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Bad intersections and constructive aspects of the Bloch–Quillen formula

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Abstract. The Bloch–Quillen formula, especially a version with Milnor K-coefficients, makes it possible to express the product of the Chow ring through an ordinary cup product in sheaf cohomology and the concatenation product of symbols. No special care is needed if cycles do not intersect properly, no moving lemma nor deformation to the normal cone. We give an explicit formula for the intersection form along this line, different from the Serre Tor-formula.

Contents

In this paper I would like to discuss a method to compute intersection multiplicities via the Bloch–Quillen formula in K-theory which — to the best of my knowledge — does not seem to be used much for concrete computations (if at all). In particular I want to confirm that actual computations are possible, even by hand.

It yields a closed formula for intersection multiplicities quite different from Serre's Tor-formula.

The basic idea is very simple. Suppose X/k is a smooth variety[1] of pure dimension n. The classical Bloch–Quillen formula is a canonical isomorphism of commutative rings

$$\mathrm{CH}^*(X) \cong \coprod_{p\geq 0} H^p(X, \mathcal{K}_p). \tag{1}$$

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[1]See §1 for our precise assumptions (and the conventions and notation we use).

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Here $\mathcal{K}_p$ denotes the Zariski sheafification of the p-th K-theory group. The product on the right-hand side is the cup product in sheaf cohomology, combined with the product in K-theory $\mathcal{K}_p \otimes_{\underline{\mathbf{Z}}} \mathcal{K}_q \to \mathcal{K}_{p+q}$, where $\underline{\mathbf{Z}}$ is the locally constant sheaf associated to $\mathbf{Z}$. For example, say X is a surface, E a divisor and we want to know the self-intersection $E \cdot E$. This is *not* a proper intersection, so by classical means we would need to take some detour to compute $E \cdot E$. Trying to avoid this, it seems all the more tempting to just use the above and compute the cup product

$$H^1(X, \mathcal{O}_X^\times) \otimes H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X, \mathcal{K}_2) \xrightarrow{?} \mathbf{Z}. \tag{2}$$

(Note that $\mathcal{K}_1 \cong \mathcal{O}_X^\times$ as sheaves.) On some small open, if e is a local section pinning down the line bundle belonging to E, this product is just

$$e \otimes e \longmapsto \{e, e\} \overset{?}{\longmapsto} \mathbf{?}$$

and so the whole difficulty in computing the self-intersection number is to understand the map "?". Or, if we want to be more precise and evaluate the self-intersection zero cycle instead of just the self-intersection number, we would need to understand the map $H^2(X, \mathcal{K}_2) \to \mathrm{CH}_0(X)$ on an explicit level. Perhaps surprisingly, it turns out that this method produces a closed formula for "?".

Moreover, the same approach works if X has higher dimension. Then an immediate drawback of the right-hand side in Equation (1) is that it is not a priori easy to write down a single explicit element of a higher K-theory group, at least not if one uses Quillen's definition in terms of homotopy groups. However, actually there are many variations of the isomorphism in Equation (1), where K-theory can be replaced by something simpler, for example Milnor K-theory. We briefly recall that for a ring A we may define

$$K_*^{\mathrm{M}}(A) := T(A^\times) / \left\langle x \otimes (1-x) \mid \text{all } x \text{ with } x, 1-x \in A^\times \right\rangle, \tag{3}$$

where $T(A^\times) := \coprod_{p \geq 0} (A^\times)^{\otimes_{\mathbf{Z}}^p}$ denotes the tensor algebra of $A^\times$ (read as a $\mathbf{Z}$-module) and we quotient out the two-sided ideal generated by the Steinberg relation (cf. [Ker10], [Mil70]). The image of a pure tensor $a_1 \otimes \cdots \otimes a_p$ is denoted by $\{a_1, \ldots, a_p\}$ (and called a *symbol*). As a consequence of the Steinberg relation, one finds several useful relations, notably

$$\{x, y\} = -\{y, x\} \qquad \{x, -x\} = 0 \qquad \{x, x\} = \{x, -1\}. \tag{4}$$

Being an ideal, these prolong to tensors in more slots, e.g., $\{x, y, z\} = -\{y, x, z\}$ or $\{x, x, x\} = \{x, -1, -1\}$. Now let $\mathcal{K}_*^{\mathrm{M}}$ denote the Zariski sheafification of Milnor K-theory (we will work with a slightly more convenient definition below). The Milnor K-theory counterpart of the product map $\mathcal{K}_p \otimes_{\underline{\mathbf{Z}}} \mathcal{K}_q \longrightarrow \mathcal{K}_{p+q}$ is strikingly simple,

$$\begin{aligned} \mathcal{K}_p^{\mathrm{M}} \otimes_{\underline{\mathbf{Z}}} \mathcal{K}_q^{\mathrm{M}} &\longrightarrow \mathcal{K}_{p+q}^{\mathrm{M}} \\ \{a_1, \ldots, a_p\} \otimes \{b_1, \ldots, b_q\} &\longmapsto \{a_1, \ldots, a_p, b_1, \ldots, b_q\} \end{aligned} \tag{5}$$

on the level of stalks. With this definition the Milnor analogue of the right-hand side in Equation (1) already looks much more amenable to actual computations.
The only remaining problem is that for any reasonable usability we probably want to work with algebraic cycle representatives in $\mathrm{CH}^p(X)$ and not with less wieldy representatives in sheaf cohomology for $H^p(X, \mathcal{K}_p^{\mathrm{M}})$. Hence, we need to understand the comparison isomorphisms

$$\alpha^p : \mathrm{CH}^p(X) \to H^p(X, \mathcal{K}_p^{\mathrm{M}}) \qquad \text{and} \qquad \beta^p : H^p(X, \mathcal{K}_p^{\mathrm{M}}) \to \mathrm{CH}^p(X)$$

on a level suitable for computations; and possibly also the map "?" of Equation (2) if we are just interested in plain intersection numbers (in general the computation of $\mathrm{CH}_0(X)$ is a hard problem). Here, it turns out that only α^p is truly difficult. The case $p = 1$ however is nice and classical,

$$\mathrm{Cl}\, X = \mathrm{CH}^1(X) \xrightarrow{\alpha^1} H^1(X, \mathcal{K}_1^{\mathrm{M}}) = H^1(X, \mathcal{O}_X^{\times}) = \mathrm{Pic}\, X, \tag{6}$$

it is the usual identification of Weil divisor classes with Cartier divisor classes/line bundles. For $p > 1$ the maps α^p are *far* less pleasant. I am not aware of any general method or even closed formula to make this morphism concrete; one always needs to make a great number of choices, e.g., if we model the sheaf cohomology using Čech theory we need to pick a good open cover, good representatives on the respective opens, etc… For the converse direction, the maps β^p admit a very pleasant description as a closed formula:

Theorem 1. *Let X/k be a smooth[2] variety of pure dimension n over a field. Suppose an element in $H^p(X, \mathcal{K}_p^{\mathrm{M}})$ is explicitly presented as a Čech cocycle $f = (f_{\beta_0 \ldots \beta_p})$ on some finite open cover $(U_i)_{i \in I}$. Then for every disjoint decomposition*

$$X = \dot{\bigcup}_{\alpha \in I} \Sigma_\alpha \qquad \textit{with} \qquad \Sigma_\alpha \subseteq U_\alpha \tag{7}$$

we have $(\beta^p f) = \coprod_{x^p \in X^p} h_{x^p}$ with multiplicities

$$h_{x^p} := \sum_{x^{p-1}, \ldots, x^0} \partial_{x^p}^{x^{p-1}} \cdots \partial_{x^1}^{x^0} f_{\alpha(x^0) \ldots \alpha(x^p)} \in \mathbf{Z}.$$

The sum runs through all chains of points $x^i \in X^i$ and $\alpha(x)$ denotes the unique index such that $x \in \Sigma_{\alpha(x)}$.

We will prove this as Proposition 2 in §2 below. The maps

$$\partial_y^x : K_q^{\mathrm{M}}(\kappa(x)) \longrightarrow K_{q-1}^{\mathrm{M}}(\kappa(y))$$

for points $x, y \in X$ such that y is of codimension one in $\overline{\{x\}}$ are the boundary maps in Milnor K-theory. We recall their definition in §1. Based on the above explicit formula, the general version of Equation (2) is given as follows:

[2]The smoothness assumption can be dropped and the formula still gives a morphism, see Remark 4 in the main body of the paper. However, it will usually not be an isomorphism.

