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Gel'fand triples and boundaries of infinite networks

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Abstract. We study the boundary theory of a connected weighted graph G from the viewpoint of stochastic integration. For the Hilbert space $\mathcal{H}_\mathcal{E}$ of Dirichlet-finite functions on G, we construct a Gel'fand triple $S \subseteq \mathcal{H}_\mathcal{E} \subseteq S'$. This yields a probability measure $\mathbb{P}$ on S' and an isometric embedding of $\mathcal{H}_\mathcal{E}$ into $L^2(S', \mathbb{P})$, and hence gives a concrete representation of the boundary as a certain class of "distributions" in S'. In a previous paper, we proved a discrete Gauss–Green identity for infinite networks which produces a boundary representation for harmonic functions of finite energy, given as a certain limit. In this paper, we use techniques from stochastic integration to make the boundary $\operatorname{bd} G$ precise as a measure space, and obtain a boundary integral representation as an integral over S'.

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1. Introduction

In this paper, we develop a boundary theory for an infinite network (connected weighted graph) G, using some techniques from the theory of stochastic integration. For the Hilbert space $\mathcal{H}_{\mathcal{E}}$ of finite-energy functions on G, we construct a Gel'fand triple $\mathcal{S}_{\mathcal{E}} \subseteq \mathcal{H}_{\mathcal{E}} \subseteq \mathcal{S}'_{\mathcal{E}}$, where both containments are strict, and the inclusion mappings are continuous. Here, $\mathcal{S}_{\mathcal{E}}$ is a space of "test functions" on the network and $\mathcal{S}'_{\mathcal{E}}$ is a class of "distributions" on the network, analogous to Schwartz's classical *functions of rapid decay* and *tempered distributions*, respectively. To our knowledge, this is a novel approach to boundary theory, and also a new application of Gel'fand triples. A key result of this paper is Theorem 5.3, which establishes an isometric embedding of $\mathcal{H}_{\mathcal{E}}$ into the Hilbert space $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on $\mathcal{S}'_{\mathcal{E}}$ such that $\mathcal{S}_{\mathcal{E}}$ indexes a Gaussian process on $\mathcal{S}'_{\mathcal{E}}$ with respect to μ.

Let $\mathcal{E}(u) = \mathcal{E}(u,u)$ denote the *energy* of u; the Dirichlet form $\mathcal{E}$ is the inner product on the Hilbert space $\mathcal{H}_{\mathcal{E}}$ (see Definition 2.5). Also, let Δ be the network Laplacian (see Definition 2.2). In a previous paper, we proved a discrete Gauss–Green identity for infinite networks which relates $\mathcal{E}$ and Δ for certain elements of $\mathcal{H}_{\mathcal{E}}$:

$$\mathcal{E}(u,v) = \sum_{G} \bar{u}\Delta v + \sum_{\operatorname{bd} G} \bar{u}\tfrac{\partial v}{\partial \mathrm{n}}. \tag{1.1}$$

This result is recalled in Theorem 2.27, where precise definitions of the symbols in (1.1) are also given. Formula (1.1) yields a boundary representation for a finite-energy harmonic function u:

$$u(x) = \sum_{\operatorname{bd} G} u \frac{\partial v_x}{\partial \mathbf{n}} + C, \tag{1.2}$$

where C is a constant and the sum is actually defined as a limit of "Riemann sums" over an increasing sequence of finite subnetworks of G; see Definition 2.26. In this paper, we make the boundary $\operatorname{bd} G$ precise as a measure space, and in Corollary 5.12 we replace the sum with an integral over $\mathcal{S}'_{\mathcal{E}}$, thus obtaining a boundary integral representation for the harmonic function u:

$$u(x) = \int_{\mathcal{S}'_{\mathcal{E}}} u(\xi) h_x(\xi)\, d\mathbb{P}(\xi) + u(o). \tag{1.3}$$

Here, $\{h_x\}_{x \in G}$ is a family of harmonic functions; in fact, a reproducing kernel for the harmonic functions of finite energy (see Lemma 2.19). Given a transient network, this allows one to identify the elements of $\mathcal{S}'_{\mathcal{E}}$ corresponding to the boundary of the network (in a manner reminiscent of the Martin boundary). Additionally, Example 6.1 presents the construction of a harmonic function of finite energy on a network with one "graph end" (in fact, a two-parameter family of such networks). The existence of such functions was first proved in [CW92], but we have never seen an explicit formula given before. We now proceed to describe these results in a bit more detail. The reader is also referred to [JP11e] which gives an general discussion of how the results of the present paper fit into a larger investigation of functions of finite energy on resistance networks, and the effective resistance metric; see also [JP09b, JP11d, JP11c, JP10d, JP10c, JP10b, JP09a, JP11a, JP10a, JP11b].

1.1. Overview. Boundary theory of harmonic functions can roughly be divided three ways: the bounded harmonic functions (Poisson theory), the nonnegative harmonic functions (Martin theory), and the finite-energy harmonic functions studied in the present paper. While the Poisson boundary is roughly a subset of the Martin boundary (more precisely, a measure-theoretic refinement), the relationship between Martin theory and the study of $\mathcal{H}_\mathcal{E}$ is more subtle. For example, there exist unbounded functions of finite energy; cf. [JP09b, Ex. 13.10]. See [Woe09, Woe00]; further results detailing the interrelations are given in [Soa94, §III.7].

Whether the focus is on the harmonic functions which are bounded, nonnegative, or finite-energy, the goals of the associated boundary theory are essentially the same:

(1) Construct a space $\bar{\mathcal{D}}$ which extends the original domain $\mathcal{D}$; this can be done by taking closure, compactifying, or similar operations.
(2) One can then identify the boundary $\operatorname{bd}\mathcal{D}$ as $\bar{\mathcal{D}}\setminus\mathcal{D}$, or (if the boundary thus obtained would be larger than necessary/practical for the application in mind), as some subset of $\bar{\mathcal{D}}\setminus\mathcal{D}$.
(3) Define a procedure for extending harmonic functions u from $\mathcal{D}$ to $\operatorname{bd}\mathcal{D}$. This extension $\widetilde{u}$ may be a measure (or other linear functional) on $\operatorname{bd}\mathcal{D}$; it might not be a function.
(4) Obtain a kernel $\Bbbk(x,\beta)$ defined on $\mathcal{D}\times\operatorname{bd}\mathcal{D}$ against which one can integrate the extension $\widetilde{u}$ so as to recover the value of u at a point in $\mathcal{D}$:
$$u(x) = \int_{\operatorname{bd}\mathcal{D}} \Bbbk(x,\beta)\widetilde{u}(d\beta), \quad \forall x \in \mathcal{D},$$
whenever u is a harmonic functions of the given class. If $\widetilde{u}$ is initially given on $\operatorname{bd}\mathcal{D}$ as boundary data, then this formula yields a solution to the Dirichlet problem at infinity.

Our approach to (1) is to use Gel'fand triples to extend the original domain, an application of this theory which is novel as far as we know. In a forthcoming work [JP10c], we will introduce an interpolation formula that uses the analytic framework developed in this paper, and which turns G into a stochastic process. For further applications, see also [JP10b, JP10d]. Another construction, more closely related to Martin boundary, is given in [Pea10].

1.2. Hilbert spaces vs. probability spaces. We are studying finite-energy harmonic functions instead of positive harmonic functions, but the construction outlined above allows us to study elements of the boundary in an analogous fashion. The difference between our boundary theory and that of Poisson and Martin is rooted in our focus on $\mathcal{H}_\mathcal{E}$ rather than ℓ^2: both classical theories concern harmonic functions with growth/decay restrictions. By contrast, provided they neither grow too wildly nor oscillate too wildly, elements of $\mathcal{H}_\mathcal{E}$ may have values tending to both $+\infty$ and $-\infty$. See [JP09b, Ex. 13.10] for a function $h \in \mathcal{H}arm$ which is unbounded in this way. *Positive* harmonic functions are naturally given to analysis based on probabilistic and potential-theoretic techniques (especially the use of order properties), and the companion study of superharmonic (or subharmonic) functions is indispensable. By contrast, the focus of the present paper is on the use of Hilbert space geometry for studying the boundary of a network: note that the finite-energy functions on a network form a Hilbert space, but the positive functions do not. In a context where one cannot assume positivity (or boundedness), one can get more mileage by considering the Dirichlet form $\mathcal{E}$ as an inner product and studying the resulting Hilbert space geometry.

Our construction is motivated by functional integrals from mathematical physics and functional analysis, see e.g., [AJL10, AAL10, AM10, HLW10, FLC08, Kru89, AKS09], and we stress the use of Gel'fand triples designed for discrete analysis. Gel'fand triples have traditionally been used in PDE and in mathematical physics, where the framework consists of spaces of smooth functions and the corresponding dual space of distributions. The Hilbert space is then $L^2(\mathbb{R}^d)$ equipped with Lebesgue measure, and this approach has successfully been combined with probability considerations in the derivation of a priori estimates in the analysis of parabolic PDEs. In the study of quantization of wave equations in mathematical physics, Gel'fand triples play an essential role in functional integration (see the cited references above).

There has been a recent interest in analysis and potential theory on infinite-dimensional spaces, and the use of stochastic integration in conjunction with reproducing kernels as in [HNS09, Xia10, CdVTU10, ZXZ09], and Gel'fand triples as in [GMŠ58, AAL08, AAL10, HLW10, BKO07]. Although our setting here is different, we are able to adapt these tools for the task at hand. This is nontrivial because we deal with discrete structures, and so we must give up differential operators, whereas in the classical case, there is a natural differentiable structure avilable, and therefore the choice of Gel'fand triple is often rather conventional. One may still create functional integrals in the discrete setting, but the Hilbert space must be chosen and constructed with some care: we use an energy space, as opposed to the more natural guess involving weighted ℓ^2-spaces.

In Martin boundary theory, elements of the boundary may understood in terms of certain minimal harmonic functions. While this remains true to some extent in the present study, we are also centrally concerned with *monopoles*, that is, finite-energy functions which are harmonic except at a single point; see Definition 2.15. In the terminology of previous network studies [Soa94, SW91, Tho90, NW59, Lyo83, KY89, Yam79, Zem91, OP96], such a function may be thought of as the potential which induces a finite flow to infinity; see [JP11a, JP10a, LP09, DS84]. In our construction, recurrent networks have no boundary, and transient networks with no nontrivial harmonic functions have exactly one boundary point (corresponding to the fact that the monopole at any given x is unique). In particular, the integer lattices $(\mathbb{Z}^d, \mathbf{1})$ each have 1 boundary point for $d \geq 3$ and 0 boundary points for $d = 1, 2$, which coincides with the Martin boundary when the transition probabilities are uniform [Woe00, Ch. IV, 25.B].

Remark 1.1. While Doob's martingale theory works well for harmonic functions in L^∞ or L^2, the situation for $\mathcal{H}_\mathcal{E}$ is different. The primary reason is that $\mathcal{H}_\mathcal{E}$ is not immediately realizable as an L^2 space.[1] A considerable advantage of the Gel'fand–Wiener–Ito construction given in Theorem 5.3 is that $\mathcal{H}_\mathcal{E}$ is isometrically embedded into $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$ in a particularly nice way: it corresponds to the polynomials of degree 1. See Remark 5.8. Another contrast is that Δ may, in general, be unbounded in our context. Recall that "the" adjoint Δ^* depends on the choice of domain, i.e., the linear subspace $\text{dom}(\Delta) \subseteq \mathcal{H}$, and the choice of inner product.

Boundary theory is a well-established subject; the deep connections between harmonic analysis, probability, and potential theory have led to several notions of boundary and we will not attempt to give complete references. However, for Martin and Poisson boundary in the discrete case, we recommend the excellent references [Woe09, Woe00, Dyn69] and also the more introductory [Saw97]. See also [DS84, LP09] for introductory material on resistance networks. Additionally, [Lyo83, Car73], and the foundational paper [NW59] provide more specific background. With regard to infinite graphs and finite-energy functions, see [Soa94, SW91, CW92, Dod06, PW90, PW88, Woe86, Tho90]. For some recent related areas, see e.g., [AL08, AAL08, AD06] for reproducing kernels, [Arv86] for Markov operators, [Cho08] for graph analysis, and [AP09] for operator theory.

1.3. Outline. In our version of the program outlined above, we follow the steps in the order (2)-(3)-(1)-(4). A brief summary is given here; further introductory material and technical details appear at the beginning of each subsection.

§2 recalls basic definitions and some previously obtained results.

§3 describes two methods for constructing a Gel'fand triple. The technique presented in §3.1 works for any network (G, c) and makes use of an orthonormal basis of $\mathcal{H}_\mathcal{E}$ derived from the energy kernel $\{v_x\}_{x \in G}$ via the

[1]Since $\mathcal{H}_\mathcal{E}$ is separable, it is clear that $\mathcal{H}_\mathcal{E} \cong \ell^2(X, \mu)$ for some X and μ, but there is no natural way to represent $\mathcal{H}_\mathcal{E}$ as $\ell^2(G, \mu)$ for any μ.

Gram–Schmidt algorithm, or equivalently, from the domain of a certain operator $\mathcal{N}$. The approach given in §3.2 works only for networks where Δ is an unbounded operator on $\mathcal{H}_\mathcal{E}$. This version of $\mathcal{S}_\mathcal{E}$ is constructed in terms of the domain of Δ.

§4 studies the structure of $\mathcal{S}_\mathcal{E}$ (the space of test functions) and $\mathcal{S}'_\mathcal{E}$ (the space of distributions) and establishes some key lemmas for later use.

