Isometric weighted composition operators

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ABSTRACT. A composition operator is an operator on a space of functions defined on the same set. Its action is by composition to the right with a fixed selfmap of that set. A composition operator followed by a multiplication operator is called a weighted composition operator. In this paper, we study when weighted composition operators on the Hilbert Hardy space of the open unit disc are isometric. We find their Wold decomposition in select cases and apply it to the computation of numerical ranges.

CONTENTS

1. Introduction

The Hardy spaces of index $1 \leq q < +\infty$, H^q, are the spaces of analytic functions f on the open unit disc $\mathbb{U}$ centered at the origin with property

$$(1) \qquad \|f\|_q := \sup\left\{ \left(\int_{\mathbb{T}} |f(ru)|^q \, dm(u) \right)^{\frac{1}{q}} : 0 \leq r < 1 \right\} < +\infty,$$

where m is the normalized arc–length measure on the unit circle $\mathbb{T} = \partial\mathbb{U}$. We denote by H^∞ the space of bounded analytic functions on $\mathbb{U}$.

For $1 \leq q < +\infty$, $\| \ \|_q$ is a Banach space norm. If $q = 2$, it is a Hilbert space norm generated by the inner product

$$(2) \qquad \langle f, g \rangle = \sum_{n=0}^{\infty} c_n \overline{d_n}, \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad g(z) = \sum_{n=0}^{\infty} d_n z^n \in H^2.$$

As is well known, if $f \in H^q$, $1 \leq q \leq +\infty$, then f has radial limits a.e. on $\mathbb{T}$. Throughout this paper, the radial limit function of some $f \in H^q$ will be denoted by the same symbol f, relying on the context in order to distinguish

between a function and its radial limit function. Using this notation, recall that, the norm $\|f\|_q$ of $f \in H^q$, can be also calculated with the formula

$$(3) \qquad \|f\|_q = \left(\int_{\mathbb{T}} |f(u)|^2 \, dm(u) \right)^{\frac{1}{q}}, \qquad f \in H^q.$$

The space H^2 is a reproducing kernel Hilbert space with kernel functions

$$K_w(z) = \frac{1}{1 - \overline{w}z}, \qquad w, z \in H^2,$$

which means that the functions K_w are in H^2 for all $w \in \mathbb{U}$ and have property

$$\langle f, K_w \rangle = f(w), \qquad w \in \mathbb{U}, \quad f \in H^2.$$

Given any analytic selfmap φ of $\mathbb{U}$, the operator

$$C_\varphi f = f \circ \varphi, \qquad f \in H^q, \quad 1 \le q \le \infty,$$

called the composition operator with symbol φ, is necessarily bounded.

Equation (3) embeds isometrically H^q into $L^q_{\mathbb{T}}(dm)$. It is therefore customary to write $H^q \subseteq L^q_{\mathbb{T}}(dm)$. We denote by P the orthogonal projection of $L^2_{\mathbb{T}}(dm)$ onto H^2. If ψ is an essentially bounded function on $\mathbb{T}$, the (necessarily bounded), operator

$$T_\psi f = P\psi f, \qquad f \in H^2$$

is called the Toeplitz operator with symbol ψ. If $\psi \in H^\infty$ then we call T_ψ an analytic Toeplitz operator. If the symbol ψ of an analytic Toeplitz operator T_ψ is an inner function (that is an analytic selfmap of $\mathbb{U}$ which has unimodular radial limits a.e. on $\mathbb{T}$), then obviously T_ψ is an isometry. If T_ψ is an analytic Toeplitz operator, then the operator $T_\psi C_\varphi$ is bounded and has the form

$$T_\psi C_\varphi f = \psi f \circ \varphi, \qquad f \in H^q, \quad 1 \le q \le \infty.$$

We call the weighted composition operator with symbols ψ and φ (in this order), the linear operator

$$T_{\psi, \varphi} f = \psi f \circ \varphi, \qquad f \in H^q, \quad 1 \le q \le \infty,$$

where we only require that ψ and φ be analytic and φ be a selfmap of $\mathbb{U}$. We are interested in this kind of operators when acting on H^2, which we will always assume, unless otherwise specified. They can be bounded even if $\psi \notin H^\infty$. To see a trivial example, consider $p \in \mathbb{U}$ and any $\psi \in H^2$. Denote also by p the selfmap of $\mathbb{U}$ constantly equal with p and note that $T_{\psi, p}$ is bounded, since one can write

$$\|T_{\psi, p} f\|_2 = |f(p)| \|\psi\|_2 = \|\psi\|_2 |\langle f, K_p \rangle| \le \|\psi\|_2 \|K_p\|_2 \|f\|_2, \qquad f \in H^2.$$

We distinguish between the two symbols of a weighted composition operator $T_{\psi, \varphi}$, by calling ψ the multiplication symbol and φ the composition symbol. In the above example, the multiplication symbol was chosen in H^2.

This choice is necessary for the boundedness of any weighted composition operator on H^2, since $T_{\psi,\varphi}(1) = \psi$.