Theorem 2. *In the situation of the previous theorem, the intersection form for Weil divisors $Z_1, \ldots, Z_p$ (with $p = n$) is given by the explicit formula*

$$(8)\quad h_{x^p} := \sum_{x^{p-1},\ldots,x^0} \partial^{x^{p-1}}_{x^p} \cdots \partial^{x^0}_{x^1} \{f^1_{\alpha(x^0)\alpha(x^1)}, f^2_{\alpha(x^1)\alpha(x^2)}, \ldots, f^p_{\alpha(x^{p-1})\alpha(x^p)}\} \in \mathbf{Z}.$$

Here $f^i = (f^i_{\alpha,\beta})_{\alpha,\beta\in I}$ with $f^i_{\alpha,\beta} \in \mathcal{O}^\times_X$ is a Čech representative of the line bundle determined by Z_i.

This will be Proposition 3. The above formula also admits a counterpart using differential forms and residues instead of symbols and boundaries (linking to the classical fact that intersection numbers can be computed via residues), we explain this in §4. We obtain a precise Čech analogue of a formula of Hübl and Yekutiely in the context of adèles [HY96]. We also explain that all this generalizes from Milnor K-theory to cycle modules (as introduced by Rost [Ros96]). We intentionally avoid this quite technical framework until this point as we have almost no use for the strength or generality of this theory.

We give an explicit computation of a negative self-intersection on a surface in §5.

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The "hands-on style" of Spencer Bloch's classic [Blo74] has served as an inspiration. So has the theory of adèles for schemes [Beĭ80], see Remark 7. I would also like to thank the anonymous referee, whose remarks led to a more streamlined presentation.

1. Recollections of the Gersten complex

A *variety* X is a scheme which is separated and finite type over a field. Once and for all let us fix a smooth variety X/k of pure dimension over a field k.
See [GS06, Ch. 8]. Let X^p denote the *set* of codimension p points. We write $\kappa(x)$ for the residue field at a point $x \in X$. We define the *Milnor K-theory sheaf* by

$$(9)\qquad K^{\mathrm{M}}_p(U) := \ker\left(\coprod_{x\in U^0} K^{\mathrm{M}}_p(\kappa(x)) \longrightarrow \coprod_{x\in U^1} K^{\mathrm{M}}_p(\kappa(x))\right)$$

for U any open in X and the Milnor K-groups of fields inside the parentheses are defined as in the introduction, cf. Equation (3). The (proven) Gersten conjecture, in the version for Milnor K-groups, is the following:

Proposition 1 (Gersten Conjecture)**.** *Let k be a field. Suppose $X = \operatorname{Spec} A$, where A is a local ring on a smooth k-scheme. Then the sequence of abelian groups*

$$0 \to K_p^{\mathrm{M}}(X) \to \textstyle\coprod_{x \in X^0} K_p^{\mathrm{M}}(\kappa(x)) \to \cdots \to \coprod_{x \in X^q} K_{p-q}^{\mathrm{M}}(\kappa(x)) \to \cdots \tag{10}$$

is exact.

This was originally proven by Gabber. For a proof see [Ros96, Thm. 6.1].

Remark 1. Instead of the definition in Equation (9) one can also sheafify the plain definition of Milnor K-groups as in Equation (3). If k is an infinite field, both definitions agree and the above proposition remains true. However, this is a nontrivial theorem. See [EVMS02], [Ker09]. For the case of k a finite field there is a remedy nonetheless, see [Ker10]

The proposition is known to fail for singular X. See [Mor12, especially Conj. 1] for a possible workaround in the context of algebraic K-theory.
The differential in the complex is the sum of all Milnor K-boundary maps, i.e., $\sum_{x \in X^p, y \in \overline{\{x\}}^1} \partial_y^x$. Applied to any element, all but finitely many summands will be zero. For the convenience of the reader we recall the definition and basic properties:

Assume for simplicity that the codimension one point $y \in \overline{\{x\}}$ is a normal point, i.e., $\mathcal{O}_{\overline{\{x\}},y}$ is normal.

Let v be the valuation coming from interpreting y as a divisor on $\overline{\{x\}}$. Then:

- For $q = 1$ the map $\partial_y^x : \kappa(x)^\times \to \mathbf{Z}$ is just the valuation v.
- For $q = 2$ the map $\partial_y^x : K_2^{\mathrm{M}}(\kappa(x)) \to \kappa(y)^\times$ is the tame symbol: $\{f, g\} \mapsto (-1)^{v(f)v(g)}\, \overline{f^{v(g)}/g^{v(f)}}$.
- In general one has the following formula:

$$\begin{aligned} \partial_y^x\{\pi, u_2, \ldots, u_q\} &:= \{\overline{u_2}, \ldots, \overline{u_q}\} \\ \partial_y^x\{u_1, \ldots, u_q\} &:= 0 \end{aligned} \tag{11}$$

  for $u_1, \ldots, u_q \in \mathcal{O}^\times_{\overline{\{x\}},y}$ and $v(\pi) = 1$ a uniformizer. Via the relations of Equations (4) any pure symbol can be rewritten as a $\mathbf{Z}$-linear combination of symbols as they occur on the left-hand side.
- There is a short exact sequence of abelian groups

$$0 \to K_q^{\mathrm{M}}(\mathcal{O}_{\overline{\{x\}},y}) \longrightarrow K_q^{\mathrm{M}}(\kappa(x)) \xrightarrow{\partial_y^x} K_{q-1}^{\mathrm{M}}(\kappa(y)) \longrightarrow 0.$$

If y is not a normal point on $\overline{\{x\}}$, all of the above facts need to replaced by something slightly more complicated:

The stalk $\mathcal{O}_{\overline{\{x\}},y}$ is a 1-dimensional local domain. The normalization

$$\operatorname{Spec} \mathcal{O}'_{\overline{\{x\}},y} \to \operatorname{Spec} \mathcal{O}_{\overline{\{x\}},y}$$

is a finite morphism. Hence, the unique closed point y in $\operatorname{Spec}\mathcal{O}_{\overline{\{x\}},y}$ has finite preimage $\{y_1',\ldots,y_r'\}\subseteq\operatorname{Spec}\mathcal{O}'_{\overline{\{x\}},y}$. By normality the localizations $(\mathcal{O}'_{\overline{\{x\}},y})_{y_i'}$ are discrete valuation rings. Define

$$\partial_y^x := \textstyle\sum_{i=1}^r \operatorname{cor}_{\kappa(y)}^{\kappa(y_i')}\circ\partial_{y_i'}^x : K_q^{\mathrm{M}}(\kappa(x))\to K_{q-1}^{\mathrm{M}}(\kappa(y)), \tag{12}$$

where $\partial_{y_i'}^x$ refers to the Milnor K-boundary map as described before (the points y_i' are normal!). The map $\operatorname{cor}_{\kappa(y)}^{\kappa(y_i')}$ is the corestriction/norm of Milnor K-theory (on K_0^{M} it is multiplication with the degree of the field extension, on K_1^{M} it is the usual norm). See [GS06, Constr. 8.1.1] or [Ros96, beginning of §2, especially Equation 2.1.0] for details regarding this mechanism.

We may now define (obviously flasque!) sheaves

$$\coprod_{x\in U^q}\underline{K_{p-q}^{\mathrm{M}}}(\kappa(x)) := \left(U\longmapsto\coprod_{x\in U^q}K_{p-q}^{\mathrm{M}}(\kappa(x))\right). \quad (\text{for } U\subseteq X \text{ open}) \tag{13}$$

By a slight abuse of language we may now interpret each entry in sequence (10) as such a sheaf and replace the initial entry by its sheafification $\mathcal{K}_p^{\mathrm{M}}$. As the exactness of a sequence of sheaves can be checked on the level of stalks, Proposition 1 can be rephrased as saying that the sheaf $\mathcal{K}_p^{\mathrm{M}}$ has a flasque resolution by the sheaves of Equation (13) (the *Gersten resolution*):

$$0\longrightarrow\mathcal{K}_p^{\mathrm{M}}\longrightarrow\coprod_{x\in U^0}\underline{K_p^{\mathrm{M}}}(\kappa(x))\longrightarrow\coprod_{x\in U^1}\underline{K_{p-1}^{\mathrm{M}}}(\kappa(x))\longrightarrow\cdots. \tag{14}$$

This also implies the *Bloch–Quillen formula*

$$\begin{aligned} H^p(X,\mathcal{K}_p^{\mathrm{M}}) &= \operatorname{coker}\left(\textstyle\coprod_{x\in X^{p-1}}K_1^{\mathrm{M}}(\kappa(x))\to\coprod_{x\in X^p}K_0^{\mathrm{M}}(\kappa(x))\right)\\ &= \mathrm{CH}^p(X). \end{aligned} \tag{15}$$

See [Blo74], [Qui73] for the original version using ordinary K-theory (the pattern of proof is the same). It remains to render this abstract map explicit.