§5 proves a key result: Theorem 5.3, which establishes the isometric embedding of $\mathcal{H}_\mathcal{E}$ into $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$ given by the Wiener transform. Applying this isometry to the energy kernel $\{v_x\}_{x \in G}$, we get a reproducing kernel $\Bbbk(x, d\mathbb{P})$ given in terms of a version of Wiener measure. The results in this section hold for any Gel'fand triple; in particular, for either of the ones constructed in §3. Then points of $\operatorname{bd} G$ correspond to limits of sequences $(\mu_{x_n})_{n=1}^\infty$ where $x_n \to \infty$, modulo a suitable equivalence relation.

2. Basic terms and previous results

We now proceed to introduce the key notions used throughout this paper: resistance networks, the energy form $\mathcal{E}$, the Laplace operator Δ, and their elementary properties. Our approach is somewhat different from existing studies of networks in the literature, and so we take this opportunity to introduce the tools we will need: an unbounded Laplace operator with dense domain in a Hilbert space, a two-point reproducing kernel for this Hilbert space, the quadratic form associated to the Laplacian, and Gelfand triples. Since these are tools not commonly used in geometric analysis, we include their definitions and some theorems from earlier papers which will be needed later. Additionally, we will use the theorems of Bochner (Theorem 2.29), and Minlos (Theorem 2.30).

Definition 2.1. A *(resistance) network* is a connected graph (G, c), where G is a graph and c is the *conductance function* which defines adjacency by $x \sim y$ iff $c_{xy} > 0$, for vertices $x, y \in G$. We assume $c_{xy} = c_{yx} \in [0, \infty)$, and write $c(x) := \sum_{y \sim x} c_{xy}$. In case of vertices of infinite degree, we require that $c(x) < \infty$, but $c(x)$ need not be a bounded function on G. The notation c is also used to indicate the multiplication operator $(cv)(x) := c(x)v(x)$, i.e., the diagonal matrix with entries $c(x)$ with respect to the (vector space) basis $\{\delta_x\}$.

In this definition, *connected* means simply that for any $x, y \in G$, there is a finite sequence $(x_i)_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \dots, n$. Conductance is the reciprocal of resistance, so one can think of (G, c) as a network of nodes G connected by resistors of resistance c_{xy}^{-1}. We may assume there is at most one edge from x to y, as two conductors c_{xy}^1 and c_{xy}^2 connected in parallel can be replaced by a single conductor with conductance $c_{xy} = c_{xy}^1 + c_{xy}^2$. Using a conductance function to define adjacency allows for comparisons between networks that share a common vertex sets, or to consider perturbations of a network; see [JP11c].

Definition 2.2. The *Laplacian* on G is the linear difference operator which acts on a function $v : G \to \mathbb{R}$ by

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)). \tag{2.1}$$

A function $v : G \to \mathbb{R}$ is *harmonic* iff $\Delta v(x) = 0$ for each $x \in G$. The domain of this operator is specified in Definition 2.17, below.

We have adopted the physics convention (so that the spectrum is nonnegative) and thus our Laplacian is the negative of the one commonly found in the PDE literature. The network Laplacian (2.1) should not be confused with the stochastically renormalized Laplace operator $c^{-1}\Delta$ which appears in the probability literature, or with the spectrally renormalized Laplace operator $c^{-1/2}\Delta c^{-1/2}$ which appears in the literature on spectral graph theory (e.g., [Chu97]).

Definition 2.3. An *exhaustion* of G is an increasing sequence of finite and connected subnetworks $(G_k)_{k=1}^\infty$, so that $G_k \subseteq G_{k+1}$ and $G = \bigcup G_k$. Since any vertex or edge is eventually contained in some G_k, there is no loss of generality in assuming they are contained in G_1, for the purposes of a specific computation. We only consider subnetworks which are *full* in the sense that the conductance on a subnetwork is obtained by restricting the domain of c to $G_k \times G_k$; this means that if $x \sim y$ in G, then any subnetwork containing both x and y also has this edge, and with the same conductance.

Definition 2.4. The notation

$$\sum_{x \in G} := \lim_{k \to \infty} \sum_{x \in G_k} \tag{2.2}$$

is used whenever the limit is independent of the choice of exhaustion (G_k) of G. This is clearly justified, for example, whenever the sum has only finitely many nonzero terms, or is absolutely convergent as in the definition of $\mathcal{E}$ in Definition 2.5.

Definition 2.5. The *energy* of functions $u, v : G \to \mathbb{C}$ is given by the (closed, bilinear) Dirichlet form

$$\mathcal{E}(u,v) := \frac{1}{2} \sum_{x \in G} \sum_{y \in G} c_{xy}(\bar{u}(x) - \bar{u}(y))(v(x) - v(y)), \tag{2.3}$$

with the energy of u given by $\mathcal{E}(u) := \mathcal{E}(u,u)$. The *domain of the energy* is

$$\operatorname{dom} \mathcal{E} = \{u : G \to \mathbb{C} \,\vdots\, \mathcal{E}(u) < \infty\}. \tag{2.4}$$

Since $c_{xy} = c_{yx}$ and $c_{xy} = 0$ for nonadjacent vertices, the initial factor of $\frac{1}{2}$ in (2.3) implies there is exactly one term in the sum for each edge in the network.

2.1. The energy space $\mathcal{H}_\mathcal{E}$. Let $\mathbf{1}$ denote the constant function with value 1 and recall that $\ker \mathcal{E} = \mathbb{C}\mathbf{1}$.

Definition 2.6. The energy form $\mathcal{E}$ is symmetric and positive definite on $\operatorname{dom}\mathcal{E}$. Then $\operatorname{dom}\mathcal{E}/\mathbb{C}\mathbf{1}$ is a vector space with inner product and corresponding norm given by

$$\langle u, v\rangle_\mathcal{E} := \mathcal{E}(u,v) \quad \text{and} \quad \|u\|_\mathcal{E} := \mathcal{E}(u,u)^{1/2}. \tag{2.5}$$

The *energy Hilbert space* $\mathcal{H}_\mathcal{E}$ is $\operatorname{dom}\mathcal{E}/\mathbb{C}\mathbf{1}$ with inner product (2.5).

Remark 2.7. Strictly speaking, the elements of $\mathcal{H}_\mathcal{E}$ are not functions, but equivalence classes of functions. Nonetheless, we refer to them as functions for ease of exposition. Most properties of functions carry over immediately by considering differences. For example, $u \in \mathcal{H}_\mathcal{E}$ is $\mathbb{R}$*-valued* iff $u(x) - u(o) \in \mathbb{R}$ for every $x \in G$, and $u \in \mathcal{H}_\mathcal{E}$ is *bounded* iff $\sup_{x\in G} |u(x) - u(o)| < \infty$.

Definition 2.8. For $v \in \mathcal{H}_\mathcal{E}$, one says that v has *finite support* iff there is a finite set $F \subseteq G$ for which $v(x) = k \in \mathbb{C}$ for all $x \notin F$, i.e., the set of functions of finite support in $\mathcal{H}_\mathcal{E}$ is represented by

$$\{u \in \operatorname{dom}\mathcal{E} \,\vdots\, u(x) = k \text{ for some } k, \text{ for all but finitely many } x \in G\}, \tag{2.6}$$

and denoted $\operatorname{span}\{\delta_x\}$, where δ_x is the Dirac mass at x, i.e., the element of $\mathcal{H}_\mathcal{E}$ containing the characteristic function of the singleton $\{x\}$. It is immediate from (2.3) that $\mathcal{E}(\delta_x) = c(x)$, whence $\delta_x \in \mathcal{H}_\mathcal{E}$. Define $\mathcal{F}in$ to be the closure of $\operatorname{span}\{\delta_x\}$ with respect to $\mathcal{E}$.

Definition 2.9. The set of harmonic functions of finite energy is denoted

$$\mathcal{H}arm := \{v \in \mathcal{H}_\mathcal{E} \,\vdots\, \Delta v(x) = 0, \text{ for all } x \in G\}. \tag{2.7}$$

Note that this is independent of choice of representative for v in virtue of (2.1).

Lemma 2.10 ([JP11a, 2.11])**.** *For any $x \in G$, one has $\langle \delta_x, u\rangle_\mathcal{E} = \Delta u(x)$.*

The next result follows easily from Lemma 2.10; cf. [JP11a, Thm. 2.15].

Theorem 2.11 (Royden decomposition)**.** *$\mathcal{H}_\mathcal{E} = \mathcal{F}in \oplus \mathcal{H}arm$.*

Definition 2.12. We denote the orthogonal projections to $\mathcal{F}in$ and $\mathcal{H}arm$ by $P_{\mathcal{F}in}$ and $P_{\mathcal{H}arm}$, respectively.

Definition 2.13. Let v_x be defined to be the unique element of $\mathcal{H}_\mathcal{E}$ for which

$$\langle v_x, u\rangle_\mathcal{E} = u(x) - u(o), \qquad \text{for every } u \in \mathcal{H}_\mathcal{E}. \tag{2.8}$$

$\{v_x\}_{x\in G}$ forms a reproducing kernel for $\mathcal{H}_\mathcal{E}$ ([JP11a, Cor. 2.6]); we call it the *energy kernel* and (2.8) shows its span is dense in $\mathcal{H}_\mathcal{E}$. Note that v_o corresponds to the 0 element of $\mathcal{H}_\mathcal{E}$ (i.e., a constant function on G), since $\langle v_o, u\rangle_\mathcal{E} = 0$ for every $u \in \mathcal{H}_\mathcal{E}$. Therefore, v_o is often ignored or omitted.

Definition 2.14. A *dipole* is any $v \in \mathcal{H}_\mathcal{E}$ satisfying the pointwise identity $\Delta v = \delta_x - \delta_y$ for some vertices $x, y \in G$. One can check that $\Delta v_x = \delta_x - \delta_o$; cf. [JP11a, Lemma 2.13].

Definition 2.15. A *monopole* at $x \in G$ is an element $w_x \in \mathcal{H}_\mathcal{E}$ which satisfies $\Delta w_x(y) = \delta_{xy}$, where δ_{xy} is Kronecker's delta. Note that, in view of its definition in terms of differences, (2.1) unambiguously defines an (honest) function $\Delta u : G \to \mathbb{R}$, for any $u \in \mathcal{H}_\mathcal{E}$. In case the network supports monopoles, let w_o always denote the unique energy-minimizing monopole at the origin.

When $\mathcal{H}_\mathcal{E}$ contains monopoles, let $\mathcal{M}_x$ denote the vector space spanned by the monopoles at x. This implies that $\mathcal{M}_x$ may contain harmonic functions; see [JP11a, Lemma 4.1]. With v_x and $f_x = P_{\mathcal{F}in} v_x$ as in Definition 2.12, we indicate the distinguished monopoles

$$w_x^v := v_x + w_o \quad \text{and} \quad w_x^f := f_x + w_o. \tag{2.9}$$

Remark 2.16. Note that $w_o \in \mathcal{F}in$, whenever it is present in $\mathcal{H}_\mathcal{E}$, and similarly that w_x^f is the energy-minimizing element of $\mathcal{M}_x$. To see this, suppose w_x is any monopole at x. Since $w_x \in \mathcal{H}_\mathcal{E}$, write $w_x = f + h$ by Theorem 2.11, and get $\mathcal{E}(w_x) = \mathcal{E}(f) + \mathcal{E}(h)$. Projecting away the harmonic component will not affect the monopole property, so $w_x^f = P_{\mathcal{F}in} w_x$ is the unique monopole of minimal energy. The Green function is $g(x, y) = w_y^o(x)$, where w_y^o is the representative of w_y^f satisfying $w_y^o(o) = 0$. Compare to [Kig03].

Definition 2.17. The subspace of $\mathcal{H}_\mathcal{E}$ spanned by monopoles (or dipoles) is

$$\mathcal{M} := \operatorname{span}\{v_x\}_{x \in G} + \operatorname{span}\{w_x^v, w_x^f\}_{x \in G}. \tag{2.10}$$

It is shown in [JP11a, Lemma 4.1] that this space is dense in $\mathcal{H}_\mathcal{E}$.

Let $\Delta_\mathcal{M}$ be the (graph) closure of the Laplacian when taken to have the dense domain $\mathcal{M}$. Since Δ agrees with $\Delta_\mathcal{M}$ pointwise, we may suppress reference to the domain for ease of notation.

Lemma 2.18 ([JP11a, Lemma 3.8])**.** *$\Delta_\mathcal{M}$ is Hermitian and semibounded, i.e.,*

$$\langle u, \Delta_\mathcal{M} u \rangle_\mathcal{E} \geq 0, \qquad \text{for all } u \in \mathcal{M}.$$

Lemma 2.19 ([JP11a, Lemma 4.1])**.** *When the network is transient, $\mathcal{M}$ contains the spaces $\operatorname{span}\{v_x\}_{x \in G}$, $\operatorname{span}\{f_x\}$, and $\operatorname{span}\{h_x\}$, where $f_x = P_{\mathcal{F}in} v_x$ and $h_x = P_{\mathcal{H}arm} v_x$ as in Definition 2.12. When the network is not transient, $\mathcal{M} = \operatorname{span}\{v_x\}_{x \in G} = \operatorname{span}\{f_x\}$.*

Remark 2.20 (Monopoles and transience)**.** The presence of monopoles in $\mathcal{H}_\mathcal{E}$ is equivalent to the transience of the simple random walk on the network with transition probabilities $p(x,y) = c_{xy}/c(x)$: note that if w_x is a monopole, then the current induced[2] by w_x is a unit flow to infinity with finite energy. It was proved in [Lyo83] that the network is transient if and only if there exists a unit current flow to infinity; see also [LP09, Thm. 2.10].