The adjoint of a bounded weighted composition operator on H^2 is known to transform kernel functions as follows:

$$(4) \qquad\qquad T_{\psi,\varphi}^* K_p = \overline{\psi(p)} K_{\varphi(p)}, \qquad p \in \mathbb{U}.$$

It is well known that, whenever φ is an analytic selfmap of $\mathbb{U}$, other than the identity or an elliptic automorphism, its sequence of iterates $\{\varphi^{[n]}\}$ converges uniformly on compacts to a constant function ω, which is called the Denjoy–Wolff point of φ, since this convergence makes the object of a well known theorem of A. Denjoy and J. Wolff [2]. Clearly ω belongs to the closed unit disc. When $\omega \in \mathbb{U}$, then ω is the unique fixed point of φ. Otherwise, that is, if $|\omega| = 1$, the Denjoy–Wolff point of φ, is a boundary fixed point of φ, that is it satisfies the equality $\lim_{r\to 1^-} \varphi(r\omega) = \omega$. Actually, ω can be identified as the only boundary fixed point of φ where the angular derivative $\varphi'(\omega)$ of φ exists and satisfies the condition $\varphi'(\omega) \le 1$. The selfmaps φ with the property $\varphi'(\omega) < 1$ are called maps of hyperbolic type, whereas those satisfying the condition $\varphi'(\omega) = 1$ are called maps of parabolic type. Finally, the maps of parabolic type come in two flavors: maps of parabolic automorphic type and maps of parabolic nonautomorphic type, depending if they have or do not have pseudohyperbolically separated orbits. We refer the reader to [2], for a thorough discussion of the Denjoy–Wolff theorem and the classification of analytic selfmaps of $\mathbb{U}$ induced by it.

As the reader must have noticed, in the current section, we introduce the main concepts used in this paper and briefly describe the main results obtained in the next sections.

The isometric operators on non-Hilbert Hardy spaces (i.e., on H^q, $q \ne 2$) are necessarily weighted composition operators of a certain kind, according to F. Forelli's paper [8]. That paper contains two theorems: a general characterization of all isometries of H^q, $q \ne 2$ (Theorem 3 in this paper), and a more specific characterization of onto isometries (Theorem 2). As observed by Forelli, his resuts do not extend to H^2 which, being a Hilbert space, has many other isometric operators (onto or not). Nevertheless, relative to Forelli's characterization of onto isometries, this author noted a recent result of Bourdon and Narayan [3], who obtained the characterization of unitary weighted composition on H^2 (Theorem 1 in this paper). Our remark is: if one substitutes q by 2 in Forelli's characterization of onto isometries of H^q, $q \ne 2$, one gets exactly Theorem 1. Thus, one can conjecture that substitution of q by 2 in Forelli's general description of the isometries of H^q would produce the if and only if characterization of the isometric weighted composition operators of H^2. We prove this conjecture is true: Theorem 5. All this is done in the second section of this paper, where we also address the issue of the Wold decomposition of isometric weighted composition operators. It should be recalled that an isometry V on a separable, complex,

infinite–dimensional, Hilbert space H is called a (unilateral) forward shift if the sequence $\{V^{*n}\}$ tends to 0 in the strong operator topology. For any isometry V on H, the space H splits into an orthogonal sum $H = H_0 \oplus H_1$ of closed reducing subspaces of V, so that $V|H_0$ is a unitary operator (called the unitary part of V) and $V|H_1$ is a forward shift. This orthogonal sum decomposition is called the Wold decomposition of V. It is well known that the space H_0 is the space of vectors which have backwards iterates of any order under V, that is $f \in H_0$ if and only if there is a sequence $\{f_n\}$ in H so that $f = V^n f_n$, $n = 1, 2, 3, \ldots$.

In the case of isometric weighted composition operators whose composition symbol has a fixed point in $\mathbb{U}$, a complete description of the Wold decomposition is obtained (Theorems 8 and 9). In the process of proving those results, we extend results in [7], a paper that inspired us to write the current paper.

Relative to the case when the multiplication symbol is fixed points free, we are able to determine the Wold decomposition of isometric weighted composition operators whose composition symbol is a disc automorphism (Theorem 10).

The third section is dedicated to finding the numerical ranges of some weighted composition operators (Theorems 13 and 15). We wish to mention that the numerical range $W(T)$ of a bounded Hilbert space operator T is the (necessarily convex), set $W(T) = \{\langle Tf, f \rangle : f \in H, \|f\| = 1\}$. As is well known, the closure of $W(T)$ contains the spectrum $\sigma(T)$ of T. The numerical range of some isometric composition operators on Hardy spaces over a half–plane is shown to coincide to $\mathbb{U}$ (Proposition 2).

2. Weighted composition isometries

The unitary weighted composition operators of H^2 were characterized in [3] as follows:

Theorem 1. *The operator $T_{\psi,\varphi}$ is unitary if and only if φ is a disc automorphism and $\psi = cK_p/\|K_p\|_2$, where $\varphi(p) = 0$ and $|c| = 1$.*

Relative to Theorem 1, recall the following well known result of F. Forelli, [8, Theorem 2]:

Theorem 2. *For $1 \leq q < \infty$, $q \neq 2$, the onto isometries of H^q are the operators V of form*

$$(5) \qquad\qquad V = T_{\lambda(\varphi')^{\frac{1}{q}},\varphi}$$

where φ is a disc automorphism and $|\lambda| = 1$.

Clearly, substitution of q by 2 in Theorem 2, will not describe all unitary operators on H^2, since H^2 is a Hilbert space. However:

Remark 1. If one substitutes q by 2 in equation (5) one obtains the description of all onto isometries of H^2 which are weighted composition operators.