2. Čech model for Chow groups

Firstly, recall that for any open cover $\mathfrak{U}=(U_\alpha)_{\alpha\in I}$ (I the index set) and Zariski sheaf $\mathcal{F}$ there are Čech cohomology groups, which we denote by $\check{H}^i(\mathfrak{U},\mathcal{F})$, defined as the cohomology of the Čech complex

$$\check{\mathrm{C}}^i(\mathfrak{U},\mathcal{F}) := \prod_{\alpha_0\ldots\alpha_r\in I^{i+1}}\mathcal{F}(U_{\alpha_0\ldots\alpha_r}),\quad \delta:\check{\mathrm{C}}^i(\mathfrak{U},\mathcal{F})\to\check{\mathrm{C}}^{i+1}(\mathfrak{U},\mathcal{F}),$$

where we denote by U_α an open in $\mathfrak{U}$, and by $U_{\alpha_0\ldots\alpha_r} := \bigcap_{i=0,\ldots,r}U_{\alpha_i}$ the respective intersections. For any refinement $\mathfrak{U}'$ of $\mathfrak{U}$, there is a canonical induced morphism $\check{H}^i(\mathfrak{U},\mathcal{F})\to\check{H}^i(\mathfrak{U}',\mathcal{F})$ (and 'unique' on $\check{\mathrm{C}}^\bullet$ only up to homotopies). Finally, $\check{H}^i(X,\mathcal{F})$ is defined as $\operatorname{colim}_{\mathfrak{U}}\check{H}^i(\mathfrak{U},\mathcal{F})$ over the diagram in which $\mathfrak{U}$ runs through all at most countably indexed open covers and arrows are the refinements. This is a filtering colimit since any two covers admit a common refinement. We have $\check{H}^i(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$ for all sheaves we work with (as they all admit a flasque resolution), so we may express sheaf cohomology this way.

2.1. Algebraic partitions of unity. In the world of real manifolds one can often patch local sections of suitable sheaves by gluing them along a partition of unity. However, in the algebraic world there is no way to "smoothly fade contributions in and out". The best possible approximation in the algebraic world are functions which only attain the values 1 and 0; characteristic functions of subsets. All sheaves which allow some sort of a multiplication with such functions then admit a similar patching mechanism. More precisely:

Lemma 1 (Algebraic "Partition of Unity"). *Suppose $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ is an open cover and $\mathcal{F}$ is a flasque sheaf such that:*

(1) *For every restriction $\operatorname{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for any two opens of the form $U_{\gamma_0\dots\gamma_s\gamma_{s+1}\dots\gamma_r} =: V \subseteq U := U_{\gamma_0\dots\gamma_s}$ we are given a morphism*
$$E^U_V : \mathcal{F}(V) \to \mathcal{F}(U).$$
(2) *For every open $V := U_{\gamma_0\dots\gamma_s}$ we have*
$$\textstyle\sum_{\alpha\in I} E^V_{V_\alpha} \circ \operatorname{res}^V_{V_\alpha} = \operatorname{id}_{\mathcal{F}(V)}\,, \qquad \text{where } V_\alpha := V \cap U_\alpha.$$
(3) *For every open $V := U_{\gamma_0\dots\gamma_s}$ and all indices $\beta \in I$ we have*
$$\textstyle\coprod_{\alpha\in I} \mathcal{F}(V_\alpha) \to \mathcal{F}(V_\beta), \qquad \text{where } V_\alpha := V \cap U_\alpha.$$
$$\textstyle\sum_{\alpha\in I} \operatorname{res}^V_{V_\beta} \circ E^V_{V_\alpha} = \sum_{\alpha\in I} E^{V_\beta}_{V_{\alpha\beta}} \circ \operatorname{res}^{V_\alpha}_{V_{\alpha\beta}}\,.$$

Define a homomorphism
$$H : \check{\mathrm{C}}^i(\mathfrak{U},\mathcal{F}) \to \check{\mathrm{C}}^{i-1}(\mathfrak{U},\mathcal{F})$$
$$(Hf)_{\beta_0\dots\beta_{i-1}} := \textstyle\sum_{\alpha\in I} E^{U_{\beta_0\dots\beta_{i-1}}}_{U_{\alpha\beta_0\dots\beta_{i-1}}} f_{\alpha\beta_0\dots\beta_{i-1}}.$$

Then H is a contracting homotopy for the Čech complex $\check{\mathrm{C}}^\bullet(\mathfrak{U},\mathcal{F})$, i.e., $H\delta + \delta H = \operatorname{id}_{\check{\mathrm{C}}^i(\mathfrak{U},\mathcal{F})}$.

The statements about the sums $\sum_{\alpha\in I}$ above are meant to imply that only finitely many summands are nonzero (otherwise they would not make sense at all).

Proof. Easy computation. Suppose we are given a cocycle $(f_{\beta_0\dots\beta_{i-1}}) \in \check{\mathrm{C}}^{i-1}(\mathfrak{U},\mathcal{F})$. For any $\alpha \in I$ we compute

$$(\delta f)_{\alpha\beta_0\dots\beta_{i-1}} = \operatorname{res}^{U_{\beta_0\dots\beta_{i-1}}}_{U_{\alpha\beta_0\dots\beta_{i-1}}} f_{\beta_0\dots\beta_{i-1}} - \sum_{k=0}^{i-1} (-1)^k \operatorname{res}^{U_{\alpha\beta_0\dots\widehat{\beta_k}\dots\beta_{i-1}}}_{U_{\alpha\beta_0\dots\beta_{i-1}}} f_{\alpha\beta_0\dots\widehat{\beta_k}\dots\beta_{i-1}}$$

and thus

$$(H\delta f)_{\beta_0\ldots\beta_{i-1}} = f_{\beta_0\ldots\beta_{i-1}} - \sum_{k=0}^{i-1}(-1)^k \sum_{\alpha\in I} E^{U_{\beta_0\cdots\beta_{i-1}}}_{U_{\alpha\beta_0\cdots\beta_{i-1}}} \operatorname{res}^{U_{\alpha\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}}_{U_{\alpha\beta_0\cdots\beta_{i-1}}} f_{\alpha\beta_0\ldots\widehat{\beta_k}\ldots\beta_{i-1}}, \tag{16}$$

where we have used property (2). Starting from $(Hf)_{\beta_0\ldots\beta_{i-2}}$, we compute

$$\begin{aligned}
&(\delta H f)_{\beta_0\ldots\beta_{i-1}} \\
&= \sum_{k=0}^{i-1}(-1)^k \operatorname{res}^{U_{\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}}_{U_{\beta_0\cdots\beta_{i-1}}} (Hf)_{\beta_0\ldots\widehat{\beta_k}\ldots\beta_{i-1}} \\
&= \sum_{k=0}^{i-1}(-1)^k \sum_{\alpha\in I} \operatorname{res}^{U_{\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}}_{U_{\beta_0\cdots\beta_{i-1}}} E^{U_{\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}}_{U_{\alpha\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}} f_{\alpha\beta_0\ldots\widehat{\beta_k}\ldots\beta_{i-1}}.
\end{aligned}$$

Defining $V := U_{\beta_0\ldots\widehat{\beta_k}\ldots\beta_{i-1}}$ the inner sum over α can be rewritten as

$$\textstyle\sum_{\alpha\in I} \operatorname{res}^{V}_{V_{\beta_k}} E^{V}_{V_\alpha},$$

so by using property (3) we obtain

$$= \sum_{k=0}^{i-1}(-1)^k \sum_{\alpha\in I} E^{U_{\beta_0\cdots\beta_{i-1}}}_{U_{\alpha\beta_0\cdots\beta_{i-1}}} \operatorname{res}^{U_{\alpha\beta_0\cdots\widehat{\beta_k}\cdots\beta_{i-1}}}_{U_{\alpha\beta_0\cdots\beta_{i-1}}} f_{\alpha\beta_0\ldots\widehat{\beta_k}\ldots\beta_{i-1}},$$

so by revisiting Equation (16) our claim follows. □

Remark 2. Suppose Y is a smooth manifold, $(U_i)_{i\in I}$ an open cover. Let (ρ_i) be a classical partition of unity subordinate to the open cover. Consider the sheaf $\mathcal{F} := \mathcal{C}^\infty(\mathbf{R})$ of smooth real-valued functions. Then

$$E^{U_{\beta_0\cdots\beta_i}}_{U_{\alpha\beta_0\cdots\beta_i}}(f) := \rho_\alpha \cdot f$$

(and prolonged by zero) satisfies the axioms of the lemma. For example,

$$\textstyle\sum_{\alpha\in I} E^V_{V_\alpha} \circ \operatorname{res}^V_{V_\alpha} f = \sum_{\alpha\in I}\rho_\alpha\cdot(f\mid_{V_\alpha}) = \left(\sum_{\alpha\in I}\rho_\alpha\right) f = f.$$

Next, we need to make sure that morphisms E^U_V as in the previous lemma exist for the flasque sheaves which occur in the Gersten resolution of our $\mathcal{K}^{\mathrm{M}}$-sheaves.