2.2. The resistance metric.

Definition 2.21. If $(G_k)_{k=1}^\infty$ is any exhaustion of G (for which $x,y \in G_0$), the *free resistance* between x and y is defined to be

$$R^F(x,y) := \lim_{k\to\infty} R_{G_k}(x,y), \tag{2.11}$$

where $R_{G_k}(x,y)$ is the voltage drop between x and y when a current of one amp is inserted into G_k at x and withdrawn at y.

The following theorem can be found in [JP10a] or [JP09b], and parts of it also appear in [LP09, Pow76, Kig01, Kig03, Kig09, Str06, Per99].

Theorem 2.22 ([JP10a, Thm. 2.14])**.** *For an infinite network G, the free resistance $R^F(x,y)$ has the following equivalent formulations:*

$$\begin{aligned} R^F(x,y) &= (v_x(x) - v_x(y)) - (v_y(x) - v_y(y)) \\ &= \mathcal{E}(v_x - v_y) \\ &= 1/\min\{\mathcal{E}(u) : u(x) = 1, u(y) = 0, u \in \operatorname{dom}\mathcal{E}\} \\ &= \inf\{\kappa \ge 0 : |v(x) - v(y)|^2 \le \kappa\mathcal{E}(v), v \in \operatorname{dom}\mathcal{E}\} \\ &= \sup\{|v(x) - v(y)|^2 : \mathcal{E}(v) \le 1, v \in \operatorname{dom}\mathcal{E}\} \end{aligned} \tag{2.12}$$

The following result is well-known; see, e.g., [JP10a, JP09b, LP09, Pow76, Kig01, Kig03, Kig09, Str06, Per99]

Theorem 2.23. *R^F is a metric.*

Remark 2.24 (Probabilistic interpretation of R^F)**.** To see the relation with probability, let X_n denote the location of the random walker on the network at time n, where the transition probabilities are given by $p(x,y) := \frac{c_{xy}}{c(x)}$ and let $\tau_a := \min\{m \ge 0 : X_m = a\}$ be the *hitting time* of $a \in G$.

If v_x is the representative of v_x with $v_x(o) = 0$, define

$$u_x := \frac{v_x}{R^F(o,x)}. \tag{2.13}$$

It can be shown (e.g., [DS84, LP09, JP10a]) that on a finite network,

$$u_x(y) := \mathbb{P}[\tau_x < \tau_o \,|\, X_0 = y]. \tag{2.14}$$

---

[2]For a potential $v \in \mathcal{H}_\mathcal{E}$, the induced current is computed by Ohm's law $V = IR$, or rather, $I(xy) := c_{xy}(v(x) - v(y))$, for any edge (xy) in the network. Note that $I(yx) = -I(xy)$.

For any network, it can additionally be shown (see [JP10a, Cor. 3.13]) that

$$R^F(x,y) = \frac{1}{c(x)\mathbb{P}[\tau_b < \tau_a^+ \mid X_0 = a]}, \tag{2.15}$$

where $\tau_a^+ := \min\{m \geq 1 : X_m = a\}$. Thus, one can consider effective resistance as the reciprocal of an integral over a path space; see [JP10a, Rem. 3.14].

2.3. The discrete Gauss–Green identity. The space $\mathcal{M}$ is introduced as a dense domain for Δ and as the scope of validity for the discrete Gauss–Green identity of Theorem 2.27.

Definition 2.25. If H is a subgraph of G, then the boundary of H is

$$\operatorname{bd} H := \{x \in H : \exists y \in H^\complement, y \sim x\}. \tag{2.16}$$

The *interior* of a subgraph H consists of the vertices in H whose neighbours also lie in H:

$$\operatorname{int} H := \{x \in H : y \sim x \Rightarrow y \in H\} = H \setminus \operatorname{bd} H. \tag{2.17}$$

For vertices in the boundary of a subgraph, the *normal derivative* of v is

$$\tfrac{\partial v}{\partial \mathbb{n}}(x) := \sum_{y \in H} c_{xy}(v(x) - v(y)), \qquad \text{for } x \in \operatorname{bd} H. \tag{2.18}$$

Thus, the normal derivative of v is computed like $\Delta v(x)$, except that the sum extends only over the neighbours of x which lie in H.

Definition 2.25 will be used primarily for subgraphs that form an exhaustion of G, in the sense of Definition 2.3.

Definition 2.26. A *boundary sum* is computed in terms of an exhaustion $(G_k)_{k=1}^\infty$ by

$$\sum_{\operatorname{bd} G} := \lim_{k \to \infty} \sum_{x \in \operatorname{bd} G_k}, \tag{2.19}$$

whenever the limit is independent of the choice of exhaustion, as in Definition 2.4.

On a finite network, all harmonic functions of finite energy are constant, so that $\mathcal{H}_\mathcal{E} = \mathcal{F}in$ by Theorem 2.11, and one has $\mathcal{E}(u,v) = \sum_{x \in G} u(x)\Delta v(x)$, for all $u, v \in \mathcal{H}_\mathcal{E}$. In fact, this remains true for recurrent infinite networks, as shown in [JP11a, Thm. 4.4]; see also [KY89]. However, the possibilities are much richer on an infinite network, as evinced by the following theorem.

Theorem 2.27 (Discrete Gauss–Green identity)**.** *If $u \in \mathcal{H}_\mathcal{E}$ and $v \in \mathcal{M}$, then*

$$\langle u, v\rangle_\mathcal{E} = \sum_G \bar{u}\Delta v + \sum_{\operatorname{bd} G} \bar{u}\tfrac{\partial v}{\partial \mathbb{n}}. \tag{2.20}$$

The discrete Gauss–Green formula (2.20) is the main result of [JP11a]; that paper contains several consequences of the formula, especially as pertains to transience of the random walk (see Remark 2.24).

2.4. Gel'fand triples and duality. Recall that our Hilbert space $\mathcal{H}_\mathcal{E}$ is defined from (G, c) and its energy form $\mathcal{E}$. This can be incorporated in a practical manner into a stochastic completion (a suitable probability space) by first passing to the corresponding transform (i.e., generating function.) Since we assume the network (G, c) is infinite, the corresponding Hilbert space $\mathcal{H}_\mathcal{E}$ is infinite-dimensional. As a result, the more familiar and classical tools, Fourier transform and Bochner's theorem (Theorem 2.29), are no longer available. Nonetheless, the construction of a Gel'fand triple (described below) enables one to make precise an infinite-dimensional transform, thus extending the classical Fourier transform. More precisely, one can overcome the obstacle posed by the nonexistence of an infinite-dimensional Lebesgue measure by passing to the canonical cylinder set measure arising from the inner product structure on a Hilbert space H (which turns out to be a measure on a larger space $S' \supseteq H$). Here a *cylinder set* $C(F, A)$ is determined by a finite subset $F \subseteq G$, and an open set $A \subseteq \mathbb{R}^{|F|}$ according to

$$C(F, A) := \{u \colon (\langle u, v_x\rangle_\mathcal{E})_{x\in F} \in A\}. \tag{2.21}$$

The desired measure is defined first for such cylinder sets (with finitely many fixed coordinates) and then extended to the entire induced σ-algebra by Kolmogorov's theorem.

By analogy with Poisson boundary theory, one would like to obtain a probability space to serve as the boundary of G. We begin by applying Minlos' theorem to obtain a Radon probability measure on the σ-algebra of cylinder sets (2.21). Minlos' theorem works for any positive semidefinite function; we will use $g(u, v) = \exp(-\frac{1}{2}\|u - v\|_\mathcal{E}^2)$ so as to obtain a *Gaussian* process on S'.

It is shown in [Gro67, Gro70, Min63] that such a measure satisfies $\mu(\mathcal{H}) = 0$ if $\mathcal{H}$ is an infinite-dimensional Hilbert space (whenever μ is σ-finite), so we turn to Minlos' theorem. Suppose one has a *Gel'fand triple* (also called a *rigged Hilbert space*): a dense subspace S of $\mathcal{H}$ with

$$S \subseteq \mathcal{H} \subseteq S', \tag{2.22}$$

where S is dense in $\mathcal{H}$ and S' is the dual of S. Then Minlos' theorem states that S' is "big enough" to support such a measure μ. We will choose S in such a way that the metric space $(G, \sqrt{R^F})$ embeds isometrically into S, where R^F is given by (2.11). (Note that if ρ is a metric, then $\rho^{1/2}$ is also a metric.)

The spaces S and S' must satisfy some technical conditions: S is a dense subspace of $\mathcal{H}$ with respect to the Hilbert norm, but also comes equipped with a strictly finer "test function" topology. When S is a Fréchet space equipped with a countable system of seminorms (stronger than the norm on $\mathcal{H}$), then the inclusion map of S into $\mathcal{H}$ is continuous; in fact, it is possible to chose the seminorms in such a way that one gets a nuclear embedding (details below). Therefore, when the dual S' is taken with respect to this finer (Fréchet) topology, one obtains a strict containment $\mathcal{H} \subsetneq S'$. As mentioned above, it turns out that S' is large enough to support a nice probability measure, even though $\mathcal{H}$ is not.

It was Gel'fand's idea to formalize this construction abstractly using a system of nuclearity axioms [GMŠ58, Min58, Min59]. Our presentation here is adapted from quantum mechanics and the goal is to realize $\operatorname{bd} G$ as a subspace of (S', μ). We will give a "test function topology" as a Fréchet topology defined via a specific sequence of seminorms, using either an onb for $\mathcal{H}_\mathcal{E}$ (in §3.1) or the domain of Δ^∞ (in §3.2). Both construction require an *unbounded* operator.

Remark 2.28 (Tempered distributions and the Laplacian)**.** There is a concrete situation when the Gel'fand triple construction is especially natural: $\mathcal{H} = L^2(\mathbb{R}, dx)$ and S is the *Schwartz space* of functions of rapid decay. That is, each $f \in S$ is C^∞ smooth functions which decays (along with all its derivatives) faster than any polynomial. In this case, S is the space of *tempered distributions* and the seminorms defining the Fréchet topology on S are

$$p_m(f) := \sup\{|x^k f^{(n)}(x)| : x \in \mathbb{R}, 0 \leq k, n \leq m\}, \qquad m = 0, 1, 2, \ldots,$$

where $f^{(n)}$ is the n^{th} derivative of f. Then S' is the dual of S with respect to this Fréchet topology. One can equivalently express S as

$$S := \{f \in L^2(\mathbb{R}) : (\widetilde{P}^2 + \widetilde{Q}^2)^n f \in L^2(\mathbb{R}), \forall n\}, \tag{2.23}$$

where $\widetilde{P} : f(x) \mapsto \frac{1}{\mathbf{i}} \frac{d}{dx}$ and $\widetilde{Q} : f(x) \mapsto x f(x)$ are Heisenberg's operators. The operator $\widetilde{P}^2 + \widetilde{Q}^2$ is often called the quantum mechanical Hamiltonian, but some others (e.g., Hida, Gross) would call it a Laplacian, and this perspective tightens the analogy with the present study. In this sense, (2.23) could be rewritten $S := \operatorname{dom} \Delta^\infty$; compare to (3.16). We discuss an operator $\mathcal{N}$ in Definition 3.7 which is unitarily equivalent to $\widetilde{P}^2 + \widetilde{Q}^2$ and hence has the same spectrum. It follows that a general network (G, c) always has a harmonic oscillator.

The duality between S and S' allows for the extension of the inner product on $\mathcal{H}$ to a pairing of S and S':

$$\langle \cdot, \cdot \rangle_\mathcal{H} : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \qquad \text{to} \qquad \langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}} : S \times S' \to \mathbb{R}.$$

In other words, one obtains a Fourier-type duality restricted to S. Moreover, it is possible to construct a Gel'fand triple in such a way that $\mu(S') = 1$ for a probability measure μ on S' such that S indexes a Gaussian process on S' with respect to μ. When applied to $\mathcal{H} = \mathcal{H}_\mathcal{E}$, the construction yields three main outcomes:

(1) the next best thing to a Fourier transform for an arbitrary graph;
(2) a concrete representation of $\mathcal{H}_\mathcal{E}$ as an L^2 measure space

$$\mathcal{H}_\mathcal{E} \cong L^2(S', \mu);$$

(3) a boundary integral representation for the harmonic functions of finite energy.

As a prelude, we begin with Bochner's Theorem, which characterizes the Fourier transform of a positive finite Borel measure on the real line. The reader may find [RS75] helpful.

Theorem 2.29 (Bochner). *Let G be a locally compact abelian group. Then there is a bijective correspondence $\mathcal{F} : \mathcal{M}(G) \to \mathcal{PD}(\hat{G})$, where $\mathcal{M}(G)$ is the collection of finite positive Borel measures on G, and $\mathcal{PD}(\hat{G})$ is the set of continuous positive definite functions on the dual group of G. Moreover, this bijection is given by the Fourier transform*

$$\mathcal{F} : \nu \mapsto \varphi_\nu \qquad \text{by} \qquad \varphi_\nu(\xi) = \int_G e^{\mathbf{i}\langle \xi, x\rangle} \, d\nu(x). \tag{2.24}$$

For our representation of the energy Hilbert space $\mathcal{H}_\mathcal{E}$ in the case of general resistance network, we will need Minlos' generalization of Bochner's theorem from [Min63, Sch73]. This important result states that a cylindrical measure on the dual of a nuclear space is a Radon measure iff its Fourier transform is continuous. In this context, however, the notion of Fourier transform is infinite-dimensional, and is dealt with by the introduction of Gel'fand triples [Lee96].

