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The λ_u-function in JB^*-algebras

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Abstract. Inspired by work of R. M. Aron and R. H. Lohman, Gert K. Pedersen introduced a geometric function, which is defined on the unit ball of a C^*-algebra and called the λ_u-function. Our goal here is to extend the notion of a λ_u-function to the context of JB^*-algebras. We study convex combinations of elements in a JB^*-algebra using the λ_u-function and earlier results we have obtained on the geometry of JB^*-algebras. A formula to compute λ_u-function is obtained for invertible elements in a JB^*-algebra. In the course of our analysis, some C^*-algebra results due to G. K. Pedersen, C. L. Olsen and M. Rørdam are extended to general JB^*-algebras.

Contents

1. Introduction and preliminaries

R. M. Aron and R. H. Lohman [2] studied a geometric function, called the λ-function for normed spaces. Subsequently, G. K. Pedersen [11] introduced a related function, namely, the λ_u-function, defined on the unit ball of a C^*-algebra, where u is a unitary element of the algebra. In this article, we study the λ_u-function in the general setting of JB^*-algebras. It may be noted here that the study of the λ_u-function can not be further extended to more general JB^*-triple systems (cf. [19] or [16]) which have no invertible (hence, no unitary) elements.

The absence of associativity in JB^*-algebras causes great difficulties in calculations. In [15], the author proved some fundamental results on unitary isotopes of JB^*-algebras and by applying these we obtained various results on representing elements as convex combinations of unitaries in a JB^*-algebra (see [14, 16, 17, 18]); our approach also provides alternative proofs to certain related results for C^*-algebras.

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We introduce below two special sets of real numbers $\mathcal{S}(x)$ and $\mathcal{V}(x)$ that turn out to be intervals and are used to study certain convex combinations of elements in JB^*-algebras. Using some of our earlier results from [15, 18], we shall investigate the relationship between the intervals $\mathcal{S}(x)$, $\mathcal{V}(x)$ and the λ_u-function for JB^*-algebras. In the process, we obtain extensions to JB^*-algebras of some C^*-algebra results that appeared in [10, 11, 12]. In the end, a formula to compute λ_u-function is obtained for invertible elements of a JB^*-algebra.

Preliminaries. We begin by recalling that if x is an element in a Jordan algebra $\mathcal{J}$, then the x-homotope of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear space structure as $\mathcal{J}$ but with a different product, denoted by " $\cdot_x$", defined by $a \cdot_x b = \{axb\}$ for all a, b in $\mathcal{J}_{[x]}$. Here, $\{pqr\}$ denotes the Jordan triple product and is defined in any Jordan algebra by $\{pqr\} = (p\circ q)\circ r - (p\circ r)\circ q + (q\circ r)\circ p$ where "$\circ$" stands for the Jordan product in the original algebra. (See [5], for instance.)

The homotopes of interest here will be obtained when $\mathcal{J}$ has a unit e and x is invertible: this means that there exists $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{\text{inv}}$. In this case, as $x \cdot_{x^{-1}} y = \{xx^{-1}y\} = y + (x^{-1} \circ y) \circ x - (x \circ y) \circ x^{-1} = y$ (see [8]), x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$.

If $x \in \mathcal{J}_{\text{inv}}$ then x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, is defined to be the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. We denote the multiplication "$\cdot_{x^{-1}}$" of $\mathcal{J}^{[x]}$ by " $\circ_x$". $\{\ ,\ ,\ \}_x$, y^{-1_x} will stand for the Jordan triple product and the multiplicative inverse (if it exists) of y in the isotope $\mathcal{J}^{[x]}$, respectively.

Isotopy may produce essentially different Jordan algebras. The x-isotope $\mathcal{J}^{[x]}$ of a Jordan algebra $\mathcal{J}$ need not be isomorphic to $\mathcal{J}$. For such examples see [9, 7]. However, some important features of Jordan algebras are unaffected by isotopy (see [15, Lemma 4.2 and Theorem 4.6], for examples).

JB*-algebras. A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|.\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|.\|)$ is a Banach space and $\|a \circ b\| \le \|a\|\|b\|$. If, in addition, $\mathcal{J}$ has unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra.

