{s \in S} A_s.$$

For each  $s \in S$  and each  $a_s \in A_s$ , denote by  $a_s \delta_s$  the element of  $\mathcal{L}(\mathcal{A})$  whose coordinates are all equal to zero except for the  $s^{\text{th}}$  coordinate which is equal to  $a_s$ . It is then clear that for a generic element  $a = (a_s)_s$  in  $\mathcal{L}(\mathcal{A})$  one has

$$a = \sum_{s \in S} a_s \delta_s.$$

In addition, this representation is clearly unique.

Define a multiplication and a star operation on  $\mathcal{L}(\mathcal{A})$  in such a way that for all  $s, t \in S$ , all  $a \in A_s$ , and all  $b \in A_t$ ,

$$(a \delta_s)(b \delta_t) = \mu_{s,t}(a \otimes b) \delta_{st} \quad \text{and} \quad (a \delta_s)^* = \text{star}_s(a) \delta_{s^*}.$$

With the aid of (2.1.i–iii) one easily proves that  $\mathcal{L}(\mathcal{A})$  is a complex associative  $*$ -algebra. Introducing a norm on  $\mathcal{L}(\mathcal{A})$  by

$$\left\| \sum_{s \in S} a_s \delta_s \right\| = \sum_{s \in S} \|a_s\|,$$

one readily checks that  $\mathcal{L}(\mathcal{A})$  is a normed  $*$ -algebra.

**3.6. Definition.** For each  $s \in S$ , let

$$\pi_s^0 : a_s \in A_s \mapsto a_s \delta_s \in \mathcal{L}(\mathcal{A}).$$

It is easy to see that  $\Pi^0 = \{\pi_s^0\}_{s \in S}$  is a pre-representation of  $\mathcal{A}$  in  $\mathcal{L}(\mathcal{A})$ . This is in fact a universal pre-representation in the following sense:

**3.7. Proposition.** *Let  $B$  be a  $*$ -algebra. If  $\Pi = \{\pi_s\}_{s \in S}$  is a pre-representation of  $\mathcal{A}$  in  $B$ , then the map  $\Phi : \mathcal{L}(\mathcal{A}) \rightarrow B$  given by*

$$\Phi \left( \sum_{s \in S} a_s \delta_s \right) = \sum_{s \in S} \pi_s(a_s)$$

*is a  $*$ -homomorphism. Conversely, given any  $*$ -homomorphism  $\Phi : \mathcal{L}(\mathcal{A}) \rightarrow B$ , consider for each  $s \in S$ , the map  $\pi_s : A_s \rightarrow B$  given by*

$$\pi_s = \Phi \circ \pi_s^0.$$

*Then  $\Pi = \{\pi_s\}_{s \in S}$  is a pre-representation of  $\mathcal{A}$  in  $B$ . In addition the correspondences  $\Pi \mapsto \Phi$ , and  $\Phi \mapsto \Pi$  described above are each other's inverse, giving bijections between the set of all  $*$ -homomorphisms from  $\mathcal{L}(\mathcal{A})$  to  $B$ , and the set of all pre-representations of  $\mathcal{A}$  in  $B$ .*

**Proof.** Left to the reader. □

**3.8. Proposition.** *Let  $B$  be a  $C^*$ -algebra. Then every  $*$ -homomorphism  $\Phi : \mathcal{L}(\mathcal{A}) \rightarrow B$  is contractive.*

**Proof.** Let  $\Pi$  be the pre-representation corresponding to  $\Phi$  according to (3.7). Given any  $a = \sum_{s \in S} a_s \delta_s \in \mathcal{L}(\mathcal{A})$ , we have that

$$\begin{aligned} \|\Phi(a)\| &= \left\| \Phi \left( \sum_{s \in S} a_s \delta_s \right) \right\| \\ &= \left\| \sum_{s \in S} \pi_s(a_s) \right\| \\ &\leq \sum_{s \in S} \|\pi_s(a_s)\| \\ &\leq \sum_{s \in S} \|a_s\| \quad (3.2) \\ &= \|a\|. \quad \square \end{aligned}$$

Observe that the pre-representation  $\Pi^0$  defined in (3.6) is not necessarily a representation since there is no reason why (3.1.iii) holds. In order to force its validity we need to mod out certain elements of  $\mathcal{L}(\mathcal{A})$ .

**3.9. Proposition.** *Let  $\mathcal{N}$  be the linear subspace of  $\mathcal{L}(\mathcal{A})$  spanned by the set*

$$\left\{ a_s \delta_s - j_{t,s}(a_s) \delta_t : s, t \in S, s \leq t, a_s \in A_s \right\}.$$

*Then  $\mathcal{N}$  is a two-sided selfadjoint ideal of  $\mathcal{L}(\mathcal{A})$ .*

**Proof.** Given  $r, s, t \in S$ , such that  $s \leq t$ , let  $a_s \in A_s$ , and  $b_r \in A_r$ . Then

$$\begin{aligned} (a_s \delta_s - j_{t,s}(a_s) \delta_t) b_r \delta_r &= \mu_{s,r}(a_s \otimes b_r) \delta_{sr} - \mu_{t,r}(j_{t,s}(a_s) \otimes b_r) \delta_{tr} \\ &= \mu_{s,r}(a_s \otimes b_r) \delta_{sr} - j_{tr,sr}(\mu_{s,r}(a_s \otimes b_r)) \delta_{tr} \quad (2.1.viii) \\ &\in \mathcal{N}, \end{aligned}$$

from where one sees that  $\mathcal{N}$  is a right ideal. A similar reasoning shows that  $\mathcal{N}$  is a left ideal as well. Also

$$\begin{aligned} (a_s \delta_s - j_{t,s}(a_s) \delta_t)^* &= \text{star}_s(a) \delta_{s^*} - \text{star}_t(j_{t,s}(a_s)) \delta_{t^*} \\ &= \text{star}_s(a) \delta_{s^*} - j_{t^*,s^*}(\text{star}_s(a_s)) \delta_{t^*} \quad (2.1.viii) \\ &\in \mathcal{N}, \end{aligned}$$

so  $\mathcal{N}$  is selfadjoint. □

The following result is self evident:

**3.10. Proposition.** *With respect to the correspondence  $\Phi \leftrightarrow \Pi$  of (3.7) one has that  $\Phi$  vanishes on  $\mathcal{N}$  if and only if  $\Pi$  is a representation.*

Observe that, by (3.2) and [1, 1.2], the enveloping  $C^*$ -algebra of  $\mathcal{L}(\mathcal{A})/\mathcal{N}$ , here denoted  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$ , exists. It is our next main goal to prove that it is  $*$ -isomorphic to  $C^*(\mathcal{A})$ .

**3.11. Definition.** We denote by  $\iota_{\mathcal{A}}$  the composition

$$\mathcal{L}(\mathcal{A}) \xrightarrow{q} \mathcal{L}(\mathcal{A})/\mathcal{N} \longrightarrow C^*(\mathcal{L}(\mathcal{A})/\mathcal{N}),$$

$\overset{\iota_{\mathcal{A}}}{\curvearrowright}$

where  $q$  is the quotient map and the unmarked arrow is the canonical map from  $\mathcal{L}(\mathcal{A})/\mathcal{N}$  to its enveloping  $C^*$ -algebra.

**3.12. Proposition.** For each  $s \in S$ , let  $\pi_s^+ = \iota_{\mathcal{A}} \circ \pi_s^0$ , where  $\pi_s^0$  was defined in (3.6), (see also Diagram (3.15)) and let  $\Pi^+ = \{\pi_s^+\}_{s \in S}$ . Then  $\Pi^+$  is a representation of  $\mathcal{A}$  in  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$ .

**Proof.** Since  $\Pi^0$  is a pre-representation of  $\mathcal{A}$  in  $\mathcal{L}(\mathcal{A})$ , it is obvious that  $\Pi^+$  is a prerepresentation of  $\mathcal{A}$  in  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$ . To prove that  $\Pi^+$  is a representation, let  $s \leq t$ , and let  $a_s \in A_s$ . Then by definition  $a_s \delta_s - j_{t,s}(a_s) \delta_t \in \mathcal{N}$ , and hence

$$\begin{aligned} 0 &= \iota_{\mathcal{A}}(a_s \delta_s - j_{t,s}(a_s) \delta_t) = \iota_{\mathcal{A}}(\pi_s^0(a_s) - \pi_t^0(j_{t,s}(a_s) \delta_t)) \\ &= \pi_s^+(a_s) - \pi_t^+(j_{t,s}(a_s)), \end{aligned}$$

proving that  $\pi_s^+ = \pi_t^+ \circ j_{t,s}$ , and hence that  $\Pi^+$  is in fact a representation.  $\square$

**3.13. Proposition.** There exists an isomorphism

$$\Theta : C^*(\mathcal{L}(\mathcal{A})/\mathcal{N}) \rightarrow C^*(\mathcal{A}),$$

such that  $\Theta \circ \pi_s^+ = \pi_s^u$ , for every  $s \in S$ .

**Proof.** In order to prove the statement it is clearly enough to prove that  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$  possesses the universal property described in (3.5) with respect to the representation  $\Pi^+$ . So we let  $\Pi = \{\pi_s\}_{s \in S}$  be any representation of  $\mathcal{A}$  in a  $C^*$ -algebra  $B$ . Let  $\Psi : \mathcal{L}(\mathcal{A}) \rightarrow B$  be given as in (3.7) in terms of  $\Pi$ . By (3.10) we have that  $\Psi$  vanishes on  $\mathcal{N}$  and hence it factors through  $\mathcal{L}(\mathcal{A})/\mathcal{N}$  giving a  $*$ -homomorphism  $\tilde{\Psi} : \mathcal{L}(\mathcal{A})/\mathcal{N} \rightarrow B$ , such that

$$(3.14) \quad \tilde{\Psi}(q(a_s \delta_s)) = \pi_s(a_s),$$

whenever  $a_s \in A_s$ . By the universality of the enveloping  $C^*$ -algebra, one has that  $\tilde{\Psi}$  further factors through  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$ , providing a map  $\Phi : C^*(\mathcal{L}(\mathcal{A})/\mathcal{N}) \rightarrow B$  such that the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{A})/\mathcal{N} & \xrightarrow{\tilde{\Psi}} & B \\ & \searrow & \nearrow \Phi \\ & C^*(\mathcal{L}(\mathcal{A})/\mathcal{N}) & \end{array}$$

commutes. In particular, for every  $a \in A_s$  we have

$$\pi_s(a_s) \stackrel{(3.14)}{=} \tilde{\Psi}(q(a_s\delta_s)) = \Phi(\iota_{\mathcal{A}}(a_s\delta_s)) = \Phi(\iota_{\mathcal{A}}(\pi_s^0(a_s))) = \Phi(\pi_s^+(a_s)),$$

as desired. That  $\Phi$  is unique follows from the fact that  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$  is generated by the union of the ranges of the  $\pi_s^+$ .  $\square$

In view of the result above we shall henceforth identify  $C^*(\mathcal{L}(\mathcal{A})/\mathcal{N})$  and  $C^*(\mathcal{A})$  bearing in mind that this identification carries  $\pi_s^+$  to  $\pi_s^u$ , for every  $s \in S$ . The following diagram shows all relevant mappings:

$$(3.15) \quad \begin{array}{ccccccc} & & \xrightarrow{\iota_{\mathcal{A}}} & & & & \\ & & \curvearrowright & & & & \\ \mathcal{L}(\mathcal{A}) & \xrightarrow{q} & \mathcal{L}(\mathcal{A})/\mathcal{N} & \longrightarrow & C^*(\mathcal{L}(\mathcal{A})/\mathcal{N}) & \xrightarrow{\Theta} & C^*(\mathcal{A}) \\ & \nwarrow \pi_s^0 & & \nearrow \pi_s^+ & & \nearrow \pi_s^u & \\ & & A_s & & & & \end{array}$$

#### 4. Fell bundles over semilattices

Recall that a semilattice is a partially ordered set  $E$  such that for every  $e, f \in E$  one has that there exists a largest element among the members of  $E$  which are smaller than both  $e$  and  $f$ . This element is denoted by  $e \wedge f$ . Viewed as a semigroup under the operation “ $\wedge$ ”, one has that  $E$  is an inverse semigroup whose elements are idempotents. Conversely any inverse semigroup consisting of idempotents is obtained as above.

We will now study the special case of Fell bundles over semilattices. So, throughout this section we will let  $\mathcal{A}$  be a Fell bundle over a semilattice  $E$ . Our main goal will be to construct a concrete representation of  $\mathcal{A}$  leading to a faithful representation of  $C^*(\mathcal{A})$ .

**4.1. Proposition.** *Given a representation  $\Pi$  of  $\mathcal{A}$  in a  $C^*$ -algebra  $B$ , let*

$$\mathcal{R}_0(\Pi) = \sum_{e \in E} \pi_e(A_e),$$

and let  $\mathcal{R}(\Pi)$  be the closure of  $\mathcal{R}_0(\Pi)$  within  $B$ . Then:

- (i)  $\mathcal{R}_0(\Pi)$  is a  $*$ -subalgebra of  $B$ ;
- (ii)  $\mathcal{R}(\Pi)$  is a closed  $*$ -subalgebra of  $B$ ;
- (iii)  $\pi_e(A_e)$  is a closed two-sided ideal of  $\mathcal{R}(\Pi)$ , for every  $e \in E$ ;
- (iv)  $\mathcal{R}_0(\Pi)$  is a two-sided selfadjoint ideal of  $\mathcal{R}(\Pi)$ .

**Proof.** The first point follows easily from (3.1.i–ii), and (ii) is an immediate consequence of (i).

In order to prove (iii) we must show that if  $a \in A_e$ , and  $x \in \mathcal{R}(\Pi)$ , then both  $\pi_e(a)x$  and  $x\pi_e(a)$  are in  $\pi_e(A_e)$ . Since  $\pi_e(A_e)$  is closed, we may

suppose without loss of generality that  $x = \pi_f(b)$ , for some  $f \in E$ , and  $b \in A_f$ . We have

$$\begin{aligned} \pi_e(a)x &= \pi_e(a)\pi_f(b) = \pi_{ef}(\mu_{e,f}(a \otimes b)) \\ &= \pi_e\left(i_{e,ef}(\mu_{e,f}(a \otimes b))\right) \in \pi_e(A_e). \end{aligned}$$

Here, the last equality follows from (3.1.iii), since  $ef \leq f$ .

That  $x\pi_e(a) \in \pi_e(A_e)$  follows in the same way. Finally, (iv) holds as a consequence of the fact that the sum of ideals is an ideal.  $\square$

Before proceeding we need an elementary fact for which we were unable to find a reference.

**4.2. Lemma.** *Let  $A$  be a  $C^*$ -algebra and let  $J$  be a (not necessarily closed) two-sided selfadjoint ideal of  $A$ . Then any  $*$ -homomorphism  $\Phi : J \rightarrow B$ , where  $B$  is a  $C^*$ -algebra, is contractive.*

**Proof.** By replacing  $A$  with its unitization we may assume that  $A$  is unital. We first claim that if  $a \in J$ , and  $0 \leq a \leq 1$ , then  $\|\Phi(a)\| \leq 1$ . For this let  $b = a\sqrt{1 - a^2}$ , so that  $b \in J$ . Observe that

$$a^4 + b^*b = a^2,$$

hence

$$\Phi(a)^4 \leq \Phi(a)^4 + \Phi(b)^*\Phi(b) = \Phi(a)^2.$$

Since  $\Phi(a)$  is selfadjoint, one has that

$$y := \Phi(a)^2$$

is a nonnegative element of  $B$  satisfying  $y^2 \leq y$ , which necessarily implies that  $\|y\| \leq 1$ . Therefore

$$\|\Phi(a)\|^2 = \|\Phi(a)^2\| = \|y\| \leq 1,$$

proving our claim.

If  $a$  is any element of  $J$ , put  $a' := \|a\|^{-2}a^*a$ , so that  $a'$  satisfies the hypotheses above and hence

$$\|a\|^{-2}\|\Phi(a)\|^2 = \|a\|^{-2}\|\Phi(a^*a)\| = \|\Phi(a')\| \leq 1,$$

from where the result follows.  $\square$

We now present a very simple characterization of the cross-sectional  $C^*$ -algebra of  $\mathcal{A}$ .