Lemma 2. *Assume the open cover $\mathfrak{U}$ is finite, i.e., I is a finite set. Suppose $\mathcal{F}$ is a sheaf of the shape*

$$\mathcal{F}(U) := \coprod_{y\in U} A_y,$$

where each A_y is some abelian group depending only on y. Fix a disjoint decomposition (always exists!)

$$X = \dot{\bigcup}_{\alpha\in I}\Sigma_\alpha \qquad \textit{with } \Sigma_\alpha \subseteq U_\alpha. \tag{17}$$

Define for any open $U := U_{\gamma_0 \ldots \gamma_s}$ *and further intersection* $V := U \cap U_\beta$ $(= U_{\gamma_0 \ldots \gamma_s \beta})$ *the homomorphism*

$$E_V^U : \mathcal{F}(V) \to \mathcal{F}(U)$$

$$\prod_{y \in V} A_y \mapsto \prod_{y \in U} A_y \qquad by \qquad \begin{cases} \mathrm{id}_{A_y} & \textit{if } y \in \Sigma_\beta \\ 0 & \textit{otherwise.} \end{cases}$$

and transitively define $E_V^U := E_{U \cap U_\beta}^U \circ E_V^{U \cap U_\beta}$ *if* $V := U \cap U_\beta \cap U_{\beta_2}$ *etc. Then the assumptions of Lemma* 1 *are satisfied.*

Remark 3. Of course such a drastic "switch-on / switch-off" definition as in this lemma would hopelessly fail for a sheaf of smooth functions on a manifold as in Remark 2.

Proof. A disjoint decomposition as in Equation (17) exists since $\mathfrak{U}$ is a cover, so $X = \bigcup U_\alpha$. For example (well-)order the set I and then let $x \in \Sigma_\alpha$ if and only if α is the (unique) smallest element of I such that $x \in U_\alpha$. It remains to prove the properties (2) and (3). For (2) observe that for every point y the map $E_{V_\alpha}^V \circ \mathrm{res}_{V_\alpha}^V$ on A_y is id_{A_y} if $y \in \Sigma_\alpha$ and zero otherwise. Since $\mathfrak{U}$ is a finite cover and the Σ_α form a *disjoint* decomposition, the equality

$$\sum_{\alpha \in I} E_{V_\alpha}^V \circ \mathrm{res}_{V_\alpha}^V = \mathrm{id}_{\mathcal{F}(V)}$$

follows, proving property (2). Next, we need to verify the equation

$$\sum_{\alpha \in I} \mathrm{res}_{V_\beta}^V \circ E_{V_\alpha}^V = \sum_{\alpha \in I} E_{V_{\alpha\beta}}^{V_\beta} \circ \mathrm{res}_{V_{\alpha\beta}}^{V_\alpha} .$$

On the one hand, for every point y the map $E_{V_{\alpha\beta}}^{V_\beta} \circ \mathrm{res}_{V_{\alpha\beta}}^{V_\alpha}$ on A_y is id_{A_y} if $y \in V_{\alpha\beta} \cap \Sigma_\alpha$. As $\mathfrak{U}$ is a finite cover and the Σ_α are pairwise disjoint, summing over $\alpha \in I$ means that $\sum_\alpha E_{V_{\alpha\beta}}^{V_\beta} \circ \mathrm{res}_{V_{\alpha\beta}}^{V_\alpha}$ on A_y is id_{A_y} if y lies in the set

$$\dot{\bigcup_{\alpha \in I}} (V_{\alpha\beta} \cap \Sigma_\alpha) = V_\beta \cap \dot{\bigcup_{\alpha \in I}} (V_\alpha \cap \Sigma_\alpha) = V_\beta \cap \dot{\bigcup_{\alpha \in I}} \Sigma_\alpha = V_\beta \cap X = V_\beta$$

since $\Sigma_\alpha \subseteq V_\alpha$, so the right-hand side on A_y is id_{A_y} if $y \in V_\beta$ and zero otherwise. On the other hand, for every point y the map $\mathrm{res}_{V_\beta}^V \circ E_{V_\alpha}^V$ on A_y is id_{A_y} if $y \in \Sigma_\alpha \cap V_\beta$. Again, for varying α these sets are pairwise disjoint and the union is all of V_β, so on the left-hand side $\sum_{\alpha \in I} \mathrm{res}_{V_\beta}^V \circ E_{V_\alpha}^V$ on A_y is id_{A_y} if $y \in V_\beta$ and zero otherwise. This proves (3). □

2.2. Explicit formula for $H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right) \to \mathrm{CH}^n(X)$. We may apply the general formalism of the previous section to the flasque sheaves which occur in the Gersten resolution of the $\mathcal{K}^M$-sheaves.

Proposition 2. *Suppose* $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ *is a finite open cover of* X *and*

$$f := (f_{\beta_0 \ldots \beta_n}) \in H^n\left(\mathfrak{U}, \mathcal{K}_n^{\mathrm{M}}\right) \to H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right)$$

a representative of a sheaf cohomology class in $H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right)$*. Fix a disjoint decomposition* $X = \dot{\bigcup}_{\alpha \in I} \Sigma_\alpha$ *as in Lemma 2. Then the image of* f *under the comparison map* $\beta^n : H^n(X, \mathcal{K}_n^{\mathrm{M}}) \to \mathrm{CH}^n(X)$ *is given by*

$$\coprod_{x^n \in X^n} h_{x^n}; \qquad h_{x^n} \in \mathbf{Z}$$

with

$$h_{x^n} := \sum_{x^{n-1} \in X^{n-1}} \cdots \sum_{x^0 \in X^0} (\partial_{x^n}^{x^{n-1}} \circ \cdots \circ \partial_{x^1}^{x^0}) f_{\alpha(x^0)\alpha(x^1)\ldots\alpha(x^{n-1})\alpha(x^n)},$$

where:

- *The sums run over all chains such that* $x^{p+1} \in \overline{\{x^p\}}$.
- $\alpha(x^p)$ *denotes the unique index in* I *such that* $x^p \in \Sigma_{\alpha(x^p)}$ *holds.*

In particular the sum has only finitely many nonzero terms. Define

$$E_0^{p,q}(\mathfrak{U}) := \check{\mathrm{C}}^q \left(\mathfrak{U}, \coprod_{x \in U^p} \underline{K_{n-p}^{\mathrm{M}}}(\kappa(x)) \right).$$

The objects $(E_0^{p,q}(\mathfrak{U}))$ can be arranged as a bicomplex. The two differentials are taken from the Gersten and Čech complex respectively. Along with it we obtain the bicomplex spectral sequence of cohomological type with differentials $(p,q) \mapsto (p+r, q-r+1)$ on the r-th page, denote its pages by ${}^{\uparrow}E_r^{\bullet,\bullet}$. The ${}^{\uparrow}E_1$-page has entries

$${}^{\uparrow}E_1^{p,q}(\mathfrak{U}) := \check{H}^q \left(\mathfrak{U}, \coprod_{x \in U^p} \underline{K_{n-p}^{\mathrm{M}}}(\kappa(x)) \right).$$

Any refinements of the open cover $\mathfrak{U}$ induce a morphism between the respective bicomplexes. Then the colimit of the ${}^{\uparrow}E_1$-page over all refinements yields

$$\operatorname{colim}_{\mathfrak{U}} {}^{\uparrow}E_1^{p,q}(\mathfrak{U}) = \begin{cases} \coprod_{x \in X^p} K_{n-p}^{\mathrm{M}}(\kappa(x)) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases}$$

since the colimit of the Čech complex computes sheaf cohomology and the above sheaf is flasque. We conclude,

$$\operatorname{colim}_{\mathfrak{U}} {}^{\uparrow}E_2^{n,0}(\mathfrak{U}) = \operatorname{coker} \left(\coprod_{x \in X^{n-1}} \kappa(x)^\times \to \coprod_{x \in X^n} \mathbf{Z} \right) = \mathrm{CH}^n(X). \tag{18}$$