To see that, recall that if φ is a disc automorphism with property $\varphi^{-1}(0) = p$, then φ must have the form $\varphi(z) = c(p - z)/(1 - \bar{p}z)$, where $|c| = 1$, and hence

$$\left(\varphi'(z)\right)^{1/2} = \sqrt{-c}K_p(z)/\|K_p\|_2, \qquad z \in \mathbb{U}.$$

Let $P(z, u)$ denote the usual Poisson kernel, that is

$$P(z, u) = \operatorname{Re}\frac{u + z}{u - z} = \frac{1 - |z|^2}{|u - z|^2}, \qquad u \in \mathbb{T}, z \in \mathbb{U}.$$

With this notation, and denoting by $\mathcal{L}$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{T}$, recall [8, Theorem 1]:

Theorem 3. *For $1 \le q < \infty$, $q \ne 2$, the isometries of H^q are the operators V of form $V = T_{\psi,\varphi}$ where φ is an inner function and ψ is an H^q-function with the property*

$$(6) \qquad \int_X |\psi|^q \, dm = \int_X \frac{dm(u)}{P(\varphi(0), \varphi(u))}, \qquad X = \varphi^{-1}(Y), \qquad Y \in \mathcal{L}.$$

Remark 1 raises the following question. If one takes $q = 2$ in (6), does one obtain the characterization of isometric weighted composition operators on H^2? We will prove the answer is affirmative. To that aim, we begin by the following:

Theorem 4. *If $T_{\psi,\varphi}$ is isometric on H^2, then φ must be an inner function and ψ must belong to H^2 and have norm equal to 1.*

Proof. As we observed, $T_{\psi,\varphi}(1) = \psi$. If $T_{\psi,\varphi}$ is isometric then, $\|\psi\|_2 = 1$, since $\|1\|_2 = 1$. To show that φ must be inner if $T_{\psi,\varphi}$ is isometric, argue by contradiction and assume φ is not inner. Then $\{u \in \mathbb{T} : |\varphi(u)| < 1\}$ is a measurable set of positive measure, hence there is $0 \le c < 1$, so that $E := \{u \in \mathbb{T} : |\varphi(u)| \le c\}$, has positive measure. Given that $T_{\psi,\varphi}$ is isometric, one has that $\|T_{\psi,\varphi}(z)\|_2 = 1$, that is $\|\psi\varphi\|_2 = 1$. This fact leads to a contradiction, since one can write

$$1 = \|\psi\varphi\|_2^2 = \int_E |\psi\varphi|^2 \, dm + \int_{\mathbb{T}\backslash E} |\psi\varphi|^2 \, dm \le c^2 \int_E |\psi|^2 \, dm + \int_{\mathbb{T}\backslash E} |\psi|^2 \, dm,$$

hence

$$1 = \int_E |\psi|^2 \, dm + \int_{\mathbb{T}\backslash E} |\psi|^2 \, dm \le c^2 \int_E |\psi|^2 \, dm + \int_{\mathbb{T}\backslash E} |\psi|^2 \, dm,$$

which implies $c \ge 1$, since $\int_E |\psi|^2 \, dm > 0$. $\qquad\square$

Proposition 1. *If $\|T_{\psi,\varphi}\| \le 1$ and $\varphi(p) = p$ for some $p \in \mathbb{U}$, then $|\psi(p)| \le 1$. If $|\psi(p)| = 1$, then K_p is a reducing eigenfunction of $T_{\psi,\varphi}$ (that is an eigenfunction of both $T_{\psi,\varphi}$ and $T_{\psi,\varphi}^*$). If $|\psi(p)| = 1$ and $T_{\psi,\varphi}$ is an isometry, then K_p is an eigenfunction associated to the eigenvalue $\psi(p)$.*

Proof. Given that $\|T_{\psi,\varphi}\| \leq 1$, equation (4) implies $|\psi(z)|\|K_p\|_2 \leq \|K_p\|_2$, hence $|\psi(p)| \leq 1$. By the Cauchy–Schwarz inequality, one can write

(7) $$|\langle T_{\psi,\varphi}f, K_p\rangle| \leq \frac{1}{\sqrt{1-|p|^2}}, \qquad f \in H^2, \; \|f\|_2 = 1.$$

Assume now $|\psi(p)| = 1$. Taking $f = K_p/\|K_p\|_2$ in (7), one gets

$$|\langle T_{\psi,\varphi}f, K_p\rangle| = \frac{|\psi(p)|}{\sqrt{1-|p|^2}} = \frac{1}{\sqrt{1-|p|^2}},$$

that is, the Cauchy–Schwarz inequality (7), is an equality for this choice of f. Then the vectors involved in the inequality are colinear, that is

$$\sqrt{1-|p|^2}T_{\psi,\varphi}K_p = \lambda K_p,$$

hence

$$\sqrt{1-|p|^2}T^*_{\psi,\varphi}T_{\psi,\varphi}K_p = \lambda T^*_{\psi,\varphi}K_p = \lambda\overline{\psi(p)}K_p.$$

If $T_{\psi,\varphi}$ is isometric, then $T^*_{\psi,\varphi}T_{\psi,\varphi} = I$, and one deduces

$$\sqrt{1-|p|^2}K_p = \lambda\overline{\psi(p)}K_p$$

hence $\lambda = \sqrt{1-|p|^2}\psi(p)$, which implies

$$T_{\psi,\varphi}K_p = \psi(p)K_p. \qquad \square$$

Given a Borel measure μ on $\mathbb{T}$ and an inner function φ, we denote by $\mu\varphi^{-1}$ the pull back measure

$$\mu\varphi^{-1}(E) = \mu(\varphi^{-1}(E)), \qquad E \subseteq \mathbb{T},$$

where E is any Borel set. If $\mu \ll m$ and $d\mu/dm = f$, we use the notation $\mu\varphi^{-1} = m_f\varphi^{-1}$. The formula

(8) $$m\varphi^{-1} = P(a, u)\, dm(u) \quad \text{where} \quad a = \varphi(0)$$

is proved in [8, pp. 725–726] and in [14]. Based on it, Forelli's condition (6) can be rewritten in an equivalent way as follows:

Lemma 1. *If ψ is an H^q-function, $1 \leq q < \infty$, then condition (6) is equivalent to*

(9) $$\int_{\varphi^{-1}(Y)} |\psi|^q\, dm = m(Y), \qquad Y \in \mathcal{L},$$

that is to

(10) $$m_{|\psi|^q}\varphi^{-1} = m.$$

If $\psi \in H^q$ satisfies the above condition then $\|\psi\|_q = 1$.

