Characteristic ideals and Iwasawa theory

Andrea Bandini, Francesc Bars and Ignazio Longhi

ABSTRACT. Let Λ be a nonnoetherian Krull domain which is the inverse limit of noetherian Krull domains Λ_d and let M be a finitely generated Λ-module which is the inverse limit of Λ_d-modules M_d. Under certain hypotheses on the rings Λ_d and on the modules M_d, we define a pro-characteristic ideal for M in Λ, which should play the role of the usual characteristic ideals for finitely generated modules over noetherian Krull domains. We apply this to the study of Iwasawa modules (in particular of class groups) in a nonnoetherian Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(\mathcal{F}/F)]]$, where F is a function field of characteristic p and $\mathrm{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$.

CONTENTS

1. Introduction

Let A be a noetherian Krull domain and M a finitely generated torsion A-module. The structure theorem for such modules provides an exact sequence

$$(1.1) \qquad 0 \longrightarrow P \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i} A \longrightarrow Q \longrightarrow 0$$

where the $\mathfrak{p}_i$'s are height 1 prime ideals of A and P and Q are pseudo-null A-modules (for more details and precise definitions of all the objects

appearing in this Introduction see Section 2). This sequence defines an important invariant of the module M, namely its *characteristic ideal*

$$\operatorname{Ch}_A(M) := \prod_{i=1}^n \mathfrak{p}_i^{e_i}.$$

Characteristic ideals play a major role in (commutative) Iwasawa theory for global fields: they provide the algebraic counterpart for the p-adic L-functions associated to Iwasawa modules (such as class groups or duals of Selmer groups). Here the Krull domain one works with is the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$, where Γ is a commutative p-adic Lie group occurring as Galois group (we shall deal mainly with the case $\Gamma \simeq \mathbb{Z}_p^d$ for some $d \in \mathbb{N}$). Even if pseudo-null modules do not contribute to characteristic ideals, they appear in the descent problem when one wants to compare the characteristic ideal of an Iwasawa module of a $\mathbb{Z}_p^d$-extension with the one of a $\mathbb{Z}_p^{d-1}$-extension contained in it. The last topic is particularly important when the global field has characteristic p, because, in this case, extensions $\mathcal{F}/F$ with $\operatorname{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$ occur quite naturally: in this situation the Iwasawa algebra is nonnoetherian and there is no guarantee one can find a sequence such as (1.1). One strategy to overcome this difficulty is to consider a filtration of $\mathbb{Z}_p^d$-extensions for $\mathcal{F}$, define the characteristic ideals at the $\mathbb{Z}_p^d$-level for all d and then pass to the limit.

To deal with the technical complications of inverse limits and projections of pseudo-null modules, in Section 2 we prove the following (see Propositions 2.7 and 2.10):

Proposition 1.1. *Let A be a noetherian Krull domain and put $B := A[[t]]$. If M is a pseudo-null B-module, then M_t (the kernel of multiplication by t) and M/tM are finitely generated torsion A-modules and*

$$(1.2) \qquad\qquad \operatorname{Ch}_A(M_t) = \operatorname{Ch}_A(M/tM).$$

Moreover, for any finitely generated torsion B-module N, we have

$$(1.3) \qquad\qquad \operatorname{Ch}_A(N_t)\pi(\operatorname{Ch}_B(N)) = \operatorname{Ch}_A(N/tN)$$

(where $\pi \colon B \to A$ is the canonical projection).

This immediately provides a criterion for an B-module to be pseudo-null (Corollary 2.11), but our main application is the definition of a nonnoetherian analogue of characteristic ideals in Iwasawa algebras.

Theorem 1.2. *Let $\{\Lambda_d\}_{d\geqslant 0}$ be an inverse system of noetherian Krull domains such that*

$$\Lambda_d \simeq \Lambda_{d+1}/\mathfrak{p}_{d+1} \quad and \quad \Lambda_{d+1} \simeq \varprojlim_n \Lambda_{d+1}/\mathfrak{p}_{d+1}^n \quad for \ any \ d \geqslant 0$$

($\mathfrak{p}_{d+1}$ *a principal prime ideal of* Λ_{d+1} *of height* 1*).* *Let* $\Lambda := \varprojlim_d \Lambda_d$ *and consider a finitely generated* Λ*-module*

$$M = \varprojlim_d M_d$$

(where each M_d *is a* Λ_d*-module). If, for every* $d \gg 1$,

  1. *the* $\mathfrak{p}_d$*-torsion submodule of* M_d *is a pseudo-null* Λ_{d-1}*-module, i.e.,* $\mathrm{Ch}_{\Lambda_{d-1}}(M_d[\mathfrak{p}_d]) = (1)$,
  2. $\mathrm{Ch}_{\Lambda_{d-1}}(M_d/\mathfrak{p}_d) \subseteq \mathrm{Ch}_{\Lambda_{d-1}}(M_{d-1})$,

then the ideals $\mathrm{Ch}_{\Lambda_d}(M_d)$ *form a projective system (with respect to the maps* $\pi_{\Lambda_{d-1}}^{\Lambda_d}$ *defining* Λ*).*

