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On shift desuspensions of Lewis–May–Steinberger spectra

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Abstract. We prove that on suspension Lewis–May–Steinberger spectra, the shift desuspension and the loop spectrum are isomorphic.

Contents

1. Introduction

The purpose of this brief note is to record an interesting technical effect in the development of equivariant spectra following [2].

To fix notation, let us review the basic definition: Let G be a compact Lie group. Then a G-universe is an infinite-dimensional orthogonal representation of G which is the direct sum of a set R of countably many finite-dimensional irreducible representations, such that R contains a trivial representation, and further contains infinitely many isomorphic copies of every representation it contains.

For a G-universe U and a cofinal set S of finite-dimensional subrepresentations of U, an S-indexed Lewis–May–Steinberger spectrum is a collection of based G-spaces E_W, $W \in S$, together with G-equivariant homeomorphisms

$$_E\rho_V^W = \rho_V^W : E_V \to \Omega^{W-V} E_W,\ V \subseteq W,\ V, W \in S$$

(where $W - V$ denotes the orthogonal complement) such that

$$\rho_V^V = \mathrm{Id},$$

$$\nu \circ \Omega^{W-V} \rho_W^{W'} \circ \rho_V^W = \rho_V^{W'}, \tag{1}$$

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where

$$\nu_{W-V,W'-W} = \nu : \Omega^{W-V}\Omega^{W'-W} \cong \Omega^{W'-V} \tag{2}$$

is the canonical isomorphism given by the adjunct of $F(\phi, ?)$ where $\phi = \phi_{W-V,W'-W}$ is the 1-point compactification of the isomorphism

$$(W - V) \oplus (W' - W) \cong (W' - V)$$

given by

$$(x, y) \mapsto x + y.$$

One defines a category of S-indexed prespectra identically with the exception that ρ_V^W are not required to be homeomorphisms. The forgetful functor from spectra to prespectra has a left adjoint, which is generically denoted by L.

It is well known ([2], Proposition I.2.4) that the categories of S-indexed Lewis–May–Steinberger spectra with different cofinal sets S are canonically equivalent. The analogous statement for prespectra is false.

A key object in our discussion is the functor of *shift desuspension of the suspension spectrum* (Definition I.4.1 of [2]). For a finite-dimensional subrepresentation V of U and a based G-space X, define

$$\Lambda^V \Sigma^\infty X = LD$$

where D is the prespectrum defined by

$$D_W = \begin{cases} \Sigma^{W-V} X & \text{when } V \subseteq W \\ * & \text{else.} \end{cases}$$

There is a more general functor of shift desuspension Λ^V from spectra to spectra ([2], Definition I.7.1) such that, as the notation suggests, $\Lambda^V \Sigma^\infty$ is isomorphic to the composition of Λ^V with the suspension spectrum functor ([2], Lemma I.7.3). The shift desuspension is a key ingredient in developing the notion of *weak equivalence of spectra* technically.

It is a somewhat notorious peculiarity of the theory that it is not known (and widely believed false, cf. [3]) that Λ^V is isomorphic to the level-wise loop functor Ω^V. The reason is that the obvious level-wise map fails to commute with the structure maps due to a switch of isomorphic representation summands. Of course an analogous statement does hold on the level of the stable category.

The main result of the present note is the following:

Proposition 1. *There is a natural isomorphism*

$$\Lambda^V \Sigma^\infty X \cong \Omega^V \Sigma^\infty X.$$

Surprisingly, this has apparently not been observed before, although a step in this direction is the Untwisting lemma 4.5 of the Appendix to [1].

2. The proof

Let E be a Lewis–May–Steinberger spectrum indexed, without loss of generality, on the set $S(U)$ of all finite-dimensional subrepresentations of U. Let V be a finite-dimensional subrepresentation of U. Let $E(V)$ be the spectrum indexed on the set $S(U,V)$ of all finite-dimensional subrepresentations of U containing V, given by

$$E(V)_W = E_{W-V} \text{ for } W \in S(U,V) \tag{3}$$

with structure maps

$$_{E(V)}\rho_W^{W'} = {}_E\rho_{W-V}^{W'-V}.$$

To get the identity (1) for $E(V)$ with V, W, W' replaced by $W \subseteq W' \subseteq W''$, $W, W', W'' \in S(U,V)$, use identity (1) for E with V, W, W' replaced by $W-V, W'-V, W''-V$.

Lemma 2. *There is a natural isomorphism* $E(V) \cong \Omega^V E$.

Proof: For $W \in S(U,V)$, define the isomorphism

$$E(V)_W \to \Omega^V E_W$$

to be the structure map

$$_E\rho_{W-V}^{W'-V} : E_{W-V} \to \Omega^V E_W.$$

To prove that this is an isomorphism of spectra, we must prove the commutativity of the following diagram for $W \subseteq W'$, $W, W' \in S(U,V)$:

$$\begin{array}{ccc}
E_{W-V} & \xrightarrow{\quad \rho_{W-V}^{W'-V} \quad} & \Omega^{W'-W} E_{W'-V} \\
\Big\downarrow \rho_{W-V}^{W} & & \Big\downarrow \Omega^{W'-W}\rho_{W'-V}^{W'} \\
\Omega^V E_W & \xrightarrow{\nu_{W'-W,V}^{-1}\nu_{V,W'-W}(\Omega^V\rho_W^{W'})} & \Omega^{W'-W}\Omega^V E_{W'}.
\end{array}$$

In the diagram, ρ means $_E\rho$. By (1), both compositions in the above diagram are equal to $\nu_{W'-W,V}^{-1}\rho_{W-V}^{W'}$. In more detail, composing $\nu_{W'-W,V}$ with the right column with the top row gives $\nu_{W'-W,V} \circ \Omega^{W'-W}\rho_{W'-V}^{W'} \circ \rho_{W-V}^{W'-V}$, which is $\rho_{W-V}^{W'}$ by (1) with V, W, W' replaced by $W-V$, $W'-V$, W', respectively. Composing $\nu_{W'-W,V}$ with the bottom row with the left column gives $\nu_{V,W'-W} \circ \Omega^V\rho_W^{W'} \circ \rho_{W-V}^{W}$ which is $\rho_{W-V}^{W'}$ by (1) with V, W, W' replaced by $W-V$, W, W', respectively. □

By cofinality, $\Lambda^V\Sigma^\infty$ is naturally isomorphic to the spectrum associated with the $S(U,V)$-indexed prespectrum

$$D'(W) = \Sigma^{W-V}X \text{ for } W \in S(U,V),$$

with the same structure maps $\phi_{W'-W,W-V}$. Since the (?)(W) construction makes sense as a functor from $S(U)$-indexed prespectra to $S(U,V)$-indexed prespectra, and commutes obviously with spectrification on inclusion prespectra (levelwise, the colimits being of isomorphic diagrams on both sides), we obtain an isomorphism

$$LD' \cong \Sigma^{\infty}(V).$$

This concludes the proof of the proposition.

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