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Diffeomorphism groups of balls and spheres

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Abstract. In this paper we discuss the relationship between groups of diffeomorphisms of spheres and balls. We survey results of a topological nature and then address the relationship *as abstract* (discrete) *groups*. We prove that the identity component of the group of smooth diffeomorphisms of an odd dimensional sphere admits no nontrivial homomorphisms to the group of diffeomorphisms of a ball of any dimension. This result generalizes theorems of Ghys and Herman. We also examine finitely generated subgroups of diffeomorphisms of spheres, and produce an example of a finitely generated torsion-free group with an action on the circle by smooth diffeomorphisms that does not extend to a C^1 action on the disc.

Contents

1. Introduction

Let M be a manifold and let $\mathrm{Diff}_0^r(M)$ denote the group of isotopically trivial C^r-diffeomorphisms of M. If M has boundary ∂M, there is a natural map

$$\pi : \mathrm{Diff}_0^r(M) \to \mathrm{Diff}_0^r(\partial M)$$

given by restricting the domain of a diffeomorphism to the boundary. The map π is surjective, as any isotopically trivial diffeomorphism f of the boundary can be extended to a diffeomorphism F of M supported on a collar neighborhood $N \cong \partial M \times I$ of ∂M by taking a smooth isotopy f_t from f to the identity, and defining F to agree with f_t on $\partial M \times \{t\}$.

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One way to measure the difference between the groups $\mathrm{Diff}^r_0(M)$ and $\mathrm{Diff}^r_0(\partial M)$ is to ask whether π admits a section. By section, we mean a map

$$\phi : \mathrm{Diff}^r_0(\partial M) \to \mathrm{Diff}^r_0(M)$$

such that $\pi \circ \phi$ is the identity on $\mathrm{Diff}^r_0(\partial M)$. There are several categories in which to ask this, namely

i) **Topological**: Require ϕ to be continuous, ignoring the group structure.
ii) **(Purely) group-theoretic**: Only require ϕ to be a group homomorphism, ignoring the topological structure on $\mathrm{Diff}^r_0(M)$.
iii) **Extensions of group actions**: In the case where no group-theoretic section exists, we ask the following *local* (in the sense of group theory) question. For which finitely generated groups Γ and a homomorphisms $\rho : \Gamma \to \mathrm{Diff}^r_0(\partial M)$ does there exist a homomorphism $\phi : \Gamma \to \mathrm{Diff}^r_0(M)$ such that $\pi \circ \phi = \rho$? If such a homomorphism exists, we say that ϕ *extends* the action of Γ on ∂M to a C^r action on M.

In this paper, we treat the case of the ball $M = B^{n+1}$ with boundary S^n. Note in the category of *homeomorphisms* rather than diffeomorphisms, there is a natural way to extend homeomorphisms of S^n to homeomorphisms of B^{n+1}. This is by "coning off" the sphere to the ball and extending each homeomorphism to be constant along rays. The result is a continuous group homomorphism

$$\phi : \mathrm{Homeo}_0(S^n) \to \mathrm{Homeo}_0(B^{n+1})$$

which is also a section of $\pi : \mathrm{Homeo}_0(B^{n+1}) \to \mathrm{Homeo}_0(S^n)$ in the sense above. We will see, however, that the question of sections for groups of *diffeomorphisms* is much more interesting!

Summary of results. Our goal in this work is to paint a relatively complete picture of known and new results for the ball B^n. Here is an outline.

Topological sections. In Section 2 we give brief survey of known results on existence and nonexistence of topological sections, and the relationship between topological sections and exotic spheres. The reader may skip this section if desired; it stands independent from the rest of this paper.

Group-theoretic sections. In contrast with the topological case, it is a theorem of Ghys that no group theoretic sections $\phi : \mathrm{Diff}^r_0(S^n) \to \mathrm{Diff}^r_0(B^{n+1})$ exist for any n or r. A close reading of Ghys' work in [Ghy91] produces *finitely generated* subgroups of $\mathrm{Diff}_0(S^{2n-1})$ that fail to extend to $\mathrm{Diff}_0(B^{2n})$ and we give an explicit presentation of such a group in Section 3. These examples rely heavily on the dynamics of finite order diffeomorphisms.

Extending actions of torsion-free groups. Building on Ghys' work and using results of Franks and Handel involving distorted elements in finite groups, in Section 4 we explicitly construct a group Γ to prove the following.

Theorem 1.1. *There exists a finitely generated, torsion-free group* Γ *and a homomorphism*

$$\rho : \Gamma \to \mathrm{Diff}^\infty(S^1)$$

that does not extend to a C^1 *action of* Γ *on* B^2*.*

Note that in contrast to Theorem 1.1, any action of $\mathbb{Z}$, of a free group, or any action of any group that is conjugate into the standard action of $\mathrm{PSL}(2,\mathbb{R})$ on S^1 *will* extend to an acton by diffeomorphisms on B^2.