**4.3. Proposition.** *Let  $\Pi$  be a representation of  $\mathcal{A}$  in a  $C^*$ -algebra  $B$  such that  $\pi_e$  is one-to-one for every  $e$  in  $E$ . Then  $\mathcal{R}(\Pi)$  is  $*$ -isomorphic to  $C^*(\mathcal{A})$ . In fact there exists a  $*$ -isomorphism*

$$\Phi : \mathcal{R}(\Pi) \rightarrow C^*(\mathcal{A}),$$

such that  $\Phi \circ \pi_e = \pi_e^u$ , for every  $e \in E$ .

**Proof.** Let  $\Pi'$  be any representation of  $\mathcal{A}$  in a  $C^*$ -algebra  $B'$ . Given  $x \in \mathcal{R}_0(\Pi')$ , write

$$x = \sum_e \pi'_e(a_e),$$

where  $a_e \in A_e$ , for each  $e \in E$ , and the set  $\{e \in E : a_e \neq 0\}$  is finite. We begin by claiming that, if  $f \in E$ , and  $b \in A_f$ , then

$$(4.4) \quad x\pi'_f(b) = \pi'_f \left( \sum_e i_{f,ef}(\mu_{e,f}(a_e \otimes b)) \right).$$

To prove it we compute

$$\begin{aligned} x\pi'_f(b) &= \sum_e \pi'_e(a_e)\pi'_f(b) \\ &= \sum_e \pi'_{ef}(\mu_{e,f}(a_e \otimes b)) \\ &= \sum_e \pi'_f(i_{f,ef}(\mu_{e,f}(a_e \otimes b))) \\ &= \pi'_f \left( \sum_e i_{f,ef}(\mu_{e,f}(a_e \otimes b)) \right). \end{aligned}$$

Observe that, since  $\mathcal{R}(\Pi')$  is generated by elements of the form  $\pi'_f(b)$ , we have that

$$x = 0 \iff (\forall f \in E, \forall b \in A_f, x\pi'_f(b) = 0).$$

From (4.4) we thus obtain the following sufficient condition for  $x$  to be zero:

$$(4.5) \quad \sum_e \pi'_e(a_e) = 0 \iff \left( \forall f \in E, \forall b \in A_f, \sum_e i_{f,ef}(\mu_{e,f}(a_e \otimes b)) = 0 \right).$$

Clearly all that has been said about  $\Pi'$  holds, mutatis mutandis, for  $\Pi$ . Moreover, since we are supposing that  $\pi_f$  is one-to-one for every  $f$ , the sufficient condition (4.5) is also necessary in the case of  $\Pi$ . This implies that the expression

$$\Psi \left( \sum_e \pi_e(a_e) \right) = \sum_e \pi'_e(a_e)$$

gives a well defined linear map from  $\mathcal{R}_0(\Pi)$  onto  $\mathcal{R}_0(\Pi')$ , which can easily be proven to be a  $*$ -homomorphism as well. Since  $\mathcal{R}_0(\Pi)$  is a two-sided selfadjoint ideal of  $\mathcal{R}(\Pi)$  by (4.1.iv), we deduce from (4.2) that  $\Psi$  extends to give a  $*$ -homomorphism from  $\mathcal{R}(\Pi)$  to  $\mathcal{R}(\Pi')$ , which clearly satisfies

$$\Psi \circ \pi_e = \pi'_e \quad \forall e \in E.$$

This shows that the conditions of (3.5) are satisfied if we replace  $C^*(\mathcal{A})$  and  $\Pi^u$  by  $\mathcal{R}(\Pi)$  and  $\Pi$ , respectively. Since the universal object is clearly unique up to isomorphism the conclusion follows.  $\square$



We thus see that, in order to obtain a concrete model for  $C^*(\mathcal{A})$ , all one needs is a faithful representation of  $\mathcal{A}$ . In the remainder of this section we shall obtain such a representation in a somewhat canonical way.

For each  $e \in E$ , consider the multiplier algebra  $M(A_e)$  and let  $B$  be the subalgebra of

$$\prod_{e \in E} M(A_e)$$

formed by the elements  $m = (m_e)_{e \in E}$ , satisfying

$$\|m\| := \sup_{e \in E} \|m_e\| < \infty.$$

It is well known that  $B$  is a  $C^*$ -algebra with the norm defined above. Given  $f \in E$  and  $b \in A_f$  let, for every  $e \in E$ ,

$$m_e^b = (L_e^b, R_e^b) \in M(A_e)$$

be given by

$$\begin{aligned} L_e^b(a) &= i_{e,fe}(\mu_{f,e}(b \otimes a)) \\ R_e^b(a) &= i_{e,ef}(\mu_{e,f}(a \otimes b)) \end{aligned}$$

for  $a \in A_e$ . We then let

$$\lambda_f(b) = (m_e^b)_{e \in E} \in B,$$

leaving to the reader the routine verification that  $\Lambda = \{\lambda_e\}_{e \in E}$  is in fact a representation of  $\mathcal{A}$ . Observe that for every  $b \in A_f$  we have that

$$(4.6) \quad \|\lambda_f(b)\| \geq \|m_f^b\| = \|b\|,$$

because  $m_f^b$  is the canonical multiplier of  $A_f$  given by left and right multiplication by  $b$ . It follows that  $\lambda_f$  is one-to-one for every  $f$ .

As a consequence we deduce the main result of this section:

**4.7. Corollary.** *Let  $\mathcal{A}$  be a Fell bundle over the semilattice  $E$ . Then the cross-sectional  $C^*$ -algebra  $C^*(\mathcal{A})$  is  $*$ -isomorphic to  $\mathcal{R}(\Lambda)$ , where  $\Lambda$  is the representation of  $\mathcal{A}$  constructed above. In fact there exists a  $*$ -isomorphism*

$$\Phi : \mathcal{R}(\Lambda) \rightarrow C^*(\mathcal{A}),$$

such that  $\Phi \circ \lambda_e = \pi_e^u$ , for every  $e \in E$ .

**Proof.** Follows from (4.3). □

In possession of a concrete representation we may now show that  $\mathcal{A}$  is faithfully represented within  $C^*(\mathcal{A})$ , at least in the case of semilattices.

**4.8. Corollary.** *Let  $\mathcal{A}$  be a Fell bundle over the semilattice  $E$ . Then for every  $e \in E$  one has that:*

- (i)  $\pi_e^u$  is isometric;
- (ii)  $\pi_e^u(A_e)$  is a closed two-sided ideal in  $C^*(\mathcal{A})$ .

**Proof.** The first point follows immediately from (4.6) and (4.7). The second point is a consequence of (4.1.iii) and (4.7).  $\square$

## 5. Support of linear functionals

In the above section we succeeded in obtaining, in a rather elementary way, a nontrivial representation of  $C^*(\mathcal{A})$ , in case  $\mathcal{A}$  is a Fell bundle over a semilattice. The case of Fell bundles over general inverse semigroups is much more involving, requiring a careful study of linear functionals defined on Fell bundles. In the present section we shall develop a few results about linear functionals on  $C^*$ -algebras which will be crucial in obtaining nontrivial representations of general Fell bundles.

The results presented here will most likely be well known by specialists in the area and are included for completeness.

**5.1. Proposition.** *Let  $A$  be a  $C^*$ -algebra, let  $J$  be a closed two-sided ideal of  $A$ , and let  $\varphi$  be a continuous linear functional on  $A$ . Then the following are equivalent:*

- (i) *For every approximate unit  $\{u_i\}_i$  for  $J$ , one has that*

$$\varphi(a) = \lim_i \varphi(au_i)$$

*for all  $a \in A$ .*

- (ii) *There exists a continuous linear functional  $\psi$  on  $J$  and an element  $b$  in  $J$  such that  $\varphi(a) = \psi(ab)$ , for all  $a \in A$ .*  
 (iii) *For every approximate unit  $\{u_i\}_i$  for  $J$ , one has that*

$$\varphi(a) = \lim_i \varphi(u_i a)$$

*for all  $a \in A$ .*

- (iv) *There exists a continuous linear functional  $\psi$  on  $J$  and an element  $b$  in  $J$  such that  $\varphi(a) = \psi(ba)$ , for all  $a \in A$ .*

**Proof.** Applying Cohen–Hewitt’s factorization theorem [10, 32.22] to the restriction  $\varphi|_J$  we deduce that there exists a continuous linear functional  $\psi$  on  $J$  and an element  $b$  in  $J$  such that  $\varphi(x) = \psi(xb)$ , for every  $x \in J$ .

Assuming (i) choose an approximate unit  $\{u_i\}_i$  for  $J$  and notice that for every  $a \in A$  we have that

$$\varphi(a) = \lim_i \varphi(au_i) = \lim_i \psi(au_i b) = \psi(ab),$$

so (ii) follows. Conversely, assuming that (ii) holds, let  $\{u_i\}_i$  be any approximate unit for  $J$ , and let  $a \in A$ . One then has that

$$\lim_i \varphi(au_i) = \lim_i \psi(au_i b) = \psi(ab) = \varphi(a),$$

proving (i). In a similar way one proves that (iii) and (iv) are equivalent.

We next prove that (ii) implies (iii). For this let  $\{u_i\}_i$  be any approximate unit for  $J$ , and let  $a \in A$ . Then, assuming (ii) and observing that  $ab \in J$ , we have

$$\lim_i \varphi(u_i a) = \lim_i \psi(u_i ab) = \psi(ab) = \varphi(a),$$

proving (iii). Similarly (iv) implies (i), and the proof is concluded.  $\square$

**5.2. Definition.** If the equivalent conditions above hold we will say that  $\varphi$  is supported on  $J$ .

**5.3. Proposition.** Let  $\varphi$  be a continuous linear functional on the  $C^*$ -algebra  $A$ , and let  $I$  and  $J$  be closed two-sided ideals of  $A$  such that  $I \subseteq J$ . If  $\varphi$  is supported on  $I$  then  $\varphi$  is also supported on  $J$ .

**Proof.** By (5.1.ii) let  $\psi$  be a continuous linear functional on  $I$  and let  $b \in I$ , be such that

$$\varphi(a) = \psi(ab) \quad \forall a \in A.$$

By Hahn–Banach’s theorem let  $\chi$  be a continuous linear extension of  $\psi$  to  $J$ . Then

$$\varphi(a) = \chi(ab) \quad \forall a \in A,$$

because  $ab \in I$ , so  $\varphi$  is supported on  $J$ .  $\square$

**5.4. Proposition.** Let  $A$  be a  $C^*$ -algebra, let  $J$  be a closed two-sided ideal of  $A$ , and let  $\varphi$  be a state on  $A$ . Denote by  $\pi$  the GNS representation of  $A$  associated to  $\varphi$ . Then the following are equivalent:

- (i)  $\varphi$  is supported on  $J$ .
- (ii) The restriction of  $\pi$  to  $J$  is nondegenerate.

**Proof.** Let  $H$  be the space of  $\pi$  and  $\xi$  be the standard cyclic vector. Assuming (i) let  $\{u_i\}_i$  be an approximate unit for  $J$ . Then, for every  $a, b \in A$  one has that

$$\lim_i \langle \pi(au_i)\xi, \pi(b)\xi \rangle = \lim_i \varphi(b^* au_i) = \varphi(b^* a) = \langle \pi(a)\xi, \pi(b)\xi \rangle.$$

Given that  $\{\pi(au_i)\xi\}_i$  is a bounded net, we see that it converges weakly to  $\pi(a)\xi$ . This shows that  $\text{span}(\pi(J)H)$  is weakly dense in  $H$ , and hence also norm-dense. Therefore  $\pi|_J$  is nondegenerate, and (ii) follows.

Assuming that  $\pi$  is nondegenerate, let once more  $\{u_i\}_i$  be an approximate unit for  $J$ . It is well known that the net  $\{\pi(u_i)\}_i$  converges strongly to the identity operator. For every  $a \in A$  we therefore have that

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle = \lim_i \langle \pi(a)\pi(u_i)\xi, \xi \rangle = \lim_i \varphi(au_i),$$

proving (i).  $\square$

**5.5. Proposition.** Let  $\varphi$  be a pure state on a  $C^*$ -algebra  $A$ , and let  $J$  be a closed two-sided ideal of  $A$ . Then either  $\varphi$  is supported on  $J$  or  $\varphi$  vanishes on  $J$ .

**Proof.** Let  $\pi$  be the GNS representation of  $A$  associated to  $\varphi$ , let  $H$  be the space of  $\pi$  and let  $\xi$  be its associated cyclic vector. If  $\varphi$  does not vanish on  $J$  then obviously neither does  $\pi$ . Thus  $\overline{\text{span}}(\pi(J)H)$  is a nonzero closed subspace of  $H$ , which is clearly invariant under  $\pi$ . Given that  $\varphi$  is pure we have that  $\pi$  is irreducible, so  $\overline{\text{span}}(\pi(J)H) = H$ , and hence  $\pi|_J$  is nondegenerate. So  $\varphi$  is supported on  $J$ .  $\square$

**5.6. Proposition.** *Let  $\varphi$  be a pure state on a  $C^*$ -algebra  $A$ , and let  $I$  and  $J$  be closed two-sided ideals of  $A$ . Suppose that  $\varphi$  is supported on both  $I$  and  $J$ . Then  $\varphi$  is supported on  $I \cap J$  as well.*

**Proof.** Let  $(\pi, \xi, H)$  be the GNS representation of  $A$  associated to  $\varphi$ , and choose approximate units  $\{u_i\}_i$  and  $\{v_j\}_j$  for  $I$  and  $J$ , respectively. As already mentioned both  $\{\pi(u_i)\}_i$  and  $\{\pi(v_j)\}_j$  converge strongly to the identity operator. In particular

$$\langle \pi(u_{i_0})\xi, \xi \rangle \neq 0,$$

for some  $i_0$ . Since

$$\lim_j \varphi(v_j u_{i_0}) = \lim_j \langle \pi(v_j)\pi(u_{i_0})\xi, \xi \rangle = \langle \pi(u_{i_0})\xi, \xi \rangle \neq 0,$$

we have that

$$\varphi(v_{j_0} u_{i_0}) \neq 0,$$

for some  $j_0$ . It follows that  $\varphi|_{I \cap J}$  is nonzero and hence  $\varphi$  is supported on  $I \cap J$  by (5.5).  $\square$

## 6. Functionals on Fell bundles

Throughout this section we fix a Fell bundle  $\mathcal{A}$  over the inverse semigroup  $S$ . Our main goal here will be to obtain a positive linear functional on  $\mathcal{L}(\mathcal{A})$ , given a state on  $A_e$ , for some idempotent  $e \in E(S)$ . This will be the first step in obtaining representations of  $C^*(\mathcal{A})$ .

If  $e \in E(S)$  and  $s \in S$  are such that  $e \leq s$ , observe that the map  $j_{s,e}$  is an isometric embedding of  $A_e$  into  $A_s$ . Therefore we may view  $A_e$  as a subspace of  $A_s$ . Moreover, since

$$e = se^*e = se \quad \text{and} \quad e = ee^*s = es.$$

we have that both  $A_s A_e$  and  $A_e A_s$  are contained in  $A_e$ .