Definition 1.3. Whenever conditions **1** and **2** of Theorem 1.2 hold, we define the *pro-characteristic ideal* of M over Λ as

$$\widetilde{\mathrm{Ch}}_\Lambda(M) := \varprojlim_d \mathrm{Ch}_{\Lambda_d}(M_d) \subseteq \Lambda.$$

The Iwasawa algebra associated with a $\mathbb{Z}_p^d$-extension of a global field F is (noncanonically) isomorphic to the Krull ring $\mathbb{Z}_p[[t_1, \ldots, t_d]]$, hence descending to a $\mathbb{Z}_p^{d-1}$-extension corresponds to the passage from $A[[t]]$ to A. So the results of Section 2 apply immediately to Iwasawa modules. In order to keep the paper short, we just consider the case of class groups. We also remark that the analogue of Proposition 1.1 for these Iwasawa algebras has been proved by T. Ochiai in [18, Section 3] and it suffices for the arithmetical applications we had in mind. Indeed in Section 3.1 we deal with a global field F of characteristic $p > 0$ (for an account of Iwasawa theory over function fields see [6] and the references there). We already mentioned that here one is naturally led to work with extremely large abelian p-adic extensions: this comes from class field theory, since, in the completion F_v of F at some place v, the group of 1-units $U_1(F_v)$ is isomorphic to $\mathbb{Z}_p^\infty$. As hinted above, our strategy to tackle $\mathcal{F}/F$ with $\mathrm{Gal}(\mathcal{F}/F) \simeq \mathbb{Z}_p^\infty$ is to work first with $\mathbb{Z}_p^d$-extensions and then use limits. The usefulness of Theorem 1.2 in this procedure is illustrated in Section 3.1.1, where we define the pro-characteristic ideal $\widetilde{\mathrm{Ch}}_\Lambda(\mathcal{A}(\mathcal{F}))$ dispensing with the crutch of the *ad hoc* hypothesis [6, Assumption 5.3]. The search for a "good" definition for it was one of the main motivations for this work.

   The arithmetic significance of our pro-characteristic ideal is ensured by a deep result of D. Burns (see [9, Theorem 3.1] and the Appendix [10]), which shows that the characteristic ideal of the class group of a $\mathbb{Z}_p^d$-extension $\mathcal{F}_d/F$ is generated by a Stickelberger element (by some language abuse we shall call *class group of* $\mathcal{F}_d$ the inverse limit of the class groups of the finite subextensions of $\mathcal{F}_d/F$). Therefore (see Corollary 3.10) our pro-characteristic ideal is generated by a Stickelberger element as well and this can be considered as an instance of Iwasawa Main Conjecture for nonnoetherian Iwasawa algebras.

Next to class groups, [6] and [4] consider the case of Selmer groups of abelian varieties: in [6, Section 3] we employed Fitting ideals of Pontrjagin duals of Selmer groups instead than characteristic ideals in order to avoid the difficulties of taking the inverse limit. With some additional work, Theorem 1.2 permits to define a pro-characteristic ideal for these modules as well, allowing to formulate a more classical Iwasawa Main Conjecture (details can be found in [7]).

Remark 1.4. If a pseudo-null $A[[t]]$-module M is finitely generated over A as well, then the statement of Proposition 1.1 is trivially deduced from the exact sequence

$$0 \to M_t \to M \xrightarrow{t} M \to M/tM \to 0$$

and the multiplicativity of characteristic ideals. As explained in [14] (see Lemma 2 and the discussion right after it), for any $\mathbb{Z}_p^d$-extension $\mathcal{F}_d/F$ and any pseudo-null $\Lambda(\mathcal{F}_d)$-module M it is always possible to find (at least one) $\mathbb{Z}_p^{d-1}$-subextension $\mathcal{F}_{d-1}$ such that M is finitely generated over $\Lambda(\mathcal{F}_{d-1})$ (where $\Lambda(\mathcal{L})$ is the Iwasawa algebra associated with the extension $\mathcal{L}/F$). Our search for a characteristic ideal via a projective limit does not allow this freedom in the choice of subextensions, hence the need for an "unconditional" result like Theorem 1.2.