Exotic homomorphisms. In Section 5, we will show that the failure of $\pi : \mathrm{Diff}^r_0(B^{n+1}) \to \mathrm{Diff}^r_0(S^n)$ to admit a section is due (at least in the case where n is odd) to a fundamental difference between the *algebraic structure* of groups of diffeomorphisms of spheres and groups of diffeomorphisms of balls. We prove

Theorem 1.2. *There is no nontrivial group homomorphism*

$$\mathrm{Diff}^\infty_0(S^{2k-1}) \to \mathrm{Diff}^1_0(B^m)$$

for any m*,* $k \geq 1$*.*

This generalizes a result of M. Herman in [Her]. Theorem 1.2 also stands in contrast to the situation with homeomorphisms of balls and spheres — any continuous foliation of B^{n+l} by n-spheres can be used to construct a continuous group homomorphism $\mathrm{Homeo}_0(S^n) \to \mathrm{Homeo}_0(B^{n+l})$.

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2. Topological sections: known results

In order to contrast our work on group-theoretic sections with the (fundamentally different) question of topological sections, we present a brief summary of known results in the topological case. Let $\mathrm{Diff}(B^n \operatorname{rel} \partial)$ denote the group of smooth diffeomorphisms of B^n that restrict to the identity on $\partial B^n = S^{n-1}$. The natural restriction map $\mathrm{Diff}(B^n) \xrightarrow{\pi} \mathrm{Diff}(S^{n-1})$ is a fibration with fiber $\mathrm{Diff}(B^n \operatorname{rel} \partial)$. Hence, asking for a topological section of π amounts to asking for a section of this fibration.

In low dimensions ($n \leq 3$), it is known that the fiber $\mathrm{Diff}(B^n \operatorname{rel} \partial)$ is contractible, so a topological section exists. The $n = 2$ case is a classical theorem of Smale [Sma59], and the $n = 3$ case a highly nontrivial theorem of Hatcher [Hat83]. Incidentally, $\mathrm{Diff}_0(B^1 \operatorname{rel} \partial)$ is also contractible and this is quite elementary — an element of $\mathrm{Diff}(B^1 \operatorname{rel} \partial)$ is a nonincreasing or nondecreasing function of the closed interval, and we can explicitly define a retraction of $\mathrm{Diff}(B^1 \operatorname{rel} \partial)$ to the identity via

$$r : \mathrm{Diff}(B^1 \operatorname{rel} \partial) \times [0, 1] \to \mathrm{Diff}(B^1 \operatorname{rel} \partial)$$

$$r(f,t)(x) = tf(x) + (1-t)x.$$

Whether $\mathrm{Diff}(B^4 \,\mathrm{rel}\, \partial)$ is contractible is an open question. To the best of the author's knowledge, whether $\mathrm{Diff}_0(B^4) \xrightarrow{\pi} \mathrm{Diff}_0(S^3)$ has a section is also open. However, in higher dimensions $\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)$ is not always contractible, giving a first obstruction to a section. This is related to the existence of exotic smooth structures on spheres.

Exotic spheres. Let $f \in \mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)$ be a diffeomorphism. We can use f to glue a copy of B^n to another copy of B^n along the boundary, producing a sphere S^n_f with a smooth structure. If f lies in the identity component of $\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)$, then S^n_f will be smoothly isotopic to the standard n-sphere S^n. If not, there is no reason that S^n_f need even be diffeomorphic to S^n. In fact, it follows from the pseudoisotopy theorem of Cerf in [Cer70] that, for $n \geq 5$, the induced map from $\pi_0(\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial))$ to the group of exotic n-spheres is injective.

Moreover — and more pertinent to our discussion — Smale's h-cobordism theorem ([Sma61]) implies the map from $\pi_0(\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial))$ to exotic n-spheres is *surjective*. In particular, this means that in any dimension n where exotic spheres exist, $\pi_0(\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)) \neq 0$. Let us now return to the fibration $\pi : \mathrm{Diff}(B^n) \to \mathrm{Diff}(S^{n-1})$ and look at the tail end of the long exact sequence in homotopy groups. If we consider the restriction of π to the identity components $\mathrm{Diff}_0(B^n) \xrightarrow{\pi} \mathrm{Diff}_0(S^{n-1})$ we have

$$\cdots \to \pi_1(\mathrm{Diff}_0(B^n)) \to \pi_1(\mathrm{Diff}_0(S^{n-1})) \to \pi_0(\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)) \to 0$$

Thus, whenever exotic spheres exist, the connecting homomorphism

$$\pi_1(\mathrm{Diff}_0(S^{n-1})) \to \pi_0(\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial))$$

is nonzero, and so no section of the bundle exists.

Question 2.1. *Does this bundle have a section in any dimensions $n \geq 5$ where exotic spheres do not exist?*

We remark that for all $n \geq 5$, it is known that $\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)$ has some nontrivial higher homotopy groups. Indeed, we learned from Allen Hatcher that recent work of Crowey and Schick [CS13] shows that $\mathrm{Diff}(B^n \,\mathrm{rel}\, \partial)$ has infinitely many nonzero higher homotopy groups whenever $n \geq 7$.