**6.1. Proposition.** *Let  $e \in E(S)$  and  $s \in S$  be such that  $e \leq s$ , and let  $\varphi_e$  be a continuous linear functional on  $A_e$ , then there exists a unique continuous linear functional  $\tilde{\varphi}_e^s$  on  $A_s$ , extending  $\varphi_e$ , with  $\|\tilde{\varphi}_e^s\| = \|\varphi_e\|$ , and such that for every approximate unit  $\{u_i\}_i$  for  $A_e$  one has that*

$$\tilde{\varphi}_e^s(x) = \lim_i \varphi_e(xu_i) = \lim_i \varphi_e(u_i x) = \lim_i \varphi_e(u_i x u_i) \quad \text{for } x \in A_s.$$

**Proof.** By Cohen–Hewitt’s factorization theorem [10, 32.22] there exists a continuous linear functional  $\psi$  on  $A_e$  and an element  $b$  in  $A_e$  such that

$$\varphi_e(a) = \psi(ba) \quad \forall a \in A_e.$$

Given  $x \in A_s$  observe that  $bx \in A_e A_s \subseteq A_e$ , so it makes sense to define

$$\tilde{\varphi}_e^s(x) = \psi(bx) \quad \forall x \in A_s.$$

Obviously  $\tilde{\varphi}_e^s$  is then a continuous linear functional on  $A_s$  extending  $\varphi_e$ , whence  $\|\tilde{\varphi}_e^s\| \geq \|\varphi_e\|$ . Given an approximate unit  $\{u_i\}_i$  for  $A_e$  one has, for every  $x \in A_s$ , that

$$\lim_i \varphi_e(u_i x) = \lim_i \psi(bu_i x) = \psi(bx) = \tilde{\varphi}_e^s(x).$$

On the other hand

$$\lim_i \varphi_e(xu_i) = \lim_i \psi(bxu_i) = \psi(bx) = \tilde{\varphi}_e^s(x),$$

because  $bx \in A_e$ , and hence  $\{bxu_i\}_i$  converges to  $bx$ . To prove the last identity of the statement one uses Cohen–Hewitt’s theorem once more to write

$$\psi(a) = \chi(ac) \quad \forall a \in A_e,$$

where  $c \in A_e$ , and  $\chi$  is a continuous linear functional on  $A_e$ . Then, for every  $x \in A_s$  one has

$$\lim_i \varphi_e(u_i x u_i) = \lim_i \psi(bu_i x u_i) = \lim_i \chi(bu_i x u_i c) = \chi(bxc) = \psi(bx) = \tilde{\varphi}_e^s(x).$$

It remains to prove that  $\|\tilde{\varphi}_e^s\| \leq \|\varphi_e\|$ . For this notice that for every  $x \in A_s$  one has that

$$|\tilde{\varphi}_e^s(x)| = \lim_i |\varphi_e(xu_i)| \leq \limsup_i \|\varphi_e\| \|x\| \|u_i\| \leq \|\varphi_e\| \|x\|,$$

concluding the proof. □

**6.2. Definition.** The functional  $\tilde{\varphi}_e^s$  obtained above will be called the *canonical extension* of  $\varphi_e$  to  $A_s$ .

In order to facilitate computations with the canonical extension it is useful to highlight the way it was constructed in the proof above:

**6.3. Proposition.** *Under the hypothesis of (6.1), whenever*

$$\varphi_e(a) = \psi(ba) \quad \forall a \in A_e,$$

*where  $\psi$  is a continuous linear functional on  $A_e$ , and  $b \in A_e$ , one has that*

$$\tilde{\varphi}_e^s(x) = \psi(bx) \quad \forall x \in A_s.$$

**Proof.** See the proof of (6.1). □

**6.4. Definition.** Given  $s, t \in S$ , we will say that  $s$  and  $t$  are:

- (i) *disjoint* if there is no  $r \in S$  such that  $r \leq s, t$ ;
- (ii)  *$\mathcal{A}$ -disjoint* if, for every  $r \in S$  such that  $r \leq s, t$ , one has that  $A_r = \{0\}$ .

Before proceeding we need the following elementary result which holds for every inverse semigroup:

**6.5. Proposition.** *Let  $s, t \in S$  and let  $e \in E(S)$ . Then the following are equivalent:*

- (i)  $e \leq s^*t$ ;
- (ii)  $se = te$ , and  $e \leq s^*s, t^*t$ .

**Proof.** (i)  $\Rightarrow$  (ii): By (i) we have that  $e = s^*te$ . So

$$s^*se = s^*s(s^*te) = s^*te = e,$$

proving that  $e \leq s^*s$ . Since  $e = e^* \leq (s^*t)^* = t^*s$ , the above reasoning gives  $e \leq t^*t$ .

Proving that  $se = te$  is equivalent to showing that  $(se)^* = et^*$ , which is to say that

- (a)  $se = se(et^*)se$ , and
- (b)  $et^* = (et^*)se(et^*)$ ,

in view of the uniqueness of the adjoint in  $S$ . Observing that

$$et^*s = (s^*te)^* = e^* = e,$$

we have that the right-hand-side of (a) equals

$$se(et^*)se = set^*se = see = se,$$

proving (a). To prove (b) notice that

$$(et^*)se(et^*) = et^*set^* = eet^* = et^*.$$

(ii)  $\Rightarrow$  (i): This follows at once from  $s^*te = s^*se = e$ .  $\square$

**6.6. Proposition.** *Let  $s$  and  $t$  be  $\mathcal{A}$ -disjoint elements in  $S$  and let  $e$  be an idempotent with  $e \leq s^*t$ . Given any continuous linear functional  $\varphi_e$  on  $A_e$  one has that*

$$\tilde{\varphi}_e^{s^*t}(a_s^*a_t) = 0,$$

for every  $a_s \in A_s$ , and  $a_t \in A_t$ .

**Proof.** By (6.5) we have that  $se = te$ . Denoting by  $r = te$ , it is then clear that  $r \leq s, t$ , so  $A_r = \{0\}$ , by hypothesis. Next observe that by (6.1) it is enough to show that  $\varphi_e(a_s^*a_tu) = 0$ , for every  $u \in A_e$ . But since

$$a_tu \in A_tA_e \subseteq A_{te} = A_r = \{0\},$$

we have that  $a_tu = 0$ , and the statement follows.  $\square$

**6.7. Proposition.** *Let  $e, s, t \in S$  be such that  $e$  is idempotent and  $e \leq s \leq t$ . Given a continuous linear functional  $\varphi_e$  on  $A_e$  one has that*

$$\tilde{\varphi}_e^s = \tilde{\varphi}_e^t \circ j_{t,s}.$$

**Proof.** Pick a continuous linear functional  $\psi$  on  $A_e$ , and  $b \in A_e$ , such that  $\varphi_e(a) = \psi(ba)$  for all  $a \in A_e$ . Then, given any  $x \in A_s$ , we have

$$\begin{aligned} \tilde{\varphi}_e^t(j_{t,s}(x)) &= \psi(bj_{t,s}(x)) && (6.3) \\ &= \psi(\mu_{e,t}(b \otimes j_{t,s}(x))) \\ &= \psi(j_{e,e}(\mu_{e,s}(b \otimes x))) && (2.1.viii) \\ &= \psi(bx) \\ &= \tilde{\varphi}_e^s(x). \end{aligned} \quad \square$$

**6.8. Proposition.** *Let  $e, f \in E(S)$  and let  $s \in S$  be such that  $e \leq f \leq s$ . Given a continuous linear functional  $\psi_f$  on  $A_f$  denote by  $\varphi_e$  its restriction to  $A_e$ . If  $\psi_f$  is supported on  $A_e$  then the canonical extensions  $\tilde{\varphi}_e^s$  and  $\tilde{\psi}_f^s$  coincide.*

**Proof.** By (5.1.iv) let  $\psi_1$  be a continuous linear functional on  $A_e$  and  $b_1 \in A_e$  be such that

$$\psi_f(a) = \psi_1(b_1a) \quad \forall a \in A_f.$$

By Hahn–Banach’s theorem we may suppose that  $\psi_1$  is the restriction to  $A_e$  of a continuous linear functional  $\psi_2$  on  $A_f$ . Since  $\varphi_e(a) = \psi_1(b_1a)$ , for all  $a \in A_e$ , we have by (6.3) that

$$\tilde{\varphi}_e^s(x) = \psi_1(b_1x) \quad \forall x \in A_s.$$

On the other hand, since  $\psi_f(a) = \psi_2(b_1a)$ , for all  $a \in A_f$ , we have, again by (6.3), that

$$\tilde{\psi}_f^s(x) = \psi_2(b_1x) \quad \forall x \in A_s.$$

However, since  $b_1x \in A_e$ , for all  $x \in A_s$ , we have that

$$\tilde{\psi}_f^s(x) = \psi_2(b_1x) = \psi_1(b_1x) = \tilde{\varphi}_e^s(x). \quad \square$$

**6.9. Proposition.** *Let  $e \in E(S)$ , and let  $\varphi_e$  be a state on  $A_e$ . For each  $s \in S$  such that  $e \leq s$ , denote by  $\tilde{\varphi}_e^s$  the canonical extension of  $\varphi_e$  to  $A_s$ , and let  $\tilde{\varphi}_e$  be defined on  $\mathcal{L}(\mathcal{A})$  by*

$$\tilde{\varphi}_e \left( \sum_{s \in S} a_s \delta_s \right) = \sum_{s \geq e} \tilde{\varphi}_e^s(a_s).$$

Then  $\tilde{\varphi}_e$  is a positive continuous linear functional on  $\mathcal{L}(\mathcal{A})$  with  $\|\tilde{\varphi}_e\| = \|\varphi_e\|$ .

**Proof.** It is clear that  $\tilde{\varphi}_e$  is linear. Moreover, if  $\sum_{s \in S} a_s \delta_s$  is a generic element of  $\mathcal{L}(\mathcal{A})$ , one has that

$$\begin{aligned} \left| \tilde{\varphi}_e \left( \sum_{s \in S} a_s \delta_s \right) \right| &\leq \sum_{s \geq e} |\tilde{\varphi}_e^s(a_s)| \\ &\leq \sum_{s \geq e} \|\tilde{\varphi}_e^s\| \|a_s\| \\ &= \|\varphi_e\| \sum_{s \geq e} \|a_s\| \\ &\leq \|\varphi_e\| \left\| \sum_{s \in S} a_s \delta_s \right\|, \end{aligned} \tag{6.1}$$

so  $\|\tilde{\varphi}_e\| \leq \|\varphi_e\|$ . Identifying  $A_e$  with  $A_e \delta_e$  one has that  $\tilde{\varphi}_e$  extends  $\varphi_e$ , so  $\|\tilde{\varphi}_e\| = \|\varphi_e\|$ . In order to prove that  $\tilde{\varphi}_e$  is positive, given

$$a = \sum_{s \in S} a_s \delta_s \in \mathcal{L}(\mathcal{A}),$$

we need to show that  $\tilde{\varphi}_e(a^*a) \geq 0$ . We have

$$\begin{aligned} (6.10) \quad \tilde{\varphi}_e(a^*a) &= \tilde{\varphi}_e \left( \sum_{s,t \in S} a_s^* a_t \delta_{s^*t} \right) \\ &= \sum_{s^*t \geq e} \tilde{\varphi}_e^{s^*t}(a_s^* a_t) \\ &= \sum_{\substack{s^*s, t^*t \geq e \\ se = te}} \tilde{\varphi}_e^{s^*t}(a_s^* a_t) \end{aligned} \tag{6.5}$$

Let  $X := \{s \in S : s^*s \geq e\}$ , and consider the equivalence relation on  $X$  defined by saying that  $s \sim t$ , if and only if  $se = te$ . Also let

$$X = \dot{\bigcup}_{\lambda \in \Lambda} X_\lambda,$$

be the decomposition of  $X$  into equivalence classes. Then, picking up from (6.10), we have that

$$\tilde{\varphi}_e(a^*a) = \sum_{\substack{s,t \in X \\ s \sim t}} \tilde{\varphi}_e^{s^*t}(a_s^* a_t) = \sum_{\lambda \in \Lambda} \sum_{s,t \in X_\lambda} \tilde{\varphi}_e^{s^*t}(a_s^* a_t).$$

To arrive at the conclusion it therefore suffices to show that

$$(6.11) \quad \sum_{s,t \in X_\lambda} \tilde{\varphi}_e^{s^*t}(a_s^* a_t) \geq 0 \quad \forall \lambda \in \Lambda.$$

Fixing  $\lambda$  choose an approximate unit  $\{u_i\}_i$  for  $A_e$  and observe that  $a_s u_i \in A_{se} = A_{r_\lambda}$ , where  $r_\lambda$  is the common value of  $se$  for all  $s \in X_\lambda$ . We may then



define an element in  $A_{r_\lambda}$  by the expression

$$y_i = \sum_{s \in X_\lambda} a_s u_i.$$

Clearly  $y_i^* y_i \in A_{r_\lambda^* r_\lambda} = A_{(se)^* se} = A_{es^* se} = A_e$ , so we have by (2.1.vi) that

$$0 \leq \varphi_e(y_i^* y_i) = \varphi_e\left(\sum_{s,t \in X_\lambda} u_i a_s^* a_t u_i\right) = \sum_{s,t \in X_\lambda} \varphi_e(u_i a_s^* a_t u_i).$$

Taking the limit as  $i \rightarrow \infty$ , we deduce that

$$0 \leq \lim_i \sum_{s,t \in X_\lambda} \varphi_e(u_i a_s^* a_t u_i) \stackrel{(6.1)}{=} \sum_{s,t \in X_\lambda} \tilde{\varphi}_e^{s^* t}(a_s^* a_t),$$

concluding the proof of (6.11). □

**6.12. Corollary.** *Let  $e \in E(S)$ , and let  $\varphi_e$  be a state on  $A_e$ . If  $s$  and  $t$  are  $\mathcal{A}$ -disjoint elements of  $S$  then  $\tilde{\varphi}_e$  vanishes on  $(A_s \delta_s)^*(A_t \delta_t)$ .*

**Proof.** Given  $a_s \in A_s$  and  $a_t \in A_t$ , one has that

$$\tilde{\varphi}_e((a_s \delta_s)^*(a_t \delta_t)) = \tilde{\varphi}_e(a_s^* a_t \delta_{s^* t}).$$

If  $e \not\leq s^* t$ , then the above vanishes by definition of  $\tilde{\varphi}_e$ . On the other hand, if  $e \leq s^* t$ , the above equals  $\tilde{\varphi}_e^{s^* t}(a_s^* a_t)$ , which vanishes by (6.6). □

### 7. Inducing pure states

In the last section we have constructed states on  $\mathcal{L}(\mathcal{A})$  from states on each  $A_e$ . These states do not necessarily vanish on the ideal  $\mathcal{N}$  of (3.9) and hence do not factor through the algebra we are mostly interested in, namely  $\mathcal{L}(\mathcal{A})/\mathcal{N}$ . Here we will improve the above construction in order to obtain states on  $\mathcal{L}(\mathcal{A})/\mathcal{N}$  and hence nontrivial representations of  $C^*(\mathcal{A})$ .

As before we fix a Fell bundle  $\mathcal{A}$  over the inverse semigroup  $S$ . Denote by  $\mathcal{E}$  the restriction of  $\mathcal{A}$  to the idempotent semilattice  $E(S)$ , and recall that  $\{\pi_e^u\}_{e \in E(S)}$  denotes the universal representation of  $\mathcal{E}$  in  $C^*(\mathcal{E})$ . Incidentally, here we shall not use the universal representation of  $\mathcal{A}$ .

By (4.8.ii) we have that  $\pi_e^u(A_e)$  is a closed two-sided ideal of  $C^*(\mathcal{E})$ , for every  $e \in E(S)$ .

**7.1. Definition.** Given a state  $\varphi$  on  $C^*(\mathcal{E})$ , the *support* of  $\varphi$  is the set

$$\text{supp}(\varphi) = \{e \in E(S) : \varphi \text{ is supported on } \pi_e^u(A_e)\}.$$

**7.2. Proposition.** *Let  $\varphi$  be a state on  $C^*(\mathcal{E})$  and let  $e, f \in E(S)$ .*

- (i) *If  $e \in \text{supp}(\varphi)$ , and  $f \geq e$ , then  $f \in \text{supp}(\varphi)$ .*
- (ii) *If  $\varphi$  is pure and  $e, f \in \text{supp}(\varphi)$ , then  $ef \in \text{supp}(\varphi)$ .*

**Proof.** If  $e \leq f$ , then

$$\pi_f^u(A_f) \supseteq \pi_f^u(j_{f,e}(A_e)) = \pi_e^u(A_e),$$

so (i) follows from (5.3). As for (ii), it follows from (5.6) and the fact that  $\pi_{ef}^u(A_{ef}) = \pi_e^u(A_e) \cap \pi_f^u(A_f)$ , according to (3.3).  $\square$

Recall that, given two elements  $e$  and  $f$  of the semilattice  $E(S)$ , one says that  $e \leq f$ , whenever  $ef = e$ . For reasons that should become clear let us introduce the *reverse order* on  $E(S)$  by declaring that

$$f \preceq e \iff e \leq f.$$

In particular one has that

$$e, f \preceq ef \quad \forall e, f \in E(S).$$

Given a pure state  $\varphi$  on  $C^*(\mathcal{E})$  observe that  $\text{supp}(\varphi)$  is a directed set under “ $\preceq$ ” by (7.2.ii). Therefore we may (and shortly will) use  $\text{supp}(\varphi)$  as the set of indices for a net.

Given  $e \in \text{supp}(\varphi)$ , the composition

$$(7.3) \quad \varphi_e := \varphi \circ \pi_e^u$$

is a state on  $A_e$ . We therefore obtain by (6.9) a positive linear functional  $\tilde{\varphi}_e$  on  $\mathcal{L}(\mathcal{A})$ , and hence  $\{\tilde{\varphi}_e\}_{e \in \text{supp}(\varphi)}$  is a net of functionals on  $\mathcal{L}(\mathcal{A})$ . This brings us to a main point.