Theorem 2.30 (Minlos). *Given a Gel'fand triple $S \subseteq \mathcal{H} \subseteq S'$, Bochner's Theorem may be extended to yield a bijective correspondence between the continuous positive definite functions on S and the Radon probability measures on S'. Moreover, for the continuous positive definite function $e^{-\frac{1}{2}\langle u,u\rangle_\mathcal{H}}$, this correspondence gives a measure μ on S' uniquely determined by the identity*

$$\int_{S'} e^{\mathbf{i}\langle u, \xi\rangle_{\widetilde{\mathcal{H}}}} \, d\mu(\xi) = e^{-\frac{1}{2}\langle u,u\rangle_\mathcal{H}}, \tag{2.25}$$

where $\langle \cdot, \cdot \rangle_\mathcal{H}$ is the original inner product on $\mathcal{H}$ and $\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}}$ is its extension to the pairing on $S \times S'$. Under this correspondence, elements of S become a Gaussian process with respect to the measure μ and covariance function given by $\langle \cdot, \cdot \rangle_\mathcal{H}$.

Minlos' identity (2.25) may be interpreted as defining the Fourier transform of μ; the function on the right-hand side is positive definite and plays a special role in stochastic integration, and its use in quantization.

3. Gel'fand triples for $\mathcal{H}_\mathcal{E}$

In this section, we describe two methods for constructing a Gel'fand triple for $\mathcal{H}_\mathcal{E}$. The first method is applicable to all networks, but relies on the choice of some enumeration of the vertices of G, and the Gram–Schmidt algorithm for producing an onb. However, we will see that the Gram–Schmidt algorithm yields a much more explicit formula than usual, in the present context. It is important to note that the construction of the Gel'fand triple requires an *unbounded* essentially self-adjoint operator. The first method constructs an operator $\mathcal{N}$ which is unbounded on any infinite network (Lemma 3.9). The second method, which uses the Laplacian, is applicable only when the Laplacian is unbounded. However, in this case the construction does not require any enumeration (or onb) and may provide for more feasible computations.

Remark 3.1. Note that $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}'_\mathcal{E}$ consist of $\mathbb{R}$-valued functions (in the sense of Remark 2.7) in this section. This technical detail is important because we do not expect the integral $\int_{S'} e^{\mathbf{i}\langle u,\cdot\rangle}\widetilde{w}\, d\mathbb{P}$ from (2.25) to converge unless it is certain that $\langle u,\cdot\rangle$ is $\mathbb{R}$-valued. After the Wiener embedding is carried out in Theorem 5.3, all results can be complexified.

3.1. Gel'fand triples via Gram–Schmidt. In this section, we describe a Gel'fand triple for $\mathcal{H}_\mathcal{E}$ where the class of test functions $\mathcal{S}_\mathcal{E}$ is described in terms of the decay properties of a certain orthonormal basis (onb) for $\mathcal{H}_\mathcal{E}$. We will see in Remark 5.6 that this onb corresponds to a system of i.i.d. random variables (which are, in fact, Gaussian with mean 0 and variance 1).

The onb $(\epsilon_n)_{n\in\mathbb{N}}$ comes by applying the Gram–Schmidt process to the reproducing kernel $(v_{x_n})_{n\in\mathbb{N}}$, where we have fixed some enumeration $(x_n)_{n\in\mathbb{N}}$ of the vertices $G\setminus\{o\}$. That is, we put $x_0 = o$ and henceforth exclude x_0 from the discussion, as it will not be relevant. Given $\{\epsilon_1,\dots,\epsilon_{n-1}\}$, one obtains ϵ_n inductively via

$$\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \vdots\\ \epsilon_n\end{bmatrix} = \begin{bmatrix}\|v_{x_1}\|_\mathcal{E}^{-1} & 0 & 0 & 0 & \dots & 0\\ M_{2,1} & M_{2,2} & 0 & 0 & \dots & 0\\ M_{3,1} & M_{3,2} & M_{3,3} & 0 & \dots & 0\\ \vdots & & & & & \\ M_{n,1} & M_{n,2} & M_{n,3} & \dots & \dots & M_{n,n}\end{bmatrix}\begin{bmatrix}v_{x_1}\\ v_{x_2}\\ v_{x_3}\\ \vdots\\ v_{x_n}\end{bmatrix}, \tag{3.1}$$

where the entries $M_{i,j}$ are computed in Lemma 3.3. Consequently, for each $N\in\mathbb{N}$, the triangular nature of M gives

$$\operatorname{span}\{v_{x_1},\dots,v_{x_N}\} = \operatorname{span}\{\epsilon_1,\dots,\epsilon_N\}, \tag{3.2}$$

a simple fact which is crucial for Lemma 3.11.

Remark 3.2. Note that the reproducing kernel gives one an explicit formula for the entries of the inverse of this particular Gram–Schmidt matrix:

$$(M^{-1})_{i,j} = \langle v_{x_i},\epsilon_j\rangle_\mathcal{E} = \epsilon_j(x_i) - \epsilon_j(o). \tag{3.3}$$

This is certainly in distinct contrast with the general case, and allows us to find a formula for the entries of M itself in Lemma 3.3.

Lemma 3.3. *The entries of the Gram–Schmidt matrix M are given by*

$$M_{i,j} = \begin{cases} (\Delta\epsilon_i)(x_j), & j \leq i \\ 0 & \textit{else}, \end{cases} \qquad \textit{for } i,j = 1,2,\ldots. \tag{3.4}$$

Proof. For $j \leq i$, an application of (2.10) gives

$$\begin{aligned} \Delta\epsilon_i(x_j) = \langle \delta_{x_j}, \epsilon_i \rangle_{\mathcal{E}} &= \left\langle \delta_{x_j}, \sum_{k\leq i} M_{i,k} v_{x_k} \right\rangle_{\mathcal{E}} \\ &= \sum_{k \leq i} M_{i,k} \langle \delta_{x_j}, v_{x_k} \rangle_{\mathcal{E}} \\ &= \sum_{k\leq i} M_{i,k} (\delta_{x_j}(x_k) - \delta_{x_j}(o)), \end{aligned}$$

where the last equality comes by the reproducing kernel property (2.8). Note that $\delta_x(y) = \delta_x(o)$ for every y except $y = x$, so the last sum above has a nonzero term only for $k = j$, and the result follows. □

From (3.3) and Lemma 3.3, we have the handy conversion formulas:

$$\epsilon_i = \sum_{j \leq i} \Delta\epsilon_i(x_j) v_{x_j} \qquad \text{and} \qquad v_{x_i} = \sum_{k\leq i} (\epsilon_k(x_i) - \epsilon_k(o))\, \epsilon_k. \tag{3.5}$$

Lemma 3.4. *We have the identity*

$$\sum_{j \leq k \leq i} (\epsilon_k(x_i) - \epsilon_k(o))\, \Delta\epsilon_k(x_j) = \delta_{i,j}, \qquad \textit{for } i,j = 1,2,\ldots. \tag{3.6}$$

Proof. By formula (3.3), the left side of (3.6) is equal to

$$\begin{aligned} \sum_{j\leq k\leq i} (\epsilon_k(x_i) - \epsilon_k(o))\, \Delta\epsilon_k(x_j) &= \Delta \left(\sum_{k \leq i} \langle v_{x_i}, \epsilon_j \rangle_{\mathcal{E}} \cdot \epsilon_k(x_j) \right) \\ &= \Delta v_{x_i}(x_j) \qquad \text{by (3.5)} \\ &= \delta_{x_i}(x_j) - \delta_o(x_j). \end{aligned}$$

Note that $\Delta\epsilon_k(x_j) = 0$ for $j > k$, so the second sum runs over all $k \leq i$. Also, note that $\delta_{x_i}(x_j) - \delta_o(x_j) = \delta_{i,j}$ (Kronecker delta) for $i, j > 0$ (and the indexing of M begins at 1, not 0). □

Lemma 3.4 can also be proven by combining the identities in (3.5).

Lemma 3.5. *Let $V_{x,y} := \langle v_x, v_y \rangle_{\mathcal{E}}$, and let $E = M^{-1}$ be defined as in* (3.3). *Then $EE^* = V$.*

Proof. Computing entrywise,

$$\begin{aligned}(EE^*)_{i,j} &= \sum_k E_{x_i,x_k} E_{x_j,x_k} \\ &= \sum_k (\epsilon_k(x_i) - \epsilon_k(o))(\epsilon_k(x_j) - \epsilon_k(o)) \\ &= \sum_k \langle v_{x_i}, \epsilon_k\rangle_{\mathcal{E}} \langle v_{x_j}, \epsilon_k\rangle_{\mathcal{E}},\end{aligned}$$

which is equal to $\langle v_{x_i}, v_{x_j}\rangle_{\mathcal{E}}$ by Parseval's identity. □

Definition 3.6. The space of test functions and the space of distributions corresponding to the onb $(\epsilon_n)_{n\in\mathbb{N}}$ are defined by

$$\mathcal{S}_{\mathcal{E}} = \left\{ s = \sum_{n\in\mathbb{N}} s_n \epsilon_n : \forall p \in \mathbb{N}, \exists C > 0 \text{ such that } |s_n| \leq C/n^p \right\}, \text{ and} \tag{3.7}$$

$$\mathcal{S}'_{\mathcal{E}} = \left\{ \xi = \sum_{n\in\mathbb{N}} \xi_n \epsilon_n : \exists p \in \mathbb{N}, \exists C > 0 \text{ such that } |\xi_n| \leq C n^p \right\}. \tag{3.8}$$

Here $s_n, \xi_n \in \mathbb{R}$, in accord with Remark 3.1. Thus,

$$\mathcal{S}_{\mathcal{E}} = \bigcap_{p\in\mathbb{N}} \{s : \|s\|_p < \infty\}$$

where the Fréchet *p-seminorm* of $s = \sum_{n\in\mathbb{N}} s_n \epsilon_n$ is

$$\|s\|_p := \left(\sum_{n\in\mathbb{N}} n^p |s_n|^2 \right)^{1/2}, \qquad s \in \mathcal{S}_{\mathcal{E}}, p \in \mathbb{N}. \tag{3.9}$$

Note that the system of seminorms (3.9) is equivalent to the system defined by

$$\|s\|_p := \sup_{n\in\mathbb{N}} n^p |s_n|, \qquad s \in \mathcal{S}_{\mathcal{E}}, p \in \mathbb{N}, \tag{3.10}$$

in the sense that both define the same Fréchet topology on $\mathcal{S}_{\mathcal{E}}$. (Each seminorm in one system is dominated by one from the other, but with a different p.) We occasionally find it more convenient to calculate with (3.10) instead of (3.9).

Definition 3.7. Let $\mathcal{V} := \text{span}\{v_x\}_{x\in G}$ and define a mapping $\mathcal{N} : \mathcal{V} \to \mathcal{H}_{\mathcal{E}}$ by

$$\mathcal{N} v_{x_n} = \sum_{k=1}^{n} k \epsilon_k(x_n) \epsilon_k. \tag{3.11}$$

Remark 3.8. From (3.11), one has

$$\|\mathcal{N} v_{x_n}\|_{\mathcal{E}}^2 = \sum_{k=1}^{n} k^2 |\epsilon_k(x_n)|^2, \quad \langle v_{x_n}, \mathcal{N} v_{x_m}\rangle_{\mathcal{E}} = \sum_{k=1}^{n\wedge m} k \epsilon_k(x_n) \epsilon_k(x_m). \tag{3.12}$$

Note that $\epsilon_k \in \mathcal{V}$ by (3.2), and that $\mathcal{N}\epsilon_k = k\epsilon_k$ for each $k \in \mathbb{N}$. We use the symbol $\mathcal{N}$ for the operator discussed in this section by way of analogy with the number operator N from quantum mechanics. Indeed, $\mathcal{N}$ can also be defined as a^*a for a certain operator a and its adjoint.

In the following lemma, $\mathcal{N}$ is an operator with domain $\operatorname{dom}\mathcal{N} = \operatorname{span}\{\epsilon_n\}$, and we use the symbol $\bar{\mathcal{N}}$ to denote the closure of the operator $\mathcal{N}$ with respect to the graph norm.

Lemma 3.9. *The operator $\mathcal{N}$ is essentially self-adjoint, and is unbounded if and only if G is infinite. Moreover, if we define the seminorms $\rho_n(u) := \|(\bar{\mathcal{N}})^n u\|_{\mathcal{E}}$, then $\{\rho_n\}$ and $\{\|\cdot\|_p\}$ induce equivalent topologies on $\mathcal{S}_{\mathcal{E}}$, so that*

$$\mathcal{S}_{\mathcal{E}} = \bigcap_{n\in\mathbb{N}} \operatorname{dom}(\bar{\mathcal{N}})^n \tag{3.13}$$

and $u \in \mathcal{S}_{\mathcal{E}}$ if and only if $\rho_n(u) < \infty$ for each $n \in \mathbb{N}$.