3. Group-theoretic sections

Recall from the introduction that a *group-theoretic section* of π is a (not necessarily continuous) group homomorphism $\phi : \mathrm{Diff}^r_0(S^n) \to \mathrm{Diff}^r_0(B^{n+1})$ such that $\pi \circ \phi$ is the identity. Recall also that, when Γ is a group and $\rho : \Gamma \to \mathrm{Diff}^r_0(S^n)$ specifies an action of Γ on S^n, we say that ρ *extends* to a C^r action on B^{n+1} if there is a homomorphism $\phi : \Gamma \to \mathrm{Diff}^r_0(B^{n+1})$ such that $\pi \circ \phi = \rho$.

The question of existence of group-theoretic sections for spheres and balls is completely answered by the following theorem of Ghys.

Theorem 3.1 ([Ghy91])**.** *There is no section of* $\mathrm{Diff}^1_0(B^{n+1}) \to \mathrm{Diff}^1_0(S^n)$*. Moreover, there is no extension of the standard embedding of* $\mathrm{Diff}^\infty_0(S^n)$ *in* $\mathrm{Diff}^1_0(S^n)$ *to a* C^1 *action of* $\mathrm{Diff}^\infty_0(S^n)$ *on* B^{n+1}*.*

We ask to what extent the failure of sections holds *locally*, i.e., for finitely generated subgroups. At one end of the spectrum, if Γ is a free group, and $\rho : \Gamma \to \mathrm{Diff}^r_0(S^n)$ is any action, we can build an extension of ρ by taking arbitrary C^r extensions of the generators of $\rho(\Gamma)$ — for instance, by using the collar neighborhood strategy sketched in the introduction. There are no relations to satisfy so this defines a homomorphism and gives a C^r action of Γ on B^{n+1}.

At the other end, a careful reading of Ghys' proof of Theorem 3.1 gives the following corollary of Theorem 3.1.

Corollary 3.2. *For any* n*, there exists a finitely generated subgroup* Γ *of* $\mathrm{Diff}^\infty_0(S^{2n-1})$ *that does not extend to a subgroup of* $\mathrm{Diff}^1_0(B^{2n})$*.*

Although this follows directly from Ghys' proof of Theorem 3.1, we outline the argument below in order to illustrate some of Ghys' techniques. We pay special attention to the $n = 1$ case because we will use part of this construction in Section 4. The reader will note that the argument is unique to odd-dimensional spheres, so does not answer the following question.

Question 3.3. *Is there a finitely generated group* Γ *and a homomorphism* $\rho : \Gamma \to \mathrm{Diff}^\infty_0(S^{2n})$ *that does not extend to a* C^1 *(or even* C^r *for some* $1 < r \leq \infty$*) action on* B^{2n+1}*?*

Sketch proof of Corollary 3.2. In the $n = 1$ case, we can take Γ to be a two-generated group as follows. Any rotation of S^1 can be written as a commutator — a nice argument for this using some hyperbolic geometry appears in Proposition 5.11 of [Ghy01] or Proposition 2.2 of [Ghy91]. So let f and g be such that their commutator $[f, g]$ is a finite order rotation, say a rotation of order 2. Using the construction in [Ghy01], we may even take f and g to be hyperbolic elements of $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Diff}^\infty_0(S^1)$. Let $\tilde{f}$ and $\tilde{g}$ be lifts of f and g to diffeomorphisms of the threefold cover of S^1. Since f and g have fixed-points, we can choose $\tilde{f}$ and $\tilde{g}$ to be the (unique) lifts that have fixed-points. Then the commutator $[\tilde{f}, \tilde{g}]$ will be rotation of the threefold cover of S^1 by $\pi/3$. Since the threefold cover of S^1 is also S^1, we can consider $\tilde{f}$ and $\tilde{g}$ as diffeomorphisms of S^1.

Let Γ be the subgroup of $\mathrm{Diff}^\infty_0(S^1)$ generated by $\tilde{f}$ and $\tilde{g}$. It has the following relations:

i) $[\tilde{f}, \tilde{g}]^6 = 1$.
ii) $[\tilde{f}, [\tilde{f}, \tilde{g}]^2] = [\tilde{g}, [\tilde{f}, \tilde{g}]^2] = 1$.

The second relation here comes from the fact that $[\tilde{f}, \tilde{g}]^2$ is the covering transformation. There may, incidentally, be other relations satisfied by Γ, but this is of no importance to us.

We claim that Γ does not extend to a subgroup of $\mathrm{Diff}^1_0(B^2)$. To see this, we argue by contradiction. Assume that there is a homomorphism $\phi : \Gamma \to \mathrm{Diff}^1_0(B^2)$ such that for any $\gamma \in \Gamma$, the restriction of $\phi(\gamma)$ to $\partial B^2 = S^1$ agrees with γ.

Let r denote rotation of S^1 by $2\pi/3$, this is the element $[\tilde{f}, \tilde{g}]^2 \in \Gamma$, and so $\phi(r)$ is an order 3 diffeomorphism of the ball acting by rotation on the boundary. In particular, it follows from Kerekjarto's theorem in [Ker19] that $\phi(r)$ is *conjugate* to an order three rotation, hence has a unique interior fixed-point x. (A reader unfamiliar with Kerekjarto's theorem on finite order diffeomorphisms may wish to consult Constantin and Kolev's proof in [CK94]).