**7.4. Proposition.** *Given a pure state  $\varphi$  on  $C^*(\mathcal{E})$ , the net  $\{\tilde{\varphi}_e\}_{e \in \text{supp}(\varphi)}$  constructed above converges pointwise to a positive linear functional  $\tilde{\varphi}$  on  $\mathcal{L}(\mathcal{A})$ , such that:*

(i) *For every  $s \in S$ , and  $a_s \in A_s$ , one has that*

$$\tilde{\varphi}(a_s \delta_s) = \begin{cases} \tilde{\varphi}_e^s(a_s), & \text{if there exists } e \in \text{supp}(\varphi), \text{ such that } e \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *For every  $e \in E(S)$ , and every  $a_e \in A_e$ , one has that*

$$\tilde{\varphi}(a_e \delta_e) = \varphi(\pi_e^u(a_e)).$$

(iii)  $\|\tilde{\varphi}\| \leq \|\varphi\|$ .

(iv)  $\tilde{\varphi}$  vanishes on the ideal  $\mathcal{N}$  defined in (3.9).

**Proof.** To prove convergence of our net it is enough to show the existence of

$$\lim_e \tilde{\varphi}_e(a_s \delta_s)$$

for every  $s \in S$ , and every  $a_s \in A_s$ .

Suppose first that there does not exist  $e \in \text{supp}(\varphi)$  such that  $e \leq s$ . By definition we have that  $\tilde{\varphi}_e(a_s \delta_s) = 0$ , for every  $e \in \text{supp}(\varphi)$ , and hence

$$(7.5) \quad \lim_e \tilde{\varphi}_e(a_s \delta_s) = 0.$$

Suppose now that  $e \leq s$ , for some  $e \in \text{supp}(\varphi)$ . We then claim that for every  $f \in \text{supp}(\varphi)$  with  $f \leq e$ , that is,  $f \succeq e$ , one has that

$$\tilde{\varphi}_f(a_s \delta_s) = \tilde{\varphi}_e(a_s \delta_s).$$

To prove it observe that

$$\varphi_f = \varphi \circ \pi_f^u = \varphi \circ \pi_e^u \circ j_{e,f} = \varphi_e \circ j_{e,f},$$

so, identifying  $A_f$  as an ideal of  $A_e$ , we have that  $\varphi_f$  coincides with the restriction of  $\varphi_e$  to  $A_f$ . It is also evident that  $\varphi_e$  is supported on  $A_f$ , so we have by (6.8) that  $\tilde{\varphi}_f^s = \tilde{\varphi}_e^s$ . Therefore

$$\tilde{\varphi}_f(a_s \delta_s) = \tilde{\varphi}_f^s(a_s) = \tilde{\varphi}_e^s(a_s) = \tilde{\varphi}_e(a_s \delta_s),$$

proving our claim, and hence that

$$(7.6) \quad \lim_f \tilde{\varphi}_f(a_s \delta_s) = \tilde{\varphi}_e(a_s \delta_s).$$

This concludes the proof of the convergence of our net. Letting  $\tilde{\varphi}$  denote the pointwise limit of the  $\tilde{\varphi}_e$ , we have that (i) follows from (7.5) and (7.6). Since each  $\tilde{\varphi}_e$  is positive by (6.9) it is clear that  $\tilde{\varphi}$  is also positive.

In order to prove (ii), suppose first that  $e \in \text{supp}(\varphi)$ . Then, using (i) and observing that the extension of  $\varphi_e$  to  $A_e$  is  $\varphi_e$  itself, we have that

$$\tilde{\varphi}(a_e \delta_e) = \tilde{\varphi}_e^e(a_e) = \varphi_e(a_e) = \varphi(\pi_e^u(a_e)).$$

Suppose next that  $e \notin \text{supp}(\varphi)$ . This means that  $\varphi$  is not supported on  $\pi_e^u(A_e)$ , which implies that  $\varphi$  vanishes on  $\pi_e^u(A_e)$  by (5.5), and hence the right-hand-side of (ii) equals zero. Because of (7.2.i), no member of the support of  $\varphi$  is dominated by  $e$ , so  $\tilde{\varphi}(a_e \delta_e) = 0$ , by (i). This concludes the proof of (ii).

Addressing (iii) notice that for every  $e \in \text{supp}(\varphi)$ , one has that

$$\|\tilde{\varphi}_e\| \stackrel{(6.9)}{=} \|\varphi_e\| = \|\varphi \pi_e^u\| \leq \|\varphi\| \|\pi_e^u\| \stackrel{(3.2)}{\leq} \|\varphi\|.$$

Taking the limit one gets  $\|\tilde{\varphi}\| \leq \|\varphi\|$ .

Let us now prove that  $\tilde{\varphi}$  vanishes on  $\mathcal{N}$ , so let  $s, t \in S$  be such that  $s \leq t$ , and let  $a_s \in A_s$ . By definition of  $\mathcal{N}$  (3.9), our task is then to prove that

$$(7.7) \quad \tilde{\varphi}(a_s \delta_s - j_{t,s}(a_s) \delta_t) = 0.$$

Consider the following statements:

- (a) There exists  $e \in \text{supp}(\varphi)$  such that  $e \leq s$ .
- (b) There exists  $e \in \text{supp}(\varphi)$  such that  $e \leq t$ .

Since  $s \leq t$ , it is clear that (a) implies (b). It is thus impossible for (a) to be true and (b) false. All other cases will be treated separately.

*Case 1:* (a) is false and (b) is true.

Choose  $e_0 \in \text{supp}(\varphi)$  such that  $e_0 \leq t$ . Then, for every  $e \leq e_0$  (i.e.,  $e \succeq e_0$ ), we have that

$$se = ts^*se = tes^*s = es^*s,$$

so  $se$  is idempotent, and clearly  $se \leq s$ . Since we are assuming (a) to be false, we deduce that  $se$  is not in  $\text{supp}(\varphi)$  and hence that  $\varphi$  is not supported in  $\pi_{se}^u(A_{se})$ . So  $\varphi$  vanishes on  $\pi_{se}^u(A_{se})$  by (5.5), and in particular

$$(7.8) \quad \varphi_e j_{e,se} = \varphi \pi_e^u j_{e,se} = \varphi \pi_{se}^u = 0.$$

Still assuming that  $e \leq e_0$ , we next claim that

$$\tilde{\varphi}_e^t(j_{t,s}(a_s)) = 0.$$

Fixing an approximate unit  $\{u_i\}_i$  for  $A_e$  we have that

$$\tilde{\varphi}_e^t(j_{t,s}(a_s)) = \lim_i \varphi_e(j_{t,s}(a_s)u_i) \tag{6.1}$$

$$= \lim_i \varphi_e(\mu_{t,e}(j_{t,s}(a_s) \otimes u_i))$$

$$= \lim_i \varphi_e(j_{e,se}(\mu_{s,e}(a_s \otimes u_i))) \tag{2.1.viii}$$

$$= 0 \tag{7.8),}$$

proving our claim. It follows that

$$\tilde{\varphi}_e(a_s \delta_s - j_{t,s}(a_s) \delta_t) \stackrel{(6.9)}{=} \tilde{\varphi}_e^t(j_{t,s}(a_s)) = 0,$$

for every  $e \succeq e_0$ , and hence (7.7) is proved.

*Case 2:* (a) and (b) are both false.

For every  $e \in \text{supp}(\varphi)$  we have that  $\tilde{\varphi}_e(a_s \delta_s - j_{t,s}(a_s) \delta_t) = 0$ , by (6.9), so (7.7) follows immediately.

*Case 3:* (a) and (b) are both true.

Choose  $e_0 \in \text{supp}(\varphi)$  such that  $e_0 \leq s$ . Then, for all  $e \leq e_0$  (i.e.,  $e \succeq e_0$ ), we have that

$$\tilde{\varphi}_e(a_s \delta_s - j_{t,s}(a_s) \delta_t) = \tilde{\varphi}_e^s(a_s) - \tilde{\varphi}_e^t(j_{t,s}(a_s)) \stackrel{(6.7)}{=} 0,$$

and hence (7.7) follows. □

It is interesting to remark that the convergence above takes place even if  $\mathbf{C}$  is given the discrete topology!

**7.9. Proposition.** *Let  $\varphi$  be a pure state on  $C^*(\mathcal{E})$ . If  $s$  and  $t$  are  $\mathcal{A}$ -disjoint elements of  $S$  then  $\tilde{\varphi}$  vanishes on  $(A_s \delta_s)^*(A_t \delta_t)$ .*

**Proof.** Since  $\tilde{\varphi}$  is the limit of the  $\tilde{\varphi}_e$ , the result follows from (6.12). □

### 8. Representations of $C^*(\mathcal{A})$ and the reduced cross-sectional algebra

As before we fix a Fell bundle  $\mathcal{A}$  over the inverse semigroup  $S$  and let  $\mathcal{E}$  be the restriction of  $\mathcal{A}$  to the idempotent semilattice  $E(S)$ .

Given a pure state  $\varphi$  on  $C^*(\mathcal{E})$  one has that the functional  $\tilde{\varphi}$  provided by (7.4) is continuous and hence extends to the Banach  $*$ -algebra completion

of  $\mathcal{L}(\mathcal{A})$ . Using [8, VI.19.3 and VI.19.5] we deduce that  $\tilde{\varphi}$  generates a  $*$ -representation  $\Upsilon_{\tilde{\varphi}}$  of (the completion of  $\mathcal{L}(\mathcal{A})$  and hence also of)  $\mathcal{L}(\mathcal{A})$  on a Hilbert space  $H_{\tilde{\varphi}}$ . To be precise,  $H_{\tilde{\varphi}}$  is the Hilbert space obtained as the Hausdorff<sup>2</sup> completion of  $\mathcal{L}(\mathcal{A})$  under the pre-inner-product

$$(8.1) \quad \langle x, y \rangle_{\tilde{\varphi}} = \tilde{\varphi}(y^*x) \quad \forall x, y \in \mathcal{L}(\mathcal{A}).$$

The representation  $\Upsilon_{\tilde{\varphi}}$  itself is given by

$$(8.2) \quad \Upsilon_{\tilde{\varphi}}(x)\hat{y} = \widehat{xy} \quad \forall x, y \in \mathcal{L}(\mathcal{A}),$$

where the “hat” notation indicates the canonical map  $y \in \mathcal{L}(\mathcal{A}) \mapsto \hat{y} \in H_{\tilde{\varphi}}$ , given by the completion process.

**8.3. Definition.** We shall refer to  $\Upsilon_{\tilde{\varphi}}$  as the *GNS representation of  $\mathcal{L}(\mathcal{A})$  associated to  $\tilde{\varphi}$* .

Notice however that, when there is no bounded approximate unit for  $\mathcal{L}(\mathcal{A})$ , the existence of a cyclic vector for  $\Upsilon_{\tilde{\varphi}}$  does not follow from the standard arguments.

Regardless of cyclic vectors we have the following criterion for the vanishing of  $\Upsilon_{\tilde{\varphi}}(x)$ , for every given  $x \in \mathcal{L}(\mathcal{A})$ :

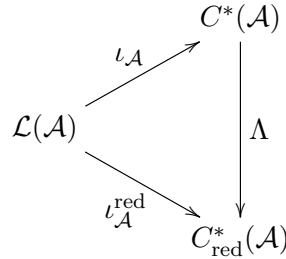
$$(8.4) \quad \Upsilon_{\tilde{\varphi}}(x) = 0 \iff \forall y, z \in \mathcal{L}(\mathcal{A}), \tilde{\varphi}(z^*xy) = 0.$$

**8.5. Definition.** By the *reduced  $C^*$ -algebra of  $\mathcal{A}$*  we shall mean the  $C^*$ -algebra  $C^*_{\text{red}}(\mathcal{A})$  obtained as the Hausdorff completion of  $\mathcal{L}(\mathcal{A})$  under the  $C^*$ -seminorm

$$\|x\| = \sup_{\varphi} \|\Upsilon_{\tilde{\varphi}}(x)\|,$$

where  $\varphi$  runs in the set of all pure states of  $C^*(\mathcal{E})$ .

**8.6. Proposition.** *There exists a surjective map  $\Lambda : C^*(\mathcal{A}) \rightarrow C^*_{\text{red}}(\mathcal{A})$ , such that the diagram*



*commutes, where  $\iota_{\mathcal{A}}^{\text{red}}$  is the canonical map arising from the completion process.*

**Proof.** If  $\varphi$  is a pure state of  $C^*(\mathcal{E})$  recall that  $\tilde{\varphi}$  vanishes on  $\mathcal{N}$  by (7.4.iv). So it follows from (8.4) that  $\Upsilon_{\tilde{\varphi}}$  also vanishes on  $\mathcal{N}$ , so  $\Upsilon_{\tilde{\varphi}}$  factors through

<sup>2</sup>Meaning that one must first mod out vectors of norm zero.

$\mathcal{L}(\mathcal{A})/\mathcal{N}$ , and hence also through its enveloping  $C^*$ -algebra, namely  $C^*(\mathcal{A})$ . It follows that  $\|\Upsilon_{\tilde{\varphi}}(x)\| \leq \|\iota_{\mathcal{A}}(x)\|$ , and hence also that

$$\|\iota_{\mathcal{A}}^{\text{red}}(x)\| = \|x\| \leq \|\iota_{\mathcal{A}}(x)\| \quad \forall x \in \mathcal{L}(\mathcal{A}),$$

from where the conclusion follows.  $\square$

**8.7. Definition.** For each  $s \in S$  we shall let  $\pi_s^{\text{red}} : A_s \rightarrow C_{\text{red}}^*(\mathcal{A})$  be the composition of maps

$$A_s \xrightarrow{\pi_s^0} \mathcal{L}(\mathcal{A}) \xrightarrow{\iota_{\mathcal{A}}^{\text{red}}} C_{\text{red}}^*(\mathcal{A}).$$

Notice that

$$\pi_s^{\text{red}} = \iota_{\mathcal{A}}^{\text{red}} \circ \pi_s^0 = \Lambda \circ \iota_{\mathcal{A}} \circ \pi_s^0 = \Lambda \circ \pi_s^u,$$

and hence  $\Pi^{\text{red}} = \{\pi_s^{\text{red}}\}_{s \in S}$  is a representation of  $\mathcal{A}$  in  $C_{\text{red}}^*(\mathcal{A})$ .

We may now finally prove a nontriviality result relating to  $C^*(\mathcal{A})$ .

**8.8. Lemma.** *Let  $n \geq 1$ , let  $s_1, \dots, s_n \in S$ , be pairwise  $\mathcal{A}$ -disjoint elements, and let  $a_i \in A_{s_i}$ , for each  $i = 1, 2, \dots, n$ . If*

$$\sum_{i=1}^n \pi_{s_i}^{\text{red}}(a_i) = 0,$$

then  $a_i = 0$ , for every  $i$ .

**Proof.** Supposing by contradiction that some  $a_k \neq 0$ , let  $e_k = s_k^* s_k$ . By (2.1.v) and (4.8.i) we have that  $\pi_{e_k}^u(a_k^* a_k) \neq 0$ , so we may choose a pure state  $\varphi$  on  $C^*(\mathcal{E})$  such that  $\varphi(\pi_{e_k}^u(a_k^* a_k)) \neq 0$ .