We also see that in the colimit the entire second page is supported in a single row, so it is clear that this page is already the same as the ${}^{\uparrow}E_\infty$-page. Moreover, for the second spectral sequence associated to the bicomplex, call it ${}^{\rightarrow}E_r^{\bullet,\bullet}$, we get

$$
\begin{aligned}
(19) \qquad {}^{\rightarrow}E_1^{0,q}(\mathfrak{U}) &= \check{\mathrm{C}}^q\left(\mathfrak{U}, \ker\Big(\coprod_{x\in U^0} \underline{K_n^{\mathrm{M}}}(\kappa(x)) \to \coprod_{x\in U^1} \underline{K_{n-1}^{\mathrm{M}}}(\kappa(x))\Big)\right) \\
&= \check{\mathrm{C}}^q\left(\mathfrak{U}, \mathcal{K}_n^{\mathrm{M}}\right).
\end{aligned}
$$

Again, after taking the colimit of all refinements, we arrive at $H^q\left(X, \mathcal{K}_n^{\mathrm{M}}\right)$ on ${}^{\rightarrow}E_2^{0,q} = {}^{\rightarrow}E_\infty^{0,q}$. All entries of the second page outside this column vanish. As a result, we may explicitly compute the comparison maps

$$H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right) \longrightarrow \mathrm{CH}^n(X).$$

Proof of Proposition 2. Suppose we start with an element in $H^n(X, \mathcal{K}_n^{\mathrm{M}})$. This element comes with an open cover $\mathfrak{U}$ on which it can be defined, say

$$(f_{\beta_0\dots\beta_n}) \in H^n\left(\mathfrak{U}, \mathcal{K}_n^{\mathrm{M}}\right) \to H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right).$$

After fixing this cover, it remains to take (arbitrary) representatives of the element in $\check{H}^n(\mathfrak{U}, \mathcal{K}_n^{\mathrm{M}}) = {}^{\rightarrow}E_2^{0,q}(\mathfrak{U})$ on the E_0-page, i.e., in ${}^{\rightarrow}E_0^{0,n}(\mathfrak{U})$. $\check{H}^q(\mathfrak{U}, \mathcal{K}_n^{\mathrm{M}}) \twoheadleftarrow {}^{\rightarrow}E_1^{0,q} \hookrightarrow {}^{\rightarrow}E_0^{0,q}$ (as it stems from Equation (19)), follow a zig-zag in the bicomplex

$$
(20) \qquad
\begin{array}{ccccccc}
E_0^{0,n} & & & & & & \\
\uparrow & & & & & & \\
E_0^{0,n-1} & \longrightarrow & E_0^{1,n-1} & & & & \\
& & \uparrow & & & & \\
& & \ddots & \longrightarrow & E_0^{n-1,0} & \longrightarrow & E_0^{n,0}
\end{array}
$$

and conclude by sending the resulting representative in ${}^{\uparrow}E_0^{n,0}(\mathfrak{U})$ along

$${}^{\uparrow}E_2^{n,0}(\mathfrak{U}) \to \mathrm{CH}^n(X).$$

The arrow comes from taking the colimit over all refinements of the cover, as in Equation (18). Write

$$f^{(p,q)}_{\beta_0\dots\beta_q|x} \in K^{\mathrm{M}}_{n-p}(\kappa(x)) \qquad \text{with } \beta_0,\dots,\beta_q \in I,\ x \in X^p$$

for the components of an element in $E_0^{p,q}(\mathfrak{U})$. Then for each step

$$
\begin{array}{ccc}
E^{p,q} & & \\
\uparrow\downarrow & & \\
E^{p,q-1} & \longrightarrow & E^{p+1,q-1}
\end{array}
$$

we may use the contracting homotopy of Lemmata 1 and 2 to find a preimage of an element in $E^{p,q}$ in $E^{p,q-1}$: So, pick a disjoint decomposition $\Sigma_\bullet$ for the fixed open cover $\mathfrak{U}$. This yields

$$f^{(p,q-1)}_{\beta_0\ldots\beta_{q-1}|x} = \sum_{\alpha\in I} E^{U_{\beta_0\cdots\beta_{q-1}}}_{U_{\alpha\beta_0\cdots\beta_{q-1}}} f^{(p,q)}_{\alpha\beta_0\ldots\beta_{q-1}|x} = f^{(p,q)}_{\alpha(x)\beta_0\ldots\beta_{q-1}|x}$$

if we agree to write $\alpha(x)$ for the (unique!) $\alpha \in I$ such that $x \in \Sigma_\alpha$. For the rightward arrow $E^{p,q-1} \to E^{p+1,q-1}$ we just need to follow the map induced by the differential of the Gersten complex. Thus, for $y \in X^{p-1}$

$$f^{(p+1,q-1)}_{\beta_0\ldots\beta_{q-1}|y} = \sum_{\{x^p\in X^p|y\in\overline{\{x\}}\}} \partial^x_y f^{(p,q)}_{\alpha(x^p)\beta_0\ldots\beta_{q-1}|x^p},$$

where ∂^x_y denotes the component of the differential in the Gersten complex going from $x \in X^p$ to $y \in X^{p+1}$. Now, use induction along the whole zig-zag in Diagram (20) (for this it is advisable to write x^p instead of x and x^{p+1} instead of y). This shows that the resulting

$$f^{(n,0)} = \left(f^{(n,0)}_{\beta_0|x^n}\right) \in E^{n,0}_0 = \check{\mathrm{C}}^0\left(\mathfrak{U}, \coprod_{x\in X^n} \underline{\mathbf{Z}}\right)$$

is given by the Čech 0-cocycle

$$f^{(n,0)}_{\beta_0|x^n} = \sum_{x^{n-1}\in X^{n-1}} \cdots \sum_{x^0\in X^0} (\partial^{x^{n-1}}_{x^n} \circ \cdots \circ \partial^{x^0}_{x^1}) f_{\alpha(x^0)\alpha(x^1)\ldots\alpha(x^{n-1})\beta_0},$$

where the sums run over all chains such that $x^{n-1} \in \overline{\{x^{n-2}\}}, \ldots, x^1 \in \overline{\{x^0\}}$ (i.e., the closures of $x^0, \ldots, x^{n-1}$ form a chain of irreducible closed subsets of X of increasing codimension). This 0-cocycle glues to a global section, so if we want to read off the x^n-component of the global section, we may use for this any open U_{β_0} such that $x^n \in U_{\beta_0}$. To make the formula as symmetric as possible, we may in particular choose $\beta_0 := \alpha(x^n)$, giving the claim. □

Remark 4 (Dropping smoothness)**.** Let us explore what happens if we drop the assumption that X be smooth. Even in the nonsmooth case we have a morphism

$$\mathcal{K}^{\mathrm{M}}_n \longrightarrow \left[\coprod_{x\in U^0} \underline{K^{\mathrm{M}}_n}(\kappa(x)) \to \coprod_{x\in U^1} \underline{K^{\mathrm{M}}_{n-1}}(\kappa(x)) \to \cdots\right]_{0,n}$$

as in Equation (14); but it need not be a quasi-isomorphism anymore. The sheaves in the complex on the right-hand side are still flasque. Thus, we still get a morphism

$$\beta^n : H^n\left(X, \mathcal{K}^{\mathrm{M}}_n\right) \longrightarrow \mathrm{CH}^{p+q}(X),$$

but it will usually neither be injective nor surjective. This should not come as a surprise, for $n = 1$ this is just the classical map from Cartier to Weil divisors. See [Gil05, §2.6] for a discussion to what extent the cohomology groups $H^n\left(X, \mathcal{K}_n^{\mathrm{M}}\right)$ provide a good replacement for Chow groups for *singular* varieties. Note that the product on the right-hand side of Equation (1) (resp. Equation (5)) still makes sense for singular varieties

$$H^p\left(X, \mathcal{K}_p^{\mathrm{M}}\right) \otimes_{\mathbf{Z}} H^q\left(X, \mathcal{K}_q^{\mathrm{M}}\right) \longrightarrow H^{p+q}\left(X, \mathcal{K}_{p+q}^{\mathrm{M}}\right) \xrightarrow{\beta^n} \mathrm{CH}^{p+q}(X),$$

while there is no natural product structure on the Chow groups.