By construction, $\tilde{f}$ and $\tilde{g}$ both commute with r so $\phi(\tilde{f})$ and $\phi(\tilde{g})$ commute with $\phi(r)$, hence fix x. The derivatives $D\phi(\tilde{f})_x$ and $D\phi(\tilde{g})_x$ commute with $D\phi(r)_x$ which acts as rotation by $2\pi/3$ on the tangent space. Moreover, $[D\phi(\tilde{f})_x, D\phi(\tilde{g})_x]^2 = D\phi(r)_x$, a rotation by $2\pi/3$. However, the centralizer of rotation by $2\pi/3$ in $\mathrm{SL}(2,\mathbb{R})$ is abelian, so writing $D\phi(r)_x$ as a commutator of elements in its centralizer is impossible. This is the desired contradiction, showing that no extension of the action of Γ exists.

The case for $n > 1$ is similar. We consider S^{2n-1} as the unit sphere

$$\left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} |z_i|^2 = 1\right\}.$$

The idea is to show that the finite order element

$$r : (z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots . \lambda_n z_n)$$

where λ_i are distinct p^{th} roots of 1, can also be expressed as a product of commutators of elements $f_1, f_2, \ldots f_k$ that each commute with a power of r. Then we can take Γ to be the subgroup generated by the diffeomorphisms f_i. Supposing again for contradiction that $\phi : \Gamma \to \mathrm{Diff}^1_0(B^{2n})$ is a section, one can show with an argument using Smith theory that the diffeomorphism $\phi(r) \in \mathrm{Diff}^1_0(B^{2n})$ has a single fixed-point x. It follows in a similar way to the $n = 1$ case that the derivative of $\phi(r)$ at x has abelian centralizer, giving a contradiction. □

4. Actions of torsion-free groups

The proof of Corollary 3.2 relied heavily on finite order diffeomorphisms. Ghys' proof of Theorem 3.1 — even in the case of even dimensional spheres — also hinges on the clever use of finite order diffeomorphisms (and the tools that they bring: Smith theory, fixed sets, derivatives in $\mathrm{SO}(n)$, etc.). Thus, we ask the following refinement of Question 3.3.

Question 4.1. *Does there exist a finitely generated, torsion-free group* Γ *and a homomorphism* $\rho : \Gamma \to \mathrm{Diff}^\infty_0(S^n)$ *that does not extend to a smooth (or even* C^r *for some* $r \geq 1$*) action on* B^{n+1}*?*

The following theorem answers this question for $n = 1$.

Theorem 1.1. *There exists a finitely generated, torsion-free group Γ and a homomorphism $\phi : \Gamma \to \mathrm{Diff}^\infty(S^1)$ that does not extend to a C^1 action on B^2.*

Our proof modifies Ghys' construction by using a *dynamical constraint* based on *algebraic structure* to force a diffeomorphism to act by rotation at a fixed-point. The algebraic structure in question is the notion of *distorted elements* and the constraint on dynamics follows from a powerful theorem of Franks and Handel. We provide a brief introduction in the following few paragraphs; a reader familiar with this work may wish to skip ahead to Corollary 4.3 and the proof of Theorem 1.1.

Distorted elements. Let Γ be a finitely generated group, and let $S = \{s_1, \dots, s_k\}$ be a symmetric generating set for Γ. For an element $g \in \Gamma$, the *word length* (or S-word length) of g is the length of the shortest word in the letters $s_1, \dots, s_k$ that represents g. We denote word length of g by $|g|$.

We say that $g \in \Gamma$ is *distorted* provided that g has infinite order and that

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0.$$

Although the word length of g^n depends on the choice of generating set S for Γ, it is not hard to see that whether g is distorted or not is independent of the choice of S.

In [FH06], Franks and Handel prove a theorem about the dynamics of actions of distorted elements in finitely generated subgroups of $\mathrm{Diff}_0(\Sigma)$, where Σ is a closed, oriented surface. The following theorem is a consequence of their main result. We use the notation $\mathrm{fix}(g)$ for the set of points x such that $g(x) = x$, and $\mathrm{per}(g)$ for the set of periodic points for g.

Theorem 4.2 (Franks–Handel, [FH06])**.** *Suppose that f is a distorted element in some finitely generated subgroup of $\mathrm{Diff}^1_0(S^2)$. Suppose also that for the smallest $n > 0$ such that $\mathrm{fix}(f^n) \neq \emptyset$, there are at least three points in $\mathrm{fix}(f^n)$. Then $\mathrm{per}(f) = \mathrm{fix}(f^n)$.*

We can derive a corresponding statement about actions on the disc.