Obviously  $\varphi$  does not vanish on the ideal  $\pi_{e_k}^u(A_{e_k})$  and consequently  $\varphi$  is supported there by (5.5). In other words,  $e_k \in \text{supp}(\varphi)$ . Since

$$0 = \sum_{i=1}^n \pi_{s_i}^{\text{red}}(a_i) = \iota_{\mathcal{A}}^{\text{red}} \left( \sum_{i=1}^n a_i \delta_{s_i} \right),$$

we must have that  $\Upsilon_{\tilde{\varphi}} \left( \sum_{i=1}^n a_i \delta_{s_i} \right) = 0$ . Given any  $u \in A_{e_k}$  we obtain from (8.4) that

$$\begin{aligned} (8.9) \quad 0 &= \tilde{\varphi} \left( (u \delta_{e_k})^* \left( \sum_{i=1}^n a_i \delta_{s_i} \right)^* \left( \sum_{i=1}^n a_i \delta_{s_i} \right) (u \delta_{e_k}) \right) \\ &= \tilde{\varphi} \left( \sum_{i,j=1}^n (a_i u \delta_{s_i e_k})^* (a_j u \delta_{s_j e_k}) \right). \end{aligned}$$

By hypothesis we have that  $s_i$  and  $s_j$  are  $\mathcal{A}$ -disjoint for  $i \neq j$ . The same is therefore also the case for  $s_i e_k$  and  $s_j e_k$ , so we have by (7.9) that

$$\tilde{\varphi} \left( (a_i u \delta_{s_i e_k})^* (a_j u \delta_{s_j e_k}) \right) = 0.$$

This means that the cross terms in (8.9) all vanish and we are left with

$$\begin{aligned}
 0 &= \tilde{\varphi} \left( \sum_{i=1}^n (a_i u \delta_{s_i e_k})^* (a_i u \delta_{s_i e_k}) \right) \\
 &\geq \tilde{\varphi} \left( (a_k u \delta_{s_k e_k})^* (a_k u \delta_{s_k e_k}) \right) \\
 &= \tilde{\varphi} (u^* a_k^* a_k u \delta_{e_k}) \\
 &= \tilde{\varphi}_{e_k}^{e_k} (u^* a_k^* a_k u) \tag{7.4.i} \\
 &= \varphi_{e_k} (u^* a_k^* a_k u) \\
 &= \varphi (\pi_{e_k}^u (u^* a_k^* a_k u)) \tag{7.3}.
 \end{aligned}$$

Letting  $u$  run through an approximate unit for  $A_{e_k}$  we deduce that

$$\varphi (\pi_{e_k}^u (a_k^* a_k)) = 0,$$

in contradiction to the choice of  $\varphi$ . Therefore  $a_k$  must be zero and the proof is concluded. □

The following is an immediate consequence:

**8.10. Corollary.** *The maps*

$$\pi_s^u : A_s \rightarrow C^*(\mathcal{A}),$$

and

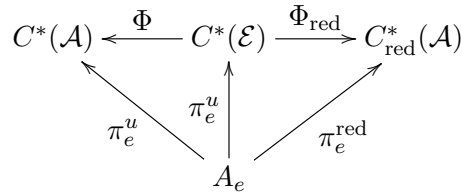
$$\pi_r^{\text{red}} : A_s \rightarrow C_{\text{red}}^*(\mathcal{A}),$$

are injective for every  $s \in S$ .

**8.11. Corollary.** *There are monomorphisms*

$$\Phi : C^*(\mathcal{E}) \rightarrow C^*(\mathcal{A}) \quad \text{and} \quad \Phi_{\text{red}} : C^*(\mathcal{E}) \rightarrow C_{\text{red}}^*(\mathcal{A}),$$

such that for every  $e \in E(S)$  the diagram



commutes.

**Proof.** Follows immediately from (8.10) and (4.3). □

Notice that in the statement of (8.11) we are denoting by  $\pi_e^u$  the canonical maps in two different contexts, namely that of the Fell bundle  $\mathcal{A}$  and that of the restricted bundle  $\mathcal{E}$ . Nevertheless by this result we may view  $C^*(\mathcal{E})$  as a subalgebra of  $C^*(\mathcal{A})$  and, once this identification is made, the two meanings of  $\pi_e^u$  are reconciled.

In a forthcoming paper we plan to further develop the theory of cross-sectional algebras, but for the time being we present the following simple fact:

**8.12. Proposition.** *Any (bounded) approximate unit for  $C^*(\mathcal{E})$  is an approximate unit for both  $C^*(\mathcal{A})$  and  $C_{\text{red}}^*(\mathcal{A})$ .*

**Proof.** Let  $\{w_i\}_i$  be an approximate unit for  $C^*(\mathcal{E})$ . Given  $s \in S$ , it follows from [11, 1.1.4] that  $\pi_s^u(A_s)$  is the closed linear span of  $\pi_s^u(A_s)\pi_s^u(A_s)^*\pi_s^u(A_s)$ . Observing that

$$\pi_s^u(A_s)\pi_s^u(A_s)^* \subseteq \pi_{ss^*}^u(A_{ss^*}) \subseteq C^*(\mathcal{E}),$$

we see that  $C^*(\mathcal{A})$  equals the closed linear span of  $C^*(\mathcal{E})C^*(\mathcal{A})$ , and therefore also of  $C^*(\mathcal{E})C^*(\mathcal{A})C^*(\mathcal{E})$ .

Incidentally, this is to say that  $C^*(\mathcal{A})$  is the closed two-sided ideal generated by  $C^*(\mathcal{E})$ . The result for  $C^*(\mathcal{A})$  then follows easily and the case of  $C_{\text{red}}^*(\mathcal{A})$  may be treated similarly.  $\square$

## Part II. Generalized Cartan subalgebras

### 9. Virtual commutants

Having followed the first few steps into the general theory of Fell bundles over inverse semigroups, we now wish to show that they appear naturally in certain situations. In fact we wish to show that, under suitable hypothesis, the inclusion of a closed  $*$ -subalgebra  $B$  of a  $C^*$ -algebra  $A$  is, in all respects, the same as the inclusion

$$C^*(\mathcal{E}) \subseteq C_{\text{red}}^*(\mathcal{A}),$$

for a Fell bundle  $\mathcal{A}$ . For a while we will forget about Fell bundles and will concentrate instead on inclusions of  $C^*$ -algebras satisfying, to begin with, the following:

**9.1. Standing Hypothesis.** From now on we assume that  $A$  is a *separable*  $C^*$ -algebra and  $B \subseteq A$  is a closed  $*$ -subalgebra containing an approximate unit  $\{w_i\}_i$  for  $A$ .

**9.2. Definition.** A *virtual commutant* of  $B$  in  $A$  is an  $A$ -valued linear map  $\varphi$  defined on a closed two-sided ideal  $J$  of  $B$ , such that

- (i)  $\varphi(bx) = b\varphi(x)$ ,
- (ii)  $\varphi(xb) = \varphi(x)b$ ,

for all  $x \in J$ , and  $b \in B$ . Should we want to highlight the domain of  $\varphi$ , we will write  $(J, \varphi)$  in place of  $\varphi$ .

Notice that, under the conditions above, both  $J$  and  $A$  are  $B$ -bimodules. Thus conditions (i) and (ii) simply say that  $\varphi$  is a  $B$ -bimodule map.

**9.3. Proposition.** *Let  $(J, \varphi)$  be a virtual commutant of  $B$  in  $A$ . Then the range of  $\varphi$  is contained in  $JAJ$ .*



**Proof.** By Cohen–Hewitt’s factorization theorem [10, 32.22] any  $x \in J$  may be written as  $x = yzw$ , with  $y, z, w \in J$ . Then

$$\varphi(x) = \varphi(yzw) = y\varphi(z)w \in JAJ. \quad \square$$

**9.4. Proposition.** *Every virtual commutant  $(J, \varphi)$  is bounded.*

**Proof.** Employing the closed graph theorem, we must prove that if

$$\{x_n\}_{n \in \mathbb{N}} \subseteq J$$

is such that  $x_n \rightarrow 0$  and  $\varphi(x_n) \rightarrow a \in A$ , then  $a = 0$ .

Notice that, since  $\varphi(x_n) \in JA$ , by (9.3), we have that  $a = \lim_n \varphi(x_n) \in \overline{JA}$ . One may then prove that  $a = \lim_i u_i a$ , for every (bounded) approximate unit  $\{u_i\}_i$  for  $J$ .

In order to prove that  $a = 0$ , it is therefore enough to prove that  $ya = 0$ , for every  $y \in J$ . We have

$$ya = \lim_{n \rightarrow \infty} y\varphi(x_n) = \lim_{n \rightarrow \infty} \varphi(yx_n) = \lim_{n \rightarrow \infty} \varphi(y)x_n = \varphi(y) \lim_{n \rightarrow \infty} x_n = 0. \quad \square$$

Here is a procedure for obtaining a fairly general example of a virtual commutant. Suppose that  $A$  is a closed  $*$ -subalgebra of some other  $C^*$ -algebra  $C$ , that is  $B \subseteq A \subseteq C$ , and that  $\tau$  is an element of  $C$  which commutes with  $B$ . Letting  $J$  be the subset of  $B$  given by

$$J = \{b \in B : \tau b \in A\},$$

it is easy to see that  $J$  is a closed two-sided ideal of  $B$ . Defining

$$\varphi(x) = \tau x \quad \forall x \in J,$$

one then has that  $(J, \varphi)$  is a virtual commutant of  $B$  in  $A$ .

The following result essentially states that the above example is the most general one.

**9.5. Theorem.** *Let  $\varphi$  be a virtual commutant of  $B$  in  $A$ . Supposing that  $A$  is faithfully represented on a Hilbert space  $H$  (in which case we shall identify  $A$  with its image within  $B(H)$ ), there exists  $\tau \in B(H)$  such that for every  $x \in J$ ,*

- (i)  $\tau x \in A$ ,
- (ii)  $\varphi(x) = \tau x$ , and
- (iii)  $\tau$  commutes with  $B$ .

**Proof.** Observe that by Cohen–Hewitt’s factorization theorem [10, 32.22], every element  $\xi \in \overline{JH}$  can be written as  $\xi = x\eta$ , for  $x \in J$ , and  $\eta \in H$ . In particular  $\overline{JH} = JH$ .

Letting  $\{u_i\}_i$  be a bounded approximate unit for  $J$ , we claim that the net  $\{\varphi(u_i)\xi\}_i$  converges to some element in  $JH$ , for every  $\xi \in JH$ . In fact, writing  $\xi = x\eta$ , as above, we have

$$\varphi(u_i)\xi = \varphi(u_i)x\eta = \varphi(u_i x)\eta \xrightarrow{i \rightarrow \infty} \varphi(x)\eta.$$

That  $\varphi(x)\eta$  lies in  $JH$  follows from (9.3). The correspondence

$$\xi \mapsto \lim_i \varphi(u_i)\xi$$

therefore defines a bounded linear map  $\tau : JH \rightarrow JH$  such that

$$\tau(x\eta) = \varphi(x)\eta \quad \forall x \in J \quad \forall \eta \in H.$$

Incidentally this shows that  $\tau$  does not depend on the choice of the approximate unit above.

Declaring  $\tau$  to be zero on  $JH^\perp$  we get an extension of  $\tau$  to  $H$  which, by abuse of language, will still be denoted by  $\tau$ .

We next claim that  $\tau$  commutes with  $B$ . Notice that since  $J$  is an ideal in  $B$ , the space  $JH$  is invariant under  $B$ , and clearly also under  $\tau$ . So, in order to prove that  $\tau b = b\tau$ , for any given  $b \in B$ , it is enough to prove that  $\tau b(\xi) = b\tau(\xi)$ , for all  $\xi \in JH$  or, equivalently, that

$$\tau b(x\xi) = b\tau(x\xi) \quad \forall x \in J \quad \forall \xi \in H.$$

We have

$$\tau b(x\xi) = \varphi(bx)\xi = b\varphi(x)\xi = b\tau(x\xi).$$

This proves (iii). We next prove (ii). Given  $\zeta \in H$ , write  $\zeta = y\xi + \eta$ , with  $y \in J$ ,  $\xi \in H$ , and  $\eta \in JH^\perp$ . Using the fact that  $\tau$  vanishes on  $JH^\perp$ , we have

$$\begin{aligned} \tau x(\zeta) &= \tau x(y\xi + \eta) = \tau(xy\xi) = \varphi(xy)\xi = \varphi(x)y\xi = \varphi(x)(y\xi + \eta) \\ &= \varphi(x)(\zeta), \end{aligned}$$

where the penultimate equality, namely that  $\varphi(x)\eta = 0$ , follows from the facts that  $\varphi(x) \in AJ$ , by (9.3), and that  $J$  vanishes on  $JH^\perp$ . So  $\varphi(x) = \tau\xi$ , taking care of (ii), and hence also of (i).  $\square$

**9.6. Definition.** We shall say that  $B$  satisfies property (MAX') if, for any virtual commutant  $\varphi$  of  $B$  in  $A$ , one has that the range of  $\varphi$  is contained in  $B$ .

As usual we will use the notation  $B' \cap A$  to refer to the relative commutant of  $B$  in  $A$ , namely

$$(9.7) \quad B' \cap A = \{a \in A : ab = ba, \forall b \in B\}.$$

**9.8. Proposition.** *If  $B$  satisfies property (MAX') then  $B' \cap A \subseteq B$ .*

**Proof.** Let  $a \in B' \cap A$ . Defining

$$\varphi : x \in B \mapsto xa \in A,$$

it is easy to see that  $(B, \varphi)$  is a virtual commutant. So the hypothesis implies that the range of  $\varphi$  lies in  $B$  or, equivalently, that  $Ba \subseteq B$ . Under (9.1) we deduce that  $a \in B$ , hence proving that  $B' \cap A \subseteq B$ .  $\square$

The converse of the above theorem is not true. To describe a counterexample let  $B$  be a (necessarily nonunital)  $C^*$ -algebra whose center reduces to  $\{0\}$ , such as the algebra of compact operators on an infinite dimensional Hilbert space. Let  $X$  be any compact Hausdorff topological space with more than one point such as the real interval  $[0, 1]$ , and let us view  $B$  as the subalgebra  $1 \otimes B \subseteq C(X) \otimes B = C(X, B)$  formed by the constant functions. It is then easy to see that  $B' \cap C(X, B) = \{0\} \subseteq B$ , but if one chooses a nonconstant function  $f \in C(X)$ , the map

$$\varphi : b \in B \mapsto f \otimes b \in C(X) \otimes B,$$

gives a virtual commutant whose range is not contained in  $B$ .

This phenomenon however does not occur in the realm of abelian algebras.

**9.9. Proposition.** *Assume that  $B$  is abelian. Then  $B$  satisfies property (MAX') if and only if  $B' \cap A \subseteq B$ , which incidentally is to say that  $B$  is maximal abelian.*

**Proof.** The implication “ $\Rightarrow$ ” follows immediately from (9.8). Conversely, let  $(J, \varphi)$  be a virtual commutant of  $B$  in  $A$ . Then, for every  $x \in J$  and  $b \in B$ , one has that

$$b\varphi(x) = \varphi(bx) = \varphi(xb) = \varphi(x)b,$$

so  $\varphi(x) \in B' \cap A \subseteq B$ , proving that  $B$  satisfies property (MAX'). □

### 10. Slices and the normalizer of $B$ in $A$

We will now begin to study the notion of normalizer in the context of  $C^*$ -algebras. This notion was introduced by Kumjian [14] to study abelian subalgebras and was also used by Renault [17].

In this section, as well as throughout the rest of this work, we keep (9.1) in force.

**10.1. Definition.** The *normalizer* of  $B$  in  $A$  is the subset

$$N(B) = \{a \in A : a^*Ba \subseteq B, aBa^* \subseteq B\}.$$

A *slice* is any closed linear subspace  $M \subseteq N(B)$ , such that both  $BM$  and  $MB$  are contained in  $M$ .

Observe that  $N(B)$  is a closed set. In addition it is closed under multiplication and under adjoint, but it is not necessarily closed under addition. On the other hand, by definition a slice is required to be closed under addition, but it is not necessarily closed under multiplication or adjoint.

**10.2. Proposition.** *Let  $M$  be a slice. Then the sets  $M^*BM$ ,  $MBM^*$ ,  $M^*M$  and  $MM^*$  are contained in  $B$ .*

**Proof.** Let  $m, n \in M$ . Then, since  $M$  is a linear subspace we have that  $m + i^k n \in M$ , for  $k = 0, 1, 2, 3$ . Therefore, given  $b \in B$ ,

$$m^*bn = \frac{1}{4} \sum_{k=0}^3 i^{-k} (m + i^k n)^* b (m + i^k n) \in B,$$

hence  $M^*BM \subseteq B$ . Applying the same reasoning to the slice  $M^*$  we deduce that  $MBM^* \subseteq B$ .

Still assuming that  $m, n \in M$ , let  $\{w_i\}_i$  be as in (9.1). Then

$$m^*n = \lim_i m^*w_i n \in B,$$

so  $M^*M \subseteq B$ , and similarly  $MM^* \subseteq B$ .  $\square$

**10.3. Corollary.** *Let  $M$  be a slice. Then:*

- (i)  $MM^*M \subseteq M$ , which is to say that  $M$  is a ternary ring of operators (cf. [21]).
- (ii)  $M^*M$  and  $MM^*$  (linear span of products) are two-sided self-adjoint ideals of  $B$ .