Proof. Let $a = \varphi(0)$ and denote $P_a(u) = P(a, u)$, $u \in \mathbb{T}$. Consider any $Y \in \mathcal{L}$ and let $X = \varphi^{-1}(Y)$. One can write using (8)

$$\int_X \frac{dm(u)}{P(\varphi(0), \varphi(u))} = \int_{\varphi^{-1}(Y)} \frac{1}{P_a \circ \varphi} \, dm = \int_Y \frac{1}{P_a} \, dm\varphi^{-1}$$

$$= \int_Y \frac{1}{P_a} P_a \, dm = m(Y).$$

If $\psi \in H^q$, note that, by taking $Y = \mathbb{T}$ in (9), one obtains $\|\psi\|_q = 1$. $\square$

We can now prove the announced partial extension to H^2 of Forelli's Theorem 3.

Theorem 5. *If φ is inner and ψ is an H^2-function, then $T_{\psi,\varphi}$ is isometric on H^2 if and only if condition (6) is satisfied by ψ, φ, and $q = 2$. If $T_{\psi,\varphi}$ is bounded, φ is inner, and $\|\psi\|_2 = 1$, then $T_{\psi,\varphi}$ is isometric if and only if $T^*_{\psi,\varphi}(\psi) = 1$. In case both ψ and φ are inner functions, then $T_{\psi,\varphi}$ is isometric if and only if $\varphi(0) = 0$.*

Proof. If (6) holds for $q = 2$, then (9) holds for $q = 2$ and so, for all $f \in H^2$, one can write

$$\|T_{\psi,\varphi} f\|_2^2 = \int_{\mathbb{T}} |\psi|^2 |f \circ \varphi|^2 \, dm = \int_{\mathbb{T}} |f|^2 \, dm_{|\psi|^2}\varphi^{-1}$$

$$= \int_{\mathbb{T}} |f|^2 \, dm = \|f\|_2^2, \qquad f \in H^2,$$

thus proving that $T_{\psi,\varphi}$ is isometric.

Conversely, if $T_{\psi,\varphi}$ is isometric, then it is bounded and, by [12, Theorem 7], $dm_{|\psi|^2}\varphi^{-1} \ll dm$, and $T^*_{\psi,\varphi} T_{\psi,\varphi}$ is the Toeplitz operator whose symbol is the Nikodym derivative $dm_{|\psi|^2}\varphi^{-1}/dm$. Given that, $T^*_{\psi,\varphi} T_{\psi,\varphi} = I = T_1$, it follows that $dm_{|\psi|^2}\varphi^{-1}/dm = 1$ that is $dm_{|\psi|^2}\varphi^{-1} = dm$. Therefore (9) holds for $q = 2$, hence (6) holds too.

Note also, that the symbol of the Toeplitz operator $T^*_{\psi,\varphi} T_{\psi,\varphi}$ is

$$T^*_{\psi,\varphi} T_{\psi,\varphi}(1) = T^*_{\psi,\varphi}(\psi).$$

If both ψ and φ are inner functions, then $T_{\psi,\varphi}$ is isometric if and only if $T^*_{\psi,\varphi} T_{\psi,\varphi} = T_{P(\varphi(0),u)} = I$. Indeed, since $|\psi| = 1$ a.e., one has

$$dm_{|\psi|^2}\varphi^{-1}(u) = dm\varphi^{-1}(u) = P(\varphi(0), u) \, dm(u),$$

by (8). On the other hand, one has $P(\varphi(0), u) = 1$ a.e., if and only if $\varphi(0) = 0$. $\square$

We take up now the problem of describing the Wold decomposition of weighted composition isometries.

If $\psi = c$ is constant and $T_{\psi,\varphi}$ isometric, then $|c| = 1$, since $\|\psi\|_2 = 1$. As we saw, φ must be inner. Then, $T_{\psi,\varphi}$ is isometric if and only if $T^*_{c,\varphi}(c) = 1$, which, by (4), is equivalent to $K_{\varphi(0)} = K_0$, that is to $\varphi(0) = 0$. We obtained as a consequence of Theorem 5, the following theorem of Nordgren [14]:

Theorem 6. *If the multiplication symbol of a weighted composition operator is constant then that operator is isometric if and only if that constant is unimodular and the composition symbol is an inner function fixing the origin.*

It is known that an isometry like the one in Theorem 6, is unitary if and only if φ is a rotation, in all other cases, its Wold decomposition being $H^2 = \mathbb{C} \oplus zH^2$, where $\mathbb{C}$ is the subspace of constant functions, [14]. The original proof establishing the aforementioned Wold decomposition is for $\psi = 1$, but visibly that proof extends with no effort to $\psi = c$, $|c| = 1$. If one of the direct summands in the Wold decomposition of an isometry is the null subspace, that isometry is either unitary or a forward shift.

