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Positive Schatten–Herz class Toeplitz operators on the ball

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Abstract. On the harmonic Bergman space of the ball, we give characterizations for an arbitrary positive Toeplitz operator to be a Schatten–Herz class operator in terms of averaging functions and Berezin transforms.

Contents

1. Introduction

For a fixed integer $n \geq 2$, let $B = B_n$ denote the open unit ball in $\mathbf{R}^n$. For $0 < p \leq \infty$, let $L^p = L^p(V)$ be the Lebesgue spaces on B where V denotes the Lebesgue volume measure on B. The harmonic Bergman space b^2 is a closed subspace of L^2 consisting of all complex-valued harmonic functions on B. By the mean value property of harmonic functions, it is easily seen that point evaluations are continuous on b^2. Thus, to each $x \in B$, there corresponds a unique $R(x, \cdot) \in b^2$ which has the following reproducing property:

$$f(x) = \int_B f(y)\overline{R(x,y)}\, dy, \qquad x \in B \tag{1.1}$$

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for all $f \in b^2$. The explicit formula of the kernel function $R(x,y)$ is well known:

$$R(x,y) = \frac{1}{|B|} \cdot \frac{1}{[x,y]^n} \left\{ \left(\frac{1-|x|^2|y|^2}{[x,y]} \right)^2 - \frac{4|x|^2|y|^2}{n} \right\}$$

for $x, y \in B$ where $[x,y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}$. Here, as elsewhere, we write $x \cdot y$ for the dot product of $x, y \in \mathbf{R}^n$ and $|E| = V(E)$ for the volume of Borel sets $E \subset B$. Hence the kernel function $R(x,y)$ is real and hence the complex conjugation in the integral of (1.1) can be removed. See [2] for more information and related facts.

Let R be the Hilbert space orthogonal projection from L^2 onto b^2. The reproducing property (1.1) yields the following integral representation of R:

$$R\psi(x) = \int_B \psi(y) R(x,y)\, dy, \qquad x \in B \tag{1.2}$$

for functions $\psi \in L^2$. It is easily seen that the projection R can be extended to an integral operator via (1.2) from L^1 into the space of all harmonic functions on B. It even extends to $\mathcal{M}$, the space of all complex Borel measures on B. Namely, for each $\mu \in \mathcal{M}$, the integral

$$R\mu(x) = \int_B R(x,y)\, d\mu(y), \qquad x \in B$$

defines a function harmonic on B. For $\mu \in \mathcal{M}$, the *Toeplitz operator* T_μ with *symbol* μ is defined by

$$T_\mu f = R(f\, d\mu)$$

for $f \in b^2 \cap L^\infty$. Note that T_μ is defined on a dense subset of b^2, because bounded harmonic functions form a dense subset of b^2.

A Toeplitz operator T_μ is called *positive* if μ is a positive finite Borel measure (hereafter we simply write $\mu \geq 0$). For positive Toeplitz operators on harmonic Bergman spaces, basic operator theoretic properties such as boundedness, compactness and the membership in the Schatten classes have been studied on various settings; see [5], [7], [11], [12] and references therein. Another aspect of positive Toeplitz operators has been recently studied. Namely, notion of the so-called Schatten–Herz classes $S_{p,q}$ (see Section 3) was introduced and studied in [10] in the holomorphic case on the unit disk. Harmonic analogues were subsequently studied in [6]. However, these earlier works are restricted to the case of $1 \leq p, q \leq \infty$. In this paper we extend the characterization in [6] for Schatten–Herz class positive Toeplitz operators to the full range of parameters p and q.

To state our results we briefly introduce some notation. Given $\mu \geq 0$, $\widehat{\mu}_r$ denotes the averaging function over pseudohyperbolic balls with radius r and $\widetilde{\mu}$ denotes the Berezin transform. See Section 2 for relevant definitions. Also, we let λ denote the measure on B defined by

$$d\lambda(x) = (1 - |x|^2)^{-n}\, dx$$

and $\mathcal{K}_q^p(\lambda)$ denote the so-called Herz spaces (see Section 3).

The next theorem is the main result of this paper. In case $1 \le p < \infty$, Theorem 1.1 (with slightly different averaging functions) below has been proved in [6]. In case $0 < p < 1$, the authors [4] have recently obtained the corresponding results on the harmonic Bergman space of the upper half-space. The cut-off point $\frac{n-1}{n}$ is sharp in the theorem below.

Theorem 1.1. *Let $0 < p \le \infty$, $0 \le q \le \infty$, $0 < r < 1$ and $\mu \ge 0$. Then the following two statements are equivalent:*

(a) $T_\mu \in S_{p,q}$;
(b) $\widehat{\mu}_r \in \mathcal{K}_q^p(\lambda)$.

Moreover, if $\frac{n-1}{n} < p \le \infty$, then the above statements are also equivalent to

(c) $\widetilde{\mu} \in \mathcal{K}_q^p(\lambda)$.

In Section 2 we investigate known results on weighted L^p-behavior of averaging functions and Berezin transforms. In Section 3, we first prove Theorem 1.1 and then provide examples indicating that the parameter range required in Theorem 1.1 is best possible.

