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On the Beurling–Lax theorem for domains with one hole

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Abstract. We consider pure subnormal operators T of the type studied in Carlsson, 2011, with the additional requirement that $\sigma(T)$ has one hole. If $\mathsf{ind}\,(T-\lambda_0)=-n$ for some λ_0 and $n\in\mathbb{N}$, we show that the operator can be decomposed as $T=\oplus_{k=1}^{n}T_k$, where each T_k satisfies $\mathsf{ind}\,(T-\lambda_0)=-1$, thus extending the classical Beurling–Lax theorem (in which $\sigma(T)$ is the unit disc). We also provide a set of unitary invariants that completely characterize T and study the model spaces for the simpler operators T_k.

Contents

1. Introduction

Let $\mathbb{T}$ denote the unit circle, let m be the arc-length measure on $\mathbb{T}$ and let $z(\zeta)=\zeta$ be the identity function on $\mathbb{C}$. We let M_z denote the operator of multiplication by z, (independent of which space we are in). Beurling's classical theorem on M_z-invariant subspaces of the Hardy space H^2 can easily be transformed into a statement about all M_z-invariant subspaces of $L^2(m)$. Let $\mathcal{H}$ be such a subspace, then either

$$\mathcal{H}=L^2_E(m)=\{f\in L^2(m):\mathsf{supp}f\subset E\}$$

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for some closed set $E \subset \mathbb{T}$, or there exists a unimodular function ϕ such that

$$\mathcal{H} = \phi H^2,$$

where we identify H^2 with a subspace of L^2 in the standard way. This dichotomy does not happen if we consider $L^2(m, l^2)$; the Hilbert space of l^2-valued measurable functions on $\mathbb{T}$ such that $\int \|f\|_{l^2}^2 dm < \infty$. Just consider $\mathcal{H} = H^2 \oplus L^2 \oplus \cdots$. P. Lax's extension of Beurling's theorem says that any M_z-invariant subspace $\mathcal{H} \subset L^2(m, l^2)$ that does not contain a reducing subspace, is of the form

$$\mathcal{H} = \Phi\left(\oplus_{k=1}^{\infty} H^2\right)$$

where Φ is a function on $\mathbb{T}$ whose value $\Phi(\xi)$ is an isometric operator at each $\xi \in \mathbb{T}$. On the other hand, if $\mathcal{H}$ is reducing then by multiplicity theory (see, e.g., [3]) there exists closed sets $E_1 \supset E_2 \supset E_3$ in $\mathbb{T}$ such that

$$\mathcal{H} = \Phi\left(\oplus_{k=1}^{\infty} L_E^2(m)\right),$$

where again Φ is an isometry at each point. Combined, this provides a characterization of all M_z-invariant subspaces of $L^2(m, l^2)$, because if $\mathcal{H}$ is such a subspace and $\mathcal{H}_1$ is the largest reducing subspace, then $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$ is M_z-invariant with no reducing subspace, (see [2]).

To have a characterization of all M_z-invariant subspaces of $L^2(m, l^2)$ is the same as understanding all subnormal operators T whose minimal normal extensions N have spectral measures whose supports are contained in $\mathbb{T}$. To see this, first note that by multiplicity theory, applying a unitary transformation we can consider N as M_z on a reducing subspace of $L^2(\mu, l^2)$, where μ is the scalar-valued spectral measure for the minimal normal extension of N, (see [4]), and $L^2(\mu, l^2)$ is the set of all l^2-valued functions such that $\int \|f(e^{it})\|_{l^2}^2 d\mu < \infty$. It is not hard to see that if μ has a singular part, then there is a reducing subspace related to that part that can easily be removed from the analysis (see [2]). But if $d\mu = w\ dm$ where $w \in L^1(\mathbb{T})$, then the map $M_{\sqrt{w}} : l^2(m, l^2) \to L^2(\mu, l^2)$ is unitary, and hence we can move the analysis to the unweighted case.

Using the Riemann mapping theorem and the tools developed in complex analysis throughout the 20th century, it is possible to move the above conclusions from $\mathbb{T}$ to any Jordan curve Γ. That is, we can characterize all M_z-invariant subspaces of $L^2(\mu, l^2)$, where μ is some measure with support on Γ. However, if Γ is a union of finitely many Jordan curves, then this will not work and we need to find new ways of analyzing M_z-invariant subspaces. This is the topic of [2], which can be seen as a predecessor to the present note. We will in this paper use results and notation from that paper without reintroduction. In particular we shall assume that the subnormal operator under investigation is pure and that $\mu = m$, where m is the arc-length measure of Γ. The main result in [2] roughly says that if $\mathcal{H} \subset L^2(m, l^2)$ is M_z-invariant and some conditions apply, $T = M_z|_{\mathcal{H}}$ is a pure operator and $\mathsf{ind}\, T - \lambda_0 = -n$ for some λ_0, then there exists $f_1, \ldots, f_n \in \mathcal{H}$ such that

$\mathcal{H} = [f_1, \dots, f_n]_{R(\sigma(T))}$. Moreover the multiplicity function of the minimal normal extension of T equals n m-a.e. on Γ. Assume now that $n < \infty$. This result reduces the study to all M_z-invariant subspaces of $L^2(m, \mathbb{C}^n)$ such that $L^2(\mu, \mathbb{C}^n) = [\mathcal{H}]_{L^\infty}$, where $[\mathcal{H}]_{L^\infty}$ is the smallest reducing subspace containing $\mathcal{H}$. Although this is a very nice result, the geometry of $\mathcal{H}$ is still far from clear. Note that Lax's result says much more; there is a unitary transformation Φ that commutes with M_z such that $\Phi^{-1}(\mathcal{H}) = \oplus_{k=1}^n H^2$, and the corresponding subnormal operator T thus transforms into n copies of the shift operator.

We will prove that a Beurling–Lax type theorem holds if $\sigma(T)$ has precisely one hole, or more precisely, if $\Gamma = \partial(\sigma(T))$ consists of 2 simple disjoint rectifiable nontrivial Jordan curves. This also follows by the results of Abrahamse and Douglas, [1], but their proofs rely on the universal cover, Forelli's and Grauert's theorems. In this paper, we construct a direct proof using function theory on $\mathcal{H}$ itself. Furthermore, we find a set of unitary invariants that completely describe the operator. More precisely, let Ω be the connected domain whose boundary is Γ, fix a point $\lambda_0 \in \Omega$ and let $\omega_{\lambda_0} dm$ denote the harmonic measure for λ_0. Recall that $R(\overline{\Omega})$ denotes the set of rational functions with poles off $\overline{\Omega}$. Also let ν be such that νdm annihilates $\mathsf{Re}(R(\overline{\Omega}))$, ($\nu$ is unique up to multiplication by constants). Let $[a_{\min}, a_{\max}]$ be the interval such that $\omega_{\lambda_0} + a\nu$ is a positive function a.e. Given $a \in [a_{\min}, a_{\max}]$ we define $G^{a,0} \subset L^2(m, \mathbb{C})$ by

$$G^{a,0} = \left[\sqrt{\omega_{\lambda_0} + a\nu}\right]_{R(\overline{\Omega})}.$$

If $a \in (a_{\min}, a_{\max})$ we also define the space $G^{a,1}$ via

$$G^{a,1} = \left[\left\{\sqrt{\omega_{\lambda_0} + a\nu}, \frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}}\right\}\right]_{R(\overline{\Omega})}.$$

We show that any pure subnormal operator T with $\mathsf{ind}\,(T - \lambda_0) = -1$ and $\sigma(T) = \overline{\Omega}$ whose minimal normal extension has spectrum on $\partial\Omega$, is unitarily equivalent with $M_z|_{G^{a,b}}$ for precisely one value of a and b. Moreover, and this is the key part, if instead the operator T has finite index $-n$, we show that there are $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that

$$\mathcal{H} = \Phi(\oplus_{k=1}^n G^{a_k, b_k}),$$

where Φ is an operator valued function such that $\Phi(\xi)$ is an isometry for each $\xi \in \Gamma$, (Theorem 5.1). Stated differently, we have that T is unitarily equivalent with M_z on $\oplus_{k=1}^n G^{a_k, b_k}$, and hence it is "orthogonally decomposed" into a sum of n operators with index -1. Moreover, given a concrete operator T one can actually calculate the a_k's and b_k's, as well as Φ. Although $M_z|_{G^{a,b}}$ is not as well understood as the shift operator, this is still a substantial simplification. We also show that the corresponding result for domains with more holes or with $n = \infty$ is not true. Moreover we investigate the spaces $G^{a,b}$ to some extent, and show for example that $G^{a,0}$ has codimension 1 in $G^{a,1}$. The paper ends with an example (Example 5.2), which the reader might be interested in looking at sooner to make the material more concrete.