The situation when a weighted composition isometry is unitary is described in Theorem 1. As for the situation when a weighted composition operator is a forward shift, we note the following result [7, Theorem 4.4]:

Theorem 7. *If $T_{\psi,\varphi}$ is isometric, and φ has a fixed point $p \in \mathbb{U}$, then $T_{\psi,\varphi}$ is a forward shift, if $|\psi(p)| < 1$.*

Based on results in this section, Theorem 7 can be extended as follows:

Theorem 8. *If $T_{\psi,\varphi}$ is isometric, and φ has a fixed point $p \in \mathbb{U}$, then $T_{\psi,\varphi}$ is a forward shift, if and only if $|\psi(p)| < 1$. If the fixed point of φ is the origin and $T_{\psi,\varphi}$ is isometric, then $T_{\psi,\varphi}$ is a forward shift if and only if ψ is nonconstant.*

Proof. As we mentioned, the if part was proved in [7]. For the only if part, note that $|\psi(p)| \leq 1$ and if $|\psi(p)| = 1$ then $T_{\psi,\varphi}$ has nonempty point spectrum (Proposition 1), hence $T_{\psi,\varphi}$ cannot be a unilateral forward shift, since, as is well known, the point spectrum of such operator is always empty. By Theorem 6 and the comment following it, if $\varphi(0) = 0$ and ψ is constant, then $T_{\psi,\varphi}$ is not a shift. On the other hand, if $\varphi(0) = 0$ and ψ is nonconstant, then the condition $\|\psi\|_2 = 1$ (which is satisfied by ψ, see Theorem 4), implies the fact that $|\psi(0)| < 1$, hence $T_{\psi,\varphi}$ is a forward shift. $\square$

To address now the situation when an isometric weighted composition operator, whose composition symbol is an inner function fixing a point, is neither unitary nor an isometry we begin by the following:

Lemma 2. *If φ is an inner function fixing $p \in \mathbb{U}$, then the following formula holds*

$$(11) \qquad dm_{P(p,u)}\varphi^{-1}(u) = P(p,u)\,dm(u), \qquad u \in \mathbb{T}.$$

Proof. By Poisson's formula, the measure $P(p,u)\,dm(u)$ has Fourier coefficients

$$\int_{\mathbb{T}} P(p,u)\overline{u}^k\,dm(u) = \overline{p}^k, \qquad k = 0, \pm 1, \pm 2, \dots.$$

One can see that the measure $dm_{P(p,u)}\varphi^{-1}(u)$ has the same Fourier coefficients, since

$$\int_{\mathbb{T}} \overline{u}^k \, dm_{P(p,u)}\varphi^{-1}(u) = \int_{\mathbb{T}} P(p,u)\overline{\varphi(u)}^k \, dm(u)$$
$$= \overline{\varphi(p)}^k = \overline{p}^k, \qquad k = 0, \pm 1, \pm 2, \ldots. \qquad \square$$

For $p \in \mathbb{U}$, denote by α_p the selfinverse disc automorphism

$$\alpha_p(z) = (p - z)/(1 - \overline{p}z), \qquad z \in \mathbb{U}.$$

Theorem 9. *If $T_{\psi,\varphi}$ is isometric, φ is an inner function, other than the identity or an elliptic automorphism, fixing some $p \in \mathbb{U}$, and $|\psi(p)| = 1$, then ψ must have the form*

(12)
$$\psi(z) = c\frac{1 - \overline{p}\varphi(z)}{1 - \overline{p}z}, \qquad z \in \mathbb{U},$$

where $c \in \mathbb{T}$. Conversely, the operators $T_{\psi,\varphi}$ with φ an inner function fixing $p \in \mathbb{U}$ and ψ given by (12) are isometric. Those operators are unitary if and only if φ is the identity or an elliptic automorphism. In all the other cases, the Wold decomposition of $T_{\psi,\varphi}$ is $H^2 = \mathrm{Span}(K_p) \oplus \alpha_p H^2$.

Proof. Assume $T_{\psi,\varphi}$ is isometric, φ is an inner function, other than the identity or an elliptic automorphism, fixing some $p \in \mathbb{U}$, and $|\psi(p)| = 1$. Then let $\psi(p) = c$ and recall that, by Proposition 1, one must have that

$$T_{\psi,\varphi}K_p = cK_p, \qquad \text{that is} \quad \frac{\psi(z)}{1 - \overline{p}\varphi(z)} = c\frac{1}{1 - \overline{p}z}, \qquad z \in \mathbb{U},$$

which implies the fact that ψ satisfies (12).

For the converse now, assume φ is an inner function fixing $p \in \mathbb{U}$ and ψ has form (12). Then, $T_{\psi,\varphi}$ is isometric, since one can write

$$\|T_{\psi,\varphi}f\|_2^2 = \int_{\mathbb{T}} \frac{|\varphi(u) - p|^2}{|u - p|^2}|f(\varphi(u))|^2 \, dm(u)$$
$$= \int_{\mathbb{T}} \frac{P(p,u)}{P(p,\varphi(u))}|f(\varphi(u))|^2 \, dm(u)$$
$$= \int_{\mathbb{T}} \frac{1}{P(p,u)}|f(u)|^2 \, dm_{P(p,u)}\varphi^{-1}(u)$$
$$= \int_{\mathbb{T}} \frac{1}{P(p,u)}|f(u)|^2 P(p,u) \, dm(u) = \|f\|_2^2, \qquad f \in H^2,$$

by formula (11).

If φ is the identity or an elliptic automorphism, then a straightforward (yet not very short), computation shows that ψ equals a unimodular multiple of the function $K_q/\|K_q\|_2$, where $\varphi(q) = 0$ and so, $T_{\psi,\varphi}$ is unitary in that case, by Theorem 1.

If φ is not the identity or an elliptic automorphism, then $T_{\psi,\varphi}$ is a nonunitary isometry whose Wold decomposition we can find in the sequel, using

the fact that the sequence $\{\varphi^{[n]}\}$ of iterates of φ tends to p uniformly on compacts. Indeed, let H_0 be the space reducing $T_{\psi,\varphi}$ to a unitary operator. That space contains $\mathrm{Span}(K_p)$ because K_p is a reducing eigenfunction associated to a unimodular eigenvalue, by Proposition 1.