2. Basic lemmas

In this section we collect several known results which will be used in our characterization.

We first recall Möbius transformations on B. All relevant details can be found in [1, pp. 17–30]. Let $a \in B$. The canonical Möbius transformation ϕ_a that exchanges a and 0 is given by

$$\phi_a(x) = a + (1 - |a|^2)(a - x^*)^*$$

for $x \in B$ (note $\phi_a = -T_a$ in the notation of [1]). Here $x^* = x/|x|^2$ denotes the inversion of x with respect to the sphere ∂B. Avoiding x^* notation, we have

$$\phi_a(x) = \frac{(1 - |a|^2)(a - x) + |a - x|^2 a}{[x, a]^2}.$$

The map ϕ_a is an involution of B, i.e., $\phi_a^{-1} = \phi_a$.

The hyperbolic distance $\beta(x, y)$ between two points $x, y \in B$ is given by

$$\beta(x, y) = \frac{1}{2} \log \frac{1 + |\phi_y(x)|}{1 - |\phi_y(x)|}.$$

As is well-known, β is Möbius invariant. Let $\rho(x, y) = |\phi_y(x)|$. This ρ is also a Möbius invariant distance on B. We shall work with this *pseudohyperbolic* distance ρ.

For $a \in B$ and $r \in (0, 1)$, let $E_r(a)$ denote the pseudohyperbolic ball with radius r and center a. A straightforward calculation shows that $E_r(a)$ is a Euclidean ball with

$$\text{(center)} = \frac{(1 - r^2)}{1 - |a|^2 r^2} a \quad \text{and} \quad \text{(radius)} = \frac{(1 - |a|^2) r}{1 - |a|^2 r^2}. \tag{2.1}$$

Given $\mu \geq 0$ and $r \in (0,1)$, the *averaging* function $\widehat{\mu}_r$ and the *Berezin transform* $\widetilde{\mu}$ are defined by

$$\widehat{\mu}_r(x) = \frac{\mu[E_r(x)]}{|E_r(x)|}$$

and

$$\widetilde{\mu}(x) = (1-|x|^2)^n \int_B |R(x,y)|^2 \, d\mu(y)$$

for $x \in B$. While it is customary to put $R(x,x)^{-1}$ in place of $(1-|x|^2)^n$ in the definition of the Berezin transform, we adopted the above definition for simplicity. For measurable functions f, we define $\widehat{f}_r$ and $\widetilde{f}$ similarly, whenever they are well defined.

Given α real, we let $L^p_\alpha = L^p(V_\alpha)$ where V_α denotes the weighted measure defined by $dV_\alpha(x) = (1-|x|^2)^\alpha \, dx$. For $\alpha = 0$, we have $L^p_0 = L^p$. Note $\lambda = V_{-n}$. Also, given a sequence $\mathfrak{a} = \{a_m\}$ in B, we let $\ell^{p,\alpha}(\mathfrak{a})$ denote the p-summable sequence space weighted by $\{(1-|a_m|^2)^\alpha\}$. For $\alpha = 0$, we let $\ell^p = \ell^{p,0}(\mathfrak{a})$.

Given an integer $k \geq 0$, we let $R_k(x,y)$ be the reproducing kernel for the weighted harmonic Bergman space with respect to the weight $(1-|x|)^k$. So, $R_0 = R$ is the harmonic Bergman kernel mentioned before. Explicit formulas for these kernels are given in [8, (3.1)].

The following lemma taken from [5, Lemma 3.1] shows that averaging functions, when radii are small enough, are dominated by Berezin transforms.

Lemma 2.1. *For an integer $k \geq 0$ and $\mu \geq 0$, there exists some $r_k \in (0,1)$ with the following property: If $0 < r \leq r_k$, then there exists a constant $C = C(n,k,r)$ such that*

$$\widehat{\mu}_r(a) \leq C(1-|a|^2)^{n+2k} \int_B |R_k(x,a)|^2 \, d\mu(x), \qquad a \in B.$$

In particular, $\widehat{\mu}_r \leq C\widetilde{\mu}$ for $0 < r \leq r_0$.

We also need the fact that the L^p_α-behavior of averaging functions of positive measures is independent of radii. In what follows, L_0 denotes the space of all functions f bounded on B and $f(x) \to 0$ as $|x| \to 1$.

Lemma 2.2. *Let $0 < p \leq \infty$, $r, \delta \in (0,1)$ and α be real. Assume $\mu \geq 0$. Then the following statements hold:*

(a) *$\widehat{\mu}_r \in L^p_\alpha$ if and only if $\widehat{\mu}_\delta \in L^p_\alpha$;*
(b) *$\widehat{\mu}_r \in L_0$ if and only if $\widehat{\mu}_\delta \in L_0$.*

Proof. See [5, Proposition 3.6]. □

Let $\{a_m\}$ be a sequence in B and $r \in (0,1)$. We say that $\{a_m\}$ is *r-separated* if the balls $E_r(a_m)$ are pairwise disjoint or simply say that $\{a_m\}$ is *separated* if it is r-separated for some r. Also, we say that $\{a_m\}$ is an *r-lattice* if it is $\frac{r}{2}$-separated and $B = \cup_m E_r(a_m)$. One can explicitly construct an r-lattice by using the same argument as in [8]. Note that any 'maximal' $\frac{r}{2}$-separated sequence is an r-lattice.