3. The cup product

We quickly recall the construction of the cup product in Čech cohomology. For general sheaves $\mathcal{F}, \mathcal{G}$ with values in abelian groups, the tensor sheaf $\mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{G}$ (where $\underline{\mathbf{Z}}$ denotes the locally constant sheaf with value $\mathbf{Z}$) has stalks

$$(\mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{G})_x = \mathcal{F}_x \otimes_{\underline{\mathbf{Z}}_x} \mathcal{G}_x = \mathcal{F}_x \otimes_{\mathbf{Z}} \mathcal{G}_x.$$

For Čech cochains on an open cover $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ one defines the $\mathbf{Z}$-bilinear pairing

$$\smile : \check{\mathrm{C}}^p(\mathfrak{U}, \mathcal{F}) \times \check{\mathrm{C}}^q(\mathfrak{U}, \mathcal{G}) \to \check{\mathrm{C}}^{p+q}(\mathfrak{U}, \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{G}) \tag{21}$$

$$(f \smile g)_{\alpha_0 \ldots \alpha_{p+q}} := {}^{\mathcal{F}}\mathrm{res}^{U_{\alpha_0 \ldots \alpha_p}}_{U_{\alpha_0 \ldots \alpha_{p+q}}} f_{\alpha_0 \ldots \alpha_p} \otimes {}^{\mathcal{G}}\mathrm{res}^{U_{\alpha_p \ldots \alpha_{p+q}}}_{U_{\alpha_0 \ldots \alpha_{p+q}}} g_{\alpha_p \ldots \alpha_{p+q}},$$

where ${}^{\mathcal{F}}\mathrm{res}$ and ${}^{\mathcal{G}}\mathrm{res}$ denote the restrictions to smaller opens of the sheaves $\mathcal{F}$, $\mathcal{G}$ respectively. The identity

$$\delta(f \smile g) = \delta f \smile g + (-1)^p f \smile \delta g$$

is easy to show and proves that Equation (21) induces a pairing of Čech cohomology groups, the *cup product*. It becomes associative on the level of cohomology groups.

Remark 5. If one defines the cup product in a derived setting as the morphism '$\cup$' in

$$\smile : \mathbf{R}\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{R}\Gamma(X, \mathcal{G}) \xrightarrow{\cup} \mathbf{R}\Gamma(X, \mathcal{F} \otimes^{\mathbf{L}}_{\underline{\mathbf{Z}}} \mathcal{G}),$$

this cup product relates (after taking the colimit over all refinements of covers) to the one in Equation (21) by composing with $\mathbf{R}\Gamma(pr)$, where $pr : \mathcal{F} \otimes^{\mathbf{L}}_{\underline{\mathbf{Z}}} \mathcal{G} \to \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{G}$ is the natural morphism.

Next, the product morphism in Milnor K-theory induces a morphism of $\mathcal{K}^{\mathrm{M}}$-sheaves

$$\cdot\; : \mathcal{K}_p^{\mathrm{M}} \otimes_{\underline{\mathbf{Z}}} \mathcal{K}_q^{\mathrm{M}} \to \mathcal{K}_{p+q}^{\mathrm{M}};$$

it is defined as the usual multiplication in the Milnor K-groups $K_p^{\mathrm{M}}(\kappa(x))$ which appear in the term $\coprod_{x \in U^0} K_p^{\mathrm{M}}(\kappa(x))$ in Equation (9). One checks easily that this is well-defined. Whenever the Milnor K-theory sheaf agrees with the plain sheafification of Milnor K-theory (as explained in Remark 1), this agrees with the multiplication as discussed in Equation (5) in the introduction. This induces a morphism of Čech cochain groups and composing this with the cup product, we get morphisms

$$\check{\mathrm{C}}^p(\mathfrak{U},\mathcal{K}_p^{\mathrm{M}})\otimes_{\mathbf{Z}}\check{\mathrm{C}}^q(\mathfrak{U},\mathcal{K}_q^{\mathrm{M}})\overset{\smile}{\longrightarrow}\check{\mathrm{C}}^{p+q}(\mathfrak{U},\mathcal{K}_p^{\mathrm{M}}\otimes_{\underline{\mathbf{Z}}}\mathcal{K}_q^{\mathrm{M}})\overset{\cdot}{\longrightarrow}\check{\mathrm{C}}^{p+q}(\mathfrak{U},\mathcal{K}_{p+q}^{\mathrm{M}}).$$

After taking the colimit over refinements of the cover, this yields cup product counterpart of the product on the Chow ring as in

$$\begin{array}{ccc} \mathrm{CH}^p(X)\otimes_{\mathbf{Z}}\mathrm{CH}^q(X) & \longrightarrow & \mathrm{CH}^{p+q}(X) \\ \downarrow & & \downarrow \\ H^p(X,\mathcal{K}_p^{\mathrm{M}})\otimes_{\mathbf{Z}}H^q(X,\mathcal{K}_q^{\mathrm{M}}) & \longrightarrow & H^{p+q}(X,\mathcal{K}_{p+q}^{\mathrm{M}}). \end{array} \tag{22}$$

For algebraic K-theory the compatibility of products was first established by Grayson [Gra78]. If X is smooth proper of pure dimension p, an inductive use of this compatibility yields the intersection form on Weil divisors (= Chow 1-cocycles)

$$\begin{aligned} \mathrm{CH}^1(X)\otimes\cdots\otimes\mathrm{CH}^1(X) &\longrightarrow \mathrm{CH}^p(X) \\ [Z_1]\otimes\cdots\otimes[Z_p] &\longmapsto [Z_1]\smile\cdots\smile[Z_p]. \end{aligned}$$

Proposition 3. *In the situation of Proposition 2 the intersection form for Weil divisors $Z_1,\ldots,Z_n$ is given by the explicit formula*

$$\langle Z_1,\ldots,Z_n\rangle=\textstyle\coprod_{x^n\in X^n}h_{x^n}\in\mathrm{CH}^n(X)$$

$$h_{x^n}:=\sum_{x^{n-1},\ldots,x^0}\partial_{x^n}^{x^{n-1}}\cdots\partial_{x^1}^{x^0}\{f^1_{\alpha(x^0)\alpha(x^1)},f^2_{\alpha(x^1)\alpha(x^2)},\ldots,f^n_{\alpha(x^{n-1})\alpha(x^n)}\}\in\mathbf{Z},$$

where $f^i=(f^i_{\alpha,\beta})_{\alpha,\beta\in I}$ with $f^i_{\alpha,\beta}\in\mathcal{O}_X^\times$ is a Čech representative of the line bundle isoclass determined by Z_i under the usual map

$$\operatorname{Div}X\to H^0\left(X,\mathcal{K}_X^\times/\mathcal{O}_X^\times\right)/H^0(\mathcal{K}_X^\times)\to H^1\left(X,\mathcal{O}_X^\times\right), \tag{23}$$

where the middle term is the group of Cartier divisor classes.

Proof. Firstly, $\mathcal{K}_1^{\mathrm{M}}\cong\mathcal{O}_X^\times$, so the passage $\alpha^1:\mathrm{CH}^1(X)\cong H^1\left(X,\mathcal{K}_1^{\mathrm{M}}\right)$ reduces to the classical comparison of Weil and Cartier divisors as in Equation (23) (or Equation (6)). Now, if each Z_i is given by a Čech 1-cocycle $(f^i_{\alpha,\beta})_{\alpha,\beta\in I}$ on a fixed open cover $\mathfrak{U}=(U_\alpha)_{\alpha\in I}$, we may unwind the lower row in diag. 22 explicitly (using Equations (5) and (21) inductively):

$$\begin{aligned} \left(f^1\smile\cdots\smile f^p\right)_{\beta_0\cdots\beta_n} &= f^1_{\beta_0\beta_1}\cdot f^2_{\beta_1\beta_2}\cdot\cdots\cdot f^n_{\beta_{n-1}\beta_n} \\ &= \{f^1_{\beta_0\beta_1},f^2_{\beta_1\beta_2},\ldots,f^n_{\beta_{n-1}\beta_n}\}\in\mathcal{K}_n^{\mathrm{M}}\left(U_{\beta_0\ldots\beta_n}\right). \end{aligned}$$

Now invoke Proposition 2 to translate this into a conventional representative for an algebraic cycle. ☐

Remark 6. If one prefers to think in terms of $H^0\left(X,\mathcal{K}_X^\times/\mathcal{O}_X^\times\right)$, let $g^i_\alpha\in\mathcal{K}_X^\times(U_\alpha)$ be a local equation cutting out the divisor Z_i. Then under the map in Equation (23) we find $f^i_{\alpha,\beta}=g^i_\beta/g^i_\alpha$.