Corollary 4.3. *Suppose that f is a distorted element in some finitely generated subgroup of $\mathrm{Diff}^1_0(B^2)$ with a periodic point on the boundary of period $k > 1$. Then $\mathrm{fix}(f)$ consists of a single point.*

Proof. Suppose f is distorted in $\Gamma \subset \mathrm{Diff}^1_0(B^2)$. By the Brouwer fixed-point theorem, f has at least one fixed-point. Since f has a periodic point on the boundary S^1, all fixed-points for f lie in the interior of B^2. Double B^2 along the boundary to produce the sphere, and double the action of Γ. This can be smoothed to a C^1 action on S^2 using the techniques of K. Parkhe in [Par12]. The smoothing construction will not change the set of fixed or periodic points. Applying Theorem 4.2 to the action on S^2, we conclude that the doubled action of Γ on the sphere can have at most two fixed-points (since there are nonfixed periodic points), so the original action of f has a single fixed-point. $\square$

With Corollary 4.3 as a tool, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Recall the group $\Gamma \subset \mathrm{Diff}_0(S^1)$ from the proof of Corollary 3.2. It is generated by two elements $\tilde{f}$ and $\tilde{g}$, satisfying the relations $[\tilde{f}, \tilde{g}]^6 = 1$ and $[\tilde{f}, [\tilde{f}, \tilde{g}]^2] = [\tilde{g}, [\tilde{f}, \tilde{g}]^2] = 1$. Let Γ' be the lift of Γ to the universal central extension $\mathrm{Diff}^{\infty}_{\mathbb{Z}}(\mathbb{R})$ of $\mathrm{Diff}^{\infty}_0(S^1)$. Explicitly, we can realize Γ' as the group of all lifts of elements of Γ to diffeomorphisms of the infinite cyclic cover $\mathbb{R}$ of S^1. For concreteness, let $\hat{f}$ and $\hat{g}$ denote the lifts of $\tilde{f}$ and $\tilde{g}$ that have fixed-points. Then Γ' is generated by $\hat{f}$, $\hat{g}$, and the central element t, and satisfies the relation $t = [\hat{f}, \hat{g}]^2$. Note that, since $\mathrm{Diff}^{\infty}_{\mathbb{Z}}(\mathbb{R})$ is torsion-free, Γ' is as well.

Finally, to complete our construction, let $\hat{\Gamma}$ be the HNN extension of Γ' obtained by adding a generator a and relation $ata^{-1} = t^4$. HNN extensions of torsion-free groups are torsion-free, so $\hat{\Gamma}$ is torsion-free also.

We now construct a homomorphism $\rho : \hat{\Gamma} \to \mathrm{Diff}^{\infty}_0(S^1)$ and show that it does not admit an extension $\phi : \hat{\Gamma} \to \mathrm{Diff}^1_0(B^2)$. The homomorphism ρ will not be faithful (and in fact the image $\rho(\hat{\Gamma})$ will have torsion), but this is besides the point — the interesting part of this question is extending ρ as an action of Γ. For example, a nonfaithful action (with torsion or not) of a free group F on S^1 always extends to the disc *as an action of a free group* just by arbitrarily extending each generator.

To define ρ, set $\rho(a) = id$, and for all $\gamma \in \tilde{\Gamma}$ let $\rho(\gamma)$ be the action of γ on the quotient $\mathbb{R}/\mathbb{Z}$, i.e., the quotient action on the original circle S^1. In other words, the image of ρ in $\mathrm{Diff}^{\infty}_0(S^1)$ *is* the group Γ of Corollary 3.2. Note that the fact that $\rho(t) = [\rho(\hat{f}), \rho(\hat{g})]^2$ is rotation by $2\pi/3$ ensures that the relation $\rho(a)\rho(t)\rho(a)^{-1} = \rho(t)^4$ is satisfied.

We claim that this action does not extend to a C^2 action on the disc. To see this, suppose for contradiction that some extension $\phi : \hat{\Gamma} \to \mathrm{Diff}^1_0(B^2)$ exists. If $\phi(t)$ has finite order, then it must be rotation by $\pi/3$, and so has a unique fixed-point x. Now we make the same argument (verbatim!) as in the proof of Corollary 3.2: since $\phi(t)$ commutes with $\phi(\hat{f})$ and $\phi(\hat{g})$, both $\phi(\hat{f})$ and $\phi(\hat{g})$ fix x and have derivatives at x in $\mathrm{SO}(2)$. This contradicts the fact that $\phi(t)$ is the commutator of $\phi(\hat{f})$ and $\phi(\hat{g})$.

If instead $\phi(t)$ has infinite order, then it is a distorted element in $\phi(\hat{\Gamma})$. We know also that the restriction of $\phi(t)$ to the boundary is rotation by $2\pi/3$. Applying Corollary 4.3, we conclude that $\phi(t)$ has a single fixed-point x, and x is again fixed by $\phi(\hat{f})$ and $\phi(\hat{g})$. If the derivative $D\phi(t)_x$ were a nontrivial rotation of order at least 3, we could again look at derivatives at x and give the same argument as in the finite order case to get a contradiction.

Thus, it remains only to show that $D\phi(t)_x$ is a rotation of order at least 3. We show that it is rotation of order 3 exactly.