The following lemma taken from [5, Theorem 3.9] gives an information on weighted L^p-behavior of averaging functions, as well as its discrete version, and Berezin transforms.

Lemma 2.3. *Let $0 < p \leq \infty$, $r, \delta \in (0, 1)$ and α be real. Let $\mu \geq 0$ and $\mathfrak{a} = \{a_m\}$ be an r-lattice. Then the following two statements are equivalent:*

(a) $\widehat{\mu}_\delta \in L^p_\alpha$;
(b) $\{\widehat{\mu}_r(a_m)\} \in \ell^{p,n+\alpha}(\mathfrak{a})$.

Moreover, if

$$\max\left\{1+\alpha, 1+\frac{\alpha}{n}, -\frac{\alpha+1}{n}\right\} < p \leq \infty,$$

then the above statements are also equivalent to

(c) $\widetilde{\mu} \in L^p_\alpha$.

For a positive compact operator T on a separable Hilbert space H, there exist an orthonormal set $\{e_m\}$ in H and a sequence $\{\lambda_m\}$ that decreases to 0 such that

$$Tx = \sum_m \lambda_m \langle x, e_m \rangle e_m$$

for all $x \in H$ where $\langle\ ,\ \rangle$ denotes the inner product on H. For $0 < p < \infty$, we say that a positive operator T belongs to the Schatten p-class $S_p(H)$ if

$$\|T\|_p := \left\{\sum_m \lambda_m^p\right\}^{1/p} < \infty.$$

More generally, given a compact operator T on H, we say that $T \in S_p(H)$ if the positive operator $|T| = (T^*T)^{1/2}$ belongs to $S_p(H)$ and we define $\|T\|_p = \|\,|T|\,\|_p$. Of course, we will take $H = b^2$ in our applications below and, in that case, we put $S_p = S_p(b^2)$. Also, for $0 < q \leq \infty$, we use the notation ℓ^q for the q summable sequence space.

We need the following characterization of Schatten class positive Toeplitz operators which is taken from [5, Theorem 4.5].

Lemma 2.4. *Let $0 < p < \infty$, $\mu \geq 0$ and assume that $\{a_m\}$ is an r-lattice. Then the following three statements are equivalent:*

(a) $T_\mu \in S_p$;
(b) $\{\widehat{\mu}_r(a_m)\} \in \ell^p$;
(c) $\widehat{\mu}_r \in L^p(\lambda)$.

Moreover, if $\frac{n-1}{n} < p$, then the above statements are also equivalent to

(d) $\widetilde{\mu} \in L^p(\lambda)$.

3. Schatten–Herz class Toeplitz operators

In this section we prove Theorem 1.1. To introduce the Herz space, we first decompose B into the disjoint union of annuli A_k given by

$$A_k = \{x \in B : 2^{-(k+1)} \le 1 - |x| < 2^{-k}\}$$

for integers $k \ge 0$. For each k, we let $\chi_k = \chi_{A_k}$. Recall that χ_E denotes the characteristic function of $E \subset B$. Also, given $\mu \in \mathcal{M}$, we let $\mu\chi_k$ stand for the restriction of μ to A_k. For $0 < p, q \le \infty$ and α real, the Herz space $\mathcal{K}_q^{p,\alpha}$ consists of functions $\varphi \in L^p_{\mathrm{loc}}(V)$ for which

$$\|\varphi\|_{\mathcal{K}_q^{p,\alpha}} := \left\| \left\{ 2^{-k\alpha} \|\varphi\chi_k\|_{L^p} \right\} \right\|_{\ell^q} < \infty.$$

Also, we let $\mathcal{K}_0^{p,\alpha}$ be the space of all functions $\varphi \in \mathcal{K}_\infty^{p,\alpha}$ such that

$$\left\{ 2^{-k\alpha} \|\varphi\chi_k\|_{L^p} \right\} \in \ell_0;$$

recall that ℓ_0 denotes the subspace of ℓ^∞ consisting of all complex sequences vanishing at ∞. Note that $\mathcal{K}_q^{p,\alpha} \subset \mathcal{K}_0^{p,\alpha}$ for all $q < \infty$. For more information on the Herz spaces, see [9] and references therein.