An important tool in Banach algebra theory is the spectrum of an element. Let $\mathcal{J}$ be a complex unital Banach Jordan algebra with unit e and let $x \in \mathcal{J}$. The spectrum of x in $\mathcal{J}$, denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\}.$$

Here, $\mathbb{C}$ denotes the field of complex numbers. When no confusion can arise, we shall write $\sigma(x)$ in place of $\sigma_{\mathcal{J}}(x)$.

We are interested in a special class of Banach Jordan algebras, called JB^*-algebras. A complex Banach Jordan algebra $\mathcal{J}$ with involution $*$ is called a JB^*-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. It follows that

$\|x^*\| = \|x\|$ for all elements x of a JB^*-algebra (see [22], for instance). The class of JB^*-algebras was introduced by Kaplansky in 1976 and it includes all C^*-algebras as a proper subclass (cf. [20]).

As usual, an element x of a JB^*-algebra $\mathcal{J}$ is said to be self-adjoint if $x^* = x$. A self-adjoint element x of $\mathcal{J}$ is said to be positive in $\mathcal{J}$ if its spectrum $\sigma_{\mathcal{J}}(x)$ is contained in the set of nonnegative real numbers. For basic theory of Banach Jordan algebras and JB^*-algebras, we refer to the sources [1, 4, 13, 19, 20, 21, 22].

Unitary Isotopes of a JB^*-algebra. Let $\mathcal{J}$ be a unital JB^*-algebra. An element $u \in \mathcal{J}$ is called unitary if $u^* = u^{-1}$. We shall denote the set of all unitary elements of $\mathcal{J}$ by $\mathcal{U}(\mathcal{J})$ and its convex hull by $co\mathcal{U}(\mathcal{J})$. If $u \in \mathcal{U}(\mathcal{J})$ then the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$.

It is well known (see [7, 3, 15]) that for any unitary element u of JB^*-algebra $\mathcal{J}$, the unitary isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra with u as its unit with respect to the original norm and the involution $*_u$ defined as below:

$$x^{*_u} = \{ux^*u\}.$$

Like invertible elements [15, Theorem 4.2 (ii)], the set of unitary elements in the (unital) JB^*-algebra $\mathcal{J}$ is invariant on passage to isotopes of $\mathcal{J}$ [15, Theorem 4.6]. Moreover, every invertible element x of JB^*-algebra $\mathcal{J}$ is positive in certain unitary isotope of $\mathcal{J}$ [15, Theorem 4.12]; this is a tricky result and its proof (appeared in [15]) involves Stone–Weierstrass Theorem and standard functional calculus. In the sequel, we shall use these results as our main tools.

2. λ_u-function

The λ_u-function, defined on the unit ball of a C^*-algebra, was introduced by Pedersen [11] where u is some unitary element of the algebra. In this section, we study the λ_u-function for general JB^*-algebras; of course, this study can not be further extended to more general JB^*-triple systems (cf. [19] or [16]) that do not have unitary elements. In the sequel, we shall introduce two special classes of sets $\mathcal{V}(x)$ and $\mathcal{S}(x)$ of real numbers. Indeed, these sets are intervals and allow us to study some special convex combinations. We investigate relationships between intervals $\mathcal{S}(x)$, $\mathcal{V}(x)$ and the λ_u-function. In doing this, some C^*-algebra results due to G. K. Pedersen, C. L. Olsen and M. Rørdam [10, 11, 12] will be extended to JB^*-algebras.

We define the set $\mathcal{V}(x)$, for each element x of the closed unit ball of a JB^*-algebra as follows:

Definition 2.1. Let $\mathcal{J}$ be a unital JB^*-algebra.

(1) For each $\delta \geq 1$, we define $co_\delta\mathcal{U}(\mathcal{J}) \subseteq co\mathcal{U}(\mathcal{J})$ by

$$co_\delta\mathcal{U}(\mathcal{J}) = \left\{ \delta^{-1}\sum_{i=1}^{n-1} u_i + \delta^{-1}(1+\delta-n)u_n : u_j \in \mathcal{U}(\mathcal{J}),\ j = 1, \ldots, n \right\}$$

where n is the integer given by $n-1 < \delta \le n$.