Proof. Unitary equivalence of $\mathcal{H}_{\mathcal{E}}$ with $\ell^2(\mathbb{Z}_+)$ is given by $U : \epsilon_n \mapsto \delta_n$, where $\delta_n(m) := \delta_{n,m}$ (Kronecker δ) for $n, m \in \mathbb{Z}_+$. Define $N_+ : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ by $N_+\delta_n = n\delta_n$ so that $N_+U = U\mathcal{N}$ holds on the dense subspace $\operatorname{span}\{\epsilon_n\}$. The rest follows by [Sim79, §2]. □

Corollary 3.10. *The inclusion mapping $\mathcal{S}_{\mathcal{E}} \hookrightarrow \mathcal{H}_{\mathcal{E}}$ is nuclear, and so $\mathcal{S}_{\mathcal{E}} \subseteq \mathcal{H}_{\mathcal{E}} \subseteq \mathcal{S}'_{\mathcal{E}}$ is a Gel'fand triple.*

Proof. When the space of test functions is defined as $\operatorname{dom} T^\infty$ for some operator T with pure point spectrum (as in (3.13)), then nuclearity follows if there is a $p \in \mathbb{Z}_+$ such that the reciprocal eigenvalues of T are p-summable; see [Sim79]. From the proof of Lemma 3.9, one can see that $\mathcal{N}$ is unitarily equivalent to N_+ and hence to the harmonic oscillator $\widetilde{P}^2+\widetilde{Q}^2$. This classical theory implies that $\mathcal{N}$ has spectrum $\mathbb{Z}_+$. Since $\sum_{n=1}^\infty n^{-p} < \infty$ for $p \geq 2$, the conclusion follows. □

Lemma 3.11. *The energy kernel $\{v_x\}_{x\in G}$ is a Fréchet-dense subset of $\mathcal{S}_{\mathcal{E}}$.*

Proof. In the expansion with respect to the onb as in (3.7), the basis element ϵ_k has coefficients $s_n = \delta_{n,k}$ (Kronecker delta). Since this sequence $\{s_n\}$ vanishes for $n \geq k$, it clearly satisfies the required decay condition $|s_n| \leq Cn^{-p}$. From (3.2), the same holds for v_{x_k}. This shows that the kernel is contained in $\mathcal{S}_{\mathcal{E}}$.

To see that $\{v_x\}$ is dense in $\mathcal{S}_{\mathcal{E}}$, it suffices by (3.2) to show that the onb $(\epsilon_n)_{n\in\mathbb{N}}$ is dense. Given any $u = \sum u_k\varepsilon_k \in \mathcal{S}_{\mathcal{E}}$ and $p \in \mathbb{N}$, there is a C such that $|u_k| \leq C/k^{p+1}$. Now if $u_N = \sum_{k=1}^N u_k\epsilon_k$ is the N^{th} truncation of u, then

$$\|u - f\|_p = \sup_k k^p|u_k - f_k| = \sup_{k>N} k^p|u_k - f_k| \leq k^p \frac{C}{k^{p+1}} \xrightarrow{k\to\infty} 0.$$

Thus, one can always approximate $u \in \mathcal{S}_{\mathcal{E}}$ with respect to the Fréchet topology by $(u_N)_{N=1}^\infty$, where $u_N \in \operatorname{span}\{\epsilon_1, \dots, \epsilon_N\}$. □

3.2. Gel'fand triples in the case when $\boldsymbol{\Delta}$ is unbounded. Denote the degree of a vertex by

$$\deg(x) = |\{y \in G : c_{xy} > 0\}|. \tag{3.14}$$

In the case when $\Delta : \mathcal{H}_\mathcal{E} \to \mathcal{H}_\mathcal{E}$ is unbounded and $\deg(x) < \infty$ at each $x \in G$ (but note that $\deg(x)$ may be unbounded), there is an alternative construction of $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}'_\mathcal{E}$, which begins by identifying a certain subspace of $\mathcal{M} = \operatorname{dom} \Delta_\mathcal{M}$ (as given in Definition 2.17) to act as the space of test functions.

Definition 3.12. Let $\mathbb{\Delta}_\mathcal{M}$ be a self-adjoint extension of $\Delta_\mathcal{M}$; since $\Delta_\mathcal{M}$ is Hermitian and commutes with conjugation (since c is $\mathbb{R}$-valued), a theorem of von Neumann's states that such an extension exists. To make this concrete, one can take $\mathbb{\Delta}_\mathcal{M}$ to be the Friedrichs extension, if desired; see [JP11b].

Let $\mathbb{\Delta}_\mathcal{M}^p u := (\mathbb{\Delta}_\mathcal{M} \mathbb{\Delta}_\mathcal{M} \dots \mathbb{\Delta}_\mathcal{M})u$ be the p-fold product of $\mathbb{\Delta}_\mathcal{M}$ applied to $u \in \mathcal{H}_\mathcal{E}$. Define $\operatorname{dom}(\mathbb{\Delta}_\mathcal{M}^p)$ inductively by

$$\operatorname{dom}(\mathbb{\Delta}_\mathcal{M}^p) := \{u : \mathbb{\Delta}_\mathcal{M}^{p-1} u \in \operatorname{dom}(\mathbb{\Delta}_\mathcal{M})\}. \tag{3.15}$$

Definition 3.13. The *(Schwartz) space of potentials of rapid decay* is

$$\mathcal{S}_\mathcal{E} := \operatorname{dom}(\mathbb{\Delta}_\mathcal{M}^\infty), \tag{3.16}$$

where $\operatorname{dom}(\mathbb{\Delta}_\mathcal{M}^\infty) := \bigcap_{p=1}^\infty \operatorname{dom}(\mathbb{\Delta}_\mathcal{M}^p)$ consists of all $\mathbb{R}$-valued functions $u \in \mathcal{H}_\mathcal{E}$ for which $\mathbb{\Delta}_\mathcal{M}^p u \in \mathcal{H}_\mathcal{E}$ for any p. The space $\mathcal{S}'_\mathcal{E}$ of *Schwartz distributions* or *tempered distributions* is the dual space of $\mathbb{R}$-valued continuous linear functionals on $\mathcal{S}_\mathcal{E}$.

Remark 3.14. Note that $\mathcal{S}_\mathcal{E}$ is dense in $\operatorname{dom}(\mathbb{\Delta}_\mathcal{M})$ with respect to the graph norm, by standard theory. For each $p \in \mathbb{N}$, there is a seminorm on $\mathcal{S}_\mathcal{E}$ defined by

$$\|u\|_p := \|\mathbb{\Delta}_\mathcal{M}^p u\|_\mathcal{E}. \tag{3.17}$$

Since $\operatorname{dom} \mathbb{\Delta}_\mathcal{M}^p$ is complete with respect to $\|\cdot\|_p$ for each $p \in \mathbb{N}$, the subspace $\mathcal{S}_\mathcal{E}$ is a Fréchet space. Note that Δ is unbounded so $\mathcal{S}_\mathcal{E}$ is a *proper* subspace of $\mathcal{H}_\mathcal{E}$.

Lemma 3.15. *For any $x \in G$, $\delta_x = c(x)v_x - \sum_{y \sim x} c_{xy} v_y$.*

Proof. Lemma 2.10 implies $\langle \delta_x, u\rangle_\mathcal{E} = \langle c(x)v_x - \sum_{y\sim x} c_{xy}v_y, u\rangle_\mathcal{E}$ for every $u \in \mathcal{H}_\mathcal{E}$, so apply this to $u = v_z$, $z \in G$. Since $\delta_x, v_x \in \mathcal{H}_\mathcal{E}$, it must also be that $\sum_{y\sim x} c_{xy} v_y \in \mathcal{H}_\mathcal{E}$. □

Lemma 3.16. *If $\deg(x)$ is finite for each $x \in G$, then one has $v_x \in \mathcal{S}_\mathcal{E}$.*

Proof. If $\deg(x) < \infty$ then Lemma 3.15 shows that $\delta_x \in \operatorname{span}\{v_x\}_{x\in G}$. □

Remark 3.17. When the hypotheses of Lemma 3.16 are satisfied, it should be noted that $\operatorname{span}\{v_x\}_{x\in G}$ is dense in $\mathcal{S}_\mathcal{E}$ with respect to $\mathcal{E}$, but *not* with respect to the Fréchet topology induced by the seminorms (3.17), nor with respect to the graph norm. One has the inclusions

$$\left\{\begin{bmatrix} v_x \\ \Delta_{\mathcal{M}} v_x \end{bmatrix}\right\} \subseteq \left\{\begin{bmatrix} s \\ \tilde{\Delta}_{\mathcal{M}} s \end{bmatrix}\right\} \subseteq \left\{\begin{bmatrix} u \\ \tilde{\Delta}_{\mathcal{M}} u \end{bmatrix}\right\} \tag{3.18}$$

where $s \in \mathcal{S}_\mathcal{E}$ and $u \in \mathcal{H}_\mathcal{E}$. The second inclusion is dense but the first is not, whenever $\Delta_{\mathcal{M}}$ is unbounded.

4. The structure of $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}'_\mathcal{E}$

From this point on, we assume that a Gel'fand triple has been chosen, using either of the methods described in the previous section. Henceforth, we use the symbol Λ to denote the operator $\bar{\mathcal{N}} = \mathcal{N}^*$ or the operator $\tilde{\Delta}_{\mathcal{M}}$, depending on how the Gel'fand triple was constructed:

$$\Lambda := \begin{cases} \bar{\mathcal{N}}, & \text{Definition 3.7} \\ \tilde{\Delta}_{\mathcal{M}}, & \text{Definition 3.12.} \end{cases} \tag{4.1}$$

Note that Λ is unbounded and essentially self-adjoint in either case.

4.1. The structure of $\mathcal{S}_\mathcal{E}$. We establish that $\mathcal{S}_\mathcal{E}$ is a dense analytic subset of $\mathcal{H}_\mathcal{E}$, and that the energy product can be extends not just to a pairing on $\mathcal{S}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$, but all the way to a pairing on $\mathcal{H}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$. Parts of this subsection closely parallel the general theory, and good references would be [Hid80, Hör03, Sim79, Str03].

Definition 4.1. Let $\chi_{[a,b]}$ denote the usual indicator function of the interval $[a,b] \subseteq \mathbb{R}$, and let $\mathfrak{S}$ be the spectral transform in the spectral representation of Λ, and let E be the associated projection-valued measure. Then define E_n to be the *spectral truncation operator* acting on $\mathcal{H}_\mathcal{E}$ by

$$E_n u := \mathfrak{S}^* \chi_{[\frac{1}{n},\, n]} \mathfrak{S} u = \int_{1/n}^n E(dt) u. \tag{4.2}$$

Lemma 4.2. *With respect to $\mathcal{E}$, $\mathcal{S}_\mathcal{E}$ is a dense analytic subspace of $\mathcal{H}_\mathcal{E}$.*

Proof. This is essentially immediate once it is clear that E_n maps $\mathcal{H}_\mathcal{E}$ into $\mathcal{S}_\mathcal{E}$. For $u \in \mathcal{H}_\mathcal{E}$, and for any $p = 1, 2, \dots,$

$$\|\Lambda^p E_n u\|_\mathcal{E}^2 = \int_{1/n}^n \lambda^{2p} \|E(d\lambda)u\|_\mathcal{E}^2 \leq n^{2p} \|u\|_\mathcal{E}^2, \tag{4.3}$$

So $E_n u \in \mathcal{S}_\mathcal{E}$. It follows that $\|u - E_n u\|_\mathcal{E} \to 0$ by standard spectral theory. □

Theorem 4.3. *$\mathcal{S}_\mathcal{E} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'_\mathcal{E}$ is a Gel'fand triple, and the energy form $\langle \cdot, \cdot \rangle_\mathcal{E}$ extends to a pairing on $\mathcal{S}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$ defined by*

$$\langle u, v \rangle := \langle \Lambda^p u, \Lambda^{-p} v \rangle_\mathcal{E}, \tag{4.4}$$

where p is any integer such that $|v(u)| \leq K \|\Delta^p u\|_\mathcal{E}$ for all $u \in \mathcal{S}_\mathcal{E}$ (for some $K > 0$), and Λ^{-p} is defined via duality (see the proof).

Proof. In combination with (3.16)–(3.17), Lemma 4.2 establishes that $\mathcal{S}_\mathcal{E} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'_\mathcal{E}$ is a Gel'fand triple. If $v \in \mathcal{S}'_\mathcal{E}$, then there is a C and p such that $|\langle s, v\rangle| \leq C\|\Lambda^p s\|_\mathcal{E}$ for all $s \in \mathcal{S}_\mathcal{E}$. Set $\varphi(\Lambda^p s) := \langle s, v\rangle$ to obtain a continuous linear functional on $\mathcal{H}_\mathcal{E}$ (after extending to the orthogonal complement of $\operatorname{span}\{\Lambda^p s\}$ by 0 if necessary). Now Riesz's lemma gives a $w \in \mathcal{H}_\mathcal{E}$ for which $\langle s, v\rangle = \langle \Lambda^p s, w\rangle_\mathcal{E}$ for all $s \in \mathcal{S}_\mathcal{E}$ and we define $\Lambda^{-p} v := w \in \mathcal{H}_\mathcal{E}$ to make the meaning of the right-hand side of (4.4) clear. □

Lemma 4.4. *The pairing on $\mathcal{S}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$ is equivalently given by*

$$\langle u, \xi\rangle = \lim_{n\to\infty} \xi(E_n u), \tag{4.5}$$

where the limit is taken in the topology of $\mathcal{S}'_\mathcal{E}$. Moreover, $\widetilde{u}(\xi) = \langle u, \xi\rangle$ is $\mathbb{R}$-valued on $\mathcal{S}'_\mathcal{E}$.