Let $0 < p < \infty$ and α be real. Then, since $1 - |x|^2 \approx 2^{-k}$ for $x \in A_k$ and $k \ge 0$, we have

$$2^{-k\alpha p} \|\varphi\chi_k\|^p_{L^p} \approx \int_{A_k} |\varphi(x)|^p (1 - |x|^2)^{\alpha p}\, dx$$

and thus

$$\|\varphi\|_{\mathcal{K}_q^{p,\alpha}} \approx \left\| \left\{ \|\varphi\chi_k\|_{L^p_{\alpha p}} \right\} \right\|_{\ell^q} \tag{3.1}$$

for $0 < q \le \infty$. In particular, we have

$$\|\varphi\|_{\mathcal{K}_q^{p,-\frac{n}{p}}} \approx \left\| \left\{ \|\varphi\chi_k\|_{L^p(\lambda)} \right\} \right\|_{\ell^q}$$

and this estimate is valid even for $p = \infty$, because $-n/p = 0$ if $p = \infty$. For this reason, we put

$$\mathcal{K}_q^p(\lambda) = \mathcal{K}_q^{p,-\frac{n}{p}}$$

for the full ranges $0 < p \le \infty$ and $0 \le q \le \infty$. Note that $\mathcal{K}_p^p(\lambda) \approx L^p(\lambda)$ for $0 < p \le \infty$. That is, these two spaces are the same as sets and have equivalent norms.

Next, we introduce a discrete version of Herz spaces. Let $\mathfrak{a} = \{a_m\}$ be an arbitrary lattice. Given a complex sequence $\xi = \{\xi_m\}$ and $k \in \mathbb{N}$, let $\xi\chi_k$ denote the sequence defined by $(\xi\chi_k)_m = \xi_m \chi_k(a_m)$. Now, given $0 < p, q \le \infty$ and α real, we let $\ell_q^{p,\alpha}(\mathfrak{a})$ be the mixed-norm space of all complex sequences ξ such that

$$\|\xi\|_{\ell_q^{p,\alpha}(\mathfrak{a})} := \left\| \{ 2^{-k\alpha} \|\xi\chi_k\|_{\ell^p} \} \right\|_{\ell^q} < \infty.$$

So, we have

$$\|\xi\|^q_{\ell^{p,\alpha}_q(\mathfrak{a})} = \sum_k \left\{ 2^{-k\alpha p} \sum_{a_m \in A_k} |\xi_m|^p \right\}^{\frac{q}{p}}, \qquad 0 < p, q < \infty.$$

Also, we say $\xi \in \ell^{p,\alpha}_0(\mathfrak{a})$ if $\{\|2^{-k\alpha}\xi\chi_k\|_{\ell^p}\} \in \ell_0$. Finally, we let $\ell^{p,0}_q(\mathfrak{a}) = \ell^p_q(\mathfrak{a})$.

We now introduce the so-called Schatten–Herz class of Toeplitz operators. Let S_∞ denote the class of all bounded linear operators on b^2 and $\|\ \|_\infty$ denote the operator norm. Given $0 < p, q \le \infty$, the *Schatten–Herz* class $S_{p,q}$ is the class of all Toeplitz operators T_μ such that $T_{\mu\chi_k} \in S_p$ for each k and the sequence $\{\|T_{\mu\chi_k}\|_p\}$ belongs to ℓ^q. The norm of $T_\mu \in S_{p,q}$ is defined by

$$\|T_\mu\|_{p,q} = \Big\| \{\|T_{\mu\chi_k}\|_p\} \Big\|_{\ell^q}.$$

Also, we say $T_\mu \in S_{p,0}$ if $T_\mu \in S_{p,\infty}$ and $\{\|T_{\mu\chi_k}\|_p\} \in \ell_0$. Note that $S_{p,q} \subset S_{p,0}$ for all $q < \infty$.

The following observation plays a key role in proving that $\mathcal{K}^{p,\alpha}_q$-behavior or $\ell^{p,\alpha}_q$-behavior of averaging functions are independent of radii.

Theorem 3.1. *Let $0 < p, q \le \infty$, $r \in (0,1)$ and α be real. Then the following statements hold for $\mu, \tau \ge 0$:*

(a) $\left\{2^{-k\alpha}\|\widehat{\mu}_r\chi_k\|_{L^p(\tau)}\right\} \in \ell^q$ *if and only if* $\left\{2^{-k\alpha}\|(\widehat{\mu\chi_k})_r\|_{L^p(\tau)}\right\} \in \ell^q$*;*

(b) $\left\{2^{-k\alpha}\|\widehat{\mu}_r\chi_k\|_{L^p(\tau)}\right\} \in \ell_0$ *if and only if* $\left\{2^{-k\alpha}\|(\widehat{\mu\chi_k})_r\|_{L^p(\tau)}\right\} \in \ell_0$*.*

Before proceeding to the proof, we note the following covering property which is a simple consequence of (2.1): if $r \in (0,1)$ and $N = N(r)$ is a positive integer such that

$$2^{N-1} \le \frac{1+r}{1-r} < 2^N, \tag{3.2}$$

then

$$E_r(z) \subset \bigcup_{j=k-N}^{k+N} A_j, \qquad z \in A_k. \tag{3.3}$$

Here and in the proof below, we let $A_j = \emptyset$ if $j < 0$.