4. Relation to residue calculus

There is a natural morphism of sheaves of abelian groups,

$$\begin{aligned} \mathrm{dlog} : \mathcal{K}_p^{\mathrm{M}} &\longrightarrow \Omega^p_{X/k} \\ \{a_1, \dots, \alpha_p\} &\longmapsto \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_p}{a_p}. \end{aligned}$$

For $p = 0$ this is supposed to mean that $\mathrm{dlog} : \underline{\mathbf{Z}} \to \mathcal{O}_X$, $1_{\mathbf{Z}} \mapsto 1_k$. The map respects the product structures on either side and the Steinberg relation due to

$$\{a, 1-a\} \mapsto \frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}\,(1-a)}{1-a} = -\frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}a}{1-a} = 0.$$

In particular, we get an induced morphism of sheaf cohomology groups

$$H^p(X, \mathcal{K}_p^{\mathrm{M}}) \xrightarrow{\mathrm{dlog}} H^p(X, \Omega^p_{X/k}).$$

The left-hand side term is only an abelian group. For the right-hand side we recall that both the categories of abelian group sheaves as well as quasi-coherent sheaves have enough injectives; one can choose a simultaneous resolution in both categories, so it does not matter in which of these two categories one computes the right-hand side (except that the k-vector space structure is not visible if one works with sheaves of abelian groups). Assume $p + q = n$. We get a pairing

$$\begin{array}{ccccc} H^p(X, \mathcal{K}_p^{\mathrm{M}}) \otimes_{\mathbf{Z}} H^q(X, \mathcal{K}_q^{\mathrm{M}}) & \longrightarrow & H^n(X, \mathcal{K}_n^{\mathrm{M}}) & \xrightarrow{\cong} & \mathrm{CH}_0(X) \\ \downarrow & & \downarrow & & \downarrow \\ H^p(X, \Omega^p_{X/k}) \otimes_{\mathbf{Z}} H^q(X, \Omega^q_{X/k}) & \longrightarrow & H^n(X, \Omega^n_{X/k}) & \xrightarrow[\cong]{} & k, \end{array}$$

where the isomorphism in the upper row is our usual comparison map $\cong \mathrm{CH}^n(X) = \mathrm{CH}_0(X)$ (since we assume X is smooth of pure dimension n), the isomorphism of the lower row is the trace map coming from residue calculus. Moreover, there are commutative squares

$$\begin{array}{ccc} K_p^{\mathrm{M}}(\kappa(x)) & \xrightarrow{\partial_y^x} & K_{p-1}^{\mathrm{M}}(\kappa(y)) \\ \downarrow & & \downarrow \\ \Omega^p_{\kappa(x)/k} & \xrightarrow[\mathrm{res}_y^x]{} & \Omega^{p-1}_{\kappa(y)/k}. \end{array}$$

This transforms the formula of Proposition 3 into

$$\begin{aligned}(24)\qquad & \operatorname{dlog} h_{x^p} \\ &= \sum_{x^{p-1}\dots x^0} \operatorname{res}_{x^p}^{x^{p-1}} \cdots \operatorname{res}_{x^1}^{x^0} \\ &\qquad \left(\operatorname{d}\log f^1_{\alpha(x^0)\alpha(x^1)} \wedge \cdots \wedge \operatorname{d}\log f^p_{\alpha(x^{p-1})\alpha(x^p)}\right) \\ &= \sum_{x^{p-1}\dots x^0} \operatorname{res}_{x^p}^{x^{p-1}} \cdots \operatorname{res}_{x^1}^{x^0} \left(\operatorname{d}\log \frac{g^1_{\alpha(x^1)}}{g^1_{\alpha(x^0)}} \wedge \cdots \wedge \operatorname{d}\log \frac{g^p_{\alpha(x^p)}}{g^p_{\alpha(x^{p-1})}}\right)\end{aligned}$$

if g^i_α denotes a local equation cutting out the divisor Z_i as in Remark 6. This is also a formula for intersection multiplicities, yet it is less precise as Proposition 3 as it does not give a zero cycle, but just an intersection numer (= degree of the zero cycle).

Remark 7. This formula is a Čech cohomology analogue of an intersection multiplicity formula due to Hübl and Yekutieli in the context of higher adèles, see [HY96, Proposition 2.6]. This in turn generalizes a formula due to Parshin [Par83, §2.2, eq. 4. & use Corollary].

Concluding this section, we shall use slightly more technology than in the previous ones, but otherwise continue our discussion seamlessly.
Following Rost [Ros96, §5] we may more generally pick a cycle module M instead of just Milnor K-theory. Denote by

$$(25)\qquad \mathcal{M}(U) := \ker\left(\coprod_{x\in U^0} M(\kappa(x)) \longrightarrow \coprod_{x\in U^1} M(\kappa(x))\right)$$

the associated Zariski sheaf. By [Ros96, Cor. 6.5] there is a canonical isomorphism $H^p(X,\mathcal{M}) \to A^p(X;M)$ (the latter group is the cohomology of Rost's cycle complex). Without any change in the argument in §2.2 we obtain an explicit description for this map as well (with the same formula!). The only change is that the boundary maps ∂ run along the graded parts of the cycle module $M \to M_{-1} \to \cdots \to M_{-p}$ where they would run down $K_p^{\mathrm{M}} \to \cdots \to K_0^{\mathrm{M}} = \mathbf{Z}$ in the above case. Picking Milnor K-theory as the cycle module one recovers precisely the discussion of §2.2. For the record:

Proposition 4. *Let M be a Rost cycle module, $\mathcal{M}$ the associated Zariski sheaf. Suppose $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ is a finite open cover of X and*

$$f := (f_{\beta_0\dots\beta_n}) \in H^n(\mathfrak{U},\mathcal{M}) \to H^n(X,\mathcal{M})$$

a representative of a sheaf cohomology class. Fix any disjoint decomposition $X = \dot{\bigcup}_{\alpha\in I}\Sigma_\alpha$ as in Lemma 2. Then the image of f under the comparison map $H^n(X,\mathcal{M}) \to A^n(X;M)$ is given by

$$\coprod_{x^n\in X^n} h_{x^n}; \qquad h_{x^n} \in M_{-n}(\kappa(x))$$

with

$$h_{x^n} := \sum_{x^{n-1}\in X^{n-1}} \cdots \sum_{x^0\in X^0} (\partial^{x^{n-1}}_{x^n} \circ \cdots \circ \partial^{x^0}_{x^1}) f_{\alpha(x^0)\alpha(x^1)\dots\alpha(x^{n-1})\alpha(x^n)},$$

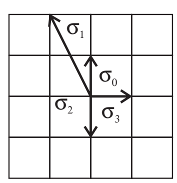
where ∂_y^x is the boundary map of the cycle module, x^i and $\alpha(x^i)$ as in Proposition 2.

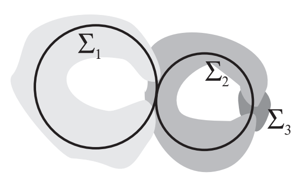
Proof. Exact copy of the proof of Proposition 2, except two details: The cokernel in Equation (18) becomes a cohomology group of the cycle complex, namely Rost's $A^n(X; M)$. The kernel in Equation (19) is precisely the definition of $\mathcal{M}$ (see Equation (25)). $\square$

5. Example with negative self-intersection

We wish to give an example of an explicit computation of a negative self-intersection number using the methods of this text. For the sake of exposition we shall use the formula based on Milnor K-theory (Proposition 3), although for this particular computation the simpler residue formula, Equation (24), would be sufficient. No moving will be used. To keep the example sufficiently simple, we consider Hirzebruch surfaces F_n ($n \in \mathbf{Z}$). They can be described either as the toric surface of the fan (see [Dan78] §1-3 for toric generalities; or [CLS11], [Ful93])

$$\sigma_0 = \mathbf{R}e_x + \mathbf{R}e_y \qquad \sigma_1 = \mathbf{R}e_y + \mathbf{R}(-e_x + ne_y)$$
$$\sigma_2 = \mathbf{R}(-e_x + ne_y) - \mathbf{R}e_y \qquad \sigma_3 = \mathbf{R}e_x - \mathbf{R}e_y$$





(depicted left for $n = 2$) or as the projectivization of the vector bundle $\pi : E \to \mathbf{P}^1_k$ given through $\mathcal{O}_{\mathbf{P}^1_k}(n) \oplus \mathcal{O}_{\mathbf{P}^1_k}$ on $\mathbf{P}^1_k$. For $n = 0$ one has $F_0 \simeq \mathbf{P}^1_k \times \mathbf{P}^1_k$. Explicitly, the smooth proper surface F_n is patched from affine opens

$$U_0 = \operatorname{Spec} k[X, Y] \qquad U_1 = \operatorname{Spec} k[X^{-1}, X^n Y]$$
$$U_2 = \operatorname{Spec} k[X^{-1}, X^{-n} Y^{-1}] \qquad U_3 = \operatorname{Spec} k[X, Y^{-1}]$$