(2) For each $x \in (\mathcal{J})_1$, we define the set $\mathcal{V}(x)$ by

$$\mathcal{V}(x) = \{\beta \ge 1 : x \in co_\beta \mathcal{U}(\mathcal{J})\}.$$

Part (a) of the next theorem extends a C^*-algebra result due to Rørdam (see [12, Proposition 3.1]); the proof follows his argument with suitable changes necessitated by the nonassociativity of Jordan algebras.

Theorem 2.2. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$.*

(a) *Let $\|\gamma x - u_o\| \le \gamma - 1$ for some $\gamma \ge 1$ and some $u_o \in \mathcal{U}(\mathcal{J})$. Let $(\alpha_2, \dots, \alpha_m) \in \Re^{m-1}$ such that $0 \le \alpha_j < \gamma^{-1}$ and $\gamma^{-1} + \sum_{j=2}^m \alpha_j = 1$. Then there exist unitaries $u_1, \dots, u_m$ in $\mathcal{J}$ such that*

$$x = \gamma^{-1} u_1 + \sum_{j=2}^{m} \alpha_j u_j. \tag{i}$$

Moreover,

$$(\gamma\, ,\, \infty) \subseteq \mathcal{V}(x). \tag{ii}$$

(b) *On the other hand, if (ii) holds then for all $r > \gamma$ there is $u_1 \in \mathcal{U}(\mathcal{J})$ such that $\|rx - u_1\| \le r - 1$.*

Proof. (a) If $\gamma = 1$, then $\|\gamma x - u_o\| \le \gamma - 1$ means that $\gamma x = u_o$ and so there is nothing left to prove (i) in this case.

Now, assume $\gamma > 1$ and put $y = (\gamma-1)^{-1}(\gamma x - u_o)$. Since $\|\gamma x - u_o\| \le \gamma - 1$, $\|y\| = (\gamma-1)^{-1}\|\gamma x - u_o\| \le (\gamma-1)^{-1}(\gamma-1) = 1$ so that $y \in (\mathcal{J})_1$. Further, since $\gamma^{-1} + \sum_{j=2}^m \alpha_j = 1$ and since $\gamma x - u_o - (\gamma-1)y = 0$, we have

$$\gamma x = u_o + (\gamma - 1)y \;=\; u_o + \gamma \sum_{j=2}^{m} \alpha_j y. \tag{iii}$$

We note that $0 \le \gamma\alpha_j < 1$ since $0 \le \alpha_j < \gamma^{-1}$ for all $j = 2, \dots, m$. Therefore, by [18, Theorem 3.6] there exist unitaries $v_1, \dots, v_m, u_2, \dots, u_m$ in $\mathcal{J}$ such that $v_1 = u_o$ and $v_{k-1} + \gamma\alpha_k y = v_k + \gamma\alpha_k u_k$ for $k = 2, \dots, m$. We put $v_m = u_1$. Then (iii) becomes that $\gamma x = v_2 + \gamma\alpha_2 u_2 + \sum_{j=3}^m \gamma\alpha_j y = \gamma\alpha_2 u_2 + v_3 + \gamma\alpha_3 u_3 + \sum_{j=4}^m \gamma\alpha_j y = \cdots = u_1 + \sum_{j=2}^m \gamma\alpha_j u_j$. This gives the representation (i) of x.

Now, let $\delta > \gamma$ and m, n be integers given by $m - 2 < (1-\gamma^{-1})\delta \le m - 1$ and $n - 1 < \delta \le n$. Then $(1-\gamma^{-1})\delta + 1 = \delta + (1 - \gamma^{-1}\delta) < \delta$ so that $m \le n$. Moreover, by setting $\alpha_m = 1 - \gamma^{-1} - (m-2)\delta^{-1}$, we have $0 < \alpha_m \le \delta^{-1} < \gamma^{-1}$. Since $\alpha_m + \gamma^{-1} + (m-2)\delta^{-1} = 1$, by the first part of this proof we get the existence of unitaries $w_1, \dots, w_n$ in $\mathcal{J}$ such that

$$\sum_{j=1}^{n} \alpha_j u_j = x = \gamma^{-1} w_1 + \delta^{-1} \sum_{k=2}^{m-1} w_k + \alpha_m w_m + 0 w_{m+1} + \cdots + 0 w_n. \tag{iv}$$