Proof. Let $\mu, \tau \ge 0$ be given. We prove the proposition only for $q < \infty$; the case $q = \infty$ is implicit in the proof below. Choose $N = N(r)$ as in (3.2) and put $\gamma_p = \gamma_{p,N} = \max\{1, (2N+1)^{1-p}\}$. By (3.3) we note

$$\widehat{\mu}_r\chi_k \le \sum_{j=k-N}^{k+N} (\widehat{\mu\chi_j})_r$$

for all k. Thus, we have

$$\|\widehat{\mu}_r\chi_k\|_{L^p(\tau)} \le \gamma_p^{\frac{1}{p}} \sum_{j=k-N}^{k+N} \|(\widehat{\mu\chi_j})_r\|_{L^p(\tau)}$$

for all k. Since $2^{-k\alpha} \le 2^{N|\alpha|}2^{-j\alpha}$ for $k-N \le j \le k+N$, it follows that

$$2^{-k\alpha}\|\widehat{\mu}_r\chi_k\|_{L^p(\tau)} \lesssim \sum_{j=k-N}^{k+N} 2^{-j\alpha}\|(\widehat{\mu\chi_j})_r\|_{L^p(\tau)}$$

for all k. This implies one direction of (b). Also, it follows that

$$2^{-k\alpha q}\|\widehat{\mu}_r\chi_k\|^q_{L^p(\tau)} \lesssim \gamma_q \sum_{j=k-N}^{k+N} 2^{-j\alpha q}\|(\widehat{\mu\chi_j})_r\|^q_{L^p(\tau)}$$

for all k. Thus, summing up both sides of the above over all k, we have

$$\sum_{k=0}^{\infty} 2^{-k\alpha q}\|\widehat{\mu}_r\chi_k\|^q_{L^p(\tau)} \lesssim (2N+1)\sum_{j=0}^{\infty} 2^{-j\alpha q}\|(\widehat{\mu\chi_j})_r\|^q_{L^p(\tau)},$$

which gives one direction of (a).

We now prove the other directions. Note that $E_r(z)$ can intersect A_k with $k \ge 0$ only when $z \in \cup_{j=k-N}^{k+N} A_j$ by (3.3). Thus we have

$$\widehat{(\mu\chi_k)}_r = \sum_{j=k-N}^{k+N} \widehat{(\mu\chi_k)}_r\chi_j \le \sum_{j=k-N}^{k+N} \widehat{\mu}_r\chi_j.$$

Thus, a similar argument yields the other directions of (a) and (b). The proof is complete. $\square$

As an immediate consequence of Lemma 2.2 and Theorem 3.1 (with $\tau = V$), we have the following Herz space version of Lemma 2.2.

Corollary 3.2. *Let $0 < p \le \infty$, $0 \le q \le \infty$, $\delta, r \in (0,1)$ and α be real. Let $\mu \ge 0$. Then $\widehat{\mu}_r \in \mathcal{K}^{p,\alpha}_q$ if and only if $\widehat{\mu}_\delta \in \mathcal{K}^{p,\alpha}_q$.*

Also, applying Theorem 3.1 with discrete measures $\tau = \sum_m \delta_{a_m}$ where δ_x denotes the point mass at $x \in B$, we have the following.

Corollary 3.3. *Let $0 < p, q \le \infty$, $r \in (0,1)$ and α be real. Let $\mu \ge 0$ and assume that $\mathfrak{a} = \{a_m\}$ is an r-lattice. Put $\xi_k = 2^{-k\alpha}\|\{\widehat{(\mu\chi_k)}_r(a_m)\}_m\|_{\ell^p}$ for $k \ge 0$. Then the following statements hold:*

(a) *$\{\widehat{\mu}_r(a_m)\} \in \ell^{p,\alpha}_q(\mathfrak{a})$ if and only if $\{\xi_k\} \in \ell^q$;*
(b) *$\{\widehat{\mu}_r(a_m)\} \in \ell^{p,\alpha}_0(\mathfrak{a})$ if and only if $\{\xi_k\} \in \ell_0$.*

We need one more fact taken from [3].

Lemma 3.4. *Let $1 \le p \le \infty$, $0 \le q \le \infty$ and α be real. Then the Berezin transform is bounded on $\mathcal{K}^{p,\alpha}_q$ if and only if $-n < \alpha + 1/p < 1$.*

We now prove the Herz space version of Lemma 2.3. The restricted range in (3.4) below can't be improved; see Example 3.9.

Theorem 3.5. *Let $0 < p \le \infty$, $0 \le q \le \infty$, $r \in (0,1)$ and α be real. Let $\mu \ge 0$ and assume that $\mathfrak{a} = \{a_m\}$ is an r-lattice. Then the following two statements are equivalent:*

(a) $\widehat{\mu}_r \in \mathcal{K}_q^{p,\alpha}$;

(b) $\{\widehat{\mu}_r(a_m)\} \in \ell_q^{p,\alpha+\frac{n}{p}}(\mathfrak{a})$.

Moreover, if

$$-n-\alpha < \frac{1}{p} < \min\left\{1-\alpha, 1-\frac{\alpha}{n}\right\}, \tag{3.4}$$

then the above statements are equivalent to

(c) $\widetilde{\mu} \in \mathcal{K}_q^{p,\alpha}$.