(all isomorphic to $\mathbf{A}^2_k$) along the intersections

$$U_{01} = \operatorname{Spec} k[X^{-1}, X, Y] \qquad U_{12} = \operatorname{Spec} k[X^{-n}Y^{-1}, X^n Y, X^{-1}]$$
$$U_{23} = \operatorname{Spec} k[X^{-1}, X, Y^{-1}] \qquad U_{03} = \operatorname{Spec} k[X, Y, Y^{-1}]$$

(all isomorphic to $\mathbf{A}^1_k \times \mathbf{G}_{m,k}$). All other intersections of two opens, e.g., U_{02} or U_{13}, are 2-tori, $\operatorname{Spec} k[X, X^{-1}, Y, Y^{-1}] \simeq \mathbf{G}^2_{m,k}$. The same holds for all triple intersections like U_{012}, U_{013}, etc. These opens U_i correspond to the toric affine opens coming from cones σ_i (associated to orbits of the torus action if we view F_n as an equivariant compactification). In particular, $\mathfrak{U} := (U_i)_{i=0,1,2,3}$ is an open cover we may use for Čech cohomology. Define

$V := F_n \setminus U_0$ as the reduced closed complement of U_0. Using the same cover as for F_n, this locally comes down to

$$V_0 = U_0 \setminus U_0 = \varnothing,$$
$$V_1 = U_1 \setminus U_0 = \operatorname{Spec} k[X^n Y],$$
$$V_2 = U_2 \setminus D_{X^{-1}(X^{-n}Y^{-1})} = \operatorname{Spec} k[X^{-1}, X^{-n}Y^{-1}]/(X^{-1}) \cdot (X^{-n}Y^{-1}),$$
$$V_3 = U_3 \setminus U_0 = \operatorname{Spec} k[X].$$

In particular $V_1 \simeq \mathbf{A}^1_k$, $V_3 \simeq \mathbf{A}^1_k$, V_2 is reducible and its irreducible components are both isomorphic to $\mathbf{A}^1_k$. The closures (in the whole surface F_n) $\overline{V_1}$ and $\overline{V_3}$ are both isomorphic to $\mathbf{P}^1_k$ (with $V_{12} \simeq \mathbf{G}_{m,k}$ and $V_{23} \simeq \mathbf{G}_{m,k}$ the overlaps of the two copies of $\mathbf{A}^1_k$), they intersect in a single closed point in V_2. Moreover, $V_{13} = \varnothing$. For the construction of the contracting homotopy in Lemma 2 we now may use the disjoint decomposition of F_n (as a set!)

$$\Sigma_0 := U_0, \quad \Sigma_1 := V \cap U_1, \quad \Sigma_2 := (V \setminus U_1) \cap U_2, \quad \Sigma_3 := (V \setminus (U_1 \cup U_2)).$$

Graphically, the decomposition of the complement V is depicted on the right in the above figure. The two circles represent $\overline{V_1}$ and $\overline{V_3}$ ($\simeq \mathbf{P}^1_k$). Summarized, F_n decomposes as follows:

- (codim. 0) the unique generic point η lies in Σ_0;
- (codim. 1) the generic points of all integral curves of U_0 are in Σ_0. Σ_1^1 contains only the codimension 1 generic point of $\overline{V_1}$, Σ_2^1 contains only the codimension 1 generic point of $\overline{V_3}$, Σ_3 does not contain any codimension 1 points;
- (codim. 2) the closed points of U_0 all lie in Σ_0. The closed points in Σ_1 are the closed points of $V_1 \simeq \mathbf{A}^1_k$ (the single additional closed point of its closure $\overline{V_1}$ lies in Σ_2 — it's the same point as the intersection $\overline{V_1} \cap \overline{V_3}$), the closed points in Σ_2 are the closed points of $V_3 \simeq \mathbf{A}^1_k$ (the single additional closed point of its closure $\overline{V_3}$ lies in Σ_3). Σ_3 consists only of this closed point.

Now we wish to study the divisor D associated to the cone spanned by e_y, i.e., $\sigma_0 \cap \sigma_1$.

Claim. We have self-intersection $D \cdot D = -n$.

The divisor D is an effective Cartier divisor in $H^0(F_n, \mathcal{K}^\times/\mathcal{O}^\times)$, which we may represent in our Čech cover through

$$c_0 = Y \qquad c_1 = X^n Y \qquad c_2 = 1 \qquad c_3 = 1$$

in $\check{H}^0(\mathfrak{U}, \mathcal{K}^\times/\mathcal{O}^\times)$. The associated line bundle is given by $(\tilde{c}_{i,j})_{i,j}$ in the group $\check{H}^1(\mathfrak{U}, \mathcal{O}^\times)$ so that

$$\tilde{c}_{01} = X^n \qquad \tilde{c}_{02} = Y^{-1} \qquad \tilde{c}_{03} = Y^{-1}$$
$$\tilde{c}_{12} = X^{-n}Y^{-1} \qquad \tilde{c}_{13} = X^{-n}Y^{-1} \qquad \tilde{c}_{23} = 1.$$

Thanks to Proposition 3 the self-intersection number $D \cdot D$ comes down to the computation of various boundaries in Milnor K-theory, namely

$$h_{x^2} = \sum_{x^1, x^0} \partial^{x^1}_{x^2} \partial^{x^0}_{x^1} \{\tilde{c}_{\alpha(x^0)\alpha(x^1)}, \tilde{c}_{\alpha(x^1)\alpha(x^2)}\}. \tag{26}$$

As the formula Equation (26) really mostly depends on the values of $\alpha(x^i)$ for various i, it is convenient to do a case-distinction depending on these values. Since $\alpha(x^0) = 0$ always (there is only one generic point and it lies in Σ_0), we are left with $\{\tilde{c}_{\alpha(x^0)\alpha(x^1)}, \tilde{c}_{\alpha(x^1)\alpha(x^2)}\} =$

$\alpha(x^2)$ / $\alpha(x^1)$	0	1	2	3
0	0	0	$*$	$*$
1	0	0	$\{Y^{-1}, X^n Y\}$	$\{Y^{-1}, X^n Y\}$
2	0	$\{X^n, X^{-n}Y^{-1}\}$	0	0
3	0	$\{X^n, X^{-n}Y^{-1}\}$	0	0,

where $*$ indicates an element of the shape $\{a, a^{-1}\}$.

Remark 8. While these elements are usually nonzero, we have $\{a, a^{-1}\} = \{a, -1\}$, so they are 2-torsion. Thus, when being mapped to an intersection number, i.e., to $\mathbf{Z}$, they necessarily vanish, so we may disregard them already here.

In Equation (26) the value $\alpha(x^1) = 3$ is impossible since Σ_3 does not contain generic points of curves. The value $\alpha(x^1) = 2$ is only possible if $x^1 = \overline{V_3}$, but the only nontrivial entry is at $\alpha(x^2) = 1$, however by the nature of our decomposition no closed points on the curve $\overline{V_3}$ lie in Σ_1. Thus, only for $\alpha(x^1) = 1$ nontrivial symbols occur. Note that $\alpha(x^1) = 1$ implies $x^1 = \overline{V_1}$ and all the closed points of $\overline{V_1}$ lie in Σ_1 and Σ_2, so the case $\alpha(x^2) = 3$ is also impossible. For $\alpha(x^2) = 2$ the closed point must be $\overline{V_1} \cap \overline{V_3}$, given by $x^2 = (X^{-1}, X^{-n}Y^{-1})$ in $U_2 = \operatorname{Spec} k[X^{-1}, X^{-n}Y^{-1}]$ and $x^1 \mid_{U_2} = \overline{V_1} \mid_{U_2} = (X^{-1})$. Hence, the whole sum of Equation (26) reduces to the single expression

$$\begin{aligned} D \cdot D &= \partial^{x^1}_{x^2} \partial^{\eta}_{x^1} \{X^n, X^{-n}Y^{-1}\} = \partial^{x^1}_{x^2} \partial^{\eta}_{x^1} (-n\{X^{-1}, X^{-n}Y^{-1}\}) \\ &= -n \partial^{x^1}_{x^2} \{\overline{X^{-n}Y^{-1}}\} = -n[x^2, 1_{\mathbf{Z}}] = -n \in \mathbf{Z} \end{aligned}$$

since x is a closed point of degree 1 on F_n. Here $[x^2, 1_{\mathbf{Z}}]$ refers to the zero cycle represented by $1_{\mathbf{Z}}$ at the closed point x^2.

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