As we mentioned in the introduction, H_0 is the subspace of H^2 consisting of functions f with backward iterates of any order under $T_{\psi,\varphi}$. To show $H_0 = \mathrm{Span}(K_p)$, we consider $f \in \alpha_p H^2$, assume f has backward iterates of any order under $T_{\psi,\varphi}$, and show f is the null function. So, there is a sequence $\{f_n\}$ in $\alpha_p H^2$, so that $T_{\psi,\varphi}^n f_n = f$, $n = 1, 2, 3, \ldots$. Given that $T_{\psi,\varphi}$ is isometric, all functions f_n have the same norm as f, thus, by the weak compactness of the closed balls of Hilbert spaces, one does not reduce the generality by assuming that $\{f_n\}$ is weakly convergent to some $g \in \alpha_p H^2$. The reason why g belongs to $\alpha_p H^2$, is the fact that weak convergence obviously implies pointwise convergence in a reproducing kernel Hilbert space, hence in H^2 as well.

Given all that, and borrowing from the proof of [7, Theorem 4.4], one can write for all $n = 1, 2, 3, \ldots$

$$(13) \qquad |f(z)| = \left| \left(T_{\psi,\varphi}^n f_n \right)(z) \right|$$

$$= \left| \psi(z)\psi(\varphi(z))\psi(\varphi^{[2]}(z))\ldots\psi(\varphi^{[n-1]}(z))f_n(\varphi^{[n]}(z)) \right|$$

$$= \left| \frac{1 - \bar{p}\varphi^{[n-1]}(z)}{1 - \bar{p}z} f_n(\varphi^{[n]}(z)) \right|.$$

Above we used (12).

Since φ is neither the identity, nor an elliptic disc automorphism, it follows (by the Denjoy–Wolff Theorem), that $\varphi^{[n]}(z) \to p$ uniformly on compact subsets of $\mathbb{U}$. Therefore

$$\frac{1 - \bar{p}\varphi^{[n-1]}(z)}{1 - \bar{p}z} \to \frac{1 - |p|^2}{1 - \bar{p}z}, \qquad z \in \mathbb{U}.$$

Also,

$$f_n(\varphi^{[n]}(z)) = \langle f_n, K_{\varphi^{[n]}(z)} \rangle \quad n = 1, 2, 3, \ldots.$$

Given that $\|K_{\varphi^{[n]}(z)} - K_p\|_2 \to 0$ and $f_n \to g$ weakly, one gets that

$$f_n(\varphi^{[n]}(z)) \to \langle g, K_p \rangle = g(p) = 0.$$

Hence, by letting $n \to \infty$ in (13), one obtains

$$|f(z)| = \frac{|g(p)|(1 - |p|^2)}{1 - \bar{p}z} = 0, \qquad z \in \mathbb{U}. \qquad \square$$

According to Theorem 1, the composition symbol of a unitary weighted composition operator needs to be a disc automorphism. In the following, we characterize all isometric weighted composition operators with automorphic composition symbol.

Theorem 10. *If φ is a disc automorphism, then $T_{\psi,\varphi}$ is isometric if and only if $T_{\psi,\varphi}$ is of the form $T_{\psi,\varphi} = T_\phi U$, where ϕ is an inner function, and U is a unitary weighted composition operator. If φ is an elliptic automorphism or the identity map, then $T_{\psi,\varphi}$ is a forward shift if and only if ϕ is nonconstant.*

Proof. The fact that weighted composition operators representable as a product of an analytic Toeplitz operator with inner symbol and a unitary weighted composition operator are isometries is evident. Let us prove the converse implication.

If φ is a disc automorphism and $T_{\psi,\varphi}$ is isometric, then, $T_{\psi,\varphi}C_\varphi^{-1} = T_\psi$ is a bounded operator, hence $\psi \in H^\infty$. Then $T_\psi^* T_\psi$ is $T_{|\psi|^2}$, the Toeplitz operator with symbol $|\psi|^2$. Since $T_{\psi,\varphi}$ is isometric, it follows that $T_\psi^* T_\psi = C_{\varphi^{-1}}^* C_{\varphi^{-1}}$. According to [11, Theorem 4], $C_{\varphi^{-1}}^* C_{\varphi^{-1}}$ is the Toeplitz operator with symbol $P(\varphi^{-1}(0), u)$, $u \in \mathbb{T}$. Denote $p = \varphi^{-1}(0)$ and recall that

$$P(p, u) = \frac{1 - |p|^2}{|u - p|^2} = \left(\frac{|K_p(u)|}{\|K_p\|_2} \right)^2, \qquad u \in \mathbb{T}.$$

We proved that

$$\frac{|K_p(u)|}{\|K_p\|_2} = |\psi(u)|, \qquad \text{a.e.}$$

Since K_p is a zero–free function, analytic on an open neigborhood of the closed unit disc, one gets that $\psi/(K_p/\|K_p\|_2)$ is a bounded analytic function whose radial limit function is unimodular a.e.. The consequence is that ψ has the representation $\psi = \phi K_p/\|K_p\|_2$, where ϕ is an inner function hence, $T_{\psi,\varphi} = T_\phi U$ where $U = T_{K_p/\|K_p\|_2, \varphi}$ is unitary, (according to Theorem 1).