Proof. (a) $\Longleftrightarrow$ (b): Let $\xi_k = 2^{-k(\frac{n}{p}+\alpha)}\|\{\widehat{(\mu\chi_k)}_r(a_m)\}\|_{\ell^p}$ for each k. Note that if $\widehat{(\mu\chi_k)}_r(a_m) > 0$, then $A_k \cap E_r(a_m) \neq \emptyset$ and thus $1-|a_m| \approx 2^{-k}$. Thus, for $p<\infty$, we have

$$\begin{aligned}\xi_k &= \|\{\widehat{(\mu\chi_k)}_r(a_m)(1-|a_m|)^{\frac{n}{p}+\alpha}\}\|_{\ell^p} \\ &= \|\{\widehat{(\mu\chi_k)}_r(a_m)\}\|_{\ell^{p,n+\alpha p}(\mathfrak{a})} \\ &\approx \|\widehat{(\mu\chi_k)}_r\|_{L^p_{\alpha p}}\end{aligned} \tag{3.5}$$

for all k by Lemma 2.3. It follows that

$$\begin{aligned}\widehat{\mu}_r \in \mathcal{K}_q^{p,\alpha} &\Longleftrightarrow \{\|\widehat{\mu}_r\chi_k\|_{L^p_{\alpha p}}\} \in \ell^q && (3.1)\\ &\Longleftrightarrow \{\|\widehat{(\mu\chi_k)}_r\|_{L^p_{\alpha p}}\} \in \ell^q && (\text{Theorem } 3.1)\\ &\Longleftrightarrow \{\xi_k\} \in \ell^q && (3.5)\\ &\Longleftrightarrow \{\widehat{\mu}_r(a_m)\} \in \ell_q^{p,\frac{n}{p}+\alpha}(\mathfrak{a}) && (\text{Corollary } 3.3),\end{aligned}$$

which completes the proof for $p<\infty$. The proof for $p=\infty$ is similar.

(c) $\Longrightarrow$ (a): This follows from Lemma 2.1 and Lemma 2.2.

We now assume (3.4) and prove that (a) or (b) implies (c). For $1 \le p \le \infty$, one may use Lemma 3.4 and [5, Lemma 3.8] to see that (a) implies (c). So, we may further assume $p<1$ in the proof below.

(b) $\Longrightarrow$ (c): Assume (b). First, consider the case $0<q<\infty$. By the proof of Lemma 2.3, we have

$$\widetilde{\mu}(x)^p \lesssim \sum_m (1-|a_m|)^{np}\widehat{\mu}_r(a_m)^p \frac{(1-|x|)^{np}}{[x,a_m]^{2np}} \tag{3.6}$$

for $x \in B$. Let j and k be given. Note that by an integration in polar coordinates and Lemma 2.4 of [5]

$$\begin{aligned}\int_{A_j} \frac{dx}{[x,a]^{2np}} &\lesssim \frac{2^{-j}}{[1-(1-2^{-j-1})|a|]^{2np-n+1}} \\ &\approx \frac{2^{-j}}{(2^{-j}+2^{-k})^{2np-n+1}}, \qquad a \in A_k;\end{aligned}$$

this estimate is uniform in j and k. Hence, if $a_m \in A_k$, then we have

$$\begin{aligned}(1-|a_m|)^{np} &\int_{A_j} \frac{(1-|x|)^{np}}{[x,a_m]^{2np}}(1-|x|)^{\alpha p}\,dx\\ &\approx 2^{-knp}2^{-jp(n+\alpha)}\int_{A_j}\frac{dx}{[x,a_m]^{2np}}\\ &\lesssim \frac{2^{-knp}2^{-j(np+\alpha p+1)}}{(2^{-j}+2^{-k})^{2np-n+1}}\\ &= \frac{2^{-knp}2^{j(np-p\alpha-n)}}{(1+2^{j-k})^{2np-n+1}}.\end{aligned}$$

Thus, setting $\xi_k^p = 2^{-kp(\frac{n}{p}+\alpha)}\sum_{a_m\in A_k}\widehat{\mu}_r(a_m)^p$ and integrating both sides of (3.6) over A_j against the measure $dV_{\alpha p}(x)$, we obtain

$$\int_{A_j}\widetilde{\mu}(x)^p(1-|x|)^{\alpha p}\,dx \lesssim \sum_k \xi_k^p\cdot\frac{2^{(j-k)(np-p\alpha-n)}}{(1+2^{j-k})^{2np-n+1}}.$$

Note that

$$\frac{2^{(j-k)(np-n-\alpha p)}}{(1+2^{j-k})^{2np-n+1}} \le 2^{(j-k)(np-n-\alpha p)} = \frac{1}{2^{(np-n-\alpha p)|k-j|}} \quad \text{for} \quad k\ge j$$

and

$$\frac{2^{(j-k)(np-n-\alpha p)}}{(1+2^{j-k})^{2np-n+1}} \approx \frac{1}{2^{(j-k)(np+\alpha p+1)}} = \frac{1}{2^{(np+\alpha p+1)|k-j|}} \quad \text{for} \quad k< j.$$

Therefore, combining these estimates, we have

$$\int_{A_j}\widetilde{\mu}(x)^p(1-|x|)^{\alpha p}\,dx \lesssim \sum_k \frac{\xi_k^p}{2^{\gamma|k-j|}} \tag{3.7}$$

for all j where $\gamma = \min\{np-n-\alpha p, np+\alpha p+1\}$. Note $\gamma>0$ by (3.4).