Lemma 4.4. *The derivative $D\phi(t)_x$ is a rotation of order 3.*

Proof. Since t is central in Γ and since $\rho(a)\rho(t)\rho(a)^{-1}x = \rho(t)^4 x = x$ implies that $\rho(a)x = x$, the whole group $\phi(\hat{\Gamma})$ fixes x. Moreover, the derivatives of $\rho(t)$ and $\rho(a)$ at x satisfy

$$D\phi(a)_x D\phi(t)_x D\phi(a)_x^{-1} = D\phi(t)_x^4.$$

This relation in $\mathrm{GL}(2,\mathbb{R})$ implies that either $D\phi(t)_x$ has a fixed tangent direction or is an order 3 rotation. Our strategy to show that it is order 3 is to compare the "rotation number" of $\phi(t)$ at the fixed-point and on the boundary.

Blow up the disc B^2 at x to get a C^0 action of $\hat{\Gamma}$ on the closed annulus, A. The action of $\hat{\Gamma}$ on one boundary component of A is the linear action on the space of tangent directions at x (so t either acts with a fixed-point or as an order 3 rotation), and on the other boundary it is the original action on ∂B^2 as an order 3 rotation.

With this setup, we can apply the notion of "linear displacement" from [FH06] and conclude that since $\rho(t)$ is distorted, it must act on each boundary component of A with the same *rotation number* and hence act as an order 3 rotation on both (See lemma 6.1 of [FH06]). But instead of defining "linear displacement" and "rotation number" here, it will be faster to give a complete, direct proof for our special case. The reader familiar with rotation numbers for circle homeomorphisms will see that it readily generalizes.

Suppose for contradiction that t acts on one boundary component of A with a fixed-point. Let $\tilde{A}$ denote the universal cover of A, identified with $\mathbb{R} \times [0,1]$ with covering transformation $T : (x_1, x_2) \mapsto (x_1 + 1, x_2)$.

Let $\tilde{t} \in \mathrm{Homeo}_0(\tilde{A})$ be the lift of the action of t to $\tilde{A}$ with a fixed-point on one boundary component; without loss of generality assume $(x_0, 1)$ is fixed. Then $\tilde{t}$ acts on $\mathbb{R} \times \{0\}$ as translation by $m + 1/3$ for some integer m. Let $\tilde{a}$ be any lift of the action of a.

Now $\tilde{a}(\tilde{t})^n\tilde{a}^{-1}$ is a lift of $(\tilde{t})^{4^n}$, so is of the form $(\tilde{t})^{4^n}T^l$ for some l. In particular, considering the distance between the images of $(x_0, 0)$ and $(x_0, 1)$ we have

$$\begin{aligned}
\|\tilde{a}(\tilde{t})^n\tilde{a}^{-1}(x_0,1) - \tilde{a}(\tilde{t})^n\tilde{a}^{-1}(x_0,0)\| &= \|(\tilde{t})^{4^n}(x_0,1) - (\tilde{t})^{4^n}(x_0,0)\| \\
&= \|(x_0,1) - (x_0 + (m+1/3)^{4n}, 1)\| \\
&\sim (m+1/3)^{4n}
\end{aligned}$$

However, the distance $\|\tilde{a}(\tilde{t})^n\tilde{a}^{-1}(x_0,1) - \tilde{a}(\tilde{t})^n\tilde{a}^{-1}(x_0,0)\|$ grows *linearly* in n — it is bounded by the maximum displacement of $\tilde{a}$ and $\tilde{t}$. Precisely, if

$$d = \max_{z \in \tilde{A}} \left\{ \max\{\|\tilde{a}(z) - z\|, \|\tilde{t}(z) - z\|\} \right\}$$

then we have

$$2(n+2)d + 1 \leq \|\tilde{a}(\tilde{t})^n \tilde{a}^{-1}(x_0, 1) - \tilde{a}(\tilde{t})^n \tilde{a}^{-1}(x_0, 0)\|$$

and this is our desired contradiction. □

Remark 4.5. It is possible to modify the construction in the proof Theorem 1.1 to avoid finite order elements. The idea is to modify $\rho(\hat{f})$ slightly so that the diffeomorphism $\rho(t) := [\rho(\hat{f}), \rho(\hat{g})]^2$ is the composition of an order 3 rotation r with an r-equivariant diffeomorphism h supported on a collection of small intervals in S^1 and conjugate to a translation on these intervals. We then modify $\rho(a)$ so that it is remains r-equivariant, but is conjugate to an expansion on the intervals of $\mathrm{supp}(h)$ — i.e., so that h and $\rho(a)$ act by a standard Baumslag–Solitar action on these intervals. Done correctly, $\rho(\hat{f})$, $\rho(\hat{g})$ and $\rho(a)$ will be infinite order diffeomorphisms, and will generate a subgroup of $\mathrm{Diff}_0^\infty(S^1)$ satisfying the relations $[\rho(t), \rho(\hat{f})] = [\rho(t), \rho(\hat{g})] = 1$ and $\rho(t)\rho(a)\rho(t)^{-1} = \rho(a)^4$. We leave the details to the reader.

5. Exotic homomorphisms: nonexistence

In [Her], Michael Herman proved the following stronger version of Theorem 3.1 in the case where $n = 1$.