Proof. E_n commutes with Λ. This is a standard result in spectral theory, as E_n and Λ are unitarily equivalent to the two commuting operations of truncation and multiplication, respectively. Therefore, for any p satisfying the hypotheses of Theorem 4.3, we have

$$\xi(E_n u) = \langle E_n u, \xi\rangle = \langle \Lambda^p E_n s, \Lambda^{-p}\xi\rangle_\mathcal{E} = \langle E_n \Lambda^p s, \Lambda^{-p}\xi\rangle_\mathcal{E} = \langle \Lambda^p s, E_n \Lambda^{-p}\xi\rangle_\mathcal{E}.$$

Standard spectral theory also gives $E_n v \to v$ in $\mathcal{H}_\mathcal{E}$, so

$$\lim_{n\to\infty} \xi(E_n u) = \lim_{n\to\infty} \langle \Lambda^p s, E_n \Lambda^{-p}\xi\rangle_\mathcal{E} = \langle \Lambda^p u, \Lambda^{-p} v\rangle_\mathcal{E}.$$

Note that the pairing $\langle \cdot\,, \cdot\rangle$ is a limit of real numbers, and hence is real. □

Corollary 4.5. *E_n extends to a mapping $\widetilde{E}_n : \mathcal{S}'_\mathcal{E} \to \mathcal{H}_\mathcal{E}$ defined via $\langle u, \widetilde{E}_n\xi\rangle_\mathcal{E} := \xi(E_n u)$. Thus, we have a pointwise extension of $\langle \cdot\,, \cdot\rangle$ to $\mathcal{H}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$ given by*

$$\langle u, \xi\rangle = \lim_{n\to\infty} \langle u, \widetilde{E}_n\xi\rangle_\mathcal{E}. \tag{4.6}$$

4.2. The structure of $\mathcal{S}'_\mathcal{E}$. There are several structure theorems in the classical theory of distributions which describe how distributions can be understood locally in terms of derivatives; see [Hör03, §2], [Str03, §6.3], or [AG92, §3.5]. This section contains some analogues of those results; note that both $\mathcal{N}$ and $\mathbb{\Delta}_\mathcal{M}$ can be interpreted as generalized differential operators in a discrete context.

Theorem 4.6. *The distribution space $\mathcal{S}'_\mathcal{E}$ is*

$$\mathcal{S}'_\mathcal{E} = \{\xi(u) = \langle \Lambda^p u, v\rangle_\mathcal{E} : \exists v \in \mathcal{H}_\mathcal{E}, p \in \mathbb{Z}^+, \forall u \in \mathcal{S}_\mathcal{E}\}. \tag{4.7}$$

Proof. It is clear from the Schwarz inequality that $\xi(u) = \langle \Lambda^p u, v\rangle_\mathcal{E}$ defines a continuous linear functional on $\mathcal{S}_\mathcal{E}$, for any $v \in \mathcal{H}_\mathcal{E}$ and nonnegative integer p. For the other direction, we use the same technique as in Lemma 4.3. Observe that if $\xi \in \mathcal{S}'_\mathcal{E}$, then there exists K, p such that $|\xi(u)| \leq K\|\Lambda^p u\|_\mathcal{E}$ for every $u \in \mathcal{S}_\mathcal{E}$. This implies that the map $\xi : \Lambda^p u \mapsto \xi(u)$ is continuous on the subspace $Y = \operatorname{span}\{\Lambda^p u : u \in \mathcal{H}_\mathcal{E}, p \in \mathbb{Z}^+\}$. This can be extended to all of $\mathcal{H}_\mathcal{E}$ by precomposing with the orthogonal projection to Y (i.e., extending by 0). Now Riesz's lemma gives a $v \in \mathcal{H}_\mathcal{E}$ for which $\xi(u) = \langle \Lambda^p u, v\rangle_\mathcal{E}$. □

We now provide two results enabling one to recognize certain elements of $\mathcal{S}'_\mathcal{E}$.

Lemma 4.7. *A linear functional $f : \mathcal{S}_\mathcal{E} \to \mathbb{C}$ is an element of $\mathcal{S}'_\mathcal{E}$ if and only if there exists $p \in \mathbb{N}$ and $F_0, F_1, \dots F_p \in \mathcal{H}_\mathcal{E}$ such that*

$$f(u) = \sum_{k=0}^{p} \langle F_k, \Lambda^k u\rangle_\mathcal{E}, \quad \forall u \in \mathcal{H}_\mathcal{E}. \tag{4.8}$$

Proof. By definition, $f \in \mathcal{S}'_\mathcal{E}$ iff $\exists p, C < \infty$ for which $|f(u)| \leq C\|u\|_p$ for every $u \in \mathcal{S}_\mathcal{E}$. Therefore, the linear functional

$$\Phi : \bigoplus_{k=0}^{p} \operatorname{dom}(\Lambda^k) \to \mathbb{C} \qquad \text{by} \qquad \Phi(u, \Lambda u, \Lambda^2 u, \dots \Lambda^p u) = f(u)$$

is continuous and Riesz's Lemma gives $F = (F_k)_{k=0}^p \in \bigoplus_{k=0}^p \mathcal{H}_\mathcal{E}$ with

$$f(u) = \langle F, (u, \Lambda u, \dots \Lambda^p u)\rangle_{\bigoplus \mathcal{H}_\mathcal{E}} = \sum_{k=0}^{p} \langle F_k, \Lambda^k u\rangle_{\bigoplus \mathcal{H}_\mathcal{E}}.$$

□

Corollary 4.8. *If $\Lambda : \mathcal{H}_\mathcal{E} \to \mathcal{H}_\mathcal{E}$ is bounded, then $\mathcal{S}'_\mathcal{E} = \mathcal{H}_\mathcal{E}$.*

Proof. We always have the inclusion $\mathcal{H}_\mathcal{E} \hookrightarrow \mathcal{S}'_\mathcal{E}$ which corresponds to taking $p = 0$. If Λ is bounded, then the adjoint Λ^* is also bounded, and (4.8) gives

$$f(u) = \left\langle \sum_{k=0}^{p} (\Lambda^*)^k F_k, u \right\rangle_{\bigoplus \mathcal{H}_\mathcal{E}}, \quad \forall u \in \mathcal{S}_\mathcal{E}. \tag{4.9}$$

Since $\mathcal{S}_\mathcal{E}$ is dense in $\mathcal{H}_\mathcal{E}$ by Lemma 4.2, we have $f = \sum_{k=0}^p (\Lambda^*)^k F_k \in \mathcal{H}_\mathcal{E}$. □

Remark 4.9. In view of Lemma 3.9, Corollary 4.8 shows that $\mathcal{S}'_\mathcal{E}$ is a proper extension of $\mathcal{H}_\mathcal{E}$ on any infinite network.

5. The Wiener embedding $\mathcal{H}_\mathcal{E} \hookrightarrow L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$

We have now obtained a Gel'fand triple $\mathcal{S}_\mathcal{E} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'_\mathcal{E}$ (from either Lemma 3.10 or Theorem 4.3), and we are ready to apply the Minlos Theorem to a particularly lovely positive definite function on $\mathcal{S}_\mathcal{E}$, in order that we may obtain a particularly nice measure on $\mathcal{S}'_\mathcal{E}$. This allows us to realize $\operatorname{bd} G$ as a subset of $\mathcal{S}'_\mathcal{E}$. Recall that $\mathcal{S}_\mathcal{E}$ contains the energy kernel; see Lemma 3.11 or Lemma 3.16.

5.1. The Wiener embedding. In [JP10a, §5], we constructed $\mathcal{H}_\mathcal{E}$ from the resistance metric by making use of the fact that it is a negative semidefinite function on $G \times G$. In the proof of our main result, Theorem 5.3, we apply Schoenberg's Theorem to the function $g(u, v) = \|u - v\|_\mathcal{E}^2$, which is negative semidefinite on $\mathcal{S}_\mathcal{E} \times \mathcal{S}_\mathcal{E}$.

Definition 5.1. A function (or matrix) $Q : X \times X \to \mathbb{R}$ is *negative semidefinite* iff for any finite subset $F \subseteq X$, and any function $f : F \to \mathbb{R}$, one has

$$\sum_{x \in F} f(x) = 0 \qquad \Rightarrow \qquad \sum_{x,y \in F} f(x)Q(x,y)f(y) \leq 0. \tag{5.1}$$

The following famous result of Schoenberg may be found in [BCR84, Def. 4.1.8, Prop. 4.3.1, Thm. 5.1.5] or originally in [SW49].

Theorem 5.2 (Schoenberg). *Let X be a set and let $Q : X \times X \to \mathbb{R}$ be a function. Then the following are equivalent.*

1. *Q is negative semidefinite.*
2. *$\forall t \in \mathbb{R}^+$, the function $p_t(x,y) := e^{-tQ(x,y)}$ is positive semidefinite on $X \times X$.*
3. *There exists a Hilbert space $\mathcal{H}$ and a function $f : X \to \mathcal{H}$ such that $Q(x,y) = \|f(x) - f(y)\|_{\mathcal{H}}^2$.*

In Theorem 5.3 and henceforth, we use the notation

$$\mathbb{E}_\xi(f) := \int_{\mathcal{S}'_{\mathcal{E}}} f(\xi)\, d\mathbb{P}(\xi), \tag{5.2}$$

so that the subscript ξ indicates integration over the probability space $(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$.

Theorem 5.3 (Wiener embedding). *The Wiener transform $\mathcal{W} : \mathcal{H}_{\mathcal{E}} \to L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ given by*

$$\mathcal{W} : v \mapsto \widetilde{v}, \quad \widetilde{v}(\xi) := \langle v, \xi \rangle, \tag{5.3}$$

is an isometry. The transformed reproducing kernel $\{\widetilde{v}_x\}_{x \in G}$ is a system of Gaussian random variables which gives the (free) effective resistance distance (2.11) by

$$R^F(x,y) = \mathbb{E}_\xi((\widetilde{v}_x - \widetilde{v}_y)^2). \tag{5.4}$$

Moreover, for any $u, v \in \mathcal{H}_{\mathcal{E}}$, the energy inner product extends directly as

$$\langle u, v \rangle_{\mathcal{E}} = \mathbb{E}_\xi \left(\overline{\widetilde{u}} \widetilde{v} \right) = \int_{\mathcal{S}'_{\mathcal{E}}} \overline{\widetilde{u}} \widetilde{v}\, d\mathbb{P}. \tag{5.5}$$

Proof. Consider the function $g(u,v) = \|u - v\|_{\mathcal{E}}^2$ on $\mathcal{S}_{\mathcal{E}} \times \mathcal{S}_{\mathcal{E}}$. To check that this function is negative semidefinite, let F be any finite subset of G and suppose $\sum_{u\in F} a_u = 0$. Then

$$
\begin{aligned}
(5.6)\qquad &\sum_{u,v\in F} \overline{a_u} g(u,v) a_v \\
&= \sum_{u,v\in F} \overline{a_u} \left(\|u\|_{\mathcal{E}}^2 - 2\langle u,v\rangle_{\mathcal{E}} + \|v\|_{\mathcal{E}}^2\right) a_v \\
&= \sum_{u\in F} \overline{a_u}\|u\|_{\mathcal{E}}^2 \cdot 0 - 2\left\langle \sum_{u\in F} a_u u, \sum_{v\in F} a_v v\right\rangle_{\mathcal{E}} + \sum_{v\in F} a_v \|v\|_{\mathcal{E}}^2 \cdot 0 \\
&= -2\left\|\sum_{u\in F} a_u u\right\|_{\mathcal{E}}^2 \leq 0,
\end{aligned}
$$

where the 0s appear in the third line of (5.6) because $\sum_{u\in F} a_u = 0$; see also [JP10a, Thm. 5.4].

Therefore, we may apply Schoenberg's theorem with $t = \frac{1}{2}$ and deduce that $\exp(-\frac{1}{2}\|u-v\|_{\mathcal{E}}^2)$ is a positive semidefinite function on $\mathcal{H}_{\mathcal{E}} \times \mathcal{H}_{\mathcal{E}}$. Consequently, an application of the Minlos correspondence to the Gel'fand triple established in Lemma 4.2 yields a probability measure $\mathbb{P}$ on $\mathcal{S}_{\mathcal{E}}'$.

Moreover, Minlos' identity (2.25) gives

$$
\mathbb{E}_\xi(e^{\mathbf{i}\langle u,\xi\rangle}) = e^{-\frac{1}{2}\|u\|_{\mathcal{E}}^2}, \tag{5.7}
$$

whence one computes

$$
\int_{\mathcal{S}_{\mathcal{E}}'} \left(1 + \mathbf{i}\langle u,\xi\rangle - \frac{1}{2}\langle u,\xi\rangle^2 + \cdots\right) d\mathbb{P}(\xi) = 1 - \frac{1}{2}\langle u,u\rangle_{\mathcal{E}} + \cdots. \tag{5.8}
$$

Now it follows that $\mathbb{E}(\widetilde{u}^2) = \mathbb{E}_\xi(\langle u,\xi\rangle^2) = \|u\|_{\mathcal{E}}^2$ for every $u \in \mathcal{S}_{\mathcal{E}}$, by comparing the terms of (5.8) which are quadratic in u. Therefore, $\mathcal{W} : \mathcal{H}_{\mathcal{E}} \to \mathcal{S}_{\mathcal{E}}'$ is an isometry, and (5.8) gives

$$
\mathbb{E}_\xi(|\widetilde{v}_x - \widetilde{v}_y|^2) = \mathbb{E}_\xi(\langle v_x - v_y, \xi\rangle^2) = \|v_x - v_y\|_{\mathcal{E}}^2, \tag{5.9}
$$

whence (5.4) follows from (2.12). Note that by comparing the linear terms, (5.8) implies $\mathbb{E}_\xi(1) = 1$, so that $\mathbb{P}$ is a probability measure, and $\mathbb{E}_\xi(\langle u,\xi\rangle) = 0$ and $\mathbb{E}_\xi(\langle u,\xi\rangle^2) = \|u\|^2$, so that $\{\langle u,\cdot\rangle\}_{u\in\mathcal{S}_{\mathcal{E}}}$ is a Gaussian process on $\mathcal{S}_{\mathcal{E}}'$.

