A supplement to the classification of flat homogeneous spaces of signature $(m, 2)$

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ABSTRACT. Duncan and Ihrig (1993) gave a classification of the flat homogeneous spaces of metric signature $(m, 2)$, provided that a certain condition on the development image of these spaces holds. In this note we show that this condition can be dropped, so that Duncan and Ihrig's classification is in fact the full classification for signature $(m, 2)$.

CONTENTS

1. Introduction

In a series of three articles, Duncan and Ihrig [2, 3, 4] developed a theory for flat pseudo-Riemannian homogeneous spaces. For geodesically complete manifolds, this theory was previously developed by Wolf [6], and his methods were extended by Duncan and Ihrig to the geodesically incomplete case.

In their first article [2], they classified the Lorentzian flat homogeneous spaces. In the second article [3], they studied isometry groups of pseudo-Riemannian flat homogeneous spaces. Underlying their investigations were the following facts (see Goldman and Hirsch [5]): Let M be a flat homogeneous pseudo-Riemannian manifold of metric signature $(n - s, s)$. If M is geodesically complete, then $M = \mathbb{R}^n_s/\Gamma$, where s is the index of a pseudo-scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, and $\Gamma \subset \mathsf{Iso}(\mathbb{R}^n_s)$ is the fundamental group of M. If M is not geodesically complete, then its universal cover $\tilde{M}$ can be mapped to an affine homogeneous domain $\mathfrak{D} \subset \mathbb{R}^n_s$ via the development map. We call $\mathfrak{D}$ the *development image* of $\tilde{M}$. The development map induces a homomorphism from the fundamental group $\pi_1(M)$ onto a group $\Gamma \subset \mathsf{Iso}(\mathfrak{D})$,

which we will call the *affine holonomy group* of M. Then the manifold M can be realised as $M = \mathfrak{D}/\Gamma$.

In their third article [4], Duncan and Ihrig classified those flat homogeneous spaces of signature $(n-2, 2)$ whose associated domain $\mathfrak{D}$ is *translationally isotropic*. This means that the set T of all translations in $\mathbb{R}_2^n$ leaving $\mathfrak{D}$ invariant contains its own orthogonal space: $T^\perp \subset T$. We will prove in this article that every development image of a flat homogeneous space of signature $(n-2, 2)$ is translationally isotropic. As a consequence, the classification by Duncan and Ihrig [4, Chapter 3] is already the full classification of flat homogeneous spaces of signature $(n-2, 2)$.

Remark 1.1. In their classification, Duncan and Ihrig [4, Theorem 3.1, case I] misquote the geodesically complete case from Wolf's article [6]. They state that in this case the affine holonomy group Γ is a discrete group of pure translations. However, it is only true that Γ is a free abelian group in this case, but as Wolf's example in Section 6 of [6] shows, the elements of Γ can have nontrivial linear parts. The classification for this case is given by Wolf [7, Theorem 3.6].

Our main result is the following theorem:

Theorem A. *Let $M = \mathfrak{D}/\Gamma$ be a flat pseudo-Riemannian homogeneous manifold, where $\mathfrak{D} \subseteq \mathbb{R}_s^n$ is an open orbit of the centraliser of Γ in $\mathsf{Iso}(\mathbb{R}_s^n)$. If M has abelian linear holonomy, then $\mathfrak{D}$ is a translationally isotropic domain.*

The proof is given in Section 3. Using Theorem 2.3 we conclude:

Corollary. *Let $M = \mathfrak{D}/\Gamma$ be a geodesically incomplete flat pseudo-Riemannian homogeneous manifold of metric signature $(n-s, s)$ with $s \in \{0, 1, 2, 3\}$. Then $\mathfrak{D}$ is a translationally isotropic domain.*

A major consequence of Theorem A for Duncan and Ihrig's classification is now:

Corollary. *Taking into account the correction in Remark 1.1, the classification of flat homogeneous manifolds of metric signature $(n-2, 2)$ with translationally isotropic domain given by Duncan and Ihrig [4, Theorem 3.1] is in fact the full classification of flat homogeneous manifolds of metric signature $(n-2, 2)$.*

Remark 1.2. For signatures with $s \geq 4$, nonabelian linear holonomy groups can occur, as Example 6.2 with signature $(4, 4)$ in Baues and Globke [1] shows. In this particular example, $\mathfrak{D}$ is translationally isotropic. However, it is not clear if the assumption of abelian linear holonomy in Theorem A can be dropped.