Theorem 5.1 ([Her])**.** *There are no nontrivial group homomorphisms*

$$\mathrm{Diff}_0^\infty(S^1) \to \mathrm{Diff}_0^1(B^2).$$

Herman's key tools are the deep fact that $\mathrm{Diff}_0^\infty(S^1)$ is simple, and the easy fact that S^1 is a finite cover of itself. We combine some of these ideas with the techniques of Ghys in [Ghy91] to prove a similar theorem for any odd dimensional sphere, with any group of diffeomorphisms of a ball as the target. This is Theorem 1.2 as stated in the introduction.

Theorem 1.2. *There are no nontrivial group homomorphisms*

$$\mathrm{Diff}_0^\infty(S^{2k-1}) \to \mathrm{Diff}_0^1(B^m)$$

for any m, $k \geq 1$.

Proof. Let $n = 2k - 1$ and identify S^n with the unit sphere

$$\left\{ (z_1, \ldots, z_k) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{k} |z_i|^2 = 1 \right\}.$$

For any prime p, there is a free $\mathbb{Z}_p$ -action on S^k generated by the map

$$f_p : (z_1, \ldots, z_k) \mapsto (\mu_1 z_1, \ldots, \mu_k z_k)$$

where μ_i are any p^{th} roots of unity.

Suppose $\phi : \mathrm{Diff}_0^\infty(S^n) \to \mathrm{Diff}_0(B^m)$ is a nontrivial homomorphism. Since $\mathrm{Diff}_0^\infty(S^n)$ is a simple group (a deep result due to Mather and Thurston, see, e.g., [Ban97] for a proof), ϕ must be injective. By the Brouwer fixed-point theorem, $\phi(f_p)$ must fix a point. Since f_p is a finite order diffeomorphism, the set $\mathrm{fix}(\phi(f_p)) \subset B^m$ of fixed-points of $\phi(f)$ is a submanifold of B^m (one way to see this is to average a metric so that f_p acts by isometries). That f_p is orientation preserving and of finite order further implies that $\mathrm{fix}(\phi(f_p))$ has codimension at least 2, this is because any finite order diffeomorphism f is an isometry with respect to some metric, and if f is nontrivial its derivative at a fixed-point is a nontrivial finite order element of $\mathrm{O}(n)$.

Let H be the group of isotopically trivial diffeomorphisms of $S^n/\langle f_p\rangle \cong S^n$. We have an exact sequence

$$0 \to \mathbb{Z}_p \to H' \to H \to 1$$

where H' is the group of all lifts of diffeomorphisms in H to f_p-equivariant diffeomorphisms of S^n.

We claim now that $\mathbb{Z}_p$ is the *only* normal subgroup of H'. To see this, suppose that $N \subset H'$ is a normal subgroup. Then the image of N in H must either be trivial or all of H. If the image is trivial, then either N is trivial or $N = \mathbb{Z}_p$ and we are done. If the image of N in H is all of H, we consider a $S^1 \times \cdots \times S^1$ subgroup of H, where the i^{th} S^1 factor is the norm 1 complex numbers mod μ_i. An element $(\lambda_1, \dots \lambda_k) \in (S^1)^k/(\mu_1, \dots, \mu_k)$ acts on $S^n/\langle f_p\rangle$ by pointwise multiplication,

$$(z_1, \dots z_k) \mapsto (\lambda_1 z_1, \dots \lambda_k z_k).$$

Consider the extension Γ as in the diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_p \arrow[r] & H' \arrow[r] & H \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z}_p \arrow[r] \arrow[u, equal] & \Gamma \arrow[r] \arrow[u, hook] & S^1 \times \cdots \times S^1 \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

Specifically, Γ is the group of all lifts of these actions

$$(z_1, \dots z_k) \mapsto (\lambda_1 z_1, \dots \lambda_k z_k)$$

to S^n, the p-fold cover of $S^n/\langle f_p\rangle$. It may be helpful for the reader to consider the $n = 1$ case, in which case we are just working with rotations of S^1 and their lifts to a p-fold cover of S^1.

Note that $N \cap \Gamma$ is a normal subgroup of Γ that projects to the full group $S^1 \times \cdots \times S^1$. In particular, since $\left(\mu_1^{\frac{1}{p}}, \dots \mu_n^{\frac{1}{p}}\right) \in S^1 \times \cdots \times S^1$, we know that some diffeormorphism g of the form

$$(z_1, \dots z_k) \overset{g}{\mapsto} \left(\mu_1^{n_1 + \frac{1}{p}} z_1, \dots \mu_k^{n_k + \frac{1}{p}} z_k\right), \quad n_i \in \mathbb{Z}$$

lies in Γ, hence in H'. It follows that $g^p = f_p$ is a generator of $\mathbb{Z}_p$, so $\mathbb{Z}_p \subset N$. Since $\mathbb{Z}_p \subset N$ and N projects to H, it follows that $N = H'$, which is what we wanted to show.