**Proof.** For (i) it is enough to notice that  $MM^*M \subseteq BM \subseteq M$ . Point (ii) is trivial.  $\square$

The following notation will be useful:

**10.4. Definition.** Given a slice  $M$  we say:

- (i) The *source* of  $M$ , denoted  $S(M)$ , is the closed two-sided ideal  $\overline{M^*M}$  (closed linear span) of  $B$ .
- (ii) The *range* of  $M$ , denoted  $R(M)$ , is the closed two-sided ideal  $\overline{MM^*}$  of  $B$ .

**10.5. Proposition.** *Every element  $a \in N(B)$  lies in some slice.*

**Proof.** We claim that  $M := \overline{BaB}$  (closed linear span) is a slice. It is clear that  $BM, MB \subseteq M$ , and in order to prove that  $M$  is contained in  $N(B)$ , it is enough to notice that  $MBM^*$  and  $M^*BM$  are contained in  $B$ . Finally observe that by (9.1) one has that  $a = \lim_i w_i a w_i \in M$ .  $\square$

## 11. Frames and conditional expectations

We will now discuss the behavior of slices under conditional expectations. We must however start by discussing the notion of frames, with which we begin the present section.

We continue to work under (9.1), observing that it is only from now on that we make use of the separability of  $A$ .

**11.1. Proposition.** *Let  $M$  be a slice. Then there exists a countable family  $\{u_i\}_{i \in \mathbb{N}}$  of elements of  $M$  such that*

$$m = \sum_{i \in \mathbb{N}} u_i u_i^* m \quad \forall m \in M,$$

where the sum converges unconditionally. Any such family will be called a frame.

**Proof.** We regard  $M$  as a right Hilbert module over  $\tilde{B}$  (unitization of  $B$ ) under the inner product

$$\langle m, n \rangle = m^*n \quad \forall m, n \in M.$$

Since  $A$  is separable it follows that  $M$  is also separable and hence, by Kasparov’s stabilization theorem [13, 3.2], [11, 1.1.24], there exists a unitary operator

$$U : \ell_2(\tilde{B}) \rightarrow M \oplus \ell_2(\tilde{B}).$$

Let  $\{e_i\}_{i \in \mathbf{N}}$  be the canonical basis of  $\ell_2(\tilde{B})$  and let  $u_i = P(U(e_i))$ , where  $P$  denotes the orthogonal projection from  $M \oplus \ell_2(\tilde{B})$  onto  $M$ . Given any  $\xi \in \ell_2(\tilde{B})$  it is easy to see that

$$\xi = \sum_{i \in \mathbf{N}} e_i \langle e_i, \xi \rangle,$$

where the sum converges unconditionally. In particular, given  $m \in M$ , we have for  $\xi = U^{-1}(m)$ , that

$$m = P(m) = P(U(\xi)) = \sum_{i \in \mathbf{N}} P(U(e_i)) \langle e_i, \xi \rangle = \sum_{i \in \mathbf{N}} u_i \langle e_i, \xi \rangle.$$

We also have that

$$\begin{aligned} \langle e_i, \xi \rangle &= \langle U(e_i), U(\xi) \rangle = \langle U(e_i), m \rangle = \langle U(e_i), P(m) \rangle \\ &= \langle P(U(e_i)), m \rangle = \langle u_i, m \rangle = u_i^* m, \end{aligned}$$

which combines with the calculation above to give the result. □

**11.2. Proposition.** *Let  $M$  be a slice and let  $\{u_i\}_{i \in \mathbf{N}}$  be a frame for  $M$ . Also let  $\mathcal{F}$  be the set of all finite subsets of  $\mathbf{N}$ , ordered by inclusion and viewed as a directed set. Then:*

- (i)  $\left\{ \sum_{i \in F} u_i u_i^* \right\}_{F \in \mathcal{F}}$  is an approximate unit for  $R(M)$ ;
- (ii)  $\| \sum_{i \in F} u_i u_i^* \| \leq 1$ , for every  $F \in \mathcal{F}$ .

**Proof.** For  $F \in \mathcal{F}$ , let  $s_F = \sum_{i \in F} u_i u_i^*$ , and if  $k \in \mathbf{N}$ , let  $s_k = s_{\{1, 2, \dots, k\}}$ . Consider the bounded operator

$$T_k : m \in M \mapsto s_k m \in M.$$

By hypothesis we have that  $T_k$  converges pointwise to the identity operator on  $M$  and hence  $\{T_k\}_k$  is a uniformly bounded set by the Banach–Steinhaus theorem. By [4, 4.7] we have that  $\|s_k\| = \|T_k\|$ , whence

$$\sup_k \|s_k\| = \sup_k \|T_k\| < \infty.$$

Given  $F \in \mathcal{F}$ , pick  $k \in \mathbf{N}$  such that  $F \subseteq \{1, 2, \dots, k\}$ , and observe that

$$0 \leq s_F = \sum_{i \in F} u_i u_i^* \leq \sum_{i \leq k} u_i u_i^* = s_k,$$

so  $\sup_{F \in \mathcal{F}} \|s_F\| < \infty$ , as well. If  $m, n \in M$  we have that

$$\lim_{F \rightarrow \infty} s_F m n^* = \sum_{i \in \mathbf{N}} u_i u_i^* m n^* = m n^*,$$

so  $\lim s_F b = b$ , for every  $b \in MM^*$ . Since the  $s_F$  are bounded, the last identity holds for every  $b$  in the closed linear span of  $MM^*$ , thus proving (i). With respect to (ii), it follows from the elementary result immediately below.  $\square$

**11.3. Proposition.** *Let  $\{v_i\}_i$  be an increasing approximate identity of a  $C^*$ -algebra  $A$  consisting of positive elements. Then  $\|v_i\| \leq 1$ , for all  $i$ .*

**Proof.** For every  $i \leq j$  and every positive element  $a$  in  $A$  one has that  $a^{1/2} v_i a^{1/2} \leq a^{1/2} v_j a^{1/2}$ . Taking the limit on  $j$  we conclude that

$$a^{1/2} v_i a^{1/2} \leq a.$$

With  $a = v_i$  we get  $v_i^2 \leq v_i$ , so  $\|v_i\| \leq 1$ .  $\square$

We now begin our study of conditional expectations.

**11.4. Proposition.** *Let  $P : A \rightarrow B$  be a conditional expectation, and let  $M$  be a slice. Then  $P(M) \subseteq R(M) \cap S(M)$ .*

**Proof.** Recall from [11, 1.1.4] that if  $\{v_i\}_i$  is an approximate identity for  $R(M)$ , then  $\lim_i v_i m = m$ , for all  $m \in M$ . Therefore

$$P(m) = \lim_i P(v_i m) = \lim_i v_i P(m) \in R(M).$$

Similarly  $P(m^*) \in R(M^*) = S(M)$ , whence

$$P(m) = P(m^*)^* \in (S(M))^* = S(M). \quad \square$$

The following is a key technical result:

**11.5. Lemma.** *Let  $P : A \rightarrow B$  be a conditional expectation, and let  $M$  be a slice. Then there exists a unique central positive element  $\tau$  in the multiplier algebra of  $R(M)$  such that  $\|\tau\| \leq 1$ , and*

$$(11.6) \quad \tau m n^* = P(m) P(n^*) \quad \forall m, n \in M.$$

Moreover, if  $\{u_i\}_{i \in \mathbf{N}}$  is any frame for  $M$ , one has that

$$\tau = \sum_{i \in \mathbf{N}} P(u_i) P(u_i^*),$$

the series being unconditionally convergent in the strict topology.

**Proof.** Since  $R(M)$  is the closure of  $MM^*$ , it is obvious that  $\tau$  is unique. As for existence let  $\{u_i\}_{i \in \mathbf{N}}$  be a frame for  $M$ . We claim that the series

$$(11.7) \quad \sum_{i \in \mathbf{N}} P(u_i)P(u_i^*)$$

converges unconditionally in the strict topology of the multiplier algebra of  $R(M)$ . This means that the net  $\{\tau_F\}_{F \in \mathcal{F}}$ , where  $\mathcal{F}$  was defined in (11.2), and

$$\tau_F = \sum_{i \in F} P(u_i)P(u_i^*) \quad \forall F \in \mathcal{F},$$

converges strictly. In turn this is to say that for every  $b \in R(M)$ , the nets  $\{b\tau_F\}_{F \in \mathcal{F}}$ , and  $\{\tau_F b\}_{F \in \mathcal{F}}$  converge in the norm topology of  $R(M)$ . We treat first the case in which  $b = mn^*$ , with  $m, n \in M$ . Given  $F \in \mathcal{F}$ , we have

$$\begin{aligned} \tau_F mn^* &= \sum_{i \in F} P(u_i)P(u_i^*)mn^* = \sum_{i \in F} P(u_i)P(u_i^*mn^*) \\ &= \sum_{i \in F} P(u_i)u_i^*mP(n^*) = \sum_{i \in F} P(u_iu_i^*m)P(n^*) \\ &= P\left(\sum_{i \in F} u_iu_i^*m\right)P(n^*). \end{aligned}$$

By (11.1) we then conclude that

$$(11.8) \quad \lim_{F \rightarrow \infty} \tau_F mn^* = P(m)P(n^*) = \lim_{F \rightarrow \infty} mn^* \tau_F,$$

where the second equality is actually a consequence of the first, by taking adjoints. To prove convergence in case  $b$  is not necessarily of the form  $mn^*$ , it is now enough to show that  $\{\tau_F\}_{F \in \mathcal{F}}$  is a bounded net, but this may be proved with the help of [18, III.3.4.iii] as follows:

$$0 \leq \tau_F = \sum_{i \in F} P(u_i)P(u_i^*) \leq \sum_{i \in F} P(u_iu_i^*) = \sum_{i \in F} u_iu_i^* \stackrel{(11.2.ii)}{\leq} 1.$$

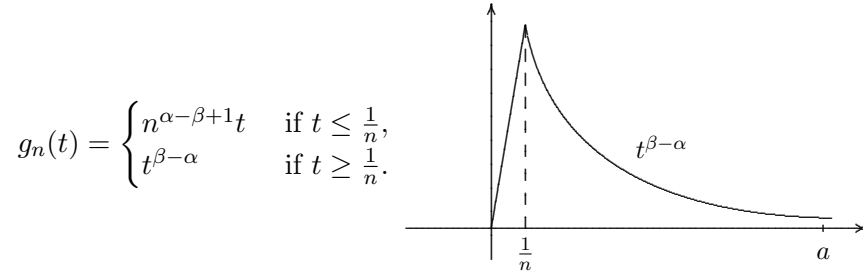
This shows that (11.7) does converge as indicated, so we let  $\tau$  be its sum. Obviously  $\|\tau\| \leq 1$ . Identity (11.6) then follows immediately from (11.8), and so does the fact that  $\tau$  is central.  $\square$

We will soon be interested in the space  $\overline{\tau^{1/2}M}$ , when we will need the following two elementary results:

**11.9. Lemma.** *Let  $a$  be a positive real number. Given  $\alpha, \beta, \varepsilon > 0$ , there exists a continuous scalar valued function  $g$  on  $[0, a]$  such that  $g(0) = 0$ , and*

$$|t^\beta - g(t)t^\alpha| < \varepsilon \quad \forall t \in [0, a].$$

**Proof.** If  $\beta > \alpha$  it is enough to take  $g(t) = t^{\beta-\alpha}$ , so we suppose that  $\beta \leq \alpha$ . For every positive integer  $n$ , let



For  $t \geq 1/n$ , it is clear that  $t^\beta = g_n(t)t^\alpha$ , while for  $t \leq 1/n$ , we have

$$|t^\beta - g_n(t)t^\alpha| = |t^\beta - n^{\alpha-\beta+1}t^{\alpha+1}| \leq t^\beta + n^{\alpha-\beta+1}t^{\alpha+1} \leq 2n^{-\beta},$$

which can be made less than  $\varepsilon$  for a suitably large  $n$ .  $\square$

**11.10. Lemma.** Let  $A$  be a  $C^*$ -algebra and let  $M$  be a left Banach module<sup>3</sup> over  $A$ . Given a self-adjoint element  $h$  in  $A$  with  $\|h\| \leq 1$ , and  $\alpha > 0$ , one has:

- (i)  $\overline{h^\alpha M} = \{m \in M : m = \lim_{k \rightarrow \infty} h^{1/k} m\}$ .
- (ii) If  $\alpha' > 0$ , then  $\overline{h^\alpha M} = \overline{h^{\alpha'} M}$ .
- (iii) If  $m \in \overline{h^\alpha M}$ , and  $h^\beta m = 0$ , for some  $\beta > 0$ , then  $m = 0$ .

**Proof.** We begin by proving (i). For this, let  $m \in \overline{h^\alpha M}$ . Given  $\varepsilon > 0$ , choose  $n \in M$  such that  $\|m - h^\alpha n\| \leq \varepsilon/2$ . Then, for every  $k \in \mathbf{N}$ , one has that

$$\begin{aligned} \|m - h^{1/k} m\| &= \|(1 - h^{1/k})m\| \\ &\leq \|(1 - h^{1/k})(m - h^\alpha n)\| + \|(1 - h^{1/k})h^\alpha n\| \\ &\leq \|1 - h^{1/k}\| \|m - h^\alpha n\| + \|h^\alpha - h^{\alpha+1/k}\| \|n\| \\ &\leq \frac{\varepsilon}{2} + \sup_{0 \leq t \leq 1} |t^\alpha - t^{\alpha+1/k}| \|n\|, \end{aligned}$$

which can be made less than  $\varepsilon$  for all sufficiently large  $k$ . Observe that we have referred to “1” above without having assumed that  $A$  is unital. Actually our use of “1” is just a calculation resource which can be avoided by expanding out all products. This shows that

$$(11.11) \quad m = \lim_{k \rightarrow \infty} h^{1/k} m,$$

as desired.

Let  $k \in \mathbf{N}$  be fixed. Plugging  $\beta = 1/k$  in (11.9) we obtain a continuous function  $g$  defined on  $[0, 1]$  such that

$$\|h^{1/k} - g(h)h^\alpha\| < \varepsilon.$$

<sup>3</sup>This means that  $M$  is a Banach space which is also a left module over  $A$ , satisfying  $\|am\| \leq \|a\|\|m\|$ , for all  $a \in A$ , and  $m \in M$ .



Given any  $m \in M$ , we then have

$$\|h^{1/k}m - g(h)h^\alpha m\| \leq \|h^{1/k} - g(h)h^\alpha\| \|m\| \leq \varepsilon \|m\|.$$

Since  $g(h)h^\alpha m = h^\alpha g(h)m$  lies in  $h^\alpha M$ , and  $\varepsilon$  is arbitrary, we conclude that  $h^{1/k}m \in \overline{h^\alpha M}$ . If we assume that  $m$  satisfies (11.11), it then follows easily that  $m \in \overline{h^\alpha M}$ , and hence (i) is proved.

Since the right-hand-side of (i) does not depend on  $\alpha$ , we see that (ii) follows.