2. Preliminaries

Assumption *T*. In the entire paper, unless explicitly stated, $\Omega \subset \mathbb{C}$ will be a bounded connected open set whose boundary consists of 2 disjoint simple closed rectifiable nontrivial Jordan curves, and m will denote the arc-length measure of $\partial\Omega$, as defined in Section 2.1, [2]. Moreover, $\mathcal{H}$ will be an M_z-invariant subspace of $L^2(m, l^2)$ such that $T = M_z|_{\mathcal{H}}$ is pure and $\sigma(T) = \overline{\Omega}$. Finally, λ_0 will be a fixed point in Ω and $\mathcal{K}_{\lambda_0} = \mathcal{H} \ominus \mathsf{Ran}(T - \lambda_0)$.

In this context, the following theorem due to Mlak is essential, [7].

Theorem 2.1. *Let T_1 and T_2 be subnormal operators as above, with minimal normal extensions N_1 and N_2 respectively. Let U be a unitary map such that $UT_1 = T_2U$. Then U has a unique extension such that $UN_1 = N_2U$.*

The result is also valid for domains with several holes, but was not needed in the predecessor [2] to this paper. Here, it will be useful due to the following observation, which is obtained by combining Theorem 2.1 with Corollary 2.5 in [2]: Denote the spaces corresponding to T_1 and T_2 by $\mathcal{H}_1$ and $\mathcal{H}_2$ and suppose that $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary such that $T_2U = UT_1$. Let $P_{[\mathcal{H}_1]_{L^\infty}}$ be the projection-valued function that defines $[\mathcal{H}_1]_{L^\infty}$ via Proposition 2.4 in [2]. Then there exists a $\mathcal{B}(l^2)$-valued SOT-measurable function u such that

$$Uf(\xi) = u(\xi)f(\xi), \quad \forall \xi \in \partial\Omega,\ f \in \mathcal{H}_1 \tag{2.1}$$

and

$$(u(\xi))^* u(\xi) = P_{[\mathcal{H}_1]_{L^\infty}}.$$

In particular, given $f, g \in \mathcal{H}_1$ we have

$$\langle Uf(\xi), Ug(\xi)\rangle_{l^2} = \langle f(\xi), g(\xi)\rangle_{l^2}, \quad \forall \xi \in \partial\Omega, \tag{2.2}$$

or, in other words, the function $f \cdot \overline{g}$ is a unitary invariant. Moreover,

$$U\big([f]_{R(\overline{\Omega})}\big) = [Uf]_{R(\overline{\Omega})}. \tag{2.3}$$

Before moving to the central theme, we recall some results from harmonic analysis. It is well known that the Dirichlet problem is solvable in Ω, i.e., each $f \in C(\partial\Omega)$ is the boundary function of a harmonic function in Ω, see, e.g., [5]. Denote the harmonic extension of f by $\tilde{f}$. Using Theorem 2.3 in [2], it is easily seen that all harmonic measures for Ω are absolutely continuous with respect to m. Given $\lambda \in \Omega$ we let ω_λ denote the function such that $\omega_\lambda m$ is the harmonic measure for λ, i.e.,

$$\int f \omega_\lambda dm = \tilde{f}(\lambda)$$

for all $f \in C(\partial\Omega)$. From [5] we get the following propositions. (See Propositions 1.6.5 and 1.6.6).

Proposition 2.2. *If $\partial\Omega$ consists of analytic curves, then ω_λ is a strictly positive C^∞-function for all $\lambda \in \Omega$.*

Let $\mathsf{Re}(R(\overline{\Omega}))$ be the set of functions $\{r + \overline{r} : r \in R(\overline{\Omega})\}$ seen as a subspace of $C(\partial\Omega)$. The following proposition gathers a few facts about its annihilator.

Proposition 2.3. *The set of annihilating measures for $\mathsf{Re}(R(\overline{\Omega}))$ is a 1-dimensional space (over $\mathbb{R}$) of m-absolutely continuous measures. Let ν be a function such that $\nu m \perp \mathsf{Re}(R(\overline{\Omega}))$. If $\partial\Omega$ consists of analytic curves, then ν is a C^∞-function.*

Proof. The statement about dimension is Theorem 4.6, ch. VI in [6], and the absolute continuity follows easily from Theorem 2.3 in [2]. The final statement follows from Theorem 4.2.3 in [5]. □

We end this section with a few technical results that will be useful later. When dealing with two domains Ω and $\tilde{\Omega}$, we will automatically denote, e.g., arc-length measure for $\partial\tilde{\Omega}$ by $\tilde{m}$.

Proposition 2.4. *Given Ω, there exists another domain $\tilde{\Omega}$ whose boundary consists of analytic curves, and an analytic bijection $\phi : \tilde{\Omega} \to \Omega$ with the property that ϕ' has nonzero nontangential limits $\tilde{m}$-a.e.. Moreover, ϕ extends to a continuous function on $\overline{\tilde{\Omega}}$,*

$$\int f dm = \int (f \circ \phi)|\phi'| d\tilde{m} \tag{2.4}$$

for all $f \in C(\partial\Omega)$, $\tilde{\omega}_{\phi^{-1}(\lambda_0)} = (\omega_{\lambda_0} \circ \phi)|\phi'|$ and $\tilde{\nu} = (\nu \circ \phi)|\phi'|$.

Proof. We only outline the details. Let $\hat{\Omega}$ be the polynomially convex hull of Ω, i.e., the domain obtained by "filling in the hole of Ω". Let $\psi : \mathbb{D} \to \hat{\Omega}$ be given by the Riemann mapping theorem and set $\tilde{\Omega} = \psi^{-1}(\Omega)$. The formula (2.4) is easily established using Theorem 2.2 in [2], but the inner boundary of $\tilde{\Omega}$ is still not analytic. Take a point λ in the "hole", apply $1/(z - \lambda)$ to $\tilde{\Omega}$ and repeat the above construction for the new domain. Then apply the inverse of $1/(z-\lambda)$ and the function ϕ is easily obtained. The properties of ϕ are easily verified (using Theorem 2.2 of [2]) and the statements concerning ω_{λ_0} and ν are easy consequences of these properties, Theorem 2.3 in [2] and formula (2.4). □

Let $a_{\max}(\lambda_0) \in \mathbb{R}$ be the largest number such that $\omega_{\lambda_0} + a_{\max}\nu$ is a positive function and let $a_{\min}(\lambda_0)$ be the smallest such number.

Proposition 2.5. *If $a \notin (a_{\min}(\lambda_0), a_{\max}(\lambda_0))$ then*

$$\int \Big| \frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} \Big|^2 dm = \infty.$$

Proof. We may assume that Ω has analytic boundaries because (with the same notation as in the previous proof) we have

$$\begin{aligned}\int_{\partial\Omega}\left|\frac{\nu}{\sqrt{(\omega_{\lambda_0}+a\nu)}}\right|^2 dm &= \int_{\partial\tilde{\Omega}}\left|\frac{\nu\circ\phi}{\sqrt{(\omega_{\lambda_0}\circ\phi+a\nu\circ\phi)}}\right|^2 |\phi'|d\tilde{m}\\ &= \int_{\partial\tilde{\Omega}}\left|\frac{\nu\circ\phi\ |\phi'|}{\sqrt{(\omega_{\lambda_0}\circ\phi+a\nu\circ\phi)|\phi'|}}\right|^2 d\tilde{m}\\ &= \int_{\partial\tilde{\Omega}}\left|\frac{\tilde{\nu}}{\sqrt{\tilde{\omega}_{\phi_0^{-1}(\lambda_0)}+a\tilde{\nu}}}\right|^2 d\tilde{m}.\end{aligned}$$