Now, for $p<q\le\infty$, we have by (3.7) and Young's inequality

$$\|\widetilde{\mu}\|^p_{\mathcal{K}^{p,\alpha}_q} \lesssim \|\{\xi_k^p\}\|_{\ell^{\frac{q}{p}}} = \|\{\xi_k\}\|^p_{\ell^q}.$$

On the other hand, for $0<q\le p$, we have again by (3.7)

$$\|\widetilde{\mu}\|^q_{\mathcal{K}^{p,\alpha}_q} \lesssim \sum_j\left\{\sum_k\frac{\xi_k^p}{2^{\gamma|k-j|}}\right\}^{\frac{q}{p}} \lesssim \sum_k \xi_k^q\sum_j\frac{1}{2^{\gamma q|k-j|/p}} \approx \|\{\xi_k\}\|^q_{\ell^q}$$

as desired. The case $q=0$ also easily follows from (3.7). The proof is complete. □

The following version of Theorem 1.1 is now a simple consequence of what we've proved so far.

Theorem 3.6. *Let $0<p\le\infty$, $0\le q\le\infty$ and $\mu\ge 0$. Assume that $\mathfrak{a}=\{a_m\}$ is an r-lattice. Then the following three statements are equivalent:*

(a) $T_\mu\in S_{p,q}$;

(b) $\widehat{\mu}_r \in \mathcal{K}^p_q(\lambda)$;
(c) $\{\widehat{\mu}_r(a_m)\} \in \ell^p_q$.

Moreover, if $\frac{n-1}{n} < p \le \infty$, *then the above statements are also equivalent to*

(d) $\widetilde{\mu} \in \mathcal{K}^p_q(\lambda)$.

Proof. The proof of Theorem 3.1 show that all the associated norms are equivalent. We have (a) $\iff$ (b) by Lemma 2.4 and Corollary 3.3. Hence the theorem follows from Theorem 3.5 (with $\alpha = -\frac{n}{p}$). □

In the rest of the paper, we show that the parameter range (3.4) is sharp. Throughout the section we consider arbitrary $0 < p \le \infty$ and α real, unless otherwise specified.

We first recall the following fact which is a consequence of various examples in [3, Section 4];

> *If* $\alpha + 1/p \le -n$ *or* $\alpha + 1/p \ge 1$, *and* $0 \le q \le \infty$, *then there exists some* $f \ge 0$ *such that* $f \in \mathcal{K}^{p,\alpha}_q$ *but* $\widetilde{f} \notin \mathcal{K}^{p,\alpha}_q$.

Moreover, proofs in [3] show that examples of functions f above also satisfy $\widehat{f}_r \in \mathcal{K}^{p,\alpha}_q$ for each $r \in (0,1)$. Note that the above take care of examples we need for the case $\alpha \ge 0$. For $\alpha < 0$, note that the parameter ranges for which we need examples in (3.4) reduce as follows:

$$1 - \frac{\alpha}{n} \le \frac{1}{p}. \tag{3.8}$$

Given $\delta > 1$, let Γ_δ be the nontangential approach region with vertex $\mathbf{e} := (1, 0, \dots, 0)$ consisting of all points $x \in B$ such that

$$|x - \mathbf{e}| < \delta(1 - |x|).$$

Also, given $\gamma \ge 0$, let f_γ be the function on B defined by

$$f_\gamma(x) = \frac{1}{(1-|x|)^n} \left(\log \frac{2}{1-|x|} \right)^{-\gamma}.$$

The source for our examples for the parameter range (3.8) will be functions of the form $f_\gamma \chi_{\Gamma_\delta}$ with γ suitably chosen. We first note the following pointwise estimates taken from [5, Lemma 5.2].

Lemma 3.7. *Given* $\delta > 1$ *and* $\gamma \ge 0$, *the function* $g = f_\gamma \chi_{\Gamma_\delta}$ *has the following properties:*

(a) *Given* $r \in (0,1)$, $f_\gamma \chi_{\Gamma_{\delta_1}} \lesssim \widehat{g}_r \lesssim f_\gamma \chi_{\Gamma_{\delta_2}}$ *for some* δ_1 *and* δ_2;
(b) *If* $\gamma > 1$, *then* $\widetilde{g} \gtrsim f_{\gamma-1} \chi_{\Gamma_{\delta_3}}$ *for some* δ_3;
(c) *If* $\gamma \le 1$, *then* $\widetilde{g} = \infty$ *on some open set.*

Next, we note the precise parameters for $f_\gamma \chi_{\Gamma_\delta}$ to belong to $\mathcal{K}^{p,\alpha}_q$.