2. Affine and linear holonomy groups

Let $M = \mathfrak{D}/\Gamma$ be a flat homogeneous pseudo-Riemannian space of signature $(n-s, s)$. The affine holonomy group Γ consists of affine transformations

of $\mathbb{R}_s^n$ leaving $\mathfrak{D}$ invariant, and the group $\mathsf{Hol}(\Gamma) \subset \mathsf{O}_{n-s,s}$ consisting of its linear parts is called the *linear holonomy group*.[1]

For M to be homogeneous, it is necessary that the centraliser of Γ in $\mathsf{Iso}(\mathfrak{D})$ acts transitively on $\mathfrak{D}$, or in other words, the centraliser of $\mathsf{C}_{\mathsf{Iso}(\mathbb{R}_s^n)}(\Gamma)$ has an open orbit $\mathfrak{D} \subset \mathbb{R}_s^n$ (see Wolf [8, Theorem 2.4.17]). We recall some properties of groups Γ with this property (see Wolf [8, Section 3.7] for the original proofs, adapted to incomplete spaces by Duncan and Ihrig [3, Section 4]).

An affine transformation $g \in \mathsf{Aff}(\mathbb{R}^n)$ is written as $g = (A, v)$, where A is the linear part of g and v is its translation part. The identity matrix is denoted by I. Let $\operatorname{im} A$ and $\ker A$ denote the image and the kernel of a matrix A, respectively. An element $v \in \mathbb{R}_s^n$ is called *isotropic* if $\langle v, v \rangle = 0$, and a vector subspace $U \subset \mathbb{R}_s^n$ is called *totally isotropic* if $\langle u, v \rangle = 0$ for all $u, v \in U$.

Lemma 2.1 (Wolf [6]). *Let $\gamma \in \Gamma$. Then $\gamma = (I + A, v)$, where $A^2 = 0$, $Av = 0$, $\operatorname{im} A$ is totally isotropic and $v \perp \operatorname{im} A$. Moreover, $\operatorname{im} A = (\ker A)^\perp$ and $\ker A = (\operatorname{im} A)^\perp$.*

We define a vector subspace of $\mathbb{R}_s^n$,

$$U_\Gamma = \sum_{(I+A,v) \in \Gamma} \operatorname{im} A, \quad \text{with } U_\Gamma^\perp = \bigcap_{(I+A,v) \in \Gamma} \ker A.$$

Then the subspace

$$(2.1) \qquad\qquad\qquad U_0 = U_\Gamma \cap U_\Gamma^\perp$$

is totally isotropic.

Proposition 2.2 (Wolf [8]). *The following are equivalent:*

(1) $\mathsf{Hol}(\Gamma)$ *is abelian.*
(2) *If $(I + A_1, v_1), (I + A_2, v_2) \in \Gamma$, then $A_1 A_2 = 0$.*
(3) *The space U_Γ is totally isotropic.*
(4) $U_0 = U_\Gamma$.

The vector space $\mathbb{R}_s^n$ decomposes as

$$\mathbb{R}_s^n = U_0 \oplus W_0 \oplus U_0^*,$$

where U_0^* is a dual space to U_0 and W_0 is orthogonal to U_0 and U_0^*. It was shown in Baues and Globke [1, Theorem 4.4] that in a basis subordinate to this decomposition, the linear part of $\gamma = (I + A, v) \in \Gamma$ takes the form

$$(2.2) \qquad\qquad A = \begin{pmatrix} 0 & -B^\top \tilde{I} & C \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix},$$

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[1]The linear holonomy group is what differential geometers simply refer to as the *holonomy group*: The group generated by parallel transports of tangent vectors around loops based at a certain point in M, see also Wolf [8, Lemma 3.4.4].

where C is skew-symmetric, and the columns of B are isotropic and mutually orthogonal with respect to $\tilde{I}$, the matrix representing the restriction of $\langle \cdot, \cdot \rangle$ to W_0.

Theorem 2.3. *Let $\Gamma \subset \mathsf{Iso}(\mathbb{R}^n_s)$ be a group acting on $\mathbb{R}^n_s$ whose centraliser in $\mathsf{Iso}(\mathbb{R}^n_s)$ has an open orbit. If $\mathsf{Hol}(\Gamma)$ is not abelian, then $s \geq 4$.*

A weaker version of this theorem was already proved in [1, Theorem 5.1], where the statement "$s \geq 4$" is replaced by "$n \geq 8$".

Proof. If $\mathsf{Hol}(\Gamma)$ is not abelian, then there exist elements

$$(I + A_1, v_1), (I + A_2, v_2) \in \Gamma$$

such that $A_1 A_2 \neq 0$ (Proposition 2.2). This means the respective blocks B_1, B_2 in (2.2) are not 0.