By Propositions 2.2 and 2.3 we get that both ω_{λ_0} and ν are C^∞-functions and moreover ω_{λ_0} is strictly positive. Clearly then $\omega_{\lambda_0}(\xi)+a\nu(\xi)=0$ for some $\xi\in\partial\Omega$ with $\nu(\xi)\neq 0$. Let γ be an analytic function defined in a neighborhood of an interval $[-T,T]$ in $\mathbb{C}$ such that $\gamma([-T,T])$ coincides with a piece of $\partial\Omega$ and $\gamma(0)=\xi$. By the construction in Proposition 2.4 it follows that we may also assume that $|\gamma'(t)|\neq 0$ for all $t\in[-T,T]$. Let C be an upper bound in $[-T,T]$ of $\big(\omega_{\lambda_0}(\gamma(t))+a\nu(\gamma(t))\big)/t$. Then

$$\begin{aligned}\int\left|\frac{\nu}{\sqrt{\omega_{\lambda_0}+a\nu}}\right|^2 dm &\geq \int_{-T}^{T}\frac{\nu^2(\gamma(t))}{\omega_{\lambda_0}(\gamma(t))+a\nu(\gamma(t))}|\gamma'(t)|dt\\ &\geq \int_{-T}^{T}\frac{\nu^2(\gamma(t))}{Ct}|\gamma'(t)|dt=\infty.\end{aligned}$$ □

3. The orthogonal decomposition

Definition 3.1. Two vectors $f_1,f_2\in L^2(\mu,l^2)$ are said to be completely orthogonal if $f_1(\zeta)\perp f_2(\zeta)$ holds at μ-a.e. ζ.

It is easily seen that f_1 is completely orthogonal to f_2 if and only if the subspaces $[f_1]_{L^\infty(\partial\Omega)}$ and $[f_2]_{L^\infty(\partial\Omega)}$ are orthogonal.

Theorem 3.2. *Let T be as in Assumption T and assume that $n=\mathsf{dim}\,\mathcal{K}_{\lambda_0}<\infty$. Then there exists a completely orthogonal basis $\{k_i:1\leq i\leq n\}$ for $\mathcal{K}_{\lambda_0}$.*

Proof. Let $\{k_i':1\leq i\leq n\}$ be any orthonormal basis for $\mathcal{K}_{\lambda_0}$. Let $r\in R(\overline{\Omega})$ be arbitrary and note that

$$\langle rk_i',k_j'\rangle=\langle r(\lambda_0)k_i',k_j'\rangle+\langle(z-\lambda_0)\frac{r(z)-r(\lambda_0)}{z-\lambda_0}k_i',k_j'\rangle=0$$

if $i\neq j$ and otherwise

$$\langle rk_i',k_i'\rangle=\ldots=r(\lambda_0).$$

Since $\mathsf{dim}\big(C(\partial\Omega)/\mathsf{Re}(R(\overline{\Omega}))\big) = 1$ by Proposition 2.3, it follows that there is an $n \times n$ self-adjoint matrix $A = (a_{i,j})$ such that

$$\begin{pmatrix} k_1' \cdot \overline{k_1'}(\cdot) & \dots & k_1' \cdot \overline{k_n'}(\cdot) \\ \vdots & \ddots & \vdots \\ k_n' \cdot \overline{k_1'}(\cdot) & \dots & k_n' \cdot \overline{k_n'}(\cdot) \end{pmatrix} = \omega_{\lambda_0}(\cdot)I + A\nu(\cdot), \tag{3.1}$$

where I denotes the $n \times n$ identity matrix and $k_i' \cdot \overline{k_j'}(\cdot) = \langle k_i'(\cdot), k_j'(\cdot)\rangle_{l^2}$. We will use the notation $\boxed{k_i'}$ for the "$\infty \times n$-matrix" with the function $k_i'(\cdot)$ as its i:th column. Then the above equation can be written shorter as

$$\boxed{k_i'}^*\boxed{k_i'} = \omega_{\lambda_0} I + A\nu.$$

Now, if $\{k_i : 1 \le i \le n\}$ is another orthonormal basis for $\mathcal{K}_{\lambda_0}$, then there is a unitary $n \times n$ matrix U (with complex entries) such that

$$\boxed{k_i} = \boxed{k_i'}U.$$

This yields

$$\boxed{k_i}^*\boxed{k_i} = U^*\boxed{k_i'}^*\boxed{k_i'}U = \omega_{\lambda_0} I + U^*AU\nu. \tag{3.2}$$

As the matrix A is self-adjoint, there exists a U such that U^*AU is a diagonal matrix D, which completes the proof. $\square$

The above proof can also be used backwards to see why the theorem is not valid for $n = \infty$ or for domains with several holes. We provide 2 examples.

Example 3.3. Let Ω be the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 1\}$, let $\lambda_0 \in \Omega$ be fixed, and let $\{e_j\}_{j=-\infty}^{\infty}$ be the standard basis for $l^2(\mathbb{Z})$. By Propositions 2.2 and 2.3, we can pick an $\epsilon > 0$ such that

$$\omega_{\lambda_0} > 2\epsilon\nu.$$

With $\alpha = \sqrt{\frac{\omega_{\lambda_0}}{2} + \sqrt{\frac{\omega_{\lambda_0}^2}{4} - \epsilon^2\nu^2}}$ and $\beta = \epsilon\nu/\alpha$, it is easily seen that

$$\begin{cases} |\alpha|^2 + |\beta|^2 = \omega_{\lambda_0} \\ \alpha\overline{\beta} = \epsilon\nu. \end{cases}$$

Thus, setting $k_i = \alpha e_i + \beta e_{i+1}$ for all $i \in \mathbb{Z}$ we get that $\{k_i\}_{i=-\infty}^{\infty}$ is an orthogonal set in $L^2(m, l^2)$. Put $\mathcal{H} = [\{k_i\}]_{R(\overline{\Omega})}$ and $T = M_z|_{\mathcal{H}}$. If $r_i \in R(\overline{\Omega})$ then

$$\begin{aligned} &\left\langle (z-\lambda_0)\sum_i r_i k_i, k_j \right\rangle \\ &\qquad = \int (\xi - \lambda_0)\big(r_{j-1} + r_{j+1}\big)(\xi)\epsilon\nu(\xi) + (\xi - \lambda_0)r_j(\xi)\omega_{\lambda_0}(\xi)dm(\xi) = 0 \end{aligned}$$

so, setting $\mathcal{K}_{\lambda_0} = \mathsf{cl}(\mathsf{Span}\{k_i\})$ it is easily seen that $\mathcal{K}_{\lambda_0} = \mathcal{H} \ominus \mathsf{Ran}(T-\lambda_0)$. Moreover, T is pure. To see this, note that by Proposition 3.1 and Corollary 2.5 in [2], we otherwise have a nonvanishing $f \in \mathcal{H}$ such that

$$\phi f \in \cap_{k\in\mathbb{N}} \mathsf{Ran}(T-\lambda_0)^k$$

for all $\phi \in L^\infty(m,\mathbb{C})$. But this implies that $f \cdot k_i(\xi) = 0$ for a.e. $\xi \in \partial\Omega$, which yields $f(\xi) = c(\xi)\big((-\alpha(\xi)/\beta(\xi))^k\big)_{k=-\infty}^{\infty}$ for some function c. Since the norm of the righthand side is ∞ we conclude that $c(\xi)m = 0$ a.e., which is a contradiction. Finally, that $\sigma(T) = \overline{\Omega}$ follows easily from the fact that $\lambda_0 \in \sigma(T)$ and Proposition 1.2 in [2].

Hence we are in the situation of Theorem 3.2, but with $\dim \mathcal{K}_{\lambda_0} = \infty$. With the obvious modifications of the notation in Theorem 3.2, we have that $\boxed{k_i(\xi)}$ is a bounded operator on l^2 for all $\xi \in \partial\Omega$. Moreover,

$$\boxed{k_i'}^* \boxed{k_i'} = \omega_{\lambda_0} I + A\nu$$

where A is given by the matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \\ \epsilon & 0 & \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon & 0 & \epsilon \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

or in other words, $A = \epsilon(S+S^*)$ where S is the bilateral forward shift. Now, if $\mathcal{K}_{\lambda_0}$ contains a basis of completely orthogonal vectors, then as in the proof of Theorem 3.2 we obtain that there exists a unitary operator U on $l^2(\mathbb{Z})$ such that U^*AU is represented by a diagonal operator, (in the standard basis). In particular, the point spectrum of A is not empty. However, this is a contradiction, because $L^2(\mathbb{T})$ and $l^2(\mathbb{Z})$ are unitarily equivalent under the Fourier transform $\mathcal{F}$, and A transforms into

$$\mathcal{F}^* A \mathcal{F} = M_{\epsilon(z+\overline{z})} = M_{2\epsilon\cos t}, \quad (z = e^{it}),$$

which clearly has an empty point spectrum.