By our previous results, if φ is an elliptic automorphism fixing $q \in \mathbb{U}$, then $T_{\psi,\varphi} = T_\phi U$ is a shift if and only if $|\psi(q)| = |\phi(q)| < 1$, a fact that happens if and only if ϕ is nonconstant. In case φ is the identity map, then $T_{\psi,\varphi} = T_\phi$ which is a unilateral forward shift if ϕ is nonconstant [11], respectively a unimodular multiple of the identity operator, if ϕ is constant. $\qquad \square$

Next, we address the issue of the Wold decomposition of the isometries in Theorem 10, whose composition symbol is a parabolic or hyperbolic disc automorphism. First, recall that invariant subspace, means closed linear manifold left invariant by an operator, and that the invariant subspace lattice $\text{Lat}T_z$ of the analytic Toeplitz operator T_z has the following description [1]:

(14) $\qquad \text{Lat}T_z = \{uH^2 : u \text{ is an inner function }\} \cup \{0\}.$

Denote by $\varphi^{[0]}$ the identity function. With this notation we state and prove:

Theorem 11. *Let φ be a parabolic or hyperbolic disc automorphism, $p = \varphi^{-1}(0)$, and ψ be a function of the form $\psi = \phi K_p/\|K_p\|_2$, where ϕ is an*

inner function. Then $T_{\psi,\varphi}$ is a forward shift if and only if

$$(15) \qquad \bigcap_{n=1}^{\infty} (\phi\phi\circ\varphi\ldots\phi\circ\varphi^{[n-1]})H^2 = \{0\}.$$

If (15) fails, and ϕ is not constant, then there is a nonconstant inner function u so that the Wold decomposition of $T_{\psi,\varphi}$ is

$$(16) \qquad H^2 = (uH^2) \oplus (uH^2)^{\perp},$$

the restriction $T_{\psi,\varphi}|uH^2$ being the unitary part of $T_{\psi,\varphi}$.

Proof. Denote $\phi_n = \phi\phi\circ\varphi\ldots\phi\circ\varphi^{[n-1]}$, $n = 1, 2, 3, \ldots$. A straightforward computation establishes the formula

$$(17) \qquad T_{\psi,\varphi}^n = T_{\phi_n} U^n \quad n = 1, 2, 3\ldots$$

where U is the unitary operator $U = T_{K_p/\|K_p\|_2,\varphi}$. Let us denote by H_0 the subspace of the unitary part in the Wold decomposition of $T_{\psi,\varphi}$. We claim that

$$(18) \qquad H_0 = \bigcap_{n=1}^{\infty} (\phi\phi\circ\varphi\ldots\phi\circ\varphi^{[n-1]})H^2 \in \mathrm{Lat}T_z.$$

Indeed, H_0 is the space of functions $f \in H^2$ which have backward iterates of any order under $T_{\psi,\varphi}$, that is $f \in H_0$ if and only if there is a sequence of functions $\{f_n\}$ in H^2 so that

$$(19) \qquad f = T_{\psi,\varphi}^n f_n, \qquad n = 1, 2, 3, \ldots.$$

If

$$f \in \bigcap_{n=1}^{\infty} (\phi\phi\circ\varphi\ldots\phi\circ\varphi^{[n-1]})H^2$$

then there is a sequence $\{g_n\}$ in H^2 so that

$$f = T_{\phi_n} g_n = T_{\phi_n} U^n (U^{*n} g_n) = T_{\psi,\varphi}^n (U^{*n} g_n) \quad n = 1, 2, 3, \ldots,$$

by (17). This proves that

$$\bigcap_{n=1}^{\infty} (\phi\phi\circ\varphi\ldots\phi\circ\varphi^{[n-1]})H^2 \subseteq H_0.$$

Conversely, if $f \in H_0$, then f satisfies condition (19) for some sequence $\{f_n\}$ in H^2, and so, one can write, using (17)

$$f = \phi_n(U^n f_n) \in \phi_n H^2, \qquad n = 1, 2, 3, \ldots,$$

which proves equality (18). Thus $H_0 \in \mathrm{Lat}T_z$ and so, $T_{\psi,\varphi}$ is a forward shift if and only if $H_0 = \{0\}$, a unitary operator if and only if $H_0 = H^2$, respectively neither of the above if and only if $\{0\} \neq H_0 \neq H^2$, in which case $H_0 = uH^2$ for some nonconstant inner function u. $\square$

According to Theorem 1, the isometries in Theorem 11 are unitary operators if and only if ϕ is a unimodular constant. Let us show by a couple of examples that in other cases, the operators in Theorem 11 can be forward shifts, respectively neither a shift, nor a unitary operator.

Example 1. It is known that, if φ is a parabolic disc automorphisms, then C_φ has nonconstant, singular, inner eigenfunctions [13]. Let v be such an eigenfunction, $p = \varphi^{-1}(0)$, and $\psi = vK_p/\|K_p\|_2$. Then $T_{\psi,\varphi}$ is a forward shift.

Indeed, the space H_0 that supports the unitary part of $T_{\psi,\varphi}$ cannot be of the form uH^2 for some inner function u because u cannot be divisible by v^n for all $n = 1, 2, 3 \ldots$ (in the sense of inner function divisibility).

Example 2. Let us consider a hyperbolic disc automorphism φ and let $p = \varphi^{-1}(0)$. The infinite Blaschke product B whose factors are all the iterates of φ (each with multiplicity 1), is convergent, since $\sum_{n=1}^\infty (1 - |\varphi(0)|) < +\infty$ (see [2]). Then $T_{K_p/\|K_p\|_2\varphi,\varphi}$ is an isometry which is neither unitary nor a forward shift.

Indeed, in this case, one can prove easily that the space H_0 that supports the unitary part of $T_{K_p/\|K_p\|_2\varphi,\varphi}$ has the description $H_0 = BH^2$.