Having shown that $\mathbb{Z}_p$ is the only normal subgroup of H', we can conclude that the action of $\phi(H')$ on $\mathrm{fix}(\phi(f_p)) \subset B^m$ is either faithful, trivial, or has kernel $\mathbb{Z}_p$. We already know that $\mathbb{Z}_p$ lies in the kernel — this is $\phi(f_p)$ acting on its fix set — so the action of $\phi(H')$ is not faithful. If the action is trivial, then for $x \in \mathrm{fix}(\phi(f_p))$, we get a representation

$$D : H' \to \mathrm{GL}(m, \mathbb{R}) \subset \mathrm{GL}(m, \mathbb{C})$$

by sending a diffeomorphism f to the derivative of $\phi(f)$ at x. Since $\phi(f_p)$ has nontrivial derivative at any point, and $\mathbb{Z}_p = \langle f_p \rangle$ is the only normal subgroup of H', the representation D must be faithful. We will show this is impossible. Indeed, it should already seem believable to the reader that H' is a "large" group and so is not linear. Here is a short, elementary argument to make this clear.

Proof that D cannot be a faithful representation. Since $D\phi(f_p)(x)$ has order p, after conjugation in $\mathrm{GL}(m, \mathbb{C})$ we may assume it is diagonal of the form

$$\begin{bmatrix} \alpha_1 I_{n_1} & 0 & \cdots & 0 \\ 0 & \alpha_2 I_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_k I_{n_k} \end{bmatrix}$$

where α_i are each distinct p^{th} roots of unity, the distinct complex eigenvalues of $D\phi(f_p)(x)$, and I_{n_i} is the $n_i \times n_i$ square identity matrix.

The centralizer of such a matrix in $\mathrm{GL}(m, \mathbb{C})$ is the set of block diagonals of the form

$$\begin{bmatrix} A_{n_1} & 0 & \cdots & 0 \\ 0 & A_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_k} \end{bmatrix}$$

with $A_{n_i} \in \mathrm{GL}(n_i, \mathbb{C})$. In other words, the centralizer is a subgroup isomorphic to $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_k, \mathbb{C})$. In particular, (after conjugation) we may view H' as a subgroup of $\mathrm{GL}(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_k, \mathbb{C})$, with $f_p \in H$ a central element.

Since $D\phi(f_p)(x)$ has order p, at least one eigenvalue is not 1. Without loss of generality, assume $\alpha_1 \neq 1$. Now consider the homomorphism $H' \to \mathbb{R}$ given by projecting $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_k, \mathbb{C})$ onto the first factor — i.e., onto $\mathrm{GL}(n_1, \mathbb{C})$ — and then taking the determinant. We may assume that we chose $p > m$, so as to ensure that the image $\alpha_1^{n_1}$ of f_p under this homomorphism is nontrivial. However, we showed above that the subgroup generated by f_p was the only normal subgroup of H'. This means that this homomorphism to $\mathbb{R}$ must be faithful — but this is impossible since H' itself is nonabelian. □

Thus, it remains only to deal with the case where H' acts on $\mathrm{fix}(\phi(f_p))$ with kernel $\mathbb{Z}_p$. In this case, we introduce an inductive argument. Consider the diffeomorphism

$$f_{p^2} : (z_1, \ldots, z_k) \mapsto (\nu_1 z_1, \ldots, \nu_k z_k)$$

where $\nu_i^2 = \mu_i$. Then f_{p^2} is an order p^2 diffeomorphism acting freely on S^n, commuting with f_p and so an element of H'. Since $f_{p^2} \notin \mathbb{Z}_p$, we know that $\phi(f_{p^2})$ acts nontrivially on $\mathrm{fix}(\phi(f_p))$. Moreover, $\mathrm{fix}(\phi(f_{p^2})) \subset \mathrm{fix}(\phi(f_p))$, and is a nonempty submanifold of codimension at least two.

As before, we consider a group of diffeomorphisms of a quotient of S^n. Let H_2 be the group of isotopically trivial diffeomorphisms of $S^n/\langle f_{p^2}\rangle$. Since $S^n/\langle f_{p^2}\rangle$ is a compact manifold, H_2 is a simple group. Let H_2' be the group of all lifts of elements of H_2 to S^n. The argument we gave above for H works (essentially verbatim) to show that $\langle f_p\rangle \cong \mathbb{Z}_p$, and $\langle f_{p^2}\rangle \cong \mathbb{Z}_{p^2}$ are the only normal subgroups of H_2'.

Now consider the action of H_2' on $\mathrm{fix}(\langle f_{p^2}\rangle)$. If the action is trivial, we get as before a global fixed-point and a linear representation $H_2' \to \mathrm{GL}(m, \mathbb{R})$. The argument using matrix centralizers above can be applied again in this case to derive a contradiction. Otherwise, the action of H_2' on $\mathrm{fix}(\phi(f_{p^2})$ is nontrivial. In this case, we can proceed inductively by considering higher powers of p and corresponding diffeomorphisms f_{p^k}. Each time we will reduce the dimension of the fix set (a finite process) or derive a contradiction. □

Note that the proof above depended on the fact that S^{2k-1} admits finite order diffeomorphisms that act freely, and so it does not readily generalize to odd dimensional spheres. We conclude with a natural follow-up problem.

Problem 5.2. *Describe all homomorphisms* $\mathrm{Diff}_0^\infty(S^{2n}) \to \mathrm{Diff}_0^1(B^m)$. *Can such a homomorphism be nontrivial?*

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