Lemma 3.8. *Let* $0 < p \le \infty$, $0 \le q \le \infty$ *and* α *be real. Given* $\delta > 1$ *and* $\gamma \ge 0$, *put* $g = f_\gamma \chi_{\Gamma_\delta}$. *Then* $g \in \mathcal{K}^{p,\alpha}_q$ *if and only if one of the following conditions holds:*

(a) $1/p > 1 - \alpha/n$;
(b) $1/p = 1 - \alpha/n$ *and* $q = \infty$;
(c) $1/p = 1 - \alpha/n$ *and* $q = 0 < \gamma$;
(d) $1/p = 1 - \alpha/n$ *and* $\gamma > \frac{1}{q} > 0$.

Proof. Note $|A_k \cap \Gamma_\delta| \approx 2^{-kn}$ for all k. Thus we have

$$2^{-k\alpha}\|g\chi_k\|_{L^p} \approx \frac{2^{-k\alpha}2^{kn}|A_k \cap \Gamma_\delta|^{\frac{1}{p}}}{(1+k)^\gamma} \approx \frac{2^{-kn(\frac{1}{p}-1+\frac{\alpha}{n})}}{(1+k)^\gamma}$$

for all k, which gives the desired result. $\square$

Now, using Lemmas 3.7 and 3.8, we have examples for the remaining parameters in (3.8) as follows.

Example 3.9. Let $0 < p \le \infty$, $0 \le q \le \infty$ and α be real. Given $\delta > 1$ and $\gamma \ge 0$, put $g = f_\gamma \chi_{\Gamma_\delta}$. Then the following statements hold for each $r \in (0,1)$:

(a) Let $1/p > 1 - \alpha/n$. If $0 \le q \le \infty$ and if $0 \le \gamma \le 1$, then $\widehat{g}_r \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{g} = \infty$ on some open set.
(b) Let $1/p = 1 - \alpha/n$.
  (b1) If $q = 0$ or $1 < q \le \infty$ and if $0 < \gamma \le 1$, then $\widehat{g}_r \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{g} = \infty$ on some open set.
  (b2) If $0 < q \le 1$ and if $1 \le 1/q < \gamma \le 1 + 1/q$, then $\widehat{g}_r \in \mathcal{K}_q^{p,\alpha}$ but $\widetilde{g} \notin \mathcal{K}_q^{p,\alpha}$.

References

[1] Ahlfors, L. V. Möbius transformations in several dimensions. Ordway Professorship Lectures in Mathematics. *Univ. of Minnesota, School of Mathematics, Minneapolis, Minn.*, 1981. MR0725161 (84m:30028), Zbl 0517.30001.

[2] Axler, S.; Bourdon, P.; Ramey, W. Harmonic function theory. Second Edition. Graduate Texts in Mathematics, 137. *Springer-Verlag, New York*, 2001. xii+259 pp. ISBN: 0-387-95218-7. MR1805196 (2001j:31001), Zbl 0959.31001.

[3] Choe, B. R. Note on the Berezin transform on Herz spaces. *RIMS Kyokuroku* **1519** (2006) 21–37.

[4] Choe, B. R.; Koo, H.; Lee, Y. J. Positive Schatten(–Herz) class Toeplitz operators on the half-space. *Potential Analysis* **27** (2007) 73–100. MR2314190 (2008k:31007), Zbl 1140.47013.

[5] Choe, B. R.; Koo, H.; Lee, Y. J. Positive Schatten class Toeplitz operators on the ball. *Studia Mathematica* **189** (2008) 65–90. MR2443376 (2009j:47057), Zbl 1155.47030.

[6] Choe, B. R.; Koo, H.; Na, K. Positive Toeplitz operators of Schatten–Herz type. *Nagoya Math. J.* **185** (2007) 31–62. MR2066107 (2005b:47055), Zbl 1167.47022.

[7] Choe, B. R.; Lee, Y.J.; Na, K. Toeplitz operators on harmonic Bergman spaces. *Nagoya Math. J.* **174** (2004) 165–186. MR2066107 (2005b:47055), Zbl 1067.47039.

[8] Coifman, R. R.; Rochberg, R. Representation theorems for holomorphic and harmonic functions in L^p. *Representation theorems for Hardy spaces*, 11-66, Astérisque, 77. *Soc. Math. France, Paris*, 1980. MR0604369 (82j:32015), Zbl 0472.46040.

[9] Herenández, E.; Yang, D. Interpolation of Herz spaces and applications. *Math. Nachr.* **205** (1999) 69–87. MR1709163 (2000e:46035), Zbl 0936.41001.

[10] Loaiza, M.; López-García, M.; Pérez-Esteva, S. Herz classes and Toeplitz operators in the disk. *Integral Equations Operator Theory* **53** (2005) 287–296. MR2187174 (2006g:47041), Zbl 1107.47019.

[11] Miao, J. Reproducing kernels for harmonic Bergman spaces of the unit ball. *Monatsh. Math.* **125** (1998) 25–35. MR1485975 (98k:46042), Zbl 0907.46020.

[12] Miao, J. Toeplitz operators on harmonic Bergman space. *Integral Equations Operator Theory* **27** (1997) 426–438. MR1442127 (98a:47028), Zbl 0902.47026.

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