We finally prove (iii). By the same reasoning above, given  $k \in \mathbf{N}$  and  $\varepsilon > 0$ , there exists a continuous function  $g$  such that

$$\|h^{1/k}m\| = \|h^{1/k}m - g(h)h^\beta m\| \leq \varepsilon \|m\|.$$

Since  $\varepsilon$  is arbitrary it follows that  $h^{1/k}m = 0$ , and hence that  $m = 0$ , by (11.11). □

**11.12. Proposition.** *Let  $P : A \rightarrow B$  be a conditional expectation, let  $M$  be a slice, and let  $\tau$  be as in (11.5). Then the sets  $\tau R(M)$  and  $P(M)$  have the same closure.*

**Proof.** For every  $m, n \in M$  we have by (11.6) that

$$\tau mn^* = P(m)P(n^*) = P(mP(n^*)) \in P(M),$$

so  $\tau MM^* \subseteq P(M)$ , and hence  $\overline{\tau R(M)} \subseteq \overline{P(M)}$ . In order to prove the reverse inclusion we first claim that

$$(11.13) \quad P(u_i) \in \overline{\tau R(M)},$$

for all  $i$ , where  $\{u_i\}_{i \in \mathbf{N}}$  is any given frame for  $M$ . Fixing  $i \in \mathbf{N}$ , let  $\sigma = P(u_i)P(u_i)^*$ , and observe that a simple computation using the  $C^*$ -identity

$$\|\sigma^t P(u_i) - P(u_i)\|^2 = \|(\sigma^t P(u_i) - P(u_i))(\sigma^t P(u_i) - P(u_i))^*\|,$$

shows that

$$\lim_{t \rightarrow 0^+} \sigma^t P(u_i) = P(u_i).$$

Clearly  $\sigma \leq \tau$ , so that  $\sigma^t \leq \tau^t$ , for all  $t \in (0, 1)$  by [16, 1.3.8], and hence

$$P(u_i)^* \sigma^t P(u_i) \leq P(u_i)^* \tau^t P(u_i) \leq P(u_i)^* P(u_i),$$

which implies that  $\lim_{t \rightarrow 0^+} P(u_i)^* \tau^t P(u_i) = P(u_i)^* P(u_i)$ . If this time one uses the  $C^*$ -identity

$$\|\tau^t P(u_i) - P(u_i)\|^2 = \|(\tau^t P(u_i) - P(u_i))^*(\tau^t P(u_i) - P(u_i))\|,$$

there should be no difficulty in proving that  $\lim_{t \rightarrow 0^+} \tau^t P(u_i) = P(u_i)$ . Since  $P(u_i) \in R(M)$ , by (11.4), we have that

$$\tau^t P(u_i) \in \tau^t R(M) \subseteq \overline{\tau^t R(M)} \stackrel{(11.10.ii)}{=} \overline{\tau R(M)},$$

from which claim (11.13) follows readily. Observing that  $\overline{\tau R(M)}$  is a closed two-sided ideal in  $R(M)$ , and hence also in  $B$ , and using (11.1), we then have for every  $m \in M$ , that

$$P(m) = \sum_{i \in \mathbf{N}} P(u_i u_i^* m) = \sum_{i \in \mathbf{N}} P(u_i) u_i^* m \in \overline{\tau R(M)},$$

proving that  $P(M) \subseteq \overline{\tau R(M)}$ , and concluding the proof.  $\square$

Let us now briefly discuss the fact that a slice  $M$  is a left module, not only over  $R(M)$ , but also over the multiplier algebra of the latter. We have already mentioned that if  $\{v_i\}_i$  is an approximate identity for  $R(M)$ , then  $\lim_i v_i m = m$ , for all  $m \in M$ . Given any multiplier  $\mu$  of  $R(M)$ , and  $m \in M$ , observe that

$$\lim_i (\mu v_i) m$$

exists, since by Cohen–Hewitt’s factorization theorem [10, 32.22] one may write  $m = an$ , for some  $a \in R(M)$ , and  $n \in M$ , in which case

$$\lim_i (\mu v_i) m = \lim_i (\mu v_i a) n = (\mu a) n.$$

This also shows that the correspondence

$$m = an \in M \longmapsto \mu m := (\mu a) n \in M$$

is well defined, thus making  $M$  a left module over the multiplier algebra of  $R(M)$ .

Regarding the element  $\tau$  of the multiplier algebra of  $R(M)$  produced by (11.5), we then have that  $\tau m \in M$ , for any  $m \in M$ , where the meaning of  $\tau m$  is given by the above module structure. The same clearly also applies to  $\tau^{1/2} m$ .

**11.14. Proposition.** *Let  $P : A \rightarrow B$  be a conditional expectation, and let  $M$  be a slice. Then there exists an isometric virtual commutant  $\varphi$  of  $B$  in  $A$  whose domain is  $P(M) = \overline{\tau R(M)}$ , and such that*

$$\varphi(P(m)) = \tau^{1/2} m \quad \forall m \in M.$$

*Moreover the range of  $\varphi$  is precisely  $\overline{\tau^{1/2} M}$ .*

**Proof.** Given  $m \in M$  we have

$$\|P(m)\|^2 = \|P(m)P(m^*)\| \stackrel{(11.6)}{=} \|\tau m m^*\| = \|\tau^{1/2} m m^* \tau^{1/2}\| = \|\tau^{1/2} m\|^2.$$

This said one sees that there exists an isometric linear map  $\varphi : P(M) \rightarrow M$ , such that  $\varphi(P(m)) = \tau^{1/2} m$ , for every  $m$  in  $M$ , which therefore may be continuously extended to the closure of  $P(M)$ . By abuse of language we will denote the extended map also by  $\varphi$ . It is then clear that the range of this extended map is  $\overline{\tau^{1/2} M}$ .

We will now prove that  $\varphi$  is a virtual commutant. Beginning with (9.2.i), let  $b \in B$ , and  $x \in \overline{P(M)}$ . It is clearly enough to consider the case in which

$x = P(m)$ , for some  $m \in M$ . Write  $m = an$ , with  $a \in R(M)$  and  $n \in M$  as above. Picking an approximate unit  $\{v_i\}_i$  for  $R(M)$ , we have that

$$\begin{aligned} b\varphi(x) &= b\varphi(P(m)) = b(\tau^{1/2}a)n = \lim_i bv_i(\tau^{1/2}a)n = \lim_i ((bv_i)\tau^{1/2})an \\ &= \lim_i (\tau^{1/2}(bv_i))an = (\tau^{1/2}(ba))n = \varphi(P(ban)) = \varphi(bP(m)) \\ &= \varphi(bx). \end{aligned}$$

The proof of (9.2.ii) is easier:

$$\varphi(x)b = \varphi(P(m))b = (\tau^{1/2}a)nb = \varphi(P(anb)) = \varphi(P(m)b) = \varphi(xb). \quad \square$$

## 12. Generalized Cartan subalgebras

In this section we introduce our proposed broadening of Renault’s notion of Cartan subalgebras and prove a result on uniqueness of conditional expectations which is a  $C^*$ -algebraic analogue of [19, IX.4.3] and also a generalization of [17, 5.7].

**12.1. Definition.** Let  $A$  be a  $C^*$ -algebra and let  $B \subseteq A$  be a closed  $*$ -subalgebra. We will say that  $B$  is a *generalized Cartan* subalgebra of  $A$  if:

- (i)  $B$  contains an approximate unit for  $A$ .
- (ii)  $B$  satisfies property (MAX') (see (9.6)).
- (iii)  $N(B)$  generates  $A$ .
- (iv) There exists a faithful<sup>4</sup> conditional expectation  $P : A \rightarrow B$ .

Notice that  $N(B)$  is closed under multiplication and under taking adjoints, and hence its linear span is always a selfadjoint subalgebra of  $A$ . Thus, to say that  $N(B)$  generates  $A$  as a  $C^*$ -algebra is the same as saying that  $N(B)$  generates  $A$  as a Banach space.

**12.2. Theorem.** Let  $A$  be a separable  $C^*$ -algebra, let  $B \subseteq A$  be a closed  $*$ -subalgebra satisfying (12.1.i–ii), and let  $M$  be a slice. Given a conditional expectation  $P$  from  $A$  to  $B$  one has that:

- (i)  $M$  is invariant under  $P$ .
- (ii)  $M = (B \cap M) \oplus (\text{Ker}(P) \cap M)$ .
- (iii) If  $B \cap M = \{0\}$ , then  $P(M) = \{0\}$ .

**Proof.** Considering the virtual commutant  $\varphi$  obtained from (11.14) one has by (12.1.ii) that the range of  $\varphi$  is contained in  $B$ . By the last sentence of (11.14) we have that  $\tau^{1/2}M \subseteq B$ , and hence also  $\tau M \subseteq B$ . So

$$\tau MM^* = MM^*\tau = M(\tau M)^* \subseteq MB^* \subseteq M.$$

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<sup>4</sup>Traditionally a conditional expectation is said to be faithful if  $P(a^*a) = 0 \Rightarrow a = 0$ . However we may adopt a weaker form of faithfulness here by requiring only that:  $(\forall b, c \ P(c^*ab) = 0) \Rightarrow a = 0$ .

Using (11.12) we then deduce that

$$P(M) \subseteq \overline{P(M)} = \overline{\tau M M^*} \subseteq M,$$

proving (i). Since the restriction of  $P$  is an idempotent operator on  $M$  it follows that  $M$  is the direct sum of  $\text{Ker}(\text{id} - P|_M)$  and  $\text{Ker}(P|_M)$ , from which (ii) follows. Finally, (iii) is an immediate consequence of (ii).  $\square$

The following is a  $C^*$ -algebraic analogue of [19, IX.4.3] and also a generalization of [17, 5.7].

**12.3. Theorem.** *Let  $A$  be a separable  $C^*$ -algebra and let  $B \subseteq A$  be a closed  $*$ -subalgebra satisfying (12.1.i–iii). Then there exists at most one conditional expectation from  $A$  onto  $B$ .*

**Proof.** Let  $P$  and  $Q$  be conditional expectations onto  $B$ . In order to show that  $Q = P$ , it is enough to show that  $Q(a) = P(a)$ , for every  $a \in N(B)$ . Using (10.5) choose a slice  $M$  with  $a \in M$ , and write  $a = a_1 + a_2$ , with  $a_1 \in B \cap M$ , and  $a_2 \in \text{Ker}(P) \cap M$ , according to (12.2.ii). Since  $a_1 \in B$ , we have that

$$P(a_1) = a_1 = Q(a_1),$$

so it suffices to prove that  $Q(a_2) = 0$  ( $= P(a_2)$ ).

Let  $N = \text{Ker}(P) \cap M$ , and observe that  $N$  is a slice with  $B \cap N = \{0\}$ . Applying (12.2.iii) to  $N$  and  $Q$  we see that  $Q(N) = \{0\}$ , so  $Q(a_2) = 0$ .  $\square$

To see the relevance of  $B$  satisfying property (MAX'), as opposed to just satisfying  $B' \cap A \subseteq B$  (compare (9.8)) in the above result, observe that in the example given after (9.8) one has that any element of  $C(X) \otimes B$  of the form  $u \otimes b$ , where  $u$  is an unimodular function, is in  $N(B)$ , and hence  $N(B)$  generates  $C(X) \otimes B$ . That is,  $B$  satisfies (12.1.iii). Clearly  $B$  also satisfies (12.1.i) and, although (12.1.ii) fails, one has that  $B' \cap (C(X) \otimes B) \subseteq B$ , as observed earlier. Nevertheless each probability measure on  $X$  would give, by integration, a different conditional expectation from  $C(X) \otimes B$  to  $B$ .

### 13. Inverse semigroups and Fell bundles from generalized Cartan subalgebras

This section is dedicated to constructing a Fell bundle over an inverse semigroup from an inclusion of  $C^*$ -algebras. In the case of generalized Cartan subalgebras we will later show that this Fell bundle contains enough information to allow for a reconstruction of the inclusion.

Our standing hypothesis here will be (9.1), namely we will assume that  $A$  is a separable  $C^*$ -algebra and  $B$  is a closed  $*$ -subalgebra containing an approximate unit for  $A$ . This will allow us to use (10.2) as in the following result:

**13.1. Proposition.** *Let  $M$  and  $N$  be slices. Then  $\overline{MN}$  (closed linear span) is a slice.*

**Proof.** Let  $x = \sum_{i=1}^k m_i n_i$ , where  $m_i \in M$  and  $n_i \in N$ . Then, for every  $i, j$  we have by (10.2) that

$$m_i^* n_i^* B n_j m_j \subseteq M^* N^* B N M \subseteq B,$$

from where we deduce that  $x^* B x \subseteq B$ . Similarly  $x B x^* \subseteq B$ , so  $x \in N(B)$ . This shows that  $\overline{MN}$  is contained in  $N(B)$ , and hence that it is a slice.  $\square$

From now on we will frequently make use of sets such as  $\overline{MN}$ , so we shall adopt the following:

**13.2. Convention.** If  $X$  and  $Y$  are linear subspaces of  $A$  we will denote by  $XY$  the *closed linear span* of the set  $\{xy : x \in X, y \in Y\}$ .

**13.3. Proposition.** *The set  $\mathfrak{S}_{A,B} = \{M \subseteq A : M \text{ is a slice}\}$  is an inverse semigroup under the operation referred to in (13.1) (and from now on denoted simply by  $MN$ , according to (13.2)).*

**Proof.** It is evident that  $\mathfrak{S}_{A,B}$  is an associative semigroup. Given  $M \in \mathfrak{S}_{A,B}$  we may view  $M$  as a left Hilbert module over  $R(M)$  by (10.3.i), so we deduce that  $MM^*M = M$  by [11, 1.1.4], and similarly  $M^*MM^* = M^*$ .

Since  $M^*M$  and  $MM^*$  (now meaning closed linear span) are closed two-sided ideals in  $B$  by (10.3.ii), they commute and hence  $\mathfrak{S}_{A,B}$  is an inverse semigroup by [15, 1.1.3].  $\square$

If  $M$  is an idempotent element of  $\mathfrak{S}_{A,B}$ , then  $M = M^*M$ , so  $M$  is a closed two-sided ideal of  $B$  by (10.3.ii). Conversely, it is easy to see that every closed two-sided ideal of  $B$  is an idempotent element of  $\mathfrak{S}_{A,B}$ . Thus

$$E(\mathfrak{S}_{A,B}) = \{J : J \text{ is a closed two-sided ideal of } B\}.$$

In particular  $B$  itself is an idempotent element, actually the unit of  $\mathfrak{S}_{A,B}$ .

The usual inverse semigroup order on  $\mathfrak{S}_{A,B}$  is defined by

$$N \leq M \iff MN^*N = N.$$

Notice that, when  $N \leq M$ , one has that  $N = MN^*N \subseteq MB \subseteq M$ , so  $N \subseteq M$ . Conversely, if  $N \subseteq M$ , then

$$N = NN^*N \subseteq MN^*N \subseteq MM^*N \subseteq BN \subseteq N,$$

so  $N = MN^*N$ . In other words we have

$$N \leq M \iff N \subseteq M.$$

In case  $N(B)$  generates  $A$ , as in (12.1.iii), notice that by (10.5) one has that

$$(13.4) \quad A = \overline{\sum_{M \in \mathfrak{S}_{A,B}} M},$$

where we use the symbol  $\overline{\sum}$  to denote the *closure* of the sum of a given family of subspaces of  $A$ .

**13.5. Proposition.** *Let  $\mathfrak{S}$  be a  $*$ -subsemigroup<sup>5</sup> of  $\mathfrak{S}_{A,B}$ . If  $A = \overline{\sum_{M \in \mathfrak{S}} M}$ , then  $B = \overline{\sum_{M \in \mathfrak{S}} M^*M}$ .*

**Proof.** Let  $B' = \overline{\sum_{M \in \mathfrak{S}} M^*M}$ , so that  $B'$  is a closed two-sided ideal of  $B$ . Also let  $A' = B'AB'$ , so  $A'$  is a  $C^*$ -subalgebra of  $A$ . For  $M \in \mathfrak{S}$  notice that

$$M = (MM^*)M(M^*M) \subseteq B'MB' \subseteq A',$$

so the hypothesis implies that  $A' = A$ . It is therefore clear that any bounded approximate unit  $\{w_i\}_i$  for  $B'$  is also an approximate unit for  $A$ . Given an arbitrary  $b \in B$ , one therefore has that

$$b = \lim_i w_i b \in B'B \subseteq B',$$

so  $B \subseteq B'$ . Since the reverse inclusion also holds, the proof is complete.  $\square$

Our next result is related to one of the main hypothesis of [5, 9.9]. See also [5, 5.4].

**13.6. Proposition.** *Let  $\mathfrak{S}$  be any  $*$ -subsemigroup of  $\mathfrak{S}_{A,B}$ . Then the following are equivalent:*

- (i) *For every  $M, N \in \mathfrak{S}$ , and every  $a \in M \cap N$ , there exists  $K \in \mathfrak{S}$  such that  $a \in K \subseteq M \cap N$ .*
- (ii) *For every  $M \in \mathfrak{S}$ , and every  $b \in M \cap B$ , there exists  $K \in \mathfrak{S}$  such that  $b \in K \subseteq M \cap B$ .*

**Proof.** (i)  $\Rightarrow$  (ii): Notice that  $M \cap B$  is a closed two-sided ideal of  $B$  so, given  $b \in M \cap B$ , we may write  $b = b_1^*b_2$ , with  $b_1, b_2 \in M \cap B$ . In particular  $b \in M^*M$ . Plugging  $N = M^*M$  in (i) we deduce that there exists  $K \in \mathfrak{S}$  such that

$$b \in K \subseteq M \cap M^*M \subseteq M \cap B.$$

(ii)  $\Rightarrow$  (i): Given  $a \in M \cap N$ , one has that

$$a^*a \in M^*M \cap M^*N \subseteq B \cap M^*N.$$

By (ii) there exists  $K \in \mathfrak{S}$ , such that  $a^*a \in K \subseteq B \cap M^*N$ , so

$$aa^*a \in MK \subseteq MB \cap MM^*N \subseteq M \cap N.$$

The proof will therefore be concluded once we show that  $a \in MK$ .