For (ii), we only have to show that $\delta \in \mathcal{V}(x)$. For this, we must find unitaries $w'_1, \ldots, w'_n$ in $\mathcal{J}$ such that

$$\text{(v)} \qquad x = \delta^{-1} \sum_{j=1}^{n-1} w'_j + \delta^{-1}(\delta + 1 - n) w'_n = \sum_{j=1}^{n} \gamma_j w'_j$$

with $\gamma_1 = \cdots = \gamma_{n-1} = \delta^{-1}$ and $\gamma_n = \delta^{-1}(\delta+1-n)$; recall that $n-1 < \delta \leq n$. The existence of such unitaries $w'_1, \ldots, w'_n$ in $\mathcal{J}$ follows from [18, Theorem 5.5] if the n-tuple $(\gamma_1, \ldots, \gamma_n) \in co\{(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) : \pi \in S_n\}$, where S_n denotes the group of all permutations on the set $\{1, \ldots, n\}$; or equivalently, if $\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \alpha_j$ for all $k = 1, \ldots, n-1$ because $0 \leq \gamma_n \leq \gamma_{n-1} = \cdots = \gamma_1$ since $\delta \leq n$ by (see [18, Lemma 5.4]). However, by (iv) and (v), for $k < m$, $\sum_{j=1}^{k} \gamma_j = k\delta^{-1} \leq \gamma^{-1} + (k-1)\delta^{-1} = \sum_{j=1}^{k} \alpha_j$ since $\delta^{-1} < \gamma^{-1}$. Also, for $k \geq m$, $\sum_{j=1}^{k} \gamma_j \leq 1 = \sum_{j=1}^{k} \alpha_j$.

(b) We suppose $(\gamma, \infty) \subseteq \mathcal{V}(x)$. If $r > \gamma$ then $x \in co_r\mathcal{U}(\mathcal{J})$ so that $x = r^{-1}(u_1 + \cdots + u_{n-1} + (1+r-n)u_n)$ for some unitaries $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{J})$ and integer n with $n-1 < r \leq n$; hence, $\|rx - u_1\| \leq n-2+(1+r-n) = r-1$. □

Corollary 2.3. *For any unital JB^*-algebra $\mathcal{J}$, $co_\gamma\mathcal{U}(\mathcal{J}) \subseteq co_\delta\mathcal{U}(\mathcal{J})$ whenever $1 \leq \gamma \leq \delta$. In particular, for each $x \in (\mathcal{J})_1$, $\mathcal{V}(x)$ is either empty or equal to $[\gamma, \infty)$ or (γ, ∞) for some $\gamma \geq 1$.*

Proof. This follows directly from Theorem 2.2 and constructions of sets $co_\delta\mathcal{U}(\mathcal{J})$, $co_\gamma\mathcal{U}(\mathcal{J})$ and $\mathcal{V}(x)$. □

Next, we define another set $\mathcal{S}(x)$ and the λ_u-function:

Definition 2.4. Let $\mathcal{J}$ be a unital JB^*-algebra. For each $x \in (\mathcal{J})_1$, we define

$$\mathcal{S}(x) = \{0 \leq \lambda \leq 1 : x = \lambda v + (1-\lambda)y \ \ \textit{with} \ \ v \in \mathcal{U}(\mathcal{J}), y \in (\mathcal{J})_1\}.$$

and the λ_u-function by

$$\lambda_u(x) \ = \ \sup \mathcal{S}(x)$$

We observe some interesting relationships between the sets $\mathcal{S}(x)$ and $\mathcal{V}(x)$:

Theorem 2.5. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$.*

(i) *If $\lambda \in \mathcal{S}(x)$ and $\lambda > 0$ then $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$.*
(ii) *If $\delta \in \mathcal{V}(x)$ then $\delta^{-1} \in \mathcal{S}(x)$.*
(iii) *$\lambda_u(x) = 0$ if and only if $\mathcal{V}(x) = \emptyset$.*
(iv) *If $\lambda_u(x) > 0$ then $\mathcal{S}(x) = [0, \lambda_u(x))$ or $[0, \lambda_u(x)]$.*
(v) *If $\lambda_u(x) > 0$ and if $0 < \lambda < \lambda_u(x)$ then $\lambda^{-1} \in \mathcal{V}(x)$.*
(vi) *If $\lambda_u(x) > 0$ then $(\inf \mathcal{V}(x))^{-1} = \lambda_u(x)$.*
(vii) *If $\inf(\mathcal{V}(x)) \in \mathcal{V}(x)$ then $\lambda_u(x) \in \mathcal{S}(x)$.*

Proof. (i) This follows from Theorem 2.2.