Finally, use polarization to compute

$$
\begin{aligned}
\langle u,v\rangle_{\mathcal{E}} &= \frac{1}{4}\left(\|u+v\|_{\mathcal{E}}^2 - \|u-v\|_{\mathcal{E}}^2\right) \\
&= \frac{1}{4}\left(\mathbb{E}_\xi\left(|\widetilde{u}+\widetilde{v}|^2\right) - \mathbb{E}_\xi\left(|\widetilde{u}-\widetilde{v}|^2\right)\right) && \text{by (5.9)} \\
&= \frac{1}{4}\int_{\mathcal{S}_{\mathcal{E}}'} |\widetilde{u}+\widetilde{v}|^2(\xi) - |\widetilde{u}-\widetilde{v}|^2(\xi)\, d\mathbb{P}(\xi) \\
&= \int_{\mathcal{S}_{\mathcal{E}}'} \overline{\widetilde{u}}(\xi)\widetilde{v}(\xi)\, d\mathbb{P}(\xi).
\end{aligned}
$$

This establishes (5.5) and completes the proof. □

It is important to note that since the Wiener transform $\mathcal{W} : \mathcal{S}_\mathcal{E} \to \mathcal{S}'_\mathcal{E}$ is an isometry, the conclusion of Minlos' theorem is stronger than usual: the isometry allows the energy inner product to be extended isometrically to a pairing on $\mathcal{H}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$ instead of just $\mathcal{S}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$.

Remark 5.4. With the embedding $\mathcal{H}_\mathcal{E} \to L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$, we obtain a maximal abelian algebra of Hermitian multiplication operators $L^\infty(\mathcal{S}'_\mathcal{E})$ acting on $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$. For a sharp contrast, note that the Hermitian multiplication operators on $\mathcal{H}_\mathcal{E}$ are trivial, by [JP10b, Lem. 3.7]. This result states that if $\varphi : G \to \mathbb{R}$ and M_φ denotes the multiplication operator defined by $(M_\varphi u)(x) = \varphi(x)u(x)$, then M_φ is Hermitian if and only if $M_\varphi = k\mathbb{I}$, for some $k \in \mathbb{R}$. See [JP10b] for more on the multiplication operators on $\mathcal{H}_\mathcal{E}$.

Remark 5.5. The reader will note that we have taken pains to keep everything $\mathbb{R}$-valued in this section (especially the elements of $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}'_\mathcal{E}$); see Remark 2.7 and Remark 3.1. This is primarily to ensure the convergence of $\int_{\mathcal{S}'} e^{\mathbf{i}\langle u,\xi\rangle}\, d\mathbb{P}(\xi)$ in (5.7). However, now that we have established the fundamental identity $\langle u, v\rangle_\mathcal{E} = \int_{\mathcal{S}'} \overline{\widetilde{u}}\widetilde{v}\, d\mathbb{P}$ in (5.5) and extended the pairing $\langle\cdot,\cdot\rangle$ to $\mathcal{H}_\mathcal{E} \times \mathcal{S}'_\mathcal{E}$, we are at liberty to complexify our results via the standard decomposition into real and complex parts: $u = u_1 + \mathbf{i}u_2$ with u_i $\mathbb{R}$-valued elements of $\mathcal{H}_\mathcal{E}$, etc.

5.1.1. Implications for $\boldsymbol{\Lambda} = \boldsymbol{\mathcal{N}}$.

Remark 5.6. The Wiener transform is an isometry, and therefore one has

$$\mathbb{E}(\widetilde{\epsilon}_x) = 0 \qquad \text{and} \qquad \mathbb{E}(\widetilde{\epsilon}_x\widetilde{\epsilon}_y) = \delta_{x,y}. \tag{5.10}$$

Since independence of *Gaussian* random variables is determined by the first two moments, it thus follows from Theorem 5.3 that $\{\widetilde{\epsilon}_x\}$ forms a system of i.i.d. Gaussian random variables with mean 0 and variance 1. This is noteworthy because while independence implies orthogonality, the converse does not hold without the additional hypothesis that the random variables are Gaussian.

5.1.2. Implications for $\boldsymbol{\Lambda} = \boldsymbol{\Delta}_\mathcal{M}$. In the case when the Gel'fand triple is constructed from the domain of $\Delta_\mathcal{M}$, as in Definition 3.12 (in §3.2), then one can extend Δ to distributions.

Definition 5.7. Extend Δ to $\mathcal{S}'_\mathcal{E}$ by defining

$$\Delta\xi(v_x) := \langle \delta_x, \xi\rangle, \tag{5.11}$$

so that $\Delta\xi(v_x) = \sum_{y\sim x} c_{xy}(\xi(v_x) - \xi(v_y))$ follows readily from Lemma 3.15.

Now extend Δ to $\widetilde{\Delta}$ defined on $\widetilde{v}_x \in L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$ by $\widetilde{\Delta}(\widetilde{v}_x)(\xi) := \widetilde{\Delta v_x}(\xi)$, so that

$$\widetilde{\Delta} : \widetilde{v}_x \mapsto c(x)\widetilde{v}_x - \sum_{y\sim x} c_{xy}\widetilde{v}_y. \tag{5.12}$$

Since $v_x \mapsto \widetilde{v}_x$ is an isometry, it is no great surprise that

$$\langle \widetilde{v}_x, \widetilde{\Delta}\widetilde{v}_y \rangle_{L^2} = \int_{\mathcal{S}'_{\mathcal{E}}} \widetilde{v}_x(\xi)\widetilde{v}_y(\Delta\xi)\, d\mathbb{P}(\xi) = \langle v_x, \Delta v_y \rangle_{\mathcal{E}}. \tag{5.13}$$

5.1.3. A mathematical physics perspective.

Remark 5.8. The polynomials are dense in $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$. More precisely, if we denote by $\varphi(t_1, t_2, \ldots, t_k)$ an ordinary polynomial in k variables, then

$$\varphi(\xi) := \varphi\big(\langle u_1, \xi \rangle, \langle u_2, \xi \rangle, \ldots \langle u_k, \xi \rangle\big) \tag{5.14}$$

is a polynomial on $\mathcal{S}'_{\mathcal{E}}$ and

$$\begin{aligned} &\mathcal{P}oly_n \\ &\quad := \big\{\varphi\big(\widetilde{u_1}(\xi), \widetilde{u_2}(\xi), \ldots \widetilde{u_k}(\xi)\big), \deg(\varphi) \leq n, \,\vdots\, u_j \in \mathcal{H}_{\mathcal{E}}, \xi \in \mathcal{S}'_{\mathcal{E}}\big\} \end{aligned} \tag{5.15}$$

is the collection of polynomials of degree at most n, and $\{\mathcal{P}oly_n\}_{n=0}^{\infty}$ is an increasing family whose union is all of $\mathcal{S}'_{\mathcal{E}}$. One can see that the monomials $\langle u, \xi \rangle$ are in $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ as follows: compare like powers of u from either side of (5.8) to see that $\mathbb{E}_\xi\left(\langle u, \xi \rangle^{2n+1}\right) = 0$ and

$$\mathbb{E}_\xi\left(\langle u, \xi \rangle^{2n}\right) = \int_{\mathcal{S}'_{\mathcal{E}}} |\langle u, \xi \rangle|^{2n}\, d\mathbb{P}(\xi) = \frac{(2n)!}{2^n n!}\|u\|_{\mathcal{E}}^{2n}, \tag{5.16}$$

and then apply the Schwarz inequality.

To see why the polynomials $\{\mathcal{P}oly_n\}_{n=0}^{\infty}$ should be dense in $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ observe that the sequence $\{P_{\mathcal{P}oly_n}\}_{n=0}^{\infty}$ of orthogonal projections increases to the identity, and therefore, $\{P_{\mathcal{P}oly_n}\widetilde{u}\}$ forms a martingale, for any $\widetilde{u} \in L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ (i.e., for any $u \in \mathcal{H}_{\mathcal{E}}$).

Denote the "multiple Wiener integral of degree n" by

$$H_n := \big(cl\,\mathrm{span}\{\langle u, \cdot \rangle^n \,\vdots\, u \in \mathcal{H}_{\mathcal{E}}\}\big) \ominus \{\langle u, \cdot \rangle^k \,\vdots\, k < n, u \in \mathcal{H}_{\mathcal{E}}\},$$

for each $n \geq 1$, and $H_0 := \mathbb{C}\mathbf{1}$ for a vector $\mathbf{1}$ with $\|\mathbf{1}\|_2 = 1$. Then we have an orthogonal decomposition of the Hilbert space

$$L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} H_n. \tag{5.17}$$

See [Hid80, Thm. 4.1] for a more extensive discussion.

A physicist would call (5.17) the (bosonic) Fock space representation of $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ with "vacuum vector" $\mathbf{1}$; note that H_n has a natural (symmetric) tensor product structure. Familiarity with these ideas is not necessary for the sequel, but the decomposition (5.17) is helpful for understanding two key things:

(i) The Wiener isometry $\mathcal{W} : \mathcal{H}_{\mathcal{E}} \to L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ identifies $\mathcal{H}_{\mathcal{E}}$ with the subspace H_1 of $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$, in particular, $L^2(\mathcal{S}'_{\mathcal{E}}, \mathbb{P})$ is not isomorphic to $\mathcal{H}_{\mathcal{E}}$. In fact, it is the second quantization of $\mathcal{H}_{\mathcal{E}}$.

(ii) The constant function $\mathbf{1}$ is an element of $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$ but does not correspond to any element of $\mathcal{H}_\mathcal{E}$. In particular, constant functions in $\mathcal{H}_\mathcal{E}$ are equivalent to 0, but this is not true in $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$.

It is somewhat ironic that we began this story by removing the constants (via modding out by the kernel of $\mathcal{E}$), only to reintroduce them with a certain amount of effort, much later. Item (ii) explains why it is not nonsense to write things like $\mathbb{P}(\mathcal{S}'_\mathcal{E}) = \int_{\mathcal{S}'_\mathcal{E}} \mathbf{1}\, d\mathbb{P} = 1$.

5.2. The resistance boundary of a transient network. With the tools developed in §3 and §5, we now construct the resistance boundary $\operatorname{bd} G$ as a set of equivalence classes of infinite paths. Recall that we began with a comparison of the Poisson boundary representation for bounded harmonic functions with the boundary sum representation recalled in (1.2):

$$u(x) = \int_{\partial\Omega} u(y)k(x,dy) \qquad \leftrightarrow \qquad u(x) = \sum_{\operatorname{bd} G} u \tfrac{\partial h_x}{\partial \mathrm{n}} + u(o).$$

In this section, we replace the sum with an integral and complete the parallel.

Remark 5.9. For $u \in \mathcal{H}arm$ and $\xi \in \mathcal{S}'_\mathcal{E}$, let us abuse notation and write u for $\widetilde{u}$. That is, $u(\xi) := \widetilde{u}(\xi) = \langle u, \xi \rangle$. Unnecessary tildes obscure the presentation and the similarities to the Poisson kernel.

Theorem 5.3 has the following immediate implication for resistance metric.

Corollary 5.10. *For $e_x(\xi) := e^{\mathbf{i}\langle v_x, \xi\rangle}$, one has $\mathbb{E}_\xi(e_x) = e^{-\frac{1}{2}R^F(o,x)}$ and hence*

$$\mathbb{E}_\xi(\overline{e_x} e_y) = \int_{\mathcal{S}'_\mathcal{E}} \overline{e_x(\xi)} e_y(\xi)\, d\mathbb{P} = e^{-\frac{1}{2}R^F(x,y)}. \tag{5.18}$$

Proof. Substitute $u = v_x$ or $u = v_x - v_y$ in (5.7) and apply (2.12). □

Remark 5.11. Free resistance is interpreted as the reciprocal of an integral over a path space in (2.15); see [JP10a, Rem. 3.14]. Corollary 5.10 provides a variation on this theme:

$$R^F(x,y) = -2 \log \mathbb{E}_\xi(\overline{e_x} e_y) = 2 \log \frac{1}{\int_{\mathcal{S}'_\mathcal{E}} \overline{e_x(\xi)} e_y(\xi)\, d\mathbb{P}}. \tag{5.19}$$

Observe that Theorem 5.3 was carried out for the free resistance, but all the arguments go through equally well for the wired resistance; note that R^W is similarly negative semidefinite by Theorem 5.2 and [JP10a, Cor. 5.5]. Thus, there is a corresponding Wiener transform $\mathcal{W} : \mathcal{F}in \to L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$ defined by

$$\mathcal{W} : v \mapsto \widetilde{f}, \qquad f = P_{\mathcal{F}in} v \text{ and } \widetilde{f}(\xi) = \langle f, \xi \rangle. \tag{5.20}$$

Again, $\{\widetilde{f}_x\}_{x \in G}$ is a system of Gaussian random variables which gives the wired resistance distance; in this case, by $R^W(x,y) = \mathbb{E}_\xi((\widetilde{f}_x - \widetilde{f}_y)^2)$ and hence also

$$R^W(x,y) = -2 \log \mathbb{E}_\xi\big(\overline{e^{\mathbf{i}\langle f_x, \xi\rangle}} e^{\mathbf{i}\langle f_y,\xi\rangle}\big) = -2 \log \mathbb{E}_\xi\big(e^{\mathbf{i}\langle f_y - f_x, \xi\rangle}\big). \tag{5.21}$$