Thus, there exists no basis of completely orthogonal vectors for $\mathcal{K}_{\lambda_0}$. But this statement easily generalizes to that there is no set of completely orthogonal generating vectors. For suppose that $\mathcal{H} = \oplus_{i=-\infty}^{\infty}[f_i]_{R(\overline{\Omega})}$ for some f_i's in $\mathcal{H}$. Then, by Theorem 4.1 [2], each subspace $[f_i]_{R(\overline{\Omega})} \ominus (z-\lambda_0)[f_i]_{R(\overline{\Omega})}$ would be spanned by precisely one vector k_i, and it easily follows that $\mathsf{Span}\{k_i\}_{i=-\infty}^{\infty} = \mathcal{H} \ominus \mathsf{Ran}(T-\lambda_0)$.

In a similar way, we now demonstrate why Theorem 3.2 is not valid for domains with several holes.

Example 3.4. Let Ω be a disc with two disjoint smaller discs removed from its interior and let m be the corresponding arc-length measure. Then, by a more general version of Proposition 2.3, $\mathsf{Re}(R(\overline{\Omega}))$ is annihilated by precisely two linearly independent C^∞-functions ν_1 and ν_2, say. Let $n \in \mathbb{N}$ be given and let $A_1,\ A_2$ be $n \times n$ self-adjoint matrices that are not mutually diagonalizable. It is not hard to see that we can pick functions $k_1, \ldots, k_n \in L^2(m, \mathbb{C}^n)$ such that

$$\boxed{k_i'}^* \boxed{k_i'} = \omega_{\lambda_0} I + \epsilon_1 A_1 \nu_1 + \epsilon_2 A_2 \nu_2,$$

if $\epsilon_1, \epsilon_2 > 0$ are chosen small enough. If a completely orthogonal basis of $\mathcal{K}_{\lambda_0}$ was to exist, then by repetition of the arguments in the proof of Proposition 3.2, there would be a unitary matrix U such that

$$\boxed{Uk_i'}^* \boxed{Uk_i'} = \omega_{\lambda_0} I + \epsilon_1 U^* A_1 U \nu_1 + \epsilon_2 U^* A_2 U \nu_2$$

is diagonal a.e., which clearly contradicts the choice of A_1 and A_2.

Due to Theorem 3.2 and the above examples, we will for the remainder assume that T is an operator as defined in Assumption T, and moreover that $-\mathsf{ind}\,(T - \lambda) = n < \infty$ for all $\lambda \in \Omega$. Our goal is to classify all such operators T up to unitary equivalence and study the typical "model spaces". More precisely, we want to associate with T a set of numbers such that another operator (of the same type) is unitarily equivalent to T if and only if the two sets of numbers coincide. It is easy to see that the numbers $\{a_1, \ldots, a_n\}$ on the diagonal of the matrix D following equation (3.2) are unitarily invariant and uniquely determined by T. (To see this, recall equations (2.2), (3.2) and the fact that $a_1, \ldots, a_n$ are the eigenvalues of A.) It will turn out that there are numbers $\{b_1, \ldots, b_n\} \subset \{0, 1\}$ such that the set $\{(a_1, b_1), \ldots, (a_n, b_n))\}$ completely determine T up to unitary equivalence. To show this, we will first introduce the model subspaces of $L^2(m, \mathbb{C})$.

4. The spaces $G^{a,b}$

As before, let λ_0 be a fixed number in Ω. Given $a \in [a_{\min}(\lambda_0), a_{\max}(\lambda_0)]$, let $G^{a,0}(\Omega, \lambda_0)$ be the subset of $L^2(m, \mathbb{C})$ defined by

$$G^{a,0}(\Omega, \lambda_0) = \left[\sqrt{\omega_{\lambda_0} + a\nu}\right]_{R(\overline{\Omega})}.$$

If $a \in (a_{\min}(\lambda_0), a_{\max}(\lambda_0))$ we also define the space $G^{a,1}(\Omega, \lambda_0)$ via

$$G^{a,1}(\Omega, \lambda_0) = \left[\left\{\sqrt{\omega_{\lambda_0} + a\nu}, \frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}}\right\}\right]_{R(\overline{\Omega})}.$$

We will most of the time keep Ω and the point λ_0 fixed, and hence omit it from the notation. Note that Proposition 2.5 explains why we do not define $G^{(a,1)}$ for $a = a_{\min}$ or $a = a_{\max}$. Whenever speaking of $G^{a,b}$, it will be implicitly assumed that (a, b) are such that it is well defined, i.e., $a \in [a_{\min}, a_{\max}]$, $b \in \{0, 1\}$ and $b = 0$ if $a = a_{\min}$ or $a = a_{\max}$.

Theorem 4.1. *The operator $T = M_z|_{G^{a,b}}$ is pure with $\sigma(T) = \overline{\Omega}$ and* $\mathsf{ind}\,(T-\lambda) = -1$ *for all $\lambda \in \Omega$. Moreover*

$$G^{a,1} = G^{a,0} \oplus \mathsf{Span}\{\nu/\sqrt{\omega_{\lambda_0} + a\nu}\}$$

whenever $a \in (a_{\min}, a_{\max})$.

Proof. Consider the subspace $\mathcal{G} \subset L^2(m,\mathbb{C})$ defined by

$$\begin{aligned}\mathcal{G} &= \overline{(z-\lambda_0)}\Big[\sqrt{\omega_{\lambda_0} + a\nu}\Big]_{\overline{R(\overline{\Omega})}} \\ &= \mathsf{cl}\Big(\mathsf{Span}\big\{\overline{(z-\lambda_0)r}\sqrt{\omega_{\lambda_0} + a\nu} : r \in R(\overline{\Omega})\big\}\Big).\end{aligned}$$

Assume first that $a \in (a_{\min}, a_{\max})$ and note that $\mathcal{G} \perp G^{a,1}$. If $f \in L^2(m,\mathbb{C})$ with $f \perp (\mathcal{G} + G^{a,0})$, then for any $r \in R(\overline{\Omega})$ we have $\langle f, r\sqrt{\omega_{\lambda_0} + a\nu}\rangle = 0$ and $\langle f, \overline{r}\sqrt{\omega_{\lambda_0} + a\nu}\rangle = 0$. To see the latter, write $r = r(\lambda_0) + (z-\lambda_0)r_1$ with $r_1 \in R(\overline{\Omega})$. In other words $f\sqrt{\omega_{\lambda_0} + a\nu} \in \mathsf{Re}R(\overline{\Omega})^{\perp}$, and hence $f\sqrt{\omega_{\lambda_0} + a\nu} = c\nu$ for some $c \in \mathbb{C}$. This implies that

$$G^{a,0} \oplus \mathsf{Span}\big\{\nu/\sqrt{\omega_{\lambda_0} + a\nu}\big\} \oplus \mathcal{G} = L^2(m,\mathbb{C}), \tag{4.1}$$

so

$$\mathcal{G}^{\perp} = \Big(G^{a,0} \oplus \mathsf{Span}\big\{\nu/\sqrt{\omega_{\lambda_0} + a\nu}\big\}\Big) \subset G^{a,1} \subset \mathcal{G}^{\perp}$$

and hence

$$G^{a,1} = G^{a,0} \oplus \mathsf{Span}\big\{\nu/\sqrt{\omega_{\lambda_0} + a\nu}\big\}.$$

If $a = a_{\min}$ or $a = a_{\max}$, then by the same reasoning as above and Proposition 2.5 one easily obtains

$$G^{a,0} \oplus \mathcal{G} = L^2(m,\mathbb{C}). \tag{4.2}$$

Now, with T as in the statement we clearly have $\sigma(T) \subset \overline{\Omega}$. If the inequality would be strict, then by Propositions 1.1 and 1.2 in [2] we get that $G^{a,b}$ would be a reducing subspace which clearly is not the case by (4.1) or (4.2), combined with the characterization of reducing subspaces in Section 2.2, [2]. For the same reason we see that T is pure and finally $\mathsf{ind}\,(T-\lambda) = -1$ follows by Theorem 4.1 in [2]. □

Corollary 4.2. *Given $f \in G^{a,b}$ we have*

$$m\big(\{\xi \in \partial\Omega : f(\xi) = 0\}\big) = 0.$$

Proof. Combine Theorem 4.1 with Proposition 3.1 in [2]. □

In particular, the above holds for $f = \omega_{\lambda_0} + a\nu$ with $a \in [a_{\min}, a_{\max}]$. There are various ways to realize $G^{a,b}$ as Hardy-type spaces, the next lemma gives one such.