(ii) Immediate from the definitions.

(iii) Suppose $\lambda_u(x) = 0$ and $\mathcal{V}(x) \neq \emptyset$. Then there is at least one (in fact, infinitely many by Corollary 2.3) $\delta \in \mathcal{V}(x)$, so $\delta^{-1} \in \mathcal{S}(x)$ by part (ii) Hence, $\lambda_u(x) \geq \delta^{-1} > 0$; a contradiction. Conversely, suppose $\mathcal{V}(x) = \emptyset$. If $\lambda_u(x) \neq 0$, there exists at least one $\lambda \in \mathcal{S}(x)$ with $\lambda > 0$. Hence, by part (i), $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$. In particular, $\mathcal{V}(x) \neq \emptyset$; a contradiction.

(iv) By definition, $0 \in \mathcal{S}(x)$. Let $\lambda \in (0, \lambda_u(x))$. By definition of $\lambda_u(x)$, there exists an increasing sequence (λ_n) in $\mathcal{S}(x)$ such that $\lim_{n\to\infty} \lambda_n = \lambda_u(x)$. As $\lambda < \lambda_u(x)$, there exists integer N such that when $n \geq N$, $\lambda_n > \lambda$. Hence, by part (i), $\lambda^{-1} \in (\lambda_n^{-1}, \infty) \subseteq \mathcal{V}(x)$ so $\lambda \in \mathcal{S}(x)$ by part (ii) So $[0, \lambda_u(x)) \subseteq \mathcal{S}(x) \subseteq [0, \lambda_u(x)]$ as $\lambda_u(x) = \sup \mathcal{S}(x)$. Thus $\mathcal{S}(x) = [0, \lambda_u(x))$ or $[0, \lambda_u(x)]$.

(v) Let $\lambda \in (0, \lambda_u(x))$. Let $\lambda_1 = \frac{1}{2}(\lambda + \lambda_u(x))$. Then $\lambda < \lambda_1 < \lambda_u(x)$, and so by part (vi), $\lambda_1 \in \mathcal{S}(x)$. Hence, $\lambda^{-1} \in (\lambda_1^{-1}, \infty) \subseteq \mathcal{V}(x)$.

(vi) Let $\lambda \in (0, \lambda_u(x))$. Then by part (v), $\lambda^{-1} \in \mathcal{V}(x)$ so $\lambda^{-1} \geq \inf \mathcal{V}(x)$. Therefore, $\lambda \leq (\inf \mathcal{V}(x))^{-1}$ and so $(\inf \mathcal{V}(x))^{-1}$ is an upper bound for $(0, \lambda_u(x))$. Hence, $\lambda_u(x) \leq (\inf \mathcal{V}(x))^{-1}$. Next, suppose $\lambda_u(x) < \delta < (\inf \mathcal{V}(x))^{-1}$. Then $\delta^{-1} > \inf \mathcal{V}(x)$ so $\delta^{-1} \in \mathcal{V}(x)$ by Corollary 2.3. Hence by part (ii), $\delta \in \mathcal{S}(x)$; a contradiction. Thus $\lambda_u(x) = (\inf \mathcal{V}(x))^{-1}$.

(vii) Follows immediately from the parts (iii), (vi) and (ii). □

Corollary 2.6. *For any $x \in (\mathcal{J})_1 \setminus \mathcal{J}_{\text{inv}}$, the following statements are equivalent:*

(i) $\operatorname{dist}(x, \mathcal{J}_{\text{inv}}) < 1 \Rightarrow \mathcal{V}(x) \neq \emptyset$.
(ii) $\lambda_u(x) = 0 \Rightarrow \operatorname{dist}(x, \mathcal{J}_{\text{inv}}) = 1$.
(iii) $\operatorname{dist}(x, \mathcal{J}_{\text{inv}}) < 1 \Rightarrow \lambda_u(x) > 0$.