Corollary 5.12 (Boundary integral representation for harmonic functions)**.** *For any $u \in \mathcal{H}arm$ and with $h_x = P_{\mathcal{H}arm} v_x$,*

$$u(x) = \int_{\mathcal{S}'_\mathcal{E}} u(\xi) h_x(\xi)\, d\mathbb{P}(\xi) + u(o). \tag{5.22}$$

Proof. Starting with (2.8), use (5.5) to compute

$$u(x) - u(o) = \langle h_x, u\rangle_\mathcal{E} = \overline{\langle u, h_x\rangle_\mathcal{E}} = \overline{\int_{\mathcal{S}'_\mathcal{E}} \overline{u} h_x \, d\mathbb{P}}, \tag{5.23}$$

where the last equality comes by substituting $v = h_x$ in (5.5). It is shown in [JP11a, Lem. 2.22] that $\overline{h_x} = h_x$. □

Remark 5.13 (A Hilbert space interpretation of $\operatorname{bd} G$)**.** In view of Corollary 5.12, we are now able to "catch" the boundary between $\mathcal{S}_\mathcal{E}$ and $\mathcal{S}'_\mathcal{E}$ by using Λ and its adjoint. The boundary of G may be thought of as (a possibly proper subset of) $\mathcal{S}'_\mathcal{E}$. Corollary 5.12 suggests that $\Bbbk(x, d\xi) := h_x(\xi) d\mathbb{P}$ is the discrete analogue in $\mathcal{H}_\mathcal{E}$ of the Poisson kernel $k(x, dy)$, and comparison of (1.2) with (5.22) gives a way of understanding a boundary integral as a limit of Riemann sums:

$$\int_{\mathcal{S}'_\mathcal{E}} u\, h_x \, d\mathbb{P} = \lim_{k\to\infty} \sum_{\operatorname{bd} G_k} u(x) \tfrac{\partial h_x}{\partial \mathbb{n}}(x). \tag{5.24}$$

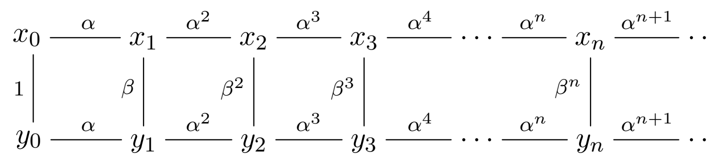
(We continue to omit the tildes as in Remark 5.9.) By a theorem of Nelson, $\mathbb{P}$ is fully supported on those functions which are Hölder-continuous with exponent $\alpha = \frac{1}{2}$, which we denote by $\operatorname{Lip}(\frac{1}{2}) \subseteq \mathcal{S}'_\mathcal{E}$; see [Nel64]. Recall from [JP10a, Cor. 2.17] that $\mathcal{H}_\mathcal{E} \subseteq \operatorname{Lip}(\frac{1}{2})$. See [Arv76a, Arv76b, Min63, Nel69].

6. Examples

Our presentation of $\operatorname{bd} G$ may appear somewhat abstract in the general case. However, we now illustrate the concept with a simple and entirely explicit example where the representation by equivalence classes given at the end of §5.2 takes on an especially concrete and visual form. Moreover, the computations can be completed without the direct construction of $\mathcal{S}_\mathcal{E}$, $\mathcal{S}'_\mathcal{E}$, or any discussion of $L^2(\mathcal{S}'_\mathcal{E}, \mathbb{P})$; we can obtain the boundary simply by constructing certain functions on the network. We feel this is an especially nice feature of our approach.

Example 6.1 (One-sided infinite ladder network)**.** Consider two copies of the nearest-neighbour graph on the nonnegative integers $\mathbb{Z}^+$, one with vertices labelled by $\{x_n\}$, and the other with vertices labelled by $\{y_n\}$. Fix two positive numbers $\alpha > 1 > \beta > 0$. In addition to the edges $c_{x_n,x_{n-1}} = \alpha^n$ and $c_{y_n,y_{n-1}} = \alpha^n$, we also add "rungs" to the ladder by defining $c_{x_n,y_n} = \beta^n$:

(6.1)



This network was suggested to us by Agelos Georgakopoulos as an example of a one-ended network with nontrivial $\mathcal{H}arm$. The function u constructed below is the first example of an explicitly computed nonconstant harmonic function of finite energy on a graph with one end (existence of such a phenomenon was proved in [CW92]). Numerical experiments indicate that this function is also bounded (and even that the sequences $\big(u(x_n)\big)_{n=0}^\infty$ and $\big(u(y_n)\big)_{n=0}^\infty$ actually converge very quickly), but we have not yet been able to prove this. Numerical evidence also suggests that Δ is not essentially self-adjoint on this network, but we have not yet proved this, either.

This graph clearly has one end. We will show that such a network has nontrivial resistance boundary if and only if $\alpha > 1$ and in this case, the boundary consists of one point for $\beta = 1$, and two points for β such that $(1 + \frac{1}{\alpha})^2 < \alpha/\beta^2$.

For presenting the construction of u, choose $\beta < 1$ satisfying $4\beta^2 < \alpha$ (at the end of the construction, we explain how to adapt the proof for the less restrictive condition $(1+\frac{1}{\alpha})^2 < \alpha/\beta^2$). We now construct a nonconstant $u \in \mathcal{H}arm$ with $u(x_0) = 0$ and $u(y_0) = -1$. If we consider the flow induced by u, the amount of current flowing through one edge determines u completely (up to a constant). Once it is clear that there are two boundary points in this case, it is clear that specifying the value of u at one (and grounding the other) determines u completely.

Due to the symmetry of the graph, we may abuse notation and write n for x_n or y_n, and $\check{n}$ for the vertex "across the rung" from n. For a function u on the ladder, denote the horizontal increments and the vertical increments by

$$\delta u(n) := u(n+1) - u(n) \qquad \text{and} \qquad \sigma u(n) := u(n) - u(\check{n}),$$

respectively. Thus, for $n \geq 1$, we can express the equation $\Delta u(n) = 0$ by

$$\Delta u(n) = \alpha^n \delta u(n-1) - \alpha^{n+1} \delta u(n) + \beta^n \sigma u(n) = 0,$$

which is equivalent to

$$\delta u(n) = \frac{1}{\alpha} \delta u(n-1) + \frac{\beta^n}{\alpha^{n+1}} \sigma u(n).$$

Since symmetry allows one to assume that $u(\check{n}) = 1 - u(n)$, we may replace $\sigma u(n)$ by $2u(n) + 1$ and obtain that any u satisfying

$$u(n+1) = u(n) + \frac{u(n) - u(n-1)}{\alpha} + \frac{2}{\alpha}\left(\tfrac{\beta}{\alpha}\right)^n u(n) + \frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^n \tag{6.2}$$

is harmonic. It remains to see that u has finite energy.

Our estimate for $\mathcal{E}(u) < \infty$ requires the assumption that $\alpha > 4\beta^2$, but numerical computations indicate that u defined by (6.2) will be both bounded and of finite energy, for any $\beta < 1 < \alpha$. First, note that $u(1) = \frac{1}{\alpha}$ and so an immediate induction using (6.2) shows that $\delta u(n) = u(n+1) - u(n) > 0$ for all $n \geq 1$, and so u is strictly increasing. Since $\beta < 1 < \alpha$, we may choose N so that

$$n \geq N \quad \Rightarrow \quad \left(\frac{\beta}{\alpha}\right)^n < \frac{\alpha - 1}{2}.$$

Then $n \geq N$ implies

$$u(n+1) \leq 2u(n) + \frac{1}{\alpha}, \tag{6.3}$$

by using (6.2) and the fact that $u(n)$ is increasing and $\frac{\beta}{\alpha} < 1$. Now use (6.2) to write

$$\begin{aligned}
\delta u(n) &= \frac{1}{\alpha}(\delta u)(n-1) + \left(\frac{2}{\alpha}u(n) + \frac{1}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^n \\
&= \frac{1}{\alpha^n}(\delta u)(0) + \sum_{k=0}^{n-1} \frac{1}{\alpha^k}\left(\frac{2}{\alpha}u(n-k) + \frac{1}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{n-k} \\
&= \frac{1}{\alpha^{n+1}} + \frac{\beta(1-\beta^n)}{\alpha^{n+1}(1-\beta)} + \frac{2}{\alpha^{n+1}}\sum_{k=1}^{n} \beta^k u(k),
\end{aligned}$$

where the second line comes by iterating the first, and the third by algebraic simplification. Applying the estimate (6.3) gives

$$\begin{aligned}
2\sum_{k=1}^{n} \beta^k u(k) &\leq 2^2 \sum_{k=1}^{n} \beta^k u(k-1) + \tfrac{2}{\alpha}\sum_{k=1}^{n} \beta^k \\
&= 2^2 \sum_{k=2}^{n} \beta^k u(k-1) + 2\tfrac{\beta}{\alpha} \cdot \tfrac{1-\beta^n}{1-\beta},
\end{aligned}$$

and iterating gives

$$\delta u(n) \leq \frac{1}{\alpha^{n+1}}\left(1 + \frac{\beta(1-\beta^n)}{1-\beta} + \frac{(2\beta)^n}{\alpha} + 2\frac{\beta}{\alpha}\sum_{k=0}^{n-1} 2^k \frac{\beta^k - \beta^n}{1-\beta}\right). \tag{6.4}$$

Now the energy $\mathcal{E}(u) = \sum_{n=0}^{\infty} \alpha^{n+1} (\delta u(n))^2$ can be estimated by using (6.4) as follows:

$$\mathcal{E}(u) \leq \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} \left(1 + \frac{\beta(1-\beta^n)}{1-\beta} + \frac{(2\beta)^n}{\alpha} + \frac{2\beta + 2\beta^{n+1} - 2^{n+2}\beta^{n+1} - 2^2\beta^{n+2} + (2\beta)^{n+2}}{\alpha(1-\beta)(2\beta-1)}\right)^2$$

and the condition $\alpha > 4\beta^2$ ensures convergence.

Note that the computations above can be slightly refined: instead of $\alpha > 4\beta^2$, one need only assume that $\alpha > (1+\frac{1}{\alpha})^2\beta^2$. Then, fix $\varepsilon > 0$ for which $\alpha/\beta^2 > (1+\frac{1}{\alpha})^2 + \varepsilon$ and choose N so that $n \geq N$ implies $(\beta/\alpha)^n < 1 + \frac{1}{\alpha} + \varepsilon(1 + 2\alpha + \alpha\varepsilon)$. Then the calculations can be repeated, with most occurrences of 2 replaced by $1 + \frac{1}{\alpha} + \varepsilon$.

Remark 6.2. [Comparison of Example 6.1 to the 1-dimensional integer lattice] In [JP11a, Ex. 6.3], we showed that the "nonnegative geometric integers" network

$$0 \overset{\alpha}{\text{——}} 1 \overset{\alpha^2}{\text{——}} 2 \overset{\alpha^3}{\text{——}} 3 \overset{\alpha^4}{\text{——}} \dots$$

supports a monopole but not a harmonic function of finite energy, for $\alpha > 1$. These conductances correspond to the biased random walk where, at each vertex, the walker has transition probabilities

$$p(n,m) = \begin{cases} \frac{1}{1+\alpha}, & m = n-1, \\ \frac{\alpha}{1+\alpha}, & m = n+1. \end{cases}$$

In particular, this is a spatially homogeneous distribution. In contrast, the random walk corresponding to Example 6.1 has transition probabilities

$$p(n,m) = \begin{cases} \frac{1}{1+\alpha+\left(\frac{\beta}{\alpha}\right)^n}, & m = n-1, \\ \frac{\alpha}{1+\alpha+\left(\frac{\beta}{\alpha}\right)^n}, & m = n+1, \\ \frac{(\beta/\alpha)^n}{1+\alpha+\left(\frac{\beta}{\alpha}\right)^n}, & m = \check{n}. \end{cases}$$

Thus, Example 6.1 is asymptotic to the nonnegative geometric integers.

One can even think of Example 6.1 as describing the *scattering theory* of the geometric half-integer model, in the sense of [LP89]; see also [JP10d]. In this theory, a wave (described by a function) travels towards an obstacle. After the wave collides with the obstacle, the original function is transformed (via the "scattering operator") and the resulting wave travels away from the obstacle. The scattering is typically localized in some sense, corresponding to the location of the collision.

To see the analogy with the present scenario, consider the current flow defined by the harmonic function u constructed in Example 6.1, i.e., induced by Ohm's law: $I(x,y) = c_{xy}(u(x) - u(y))$. With

$$\operatorname{div}_{|I|}(x) := \frac{1}{2} \sum_{\{z \,:\, I(x,z)>0\}} |I(x,z)|,$$

this current defines a Markov process with transition probabilities

$$P(x,y) = \frac{I(x,y)}{\operatorname{div}_{|I|}(x)}, \quad \text{if} \quad I(x,y) > 0,$$

and $P(x,y) = 0$ otherwise; see [JP09b, JP09a]. This describes a random walk where a walker started on the bottom edge of the ladder will tend to step leftwards, but with a geometrically increasing probability of stepping to the upper edge, and then walking rightwards off towards infinity. The walker corresponds to the wave, which is scattered as it approaches the geometrically localized obstacle at the origin.

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