Lemma 4.3. *Given $a \in (a_{\min}, a_{\max})$ and $f \in G^{a,0}$ there exists an analytic function $\tilde{f}$ on Ω with nontangential limits m-a.e. and*

$$\mathop{\mathsf{nt\text{-}lim}}_{\lambda \to \xi} \tilde{f}(\lambda) = \frac{f(\xi)}{\sqrt{\omega_{\lambda_0}(\xi) + a\nu(\xi)}}$$

for m-a.e. $\xi \in \partial\Omega$.

Proof. First, one proves that the problem can be reduced to the case when Ω has analytic boundaries. This involves Proposition 2.4 and the results in Section 2.1 of [2] along with showing that $\Phi : L^2(m, \mathbb{C}) \to L^2(\tilde{m}, \mathbb{C})$ defined via

$$\Phi(f) = f \circ \phi \sqrt{|\phi'|}$$

is unitary and $\Phi(\sqrt{\omega_{\lambda_0} + a\nu}) = \sqrt{\tilde{\omega}_{\phi^{-1}(\lambda_0)} + a\tilde{\nu}}$. We omit the details and assume from now that Ω has analytic boundary. By Cauchy's theorem we clearly have

$$\begin{aligned} 2\pi i \; r(\lambda) &= \int_{\partial\Omega} \frac{r}{z - \lambda} dz = \int_{\partial\Omega} \frac{r\sqrt{\omega_{\lambda_0} + a\nu}\,\alpha}{(z-\lambda)\sqrt{\omega_{\lambda_0} + a\nu}} dm(z) \\ &= \left\langle r\sqrt{\omega_{\lambda_0} + a\nu}, \overline{\frac{\overline{\alpha}}{(\cdot - \lambda)\sqrt{\omega_{\lambda_0} + a\nu}}} \right\rangle_{L^2(m,\mathbb{C})} \end{aligned}$$

for all $r \in R(\overline{\Omega})$ and $\lambda \in \Omega$. (Note that

$$\overline{\alpha/(z-\lambda)\sqrt{\omega_{\lambda_0} + a\nu}} \in L^2(m, \mathbb{C})$$

since $\sqrt{\omega_{\lambda_0} + a\nu}$ is continuous and positive by Propositions 2.2 and 2.3). Given an $f \in G^{a,0}$ and a sequence $r_1, r_2, \ldots \in R(\overline{\Omega})$ such that

$$\lim_i r_i \sqrt{\omega_{\lambda_0} + a\nu} = f,$$

this implies that $(r_i)_{i=1}^{\infty}$ converges uniformly on compacts in Ω to a unique analytic function which we denote $\tilde{f}$. Moreover we clearly have

$$\tilde{f}(\lambda) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{(z-\lambda)\sqrt{\omega_{\lambda_0} + a\nu}} dz$$

for all $\lambda \in \Omega$. If $\lambda \in \mathbb{C} \setminus \overline{\Omega}$ one obtains similarly that the righthand side above is 0, and hence the desired statement follows from Theorem 2.1 in [2]. $\square$

Define the function β on $(a_{\min}, a_{\max})$ via

$$\begin{aligned} \beta(a) &= \left\langle M_z \frac{\nu/\sqrt{\omega_{\lambda_0} + a\nu}}{\|\nu/\sqrt{\omega_{\lambda_0} + a\nu}\|}, \frac{\nu/\sqrt{\omega_{\lambda_0} + a\nu}}{\|\nu/\sqrt{\omega_{\lambda_0} + a\nu}\|} \right\rangle \\ &= \int \frac{z\nu^2}{\omega_{\lambda_0} + a\nu} dm \Big/ \int \frac{\nu^2}{\omega_{\lambda_0} + a\nu} dm. \end{aligned}$$

β will play a key role when proving uniqueness of the model in the next section.

Proposition 4.4. *The function β is injective and $\mathsf{Im}\beta \subset \Omega$. Moreover*

$$G^{a,0} = \big(M_z - \beta(a)\big) G^{a,1}.$$

Proof. Set $T = M_z|_{G^{a,1}}$. Given $c \in \mathbb{C}$ and $r \in R(\overline{\Omega})$ we get

$$\left\langle \frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}}, \big(M_z - \beta(a)\big) \left(c\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} + r\sqrt{\omega_{\lambda_0} + a\nu} \right) \right\rangle = 0$$

which implies that $\frac{\nu}{\sqrt{\omega_{\lambda_0}+a\nu}} \perp \mathsf{Ran}(T-\beta(a))$. By Theorem 4.1 we thus infer that $\beta(a) \in \Omega$. To see that $G^{a,0} = \big(T - \beta(a)\big)G^{a,1}$, note that

$$\mathsf{codim}(\mathsf{Ran}(T - \beta(a))) = 1$$

by Theorem 4.1 so

$$\mathsf{Ran}(T - \beta(a)) = \left\{ \frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} \right\}^{\perp} = \big[\sqrt{\omega_{\lambda_0} + a\nu}\big]_{R(\overline{\Omega})} = G^{a,0}. \tag{4.3}$$

It remains to prove that β is injective. Assume not and let $\lambda = \beta(a) = \beta(a')$ for some $a \neq a'$. By (4.3) and Lemma 4.3 we have that there are functions h, h' on $\partial\Omega$ which are boundary values of analytic functions $\tilde{h}$ and $\tilde{h}'$ such that

$$\begin{aligned} (z - \lambda)\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} &= h\sqrt{\omega_{\lambda_0} + a\nu}, \\ (z - \lambda)\frac{\nu}{\sqrt{\omega_{\lambda_0} + a'\nu}} &= h'\sqrt{\omega_{\lambda_0} + a'\nu}, \end{aligned} \tag{4.4}$$

holds a.e. on $\partial\Omega$. This in turn implies that

$$\frac{h}{(z - \lambda) - ah} = \frac{\nu}{\omega_{\lambda_0}} = \frac{h'}{(z - \lambda) - a'h'}$$

a.e. on $\partial\Omega$. By Privalov's theorem we obtain that $(z-\lambda)(\tilde{h}-\tilde{h}') = (a'-a)\tilde{h}\tilde{h}'$. We conclude that either $\tilde{h}$ or $\tilde{h}'$ has a zero at λ. Say that $\tilde{h}(\lambda) = 0$. Using Theorem 4.1 and Lemma 4.3 it is then easy to show that $h\sqrt{\omega_{\lambda_0} + a\nu} = (z - \lambda)g$ for some $g \in G^{a,0}$. By inserting this in (4.4) we get

$$\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} = g \in G^{a,0}$$

which contradicts Theorem 4.1. □

5. The representation

We are now ready for the main theorem of this paper.

Theorem 5.1. *Let T be as in Assumption T, and assume that* $\mathsf{dim}\,\mathcal{K}_{\lambda_0} = n < \infty$. *Then there is a unique collection of pairs*

$$\{(a_i, b_i)\}_{i=1}^n \subset [a_{\min}, a_{\max}] \times \{0, 1\}$$

such that T is unitarily equivalent with M_z on $\oplus_{i=1}^n G^{a_i,b_i}$. Moreover, given a unitary map $U : \mathcal{H} \to \oplus_{i=1}^n G^{a_i,b_i}$ such that $UT = M_zU$, there are completely *orthogonal functions* $u_1, \dots, u_n \in L^2(m, l^2)$ *such that*

$$U^{-1}(f_1, \dots, f_n) = \sum_{i=1}^{n} f_i u_i. \tag{5.1}$$

Proof. The proof is rather long so we separate the various statements.