3. Numerical ranges of some weighted composition operators

One of the applications of the Wold decomposition is reducing the computation of the numerical range of a nonunitary isometry to finding the eigenvalues (if any), of its unitary part. To explain what we mean, let us denote by $\sigma_p(T)$ the point spectrum of any operator T.

Theorem 12. *If V is a nonunitary isometry with Wold decomposition $V = U \oplus S$ where U is unitary and S is a unilateral forward shift, then*

$$W(V) = \mathbb{U} \cup \sigma_p(U).$$

This fact is proved in [10, Proposition 2.3] in the particular case of nonunitary, isometric, composition operators. The same proof works for arbitrary nonunitary isometries.

In [7], the authors take up the problem of describing the numerical ranges of weighted composition operators with nonconstant multiplication symbol. In the following we will present some results related to that topic.

Theorem 13. *If $T_{\psi,\varphi}$ is a nonunitary isometry whose composition symbol is an inner function fixing some $p \in \mathbb{U}$, then $W(T_{\psi,\varphi}) = \{\psi(p)\} \cup \mathbb{U}$ if $|\psi(p)| = 1$.*

Proof. If $|\psi(p)| = 1$, then by Proposition 1 and Theorem 9 , the unitary part of $T_{\psi,\varphi}$ is $\psi(p)I$ where I is the identity operator acting on the 1-dimensional subspace $\mathrm{Span}(K_p)$, and so $\psi(p)$ is the only eigenvalue of that operator. Then, by Theorem 12, one has that $W(T_{\psi,\varphi}) = \{\psi(p)\} \cup \mathbb{U}$. $\qquad\square$

Given that, the numerical range of a unitary weighted composition operator whose composition symbol fixes a point in $\mathbb{U}$ was determined in [7], the description of numerical ranges of weighted composition isometries whose multiplication symbol has a fixed point in $\mathbb{U}$ is completely known now.

Recall that a subset E of the complex plane is said to have circular symmetry about the origin if $\mathbb{T}E \subseteq E$. If E is a convex set, then, it can have circular symmetry about the origin, if and only if it is a circular disc (open or closed) about the origin, possibly reduced to its center.

As observed in [13], if a composition operator C_φ has inner eigenvalues associated to all numbers in $\mathbb{T}$ then $W(C_\varphi)$ has circular symmetry. The same is true for weighted composition operators, the proof being nearly identical to the proof of [13, Proposition 1], (a result inspired from [4]):

Theorem 14. *If $T_{\psi,\varphi}$ is bounded , $\mathbb{T} \subseteq \sigma_p(C_\varphi)$, and C_φ has inner eigenfunctions associated to each eigenvalue $\lambda \in \mathbb{T}$, then $W(T_{\psi,\varphi})$ has circular symmetry.*

This leads to the following:

Theorem 15. *If $T_{\psi,\varphi}$ is bounded and φ is an inner function of parabolic automorphic or of hyperbolic type, then $W(T_{\psi,\varphi})$ is a circular disc centered at the origin. In particular, all isometric weighted composition operators whose composition symbol is an inner function of parabolic automorphic or of hyperbolic type, have numerical range equal to $\mathbb{U}$.*

Proof. If φ is an inner function of parabolic automorphic or of hyperbolic type, then C_φ has inner eigenvalues associated to all complex numbers in $\mathbb{T}$ [13], hence $W(T_{\psi,\varphi})$ is a circular disc centered at the origin, whenever $T_{\psi,\varphi}$ is bounded. If $T_{\psi,\varphi}$ is isometric, then necessarily $W(T_{\psi,\varphi}) = \mathbb{U}$. Indeed, borrowing an idea from [7], note that unitary operators acting on separable Hilbert spaces have at most countable point spectrum. Indeed, as is well known, two eigenvectors of a unitary operator corresponding to distinct eigenvalues, must be perpendicular. Then $W(T_{\psi,\varphi})$ cannot be the closure of $\mathbb{U}$, because, by Theorem 12, that fact would imply that the unitary part of $T_{\psi,\varphi}$ had uncountably many eigenvalues. Thus, $W(T_{\psi,\varphi}) = \mathbb{U}$.   □

Let Π^+ be the right open half–plane and $H^2(\Pi^+)$ the Hilbert Hardy space over Π^+. This means that $H^2(\Pi^+)$ consists of all analytic functions on Π^+ with the property

$$\sup \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^2 \, dy : x > 0 \right\} < +\infty.$$

The unitary invariant properties of the composition operators on $H^2(\Pi^+)$ can be reduced to the study of the corresponding properties for weighted composition operators on H^2 as follows. Let ϕ be an analytic selfmap of Π^+, $\gamma(z) = (1+z)/(1-z)$, the Kayley transform of $\mathbb{U}$ onto Π^+, and $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ the conformal conjugate of ϕ via Kayley's transform. Our tool

in the following is the fact that C_ϕ is unitarily equivalent to $T_{\psi,\varphi}$, $\psi(z) = (1 - \varphi(z))/(1 - z)$, $z \in \mathbb{U}$, a weighted composition operator acting on H^2 [9].

An application of Theorem 15 is:

Proposition 2. *If $C_\phi \neq I$ is an isometric composition operator on $H^2(\Pi^+)$, then $W(C_\phi) = \mathbb{U}$.*

Proof. This is a direct consequence of Theorem 15 combined with the fact that C_ϕ is an isometry other than the identity if and only if its conformal conjugate φ is an inner function of parabolic type with Denjoy–Wolff point $\omega = 1$, [12, Proposition 4]. □

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