Proof. Immediate from Theorem 2.5(iii). □

We now look for the elements with $\mathcal{V}(x) \cap [1, 2) \neq \emptyset$.

Theorem 2.7. *Let $0 \leq \alpha < \frac{1}{2}$. Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$. Then the following statements are equivalent:*

(i) $\operatorname{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$.
(ii) $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$.
(iii) $(1-\alpha)^{-1} \in \mathcal{V}(x)$.
(iv) $(1-\alpha) \in \mathcal{S}(x)$.

Proof. (i)⇒(ii): See the proof of [18, Corollary 4.4].

(ii)⇒(iii): If $x = \alpha u_1 + (1-\alpha)u_2$ where $u_1, u_2 \in \mathcal{U}(\mathcal{J})$ then as $\alpha < \frac{1}{2}$ we have $1 \leq (1-\alpha)^{-1} < 2$. Therefore, $x = (1-\alpha)u_2 + (1-\alpha)(1 + \frac{1}{1-\alpha} - 2)u_1$ so that $(1-\alpha)^{-1} \in \mathcal{V}(x)$.

(iii)⇒(iv): Clear from part (ii) of Theorem 2.5.

(iv)⇒(i): Suppose $x = (1-\alpha)u_1 + \alpha y_1$ where $u_1 \in \mathcal{U}(\mathcal{J})$ and $y_1 \in (\mathcal{J})_1$. Then $\|x - u_1\| = \alpha\|y_1 - u_1\| \leq 2\alpha$. Hence, $\operatorname{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\alpha$. □

Corollary 2.8. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in (\mathcal{J})_1$.*

(a) *The following statements are equivalent:*

(i) *x is invertible.*
(ii) *$x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$ for some $0 \le \alpha < \frac{1}{2}$.*
(iii) *$\operatorname{dist}(x,\mathcal{U}(\mathcal{J})) \le 2\alpha$ for some $0 \le \alpha < \frac{1}{2}$.*
(iv) *$(1-\alpha) \in \mathcal{S}(x)$ for some $0 \le \alpha < \frac{1}{2}$.*
(v) *$(1-\alpha)^{-1} \in \mathcal{V}(x)$ for some $0 \le \alpha < \frac{1}{2}$.*
(vi) *$\lambda \in \mathcal{V}(x)$ for some $1 \le \lambda < 2$.*

(b) *Moreover, if x is invertible then $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1}$ and $\mathcal{V}(x) = [2(1 + \|x^{-1}\|^{-1})^{-1}, \infty)$.*

Proof. (a) By Theorem 2.7, (ii)⇔(iii)⇔(iv)⇔(v) with the same α.

(i)⇒(ii): x being invertible is positive in some unitary isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$ by [15, Theorem 4.12], and $\sigma(x) \subseteq [-1,1] \setminus (2\alpha-1, 1-2\alpha)$ for some $\alpha > 0$ as $0 \notin \sigma(x)$. Then by [18, Lemma 3.4], $x \in \alpha\mathcal{U}(\mathcal{J}^{[u]}) + (1-\alpha)\mathcal{U}(\mathcal{J}^{[u]})$ and hence $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$ by [15, Theorem 4.6].

(ii)⇒(i): Follows from [15, lemmas 2.2(iii) and 4.2(ii)].

(v)⇔(vi): Follows from the fact that $0 \le \alpha < \frac{1}{2}$ iff $1 \le (1-\alpha)^{-1} < 2$.

(b) By [15, Theorem 14.2], x being invertible is positive invertible in certain unitary isotope $\mathcal{J}^{[u]}$. Hence by [18, Lemma 3.4] and [15, Theorem 4.2], for any $0 \le \beta < \frac{1}{2}$,

$$x \in \beta\mathcal{U}(\mathcal{J}) + (1-\beta)\mathcal{U}(\mathcal{J}) \quad \text{iff} \quad \inf \sigma_{\mathcal{J}^{[u]}}(x) \ge 1 - 2\beta\ .$$