Using (11.9), for every  $n \in \mathbf{N}$ , let  $h_n$  be a continuous real valued function on the interval  $[0, \|a\|^2]$  such that

$$|h_n(t)t - t^{1/n}| < \frac{1}{n} \quad \forall t \in [0, \|a\|^2],$$

so that  $\|h_n(aa^*)aa^* - (aa^*)^{1/n}\| < 1/n$ . We claim that

$$\lim_{n \rightarrow \infty} h_n(aa^*)aa^*a = a.$$

<sup>5</sup>By a  $*$ -subsemigroup of  $\mathfrak{S}_{A,B}$  we shall mean a subsemigroup  $\mathfrak{S} \subseteq \mathfrak{S}_{A,B}$  such that  $\mathfrak{S}^* = \mathfrak{S}$ , so that  $\mathfrak{S}$  is an inverse semigroup in its own right.

In fact

$$\begin{aligned} \|h_n(aa^*)aa^*a - a\| &\leq \|h_n(aa^*)aa^*a - (aa^*)^{1/n}a\| + \|(aa^*)^{1/n}a - a\| \\ &\leq \frac{\|a\|}{n} + \left\| \left( (aa^*)^{1/n}a - a \right) \left( (aa^*)^{1/n}a - a \right)^* \right\|^{1/2} \\ &= \frac{\|a\|}{n} + \left\| (aa^*)^{1+2/n} - 2(aa^*)^{1+1/n} + aa^* \right\|^{1/2}, \end{aligned}$$

which converges to zero as  $n \rightarrow \infty$ , proving our claim. Since  $h_n(aa^*)aa^*a \in MK$ , we deduce that  $a \in MK$ , concluding the proof.  $\square$

**13.7. Definition.** Let  $\mathfrak{S}$  be a  $*$ -subsemigroup of  $\mathfrak{S}_{A,B}$ . We will say that  $\mathfrak{S}$  is an *admissible semigroup* for the  $C^*$ -algebra inclusion “ $A \subseteq B$ ” if:

- (i)  $A = \overline{\sum_{M \in \mathfrak{S}} M}$ .
- (ii)  $\mathfrak{S}$  satisfies the equivalent conditions of (13.6).

If  $N(B)$  generates  $A$ , we have seen that (13.4) holds. Moreover  $\mathfrak{S}_{A,B}$  is clearly closed under intersections, and hence  $\mathfrak{S}_{A,B}$  satisfies (13.6.i). In other words,  $\mathfrak{S}_{A,B}$  is an admissible semigroup. It is however likely to be very big, perhaps even uncountable, so one might wonder whether a countable admissible semigroup exists.

**13.8. Proposition.** *If  $(A, B)$  is a generalized Cartan pair with separable  $A$ , then there exists a countable admissible semigroup.*

**Proof.** If  $A$  is separable then so is  $N(B)$ , hence we may choose a countable dense subset  $\{a_n : n \in \mathbf{N}\}$  of  $N(B)$ . Using (10.5), for each  $n \in \mathbf{N}$  pick a slice  $M_n$  containing  $a_n$ . The smallest  $*$ -subsemigroup of  $\mathfrak{S}_{A,B}$  closed under pairwise intersections, and containing all of the  $M_n$  satisfies the conditions in the statement.  $\square$

### 14. Fell bundle models for generalized Cartan subalgebras

In this section we present our main result giving a sufficient condition for an inclusion of  $C^*$ -algebras to be modeled by the monomorphism  $\Phi_{\text{red}} : C^*(\mathcal{E}) \rightarrow C^*_{\text{red}}(\mathcal{A})$  of (8.11).

Throughout this section we let  $A$  be a separable  $C^*$ -algebra and  $B$  be a generalized Cartan subalgebra of  $A$ . We will also fix any admissible semigroup  $\mathfrak{S} \subseteq \mathfrak{S}_{A,B}$ . Observe that a countable such  $\mathfrak{S}$  exists by (13.8).

We thus get a Fell bundle  $\mathcal{A}$  over  $\mathfrak{S}$  by setting  $A_M = M$ , for every  $M \in \mathfrak{S}$ . As in Section (7) we let  $\mathcal{E}$  denote the restriction of  $\mathcal{A}$  to the idempotent semilattice  $E(\mathfrak{S})$ . Observe that, if for each  $J \in E(\mathfrak{S})$  we consider the inclusion mapping

$$\iota_J : J \hookrightarrow B,$$

then  $\iota = \{\iota_J\}_{J \in E(\mathfrak{S})}$  is a representation of  $\mathcal{E}$  in  $B$  satisfying the hypothesis of (4.3). Since  $\mathcal{R}(\iota)$  coincides with  $B$  by (13.5) we deduce that  $C^*(\mathcal{E})$  is

isomorphic to  $B$ . We shall henceforth identify  $C^*(\mathcal{E})$  and  $B$ , observing that the identification puts in correspondence the canonical mappings

$$(14.1) \quad \pi_J^u \simeq \iota_J.$$

Turning our attention to  $\mathcal{A}$  let us extend the above representation  $\iota$  to the whole of  $\mathcal{A}$  by considering the inclusion mappings

$$(14.2) \quad \iota_M : M \hookrightarrow A,$$

for each  $M \in \mathcal{A}$ . It is then evident that  $\iota = \{\iota_M\}_{M \in \mathfrak{S}}$  is a representation of  $\mathfrak{S}$  in  $A$ . Let

$$(14.3) \quad \Phi : \mathcal{L}(\mathcal{A}) \rightarrow A$$

be the  $*$ -homomorphism associated to  $\iota$ , as in (3.7). By (3.10) one has that  $\Phi$  vanishes on  $\mathcal{N}$ .

We next want to give an explicit description of the state  $\tilde{\varphi}$  on  $\mathcal{L}(\mathcal{A})$  obtained from (7.4) for each given pure state  $\varphi$  on  $C^*(\mathcal{E})$ .

**14.4. Proposition.** *Let  $\varphi$  be a pure state on  $B = C^*(\mathcal{E})$  and let  $\tilde{\varphi}$  be the state on  $\mathcal{L}(\mathcal{A})$  given by (7.4) in terms of  $\varphi$ . Then*

$$\tilde{\varphi} = \varphi \circ P \circ \Phi.$$

**Proof.** It is clearly enough to verify that

$$(14.5) \quad \tilde{\varphi}(a_M \delta_M) = \varphi \circ P \circ \Phi(a_M \delta_M),$$

for all  $M \in \mathfrak{S}$ , and every  $a_M \in M$ .

*Case 1:* Assume that  $a_M \in \text{Ker}(P) \cap M$ . The right-hand-side of (14.5) then becomes

$$\varphi \circ P \circ \Phi(a_M \delta_M) = \varphi(P(a_M)) = 0.$$

Using the description of  $\tilde{\varphi}$  given in (7.4.i) suppose first that there exists  $J \in \text{supp}(\varphi)$  with  $J \leq M$ . Then the left-hand-side of (14.5) becomes

$$(14.6) \quad \tilde{\varphi}(a_M \delta_M) = \tilde{\varphi}_J^M(a_M) \stackrel{(6.1)}{=} \lim_i \varphi(a_M w_i),$$

where  $\{w_i\}_i$  is any approximate identity for  $J$ . Notice however that since  $J \leq M$ , we have that  $MJ = J$ , so  $a_M w_i \in MJ = J \subseteq B$ . Thus

$$a_M w_i = P(a_M w_i) = P(a_M) w_i = 0,$$

so (14.6) vanishes.

Suppose next that there exists no  $J \in \text{supp}(\varphi)$  with  $J \leq M$ . Then  $\tilde{\varphi}(a_M \delta_M) = 0$ , by the second clause of (7.4.i), and hence (14.5) is proved under our first case.

*Case 2:* Assuming that  $a_M \in B \cap M$ , the right-hand-side of (14.5) becomes

$$\varphi \circ P \circ \Phi(a_M \delta_M) = \varphi(P(a_M)) = \varphi(a_M).$$

Recall that  $\mathfrak{S}$  is an admissible semigroup and hence that (13.6.ii) holds. Therefore there exists  $J \in \mathfrak{S}$  such that  $a_M \in J \subseteq M \cap B$ . Since  $J \subseteq B$  we



see that  $J$  is in  $E(\mathfrak{S})$ , and since  $J \subseteq M$  we have that  $J \leq M$ . Therefore  $a_M \delta_M \equiv a_M \delta_J$ , modulo  $\mathcal{N}$ , and since  $\tilde{\varphi}$  vanishes on  $\mathcal{N}$  by (7.4.iv), the left-hand-side of (14.5) becomes

$$\tilde{\varphi}(a_M \delta_M) = \tilde{\varphi}(a_M \delta_J) \stackrel{(7.4.ii)}{=} \varphi(\pi_J^u(a_M)) \stackrel{(14.1)}{=} \varphi(a_M),$$

hence (14.5) is proved under Case 2.

In view of (12.2.ii) we see that the general case of (14.5) follows from the two cases already treated and the proof is then complete.  $\square$

We now come to our main result:

**14.7. Theorem.** *Let  $A$  be a separable  $C^*$ -algebra and let  $B$  be a generalized Cartan subalgebra of  $A$ . Choose any admissible subsemigroup  $\mathfrak{S}$  of  $\mathfrak{S}_{A,B}$ , such as given by (13.8), and consider the Fell bundle  $\mathcal{A}$  over  $\mathfrak{S}$  given by  $A_M = M$ , for every  $M \in \mathfrak{S}$ . Then  $A$  is isomorphic to  $C_{\text{red}}^*(\mathcal{A})$  via an isomorphism  $\Psi : C_{\text{red}}^*(\mathcal{A}) \rightarrow A$  such that*

$$\Psi \circ \pi_M^{\text{red}}(m) = m \quad \forall M \in \mathfrak{S} \quad \forall m \in M.$$

*In addition  $\Psi(C^*(\mathcal{E})) = B$ , where  $\mathcal{E}$  is the restriction of  $\mathcal{A}$  to the idempotent semilattice of  $\mathfrak{S}$ .*

**Proof.** Let  $\Phi : \mathcal{L}(\mathcal{A}) \rightarrow A$  be as in (14.3). We first claim that

$$(14.8) \quad \|\Phi(x)\| = \|x\| \quad \forall x \in \mathcal{L}(\mathcal{A}),$$

where  $\|\cdot\|$  was defined in (8.5). With this goal in mind let  $\varphi$  be any pure state on  $B$  and consider the state  $\psi = \varphi \circ P$  on  $A$ . Also let  $(\Pi_\psi, H_\psi, \xi_\psi)$  be the GNS representation of  $A$  associated to  $\psi$ . Consider the map

$$U : x \in \mathcal{L}(\mathcal{A}) \mapsto \Pi_\psi(\Phi(x))\xi_\psi \in H_\psi.$$

For  $x, y \in \mathcal{L}(\mathcal{A})$  we have

$$\begin{aligned} \langle U(x), U(y) \rangle &= \langle \Pi_\psi(\Phi(x))\xi_\psi, \Pi_\psi(\Phi(y))\xi_\psi \rangle = \langle \Pi_\psi(\Phi(y^*x))\xi_\psi, \xi_\psi \rangle \\ &= \psi(\Phi(y^*x)) = \varphi \circ P \circ \Phi(y^*x) \stackrel{(14.4)}{=} \tilde{\varphi}(y^*x), \end{aligned}$$

where  $\tilde{\varphi}$  is the extension of  $\varphi$  to  $\mathcal{L}(\mathcal{A})$  provided by (7.4). This says that

$$\langle U(x), U(y) \rangle = \langle x, y \rangle_{\tilde{\varphi}},$$

where  $\langle \cdot, \cdot \rangle_{\tilde{\varphi}}$  was defined in (8.1). The map  $U$  thus defines an isometry (also denoted  $U$  by abuse of language)

$$U : H_{\tilde{\varphi}} \rightarrow H_\psi,$$

such that  $U(\hat{x}) = \Pi_\psi(\Phi(x))\xi_\psi$ , for every  $x \in \mathcal{L}(\mathcal{A})$ . By (13.7.i) we have that  $\Phi$  has dense image and since  $\Pi_\psi$  is cyclic we deduce that  $U$  is in fact a

unitary operator. We next claim that the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{A}) & \xrightarrow{\Phi} & A \\ \Upsilon_{\tilde{\varphi}} \downarrow & & \downarrow \Pi_{\psi} \\ B(H_{\tilde{\varphi}}) & \xrightarrow{\text{Ad}_U} & B(H_{\psi}) \end{array}$$

commutes. Proving this amounts to checking that

$$\Pi_{\psi}(\Phi(x))U|_{\hat{y}} = U\Upsilon_{\tilde{\varphi}}(x)|_{\hat{y}} \quad \forall x, y \in \mathcal{L}(\mathcal{A}).$$

We have

$$\begin{aligned} \Pi_{\psi}(\Phi(x))U|_{\hat{y}} &= \Pi_{\psi}(\Phi(x))\Pi_{\psi}(\Phi(y))\xi_{\psi} \\ &= \Pi_{\psi}(\Phi(xy))\xi_{\psi} = U(\widehat{xy}) = U\Upsilon_{\tilde{\varphi}}(x)|_{\hat{y}}. \end{aligned}$$

This shows that our diagram indeed commutes and hence we deduce that, for every  $x \in \mathcal{L}(\mathcal{A})$ ,

$$\|x\| = \sup_{\varphi} \|\Upsilon_{\tilde{\varphi}}(x)\| = \sup_{\varphi} \|\Pi_{\psi}(\Phi(x))\| = \sup_{\varphi} \|\Pi_{\varphi \circ P}(\Phi(x))\|,$$

where  $\varphi$  runs in the set of all pure states of  $B$ . To prove (14.8) it now suffices to prove that

$$\sup_{\varphi} \|\Pi_{\varphi \circ P}(a)\| = \|a\| \quad \forall a \in A,$$

which is in turn equivalent to proving that  $\bigoplus_{\varphi} \Pi_{\varphi \circ P}$  is a faithful representation of  $A$ .

If  $a \in A$  is nonzero then there are  $b, c \in A$  such that  $P(c^*ab) \neq 0$  (see the footnote in Definition (12.1)), and hence  $\varphi(P(c^*ab)) \neq 0$ , for some pure state  $\varphi$  on  $B$ , which implies that  $\Pi_{\varphi \circ P}(a) \neq 0$ . This proves the faithfulness of  $\bigoplus_{\varphi} \Pi_{\varphi \circ P}$ , and hence concludes the proof of (14.8).

It is now clear the the correspondence

$$\iota_{\mathcal{A}}^{\text{red}}(x) \in C_{\text{red}}^*(\mathcal{A}) \longmapsto \Phi(x) \in A$$

is well defined and extends to an isomorphism  $\Psi : C_{\text{red}}^*(\mathcal{A}) \rightarrow A$ , satisfying  $\Psi \circ \iota_{\mathcal{A}}^{\text{red}} = \Phi$ . That  $\Psi$  is onto follows from the already mentioned fact that the range of  $\Phi$  is dense in  $A$ . Given  $M \in \mathfrak{S}$  we have that

$$\Psi \circ \pi_M^{\text{red}} \stackrel{(8.7)}{=} \Psi \circ \iota_{\mathcal{A}}^{\text{red}} \circ \pi_M^0 = \Phi \circ \pi_M^0 = \iota_M.$$

This is to say that the identity displayed in the statement holds.

Viewing  $C^*(\mathcal{E})$  as a subalgebra of  $C_{\text{red}}^*(\mathcal{A})$  according to (8.11), we have that  $C^*(\mathcal{E}) = \overline{\sum_{J \in E(\mathfrak{S})} \pi_J^{\text{red}}(J)}$ , so we deduce that

$$\Psi(C^*(\mathcal{E})) = \overline{\sum_{J \in E(\mathfrak{S})} \Psi(\pi_J^{\text{red}}(J))} = \overline{\sum_{J \in E(\mathfrak{S})} J} \stackrel{(13.5)}{=} B. \quad \square$$

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