According to rule (1) in Baues and Globke [1, Section 5], the columns of B_1 and B_2 contain at least four linearly independent isotropic vectors $b_1^1, b_1^2, b_2^1, b_2^2 \in W_0$ satisfying $\langle b_1^i, b_1^j \rangle_{W_0} = 0$ and $\langle b_1^i, b_2^j \rangle_{W_0} \neq 0$ for $i \neq j$. In particular, b_1^1 and b_1^2 span a 2-dimensional totally isotropic subspace $W' \subset W_0$.

Moreover, this implies that $\mathrm{rk}\, B_1 = \mathrm{rk}\, B_1^\top \geq 2$, so that $\dim U_0 \geq 2$. As $W' \perp U_0$, the space $W' \oplus U_0$ is a totally isotropic subspace of dimension ≥ 4. But then $s \geq 4$ holds, because s is an upper bound for the dimension of totally isotropic subspaces. $\qquad \square$

3. Proof of the main result

Let $\mathfrak{D} \subset \mathbb{R}^n_s$ be an open domain. Its isometry group is

$$\mathsf{Iso}(\mathfrak{D}) = \{ g \in \mathsf{Iso}(\mathbb{R}^n_s) \mid g.\mathfrak{D} \subseteq \mathfrak{D} \},$$

and by $T = \mathsf{Iso}(\mathfrak{D}) \cap \mathbb{R}^n_s$ we denote the set of translations leaving $\mathfrak{D}$ invariant, $T + \mathfrak{D} = \mathfrak{D}$. If $T^\perp \subset T$, then we call $\mathfrak{D}$ *translationally isotropic.*

Lemma 3.1. *Let U be a totally isotropic subspace in $\mathbb{R}^n_s$. If $U^\perp \subseteq T$, then $\mathfrak{D}$ is translationally isotropic.*

Proof. Assume $U^\perp \subseteq T$. If some vector v satisfies $v + \mathfrak{D} \not\subset \mathfrak{D}$, then $v \notin U^\perp$. But then $v \not\perp U \subset T$. In particular, $v \notin T^\perp$. So any vector in $u \in T^\perp$ satisfies $u + \mathfrak{D} \subseteq \mathfrak{D}$, which means that $\mathfrak{D}$ is translationally isotropic. $\qquad \square$

In the following, we will assume $\mathfrak{D}$ is the development image for a flat pseudo-Riemannian homogeneous space. So let $\Gamma \subseteq \mathsf{Iso}(\mathbb{R}^n_s)$ be a discrete group acting freely and properly discontinuously on $\mathbb{R}^n_s$, and $\mathfrak{D} \subset \mathbb{R}^n_s$ is an open orbit of its centraliser $C = \mathrm{C}_{\mathsf{Iso}(\mathbb{R}^n_s)}(\Gamma)$ (all this holds if Γ is the affine holonomy group of $M = \mathfrak{D}/\Gamma$). Let T be the set of translations in $\mathbb{R}^n_s$ satisfying $T + \mathfrak{D} = \mathfrak{D}$, and let U_Γ and U_0 be the subspaces of $\mathbb{R}^n_s$ defined in Section 2.

We prove that if Γ has abelian linear holonomy, then $\mathfrak{D}$ is translationally isotropic.

Lemma 3.2. *Identify* U_0 *with the group of translations by vectors in* U_0. *Then* $U_0 \subset C \cap T$.

Proof. A translation by $u \in U_0$ is represented by (I, u). If $(I + A, v) \in \Gamma$, then

$$(I + A, v)(I, u) = (I + A, u + Au + v) = (I + A, u + v) = (I, u)(I + A, v),$$

where we used the fact that $U_0 \subset \ker A$ by definition. So $(I, u) \in C$ and thus (I, u) is a translation leaving C-orbits invariant, implying $(I, u) \in C \cap T$. $\quad\square$

Lemma 3.3. $\mathsf{Hol}(\Gamma)$ *is abelian if and only if* $U_0^\perp \subseteq C$. *If this holds, then* $\mathfrak{D}$ *is translationally isotropic.*

Proof. Let $u \in U_0^\perp$. Then

$$(I + A, v)(I, u) = (I + A, u + Au + v) = (I + A, u + v) = (I, u)(I + A, v)$$

for all $(I + A, v) \in \Gamma$ if and only if $Au = 0$ for all $(I + A, v) \in \Gamma$. But this is equivalent to

$$u \in \bigcap_A \ker A = U_\Gamma^\perp \subset U_0^\perp.$$

As u is arbitrary, this means $U_\Gamma^\perp = U_0^\perp$, which again is equivalent to the linear holonomy of Γ being abelian by part (4) of Proposition 2.2. In this case, $\mathfrak{D}$ is translationally isotropic by Lemma 3.1. $\quad\square$

This concludes the proof of Theorem A.

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