By [15, Lemma 4.2(iii)], $x^{-1_u} = \{ux^{-1}u\}$. Since $x^{-1} = \{u^*\{ux^{-1}u\}u^*\}$ and $\|x^{-1}\| = \|\{u^*\{ux^{-1}u\}u^*\}\| \le \|\{ux^{-1}u\}\| \le \|x^{-1}\|$, it follows that $\|x^{-1_u}\| = \|x^{-1}\|$. Thus, by the functional calculus for positive elements,

$$\inf \sigma_{\mathcal{J}^{[u]}}(x) \;=\; \|x^{-1_u}\|^{-1} \;=\; \|x^{-1}\|^{-1}.$$

Therefore, $x \in \beta\mathcal{U}(\mathcal{J}) + (1-\beta)\mathcal{U}(\mathcal{J})$ if and only if $\|x^{-1}\|^{-1} \ge 1 - 2\beta$. However, by part (a) (as x is invertible) there exists $\lambda \in \mathcal{V}(x)$ with $1 \le \lambda < 2$ so that $(1-\lambda^{-1}) \in \mathcal{V}(x)$ with $0 < \lambda^{-1} < \frac{1}{2}$, hence by Theorem 2.7 we get $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$ with $\alpha = 1 - \lambda^{-1}$. It follows that $\|x^{-1}\|^{-1} \ge 1 - 2\alpha = 2\lambda^{-1} - 1$. Hence $\inf \mathcal{V}(x) \ge 2(1 + \|x^{-1}\|^{-1})^{-1}$.

Since x is positive in $\mathcal{J}^{[u]}$, setting $\alpha = 1 - \frac{1}{2}(1 + \inf \sigma_{\mathcal{J}^{[u]}}(x))$ we have $0 \le \alpha < \frac{1}{2}$ and $\inf \sigma_{\mathcal{J}^{[u]}}(x) = 1 - 2\alpha$ so that $\sigma_{\mathcal{J}^{[u]}}(x) \subseteq [1-2\alpha, 1]$. Hence by [18, Lemma 3.4] and [15, Theorem 4.6], $x \in \alpha\mathcal{U}(\mathcal{J}) + (1-\alpha)\mathcal{U}(\mathcal{J})$. Thus, by Theorem 2.7, $2(1 + \|x^{-1}\|^{-1})^{-1} = (1-\alpha)^{-1} \in \mathcal{V}(x)$. □

Corollary 2.9. *Let $\mathcal{J}$ be a unital JB^*-algebra and for $1 \le \delta < 2$, the set $K = \{x \in (\mathcal{J})_1 \cap \mathcal{J}_{\text{inv}} : \|x^{-1}\| \le (2\delta^{-1} - 1)^{-1}\}$. Then $co_\delta\mathcal{U}(\mathcal{J}) = K$ and K is closed.*

Proof. The first part is immediate from Corollary 2.8. For the other part, let $\{y_n\}$ be a sequence in K which converges to $y \in \mathcal{J}$. Clearly, $y \in (\mathcal{J})_1$. By part (ai), $\delta \in \mathcal{V}(y_n)$ for all n. Let $\alpha = 1 - \delta^{-1}$. Then $0 \le \alpha < \frac{1}{2}$ and $\operatorname{dist}(y_n,\mathcal{U}(\mathcal{J})) \le 2\alpha$ for all n by Corollary 2.8(a). Hence by the continuity of the distance, we get $\operatorname{dist}(y,\mathcal{U}(\mathcal{J})) \le 2\alpha$. Therefore, y is invertible and hence $\|y^{-1}\| = \lim_{n\to\infty} \|y_n^{-1}\| \le (2\delta^{-1} - 1)^{-1}$. □

Corollary 2.10. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in (\mathcal{J})_1$ be invertible. Then $\lambda_u(x) = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ and there exist unitaries u_1, u_2 in $\mathcal{U}(\mathcal{J})$ such that $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$.*

Proof. By Corollary 2.8(b), $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1} \in \mathcal{V}(x)$. So, by Theorem 2.5, $\lambda_u(x) = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ and $\frac{1}{2} < \lambda_u(x) \leq 1$. Thus, $0 \leq 1 - \lambda_u(x) < \frac{1}{2}$, so there exist unitaries u_1, u_2 in $\mathcal{U}(\mathcal{J})$ such that $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$, by Theorem 2.7. □

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