There exists a map U *with the above properties.* Use Theorem 3.2 to pick a completely orthogonal basis $k_1, \dots, k_n$ for $\mathcal{K}_{\lambda_0}$ such that, (using the notation from the proof of Theorem 3.2), we have

$$\boxed{k_i}^* \boxed{k_i} = \omega_{\lambda_0} I + A\nu$$

where

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}. \tag{5.2}$$

By Theorem 4.1 [2] we have that $\mathcal{H} \subset [\mathcal{K}_{\lambda_0}]_{L^\infty}$, by Proposition 3.1 of [2] we have that $k_i(\xi) \neq 0$ a.e. for $1 \leq i \leq n$ so by the complete orthogonality of the k_i's we can write each $f \in [\mathcal{K}_{\lambda_0}]_{L^\infty}$ as

$$f = \sum_{i=1}^{n} f_i \frac{k_i}{|k_i|}$$

with $f_i \in L^2(m, \mathbb{C})$ and $\|f\|^2 = \sum_{i=1}^{n} \|f_i\|^2$, (recall that we use the notation $|f| = \|f(\cdot)\|_{l^2}$ and $\|f\| = \|f\|_{L^2(m,l^2)}$). Let

$$U_1 : [\mathcal{K}_{\lambda_0}]_{L^\infty} \to \oplus_{i=1}^{n} L^2(m, \mathbb{C})$$

be the unitary operator given by $U_1(f) = (f_1, \dots, f_n)$. It is easy to see that

$$U_1(k_i) = \sqrt{\omega_{\lambda_0} + a_i \nu} \tag{5.3}$$

and clearly $a_i \in [a_{\min}, a_{\max}]$. For simplicity of notation we shall assume that $a_i \in (a_{\min}, a_{\max})$ for all i. The proof can be modified to include $a_{\min}$ and $a_{\max}$ as well, but we omit the details. As usual we have that

$$[\overline{(z - \lambda_0)} \mathcal{K}_{\lambda_0}]_{\overline{R(\overline{\Omega})}} \subset \mathcal{H}^\perp$$

and by equations (4.1) and (5.3) it follows that

$$\Big(U_1 \Big([\overline{(z - \lambda_0)} \mathcal{K}_{\lambda_0}]_{\overline{R(\overline{\Omega})}} \Big) \Big)^\perp = \oplus_{i=1}^{n} G^{a_i,1}.$$

Since U_1 is unitary we conclude that

$$\oplus_{i=1}^{n} G^{a_i,0} \subset U_1(\mathcal{H}) \subset \oplus_{i=1}^{n} G^{a_i,1}.$$

Set

$$\mathcal{G} = U_1 \Big(\mathcal{H} \ominus [\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})} \Big) = U_1(\mathcal{H}) \ominus \Big(\oplus_{i=1}^{n} G^{a_i,0} \Big). \tag{5.4}$$

Moreover let $V : \mathbb{C}^n \to \oplus_{i=1}^n L^2(m, \mathbb{C})$ be given by

$$V(a_i) = \sum_{i=1}^n a_i \frac{\nu}{\sqrt{\omega_{\lambda_0} + a_i \nu}} e_i,$$

where e_i is the standard basis for $\mathbb{C}^n$. Note that $\mathcal{G} \subset \mathsf{Ran} V$ and that the existence of a U with the desired properties follows if we show that there are integers $i_1 < i_2 < \ldots < i_{\dim \mathcal{G}}$ such that

$$V^{-1}(\mathcal{G}) = \mathsf{Span}\{e_{i_j}\}_{j=1}^{\dim \mathcal{G}}. \tag{5.5}$$

Indeed, setting

$$b_i = \begin{cases} 1 & \text{if } i \in \{i_j\}_{j=1}^{\dim \mathcal{G}} \\ 0 & \text{otherwise} \end{cases}$$

we would then have that

$$U_1(\mathcal{H}) = \Big(\oplus_{i=1}^n G^{a_i,0} \Big) \oplus \mathcal{G} = \oplus_{i=1}^n G^{a_i,b_i}$$

so $U = U_1|_{\mathcal{H}}$ is the desired unitary operator.

We thus have to prove that (5.5) holds. Let $P_{\mathsf{Ran} V}$ denote the orthogonal projection onto $\mathsf{Ran} V$ in $\oplus_{i=1}^n L^2(m, \mathbb{C})$. A short calculation shows that the operator $V^{-1} P_{\mathsf{Ran} V} M_z V$ is given by the matrix

$$B = \begin{pmatrix} \beta(a_1) & 0 & \ldots & 0 \\ 0 & \beta(a_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta(a_n) \end{pmatrix}.$$

It is also not hard to see that $V^{-1}(\mathcal{G})$ is B-invariant. If all a_i's are distinct, then the $\beta(a_i)$'s are distinct by Lemma 4.4. In this case, it is now easy to see that (5.5) indeed holds because all B-invariant subspaces have the desired form, by basic spectral theory for matrices.

In the general case, it is not hard to see that (5.5) may actually fail to hold, so we will have to find a way around this. There clearly exist disjoint subsets $S_1, \ldots, S_I \subset \{1, \ldots, n\}$ such that

$$\cup_{i=1}^I S_i = \{1, \ldots, n\} \tag{5.6}$$

and $a_{i_1} = a_{i_2}$ if and only if i_1 and i_2 belong to the same subset S_i. Again, the fact that $V^{-1}(\mathcal{G})$ is B-invariant combined with spectral theory for matrices, implies that

$$\mathcal{G} = \oplus_{i=1}^I \mathcal{G} \cap [\{U_1(k_j) : j \in S_i\}]_{L^\infty},$$

and hence we have

$$\mathcal{H} = \oplus_{i=1}^I \mathcal{H} \cap [\{k_j : j \in S_i\}]_{L^\infty}$$

as well. But this implies that it is sufficient to consider the case when all a_i's are the same. We thus now assume that $a = a_i$ for some a and all i's. Let $w_1, \ldots, w_{\dim \mathcal{G}} \in \mathbb{C}^n$ be an orthonormal basis for $V^{-1}(\mathcal{G})$ and pick

$w_{\mathsf{dim}\,\mathcal{G}+1}, \ldots, w_n \in \mathbb{C}^n$ so that $\{w_i\}$ becomes an orthonormal basis for $\mathbb{C}^n$. Set

$$k_i' = \sum_{j=1}^{n} w_i(j) k_j \in \mathcal{H}$$

and note that

$$k_{i_1}' \cdot \overline{k_{i_2}'} = \sum_j w_{i_1}(j)\overline{w_{i_2}(j)}|k_j|^2 = \big\langle w_{i_1}, w_{i_2}\big\rangle\big(\omega_{\lambda_0} + a\nu\big).$$

Thus $\{k_i'\}$ form a completely orthogonal basis for $\mathcal{K}_{\lambda_0}$. Moreover,

$$U_1(k_i') = (w_i(1)\sqrt{\omega_{\lambda_0} + a\nu}, \ldots, w_i(n)\sqrt{\omega_{\lambda_0} + a\nu}),$$
$$V(w_i) = \left(w_i(1)\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}}, \ldots, w_i(n)\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}}\right),$$

so we see that

$$\mathsf{dim}\Big(\mathcal{G} \cap \big[U_1(k_i')\big]_{L^\infty}\Big) = 1$$

for all $i \le \mathsf{dim}\,\mathcal{G}$. Stated differently, we have

$$\mathsf{dim}\,\Big(\Big(\mathcal{H} \ominus [\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})}\Big) \cap \big[k_i'\big]_{L^\infty}\Big) = 1$$

for all $i \le \mathsf{dim}\,\mathcal{G}$. We now start again from the beginning of the proof, but using $\{k_i'\}$ as a basis for $\mathcal{K}_{\lambda_0}$ instead of $\{k_i\}$. When we reach (5.4), we have $\frac{\nu}{\sqrt{\omega_{\lambda_0}+a\nu}} e_i \in \mathcal{G}$ for all $i \le \mathsf{dim}\,\mathcal{G}$, and therefore we have

$$\mathcal{G} = \mathsf{Span}\left\{\frac{\nu}{\sqrt{\omega_{\lambda_0} + a\nu}} e_i : 1 \le i \le \mathsf{dim}\,\mathcal{G}\right\}$$

or, in other words,

$$U_1(\mathcal{H}) = \Big(\oplus_{i=1}^{\mathsf{dim}\,\mathcal{G}}\, G^{a,1}\Big) \oplus \Big(\oplus_{i=(\mathsf{dim}\,\mathcal{G}+1)}^{n}\, G^{a,0}\Big)$$

and hence the first part of the proof is complete.

