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On principal left ideals of βG

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Abstract. Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group written additively, let $G = \bigoplus_{\alpha<\kappa} G_\alpha$, and let $H_\alpha = \{x \in G : x(\gamma) = 0 \text{ for all } \gamma < \alpha\}$. Let βG be the Stone–Čech compactification of G as a discrete semigroup and define a closed subsemigroup $T \subseteq \beta G$ by $T = \bigcap_{\alpha<\kappa} \mathrm{cl}_{\beta G}(H_\alpha \setminus \{0\})$. We show that, for every $p, q \in T$, if $(\beta G + p) \cap (\beta G + q) \neq \emptyset$, then either $p \in \beta G + q$ or $q \in \beta G + p$.

Let S be a discrete semigroup with Stone–Čech compactification βS. The operation on S extends to one on βS in such a way that for each $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous, and for each $q \in \beta S$, the right translation

$$\beta S \ni x \mapsto xq \in \beta S$$

is continuous.

We take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S. The topology of βS is generated by taking as a base the subsets of the form

$$\bar{A} = \{p \in \beta S : A \in p\},$$

where $A \subseteq S$. For $p, q \in \beta S$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup_{x \in A} xB_x,$$

where $A \in p$ and $B_x \in q$.

The semigroup βS is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to βS can be found in [1].

In the study of algebraic structure of βS an important role is played by the following fact.

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Theorem 1 ([1, Corollary 6.20])**.** *Let G be a countable group. For every $p, q \in \beta G$, if $((\beta G)p) \cap ((\beta G)q) \neq \emptyset$, then either $p \in (\beta G)q$ or $q \in (\beta G)p$.*

Theorem 1 tells us that for any countable group G and for any two principal left ideals of βG, either one of them is contained in another or they are disjoint.

In this note we prove the following extension of Theorem 1.

Theorem 2. *Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group written additively, let $G = \bigoplus_{\alpha<\kappa} G_\alpha$, and let*

$$H_\alpha = \{x \in G : x(\gamma) = 0 \text{ for all } \gamma < \alpha\}.$$

Define a closed subsemigroup $T \subseteq \beta G$ by $T = \bigcap_{\alpha<\kappa} \overline{H_\alpha \setminus \{0\}}$. For every $p, q \in T$, if $(\beta G + p) \cap (\beta G + q) \neq \emptyset$, then either $p \in \beta G + q$ or $q \in \beta G + p$.

Before proving Theorem 2, let us check that T is indeed a subsemigroup. It suffices to show that for every $p, q \in T$ and $\alpha < \kappa$, $p + q \in \overline{H_\alpha \setminus \{0\}}$, equivalently $H_\alpha \setminus \{0\} \in p + q$. Define $A \in p$ and $B_x \in q$ for every $x \in A$ by $A = H_\alpha$ and $B_x = H_\alpha \setminus \{-x\}$. Then $x + B_x \subseteq H_\alpha \setminus \{0\}$, and so $\bigcup_{x\in A}(x+B_x) \subseteq H_\alpha \setminus \{0\}$. But $\bigcup_{x\in A}(x+B_x) \in p+q$. Hence, $H_\alpha \setminus \{0\} \in p+q$.

Proof of Theorem 2. Assume on the contrary that $p \notin \beta G + q$ and $q \notin \beta G + p$ for some $p, q \in T$. We shall show that

$$(\beta G + p) \cap (\beta G + q) = \emptyset,$$

which is a contradiction.

Since $p \notin \beta G + q$, there are $P \in p$ and $Q_x \in q$ for every $x \in G$ such that

$$P \cap \bigcup_{x\in G}(x + Q_x) = \emptyset.$$

And since $q \notin \beta G + p$, there are $Q \in q$ and $P_x \in p$ for every $x \in G$ such that

$$Q \cap \bigcup_{x\in G}(x + P_x) = \emptyset.$$

For every $x \in G \setminus \{0\}$, let

$$\phi(x) = \max \operatorname{supp}(x) \text{ and } \theta(x) = \min \operatorname{supp}(x).$$

As usual, $\operatorname{supp}(x) = \{\alpha < \kappa : x(\alpha) \neq 0\}$. Also let

$$\phi(0) = -1 \text{ and } \theta(0) = \kappa.$$

Define partial orders $\leq_L$ and $\leq_R$ on G by

$$x \leq_L y \text{ if and only if } x(\alpha) = y(\alpha) \text{ for each } \alpha \leq \phi(x),$$
$$x \leq_R y \text{ if and only if } x(\alpha) = y(\alpha) \text{ for each } \alpha \geq \theta(x).$$

Now for every $x \in G$, define $A_x \in p$ and $B_x \in q$ by

$$A_x = \left(\bigcap_{y \le_R x} P_y\right) \cap P \cap H_{\phi(x)+1},$$

$$B_x = \left(\bigcap_{y \le_R x} Q_y\right) \cap Q \cap H_{\phi(x)+1}.$$

(Notice that $\{y \in G : y \le_R x\}$ is finite.) It then follows that:

(1) $\left(\bigcup_{x\in G}(x + A_x)\right) \cap B_0 = \emptyset$ and $\left(\bigcup_{x\in G}(x + B_x)\right) \cap A_0 = \emptyset$.

(2) For every $x \in G$ and for every $y \le_R x$, one has $A_x \subseteq A_y$ and $B_x \subseteq B_y$, in particular, $A_x \subseteq A_0$ and $B_x \subseteq B_0$.

(3) For every $x \in G$, one has $A_x, B_x \subseteq H_{\phi(x)+1}$.

We claim that

$$\left(\bigcup_{x\in G}(x + A_x)\right) \cap \left(\bigcup_{x\in G}(x + B_x)\right) = \emptyset.$$

To see this, let $x, y \in G$. We have to show that $(x + A_x) \cap (y + B_y) = \emptyset$. Consider two cases.

Case 1: neither $x \le_L y$ nor $y \le_L x$. Then $(x+H_{\phi(x)+1})\cap(y+H_{\phi(y)+1}) = \emptyset$. Consequently by (3), $(x + A_x) \cap (y + B_y) = \emptyset$.

Case 2: either $x \le_L y$ or $y \le_L x$. Let $y \le_L x$. By (1), $(x-y+A_{x-y})\cap B_0 = \emptyset$, so $(x + A_{x-y}) \cap (y + B_0) = \emptyset$. But $x - y \le_R x$ and $0 \le_R y$. Consequently by (2), again $(x + A_x) \cap (y + B_y) = \emptyset$.

Since the subsets

$$U = \bigcup_{x\in G}(x + A_x) \quad \text{and} \quad V = \bigcup_{x\in G}(x + B_x)$$

of G are disjoint, the subsets $\bar{U}$ and $\bar{V}$ of βG are also disjoint. But $\beta G + p \subseteq \bar{U}$ and $\beta G + q \subseteq \bar{V}$. Hence, $(\beta G + p) \cap (\beta G + q) = \emptyset$. □

Remark 1. Theorem 2 was inspired by [2, Proposition 3.4].

Remark 2. The semigroup T from Theorem 2 depends only on two cardinals: κ and $\lambda = \min\{|\bigoplus_{\gamma\le\alpha<\kappa} G_\alpha| : \gamma < \kappa\}$ [3].

We conclude this note with the following question.

Question. Is it true that for any (Abelian) group G and for any two principal left ideals of βG, either one of them is contained in another or they are disjoint?

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