Uniqueness of $\{(a_i, b_i)\}$. We now prove that if we have two collections of pairs $\{(a_i, b_i)\}_{i=1}^n$ and $\{(a_i', b_i')\}_{i=1}^n$ and a unitary operator

$$U : \oplus_{i=1}^n G^{a_i,b_i}(\Omega, \lambda_0) \longrightarrow \oplus_{i=1}^n G^{a_i',b_i'}(\Omega, \lambda_0)$$

such that $M_z U = U M_z$, then

$$\{(a_i, b_i)\}_{i=1}^n = \{(a_i', b_i')\}_{i=1}^n.$$

We indicate all objects in $G^{a_i',b_i'}$ by adding an $'$, e.g., we write $\mathcal{K}_{\lambda_0}$ for $\oplus_{i=1}^n G^{a_i,b_i} \ominus (z - \lambda_0) \oplus_{i=1}^n G^{a_i,b_i}$ and $\mathcal{K}_{\lambda_0}'$ for the corresponding object in $\oplus_{i=1}^n G^{a_i',b_i'}$. First note that if $\{k_i\}_{i=1}^n$ is an orthonormal basis for $\mathcal{K}_{\lambda_0}$ then, by the proof of Theorem 3.2, we have

$$\boxed{k_i}^* \boxed{k_i} = \omega_{\lambda_0} I + A\nu,$$

where the eigenvalues of the matrix A are independent of the choice of basis $\{k_i\}$. In particular, choosing the basis $\sqrt{\omega_{\lambda_0} + a_i\nu}\, e_i$ we see that the collection of eigenvalues equals $\{a_i\}_{i=1}^n$. Now, note that $\mathcal{K}'_{\lambda_0} = U\mathcal{K}_{\lambda_0}$, so by (2.2) we have that

$$\boxed{U(k_i)}^* \boxed{U(k_i)} = \boxed{k_i}^* \boxed{k_i} = \omega_{\lambda_0} I + A\nu.$$

Summing up, this implies that

$$\{a_i\}_{i=1}^n = \{a'_i\}_{i=1}^n.$$

We may thus assume that $a_i = a'_i$ for all i. Now,

$$\begin{aligned}\mathsf{Span}&\left\{b_i \frac{\nu}{\sqrt{\omega_{\lambda_0} + a_i\nu}} e_i\right\} \\ &= \left(\oplus_{i=1}^n G^{a_i,b_i}\right) \ominus \left(\oplus_{i=1}^n G^{a_i,0}\right) = \left(\oplus_{i=1}^n G^{a_i,b_i}\right) \ominus [\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})},\end{aligned}$$

(and clearly an equivalent equation holds with $'$'s) and, using (2.3), we have

$$U\Big(\left(\oplus_{i=1}^n G^{a_i,b_i}\right) \ominus [\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})}\Big) = \left(\oplus_{i=1}^n G^{a_i,b'_i}\right) \ominus [\mathcal{K}'_{\lambda_0}]_{R(\overline{\Omega})},$$

which combine to give that

$$U\left(\mathsf{Span}\left\{b_i \frac{\nu}{\sqrt{\omega_{\lambda_0} + a_i\nu}} e_i\right\}\right) = \mathsf{Span}\left\{b'_i \frac{\nu}{\sqrt{\omega_{\lambda_0} + a_i\nu}} e_i\right\} \tag{5.7}$$

Set $k_i = k'_i = \sqrt{\omega_{\lambda_0} + a_i\nu}\, e_i$. As noted earlier, $\mathcal{K}'_{\lambda_0} = U(\mathcal{K}_{\lambda_0})$ so there is an $n \times n$-matrix V such that

$$\boxed{U(k_i)} = \boxed{k'_i} V$$

By the same calculation as in (3.2) we deduce that

$$A = V^* A V$$

where A is the matrix given in (5.2). Pick disjoint subsets $S_1, \ldots, S_I \subset \{1, \ldots, n\}$ as in (5.6). By more standard matrix theory, the above equation then implies that the subspaces $\mathsf{Span}\{e_i : i \in S_j\}$ are invariant under V. Let u be the matrix-valued function representing U as in (2.1). For a.e. $\xi \in \partial\Omega$ we then have

$$u(\xi)\boxed{k_i(\xi)} = \boxed{U(k_i)(\xi)} = \boxed{k'_i(\xi)} V = \boxed{k_i(\xi)} V$$

which by Corollary 4.2 gives

$$u(\xi) = \boxed{k_i(\xi)} V \boxed{k_i(\xi)}^{-1}. \tag{5.8}$$

Since the subspaces $\mathsf{Span}\{e_i : i \in S_j\}$ clearly are invariant for $\boxed{k_i(\xi)}$ as well, and since these matrixes act as a constant times the identity on each such subspace, we conclude that

$$u(\xi) = V, \quad \text{for } m - a.e.\ \xi \in \partial\Omega.$$

This observation combined with (5.7) implies that

$$U\left(\mathsf{Span}\left\{b_i\frac{\nu}{\sqrt{\omega_{\lambda_0}+a_i\nu}}e_i : i\in S_j\right\}\right)=\mathsf{Span}\left\{b_i'\frac{\nu}{\sqrt{\omega_{\lambda_0}+a_i\nu}}e_i : i\in S_j\right\}$$

for all $j=1,\dots,j_1$. Consequently,

$$\sum_{i\in S_j} b_i=\sum_{i\in S_j} b_i'$$

and hence

$$\{(a_i,b_i)\}_{i=1}^n=\{(a_i',b_i')\}_{i=1}^n,$$

as desired.

The form of U. It only remains to prove that any unitary map $U:\mathcal{H}\to\oplus_{i=1}^n G^{a_i,b_i}$ such that $UT=M_zU$ has the form (5.1). Let U_1 denote the map constructed in the first part of the proof, and recall that it does satisfy (5.1). By the second part of the proof, it follows that the map $U_1U^{-1}=M_V$, where V is a constant $n\times n$ matrix. The desired conclusion is now immediate. $\square$

We note that in the case $\Omega=\mathbb{D}$, one can "strip" the above proof and that of Theorem 3.2 of everything related to ν, to give a fairly simple proof of the Beurling–Lax theorem. We conclude this paper with an example, which in particular shows that the b_i's are dependent of the choice of λ_0.

Example 5.2. Set $\Omega=\{\zeta\in\mathbb{C}:1<|\zeta|<2\}$. Set $R=0.35$ and define $g\in L^2(m,\mathbb{C})$ by

$$g(\xi)=\begin{cases}\sqrt{1/2\pi}, & |\xi|=1\\ \sqrt{R/4\pi}, & |\xi|=2.\end{cases}$$

Set $\mathcal{H}=[g]_{R(\overline{\Omega})}$. It is easy to see that each $h\in\mathcal{H}$ can be written as

$$h(\xi)=g(\xi)\sum_{-\infty}^{\infty}a_k\xi^k$$

where $\|h\|^2=\sum_{k=-\infty}^{\infty}(1+R4^k)|a_k|^2$. In particular, we can realize $\mathcal{H}$ as the space of analytic functions on Ω given by $\sum_{k=-\infty}^{\infty}a_k\zeta^k$, $(\zeta\in\Omega)$, endowed with the above norm. The reproducing kernel for a given $\lambda\in\Omega$ is then given by

$$k_\lambda(\zeta)=\sum_{k=-\infty}^{\infty}\frac{\overline{\lambda}^k\zeta^k}{1+R4^k}.$$

It is clear by the results in this paper and [2] that $T=M_z|_{\mathcal{H}}$ is a pure subnormal operator with $\mathsf{ind}\,(T-\lambda)=-1$ for all $\lambda\in\Omega$. Hence, given any $\lambda_0\in\Omega$, T is unitarily equivalent with M_z on a unique space $G^{a,b}(\Omega,\lambda_0)$. By the rotational symmetry of Ω it is clear that a and b only depend on $|\lambda_0|$. Set $|\lambda_0|=r$. Moreover,

$$b(r)=\mathsf{dim}\left(\mathcal{H}\ominus[k_r]_{R(\overline{\Omega})}\right).$$

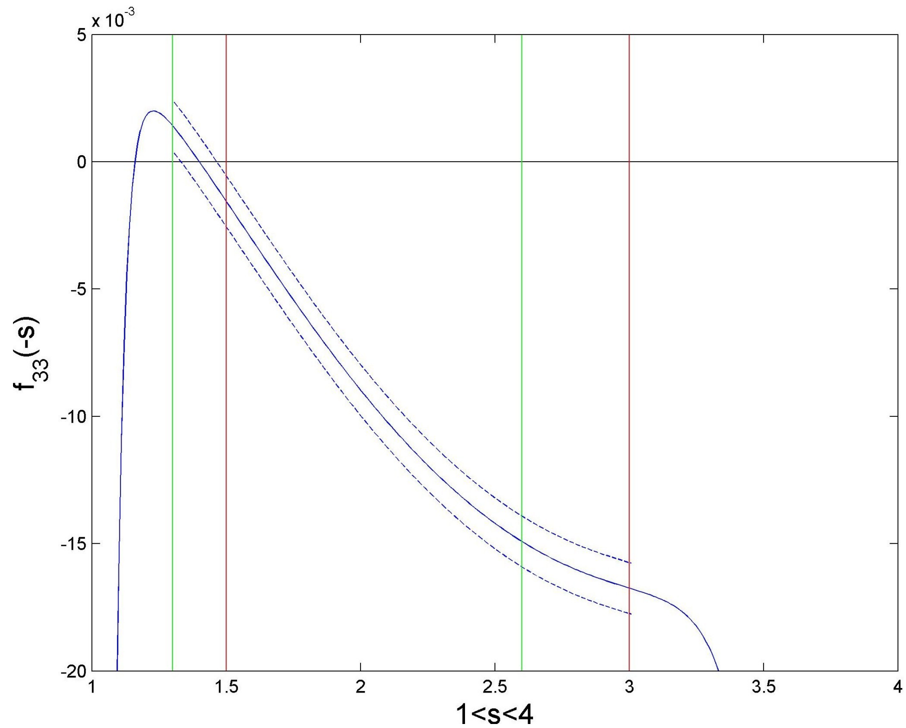

FIGURE 1. Blue graph: the function $f_{33}(-s)$, $1 < s < 4$. The function f lies between the dotted lines in the interval $(1.3, 3)$. The green lines show $s = 1.3$ and $s = 2.6$, so $k_{1.3}((-2, -1))$ takes values in the area between the dotted lines and the green lines. Analogously the red lines correspond to $k_{1.5}$. Note that the sharp turn of f_{33} for $s < 1.3$ is due to the truncation of (5.9), since f can not have two zeros.

By Theorem 2.7, Proposition 3.4 in [2] and the fact that $\mathcal{H} \ominus [k_r]_{R(\overline{\Omega})} < \infty$, we see that $b(r)$ equals the amount of zeroes of k_r in Ω. By the symmetry of k_r, $k_r(x + iy) = 0$ if and only if $k_r(x - iy) = 0$ which, since $b(r) \leq 1$, implies that any zero of k_r lies on $\mathbb{R}$. Moreover it has to lie on $(-2, -1)$ since clearly $k_r(t) > 0$ for $t \in (1, 2)$. Consider the function $f : \{\zeta \in \mathbb{C} : 1 < |\zeta| < 4\} \to \mathbb{R}$ defined by

$$f(\zeta) = \sum_{k=-\infty}^{\infty} \frac{\zeta^k}{1 + R4^k}, \tag{5.9}$$

and note that $k_r(\zeta) = f(r\zeta)$. With $f_N(\zeta) = \sum_{k=-N}^{N} \frac{\zeta^k}{1+R4^k}$, standard estimates show that

$$|f(\zeta) - f_{33}(\zeta)| < 10^{-3}, \quad 1.3 < |\zeta| < 3. \tag{5.10}$$

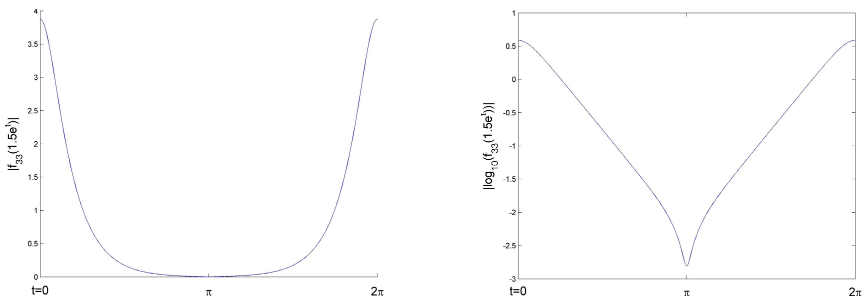

FIGURE 2. Plots of $|f_{33}(1.5e^{it})|$ and $\log_{10}|f_{33}(1.5e^{it})|$, $0 < t < 2\pi$. Due to (5.10), these plots show that $k_{1.5}(\xi)$ has no zeros on $|\xi| = 1$.

Figure 1 shows a plot of $f_{33}(-s)$, $(1 < s < 4)$ with negligible error margin. It is clear that

$$k_{1.3}(-t) = f(-1.3t), \quad 1 < t < 2,$$

has a zero whereas $k_{1.5}(-t) = f(-1.5t)$ does not. Thus $b(1.3) = 1$ and $b(1.5) = 0$.

Moreover, $a(1.5) \neq a_{\min}(1.5)$ and $a(1.5) \neq a_{\max}(1.5)$. To see this, note that by (3.1) and some simple computations, there are constants $C_1, C_2 \in \mathbb{R}$ such that $\omega_{1.5} + a(1.5)\nu$ equals $C_1|k_{1.5}|^2$ on the inner circle and $C_2|k_{1.5}|^2$ on the outer circle. By Figure 2 it thus follows that $\omega_{1.5} + a(1.5)\nu$ has no zero on the inner circle, and a similar plot (which is omitted) shows that the same is true for the outer circle as well. By the material in Section 2, the claim is now easily established. However, by Figure 1 we also conclude that $a(r) = a_{\min}(r)$ or $a(r) = a_{\max}(r)$ for some value of $r \in (1.3, 1.5)$, and this must be the point where $b(r)$ changes value.

References

[1] Abrahamse, M. B.; Douglas, R. G. A class of subnormal operators related to multiply-connected domains. *Advances in Math.* **19** (1976) 106–148. MR0397468 (53 #1327), Zbl 0321.47019.

[2] Carlsson, M. On subnormal operators whose spectrum are multiply connected domains. *New York J. Math.* **17a** (2011) 177–191.

[3] Conway, J. B. A course in functional analysis. Second edition. Graduate Texts in Mathematics, 96. *Springer-Verlag, New York*, 1990. xvi+399 pp. ISBN: 0-387-97245-5. MR1070713 (91e:46001), Zbl 0706.46003.

[4] Conway, J. B. The theory of subnormal operators. Mathematical Surveys and Monographs, 36. *American Mathematical Society, Providence, RI*, 1991. xvi+436 pp. ISBN: 0-8218-1536-9. MR1112128 (92h:47026), Zbl 0743.47012.

[5] Fisher, S. D. Function theory on planar domains. A second course in complex analysis. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. *John*

Wiley & Sons, Inc., New York, 1983. xiii+269 pp. ISBN: 0-471-87314-4. MR0694693 (85d:30001), Zbl 0511.30022.

[6] Gamelin, T. W. Uniform algebras. *Prentice-Hall, Inc., Englewood Cliffs, N. J.*, 1969. xiii+257 pp. MR0410387 (53 #14137), Zbl 0213.40401.

[7] Mlak, W. Intertwining operators. *Studia Math.* **43** (1972) 219–233. MR0322539 (48 #901), Zbl 0257.46081.

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