Acknowledgments. We thank the referee for his or her very prompt report and for suggesting a more elegant reasoning in one of our arguments.

2. Pseudo-null modules and characteristic ideals

2.1. Krull domains. We begin by reviewing some basic facts and definitions we are going to need. A comprehensive reference is [8, Chapter VII].

An integral domain A is called a Krull domain if $A = \cap A_{\mathfrak{p}}$ (where $\mathfrak{p}$ varies among prime ideals of height 1 and $A_{\mathfrak{p}}$ denotes localization), all $A_{\mathfrak{p}}$'s are discrete valuation rings and any $x \in A - \{0\}$ is a unit in $A_{\mathfrak{p}}$ for almost all $\mathfrak{p}$.[1] In particular, one attaches a discrete valuation to any height 1 prime ideal. Furthermore, a ring is a unique factorization domain if and only if it is a Krull domain and all height 1 prime ideals are principal ([8, VII, §3.2, Theorem 1]).

2.1.1. Torsion modules. Let A be a noetherian Krull domain. A finitely generated torsion A-module is said to be *pseudo-null* if its annihilator ideal has height at least 2. A morphism with pseudo-null kernel and cokernel is called a pseudo-isomorphism: being pseudo-isomorphic is an equivalence relation between finitely generated torsion A-modules (torsion is essential here[2]) and we shall denote it by $\sim_A$. If M is a finitely generated torsion

---

[1]This is not the definition in [8], but it is equivalent to it: see [8, VII, §1.6, Theorem 4].

[2]For example the map $(p, t) \hookrightarrow \mathbb{Z}_p[[t]]$ is a pseudo-isomorphism, but there is no such map from $\mathbb{Z}_p[[t]]$ to (p, t).

A-module then there is a pseudo-isomorphism

$$(2.1) \qquad M \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i}$$

where the $\mathfrak{p}_i$'s are height 1 prime ideals of A (not necessarily distinct) and the $\mathfrak{p}_i$'s, n and the e_i's are uniquely determined by M (see, e.g., [8, VII, §4.4, Theorem 5]). A module like the one on the right-hand side of (2.1) will be called *elementary A-module* and

$$E(M) := \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i} \sim_A M$$

is the *elementary module attached to M*.

Definition 2.1. Let M be a finitely generated A-module: its *characteristic ideal* is

$$\mathrm{Ch}_A(M) := \begin{cases} 0 & \text{if } M \text{ is not torsion;} \\ \displaystyle\prod_{i=1}^{n} \mathfrak{p}_i^{e_i} & \text{if } M \sim_A \displaystyle\bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{e_i}. \end{cases}$$

In particular, M is pseudo-null if and only if $\mathrm{Ch}_A(M) = A$.

We shall denote by $\mathbf{Fgt}_A$ the category of finitely generated torsion A-modules.

Remarks 2.2.

    1. An equivalent definition of pseudo-null is that all localizations at primes of height 1 are zero. If $\mathfrak{p}$ and $\mathfrak{q}$ are two different primes of height 1 (and M is a torsion A-module) we have $M \otimes_A A_\mathfrak{p} \otimes_A A_\mathfrak{q} = 0$. By the structure theorem recalled in (2.1) it follows immediately that for a finitely generated torsion A-module M, the canonical map

$$(2.2) \qquad M \longrightarrow \bigoplus_{\mathfrak{p}} \left(M \otimes_A A_\mathfrak{p} \right)$$

    (where the sum is taken over all primes of height 1) is a pseudo-isomorphism. Actually, the right-hand side of (2.2) can be used to compute $\mathrm{Ch}_A(M)$: a prime $\mathfrak{p}$ appears in $\mathrm{Ch}_A(M)$ with exponent the length of the module $M \otimes_A A_\mathfrak{p}$.

    2. The previous remark suggests a generalization of the definition of characteristic ideal by means of *supernatural divisors*.[3] Let M be any torsion A-module (we drop the finitely generated assumption) and define

$$\mathrm{Ch}_A(M) := \prod_{\mathfrak{p}} \mathfrak{p}^{l_\mathfrak{p}(M \otimes_A A_\mathfrak{p})}$$

---

[3]We recall that the group of divisors of A is the free abelian group generated by the prime ideals of height 1 in A (see [8, VII, §1]).

where the product is taken over all primes of height 1 and the exponent of $\mathfrak{p}$ (i.e., the *length* of the module $M \otimes_A A_\mathfrak{p}$) is a supernatural number (i.e., belongs to $\mathbb{N} \cup \{\infty\}$). More precisely, for N a finitely generated torsion $A_\mathfrak{p}$-module let $l_\mathfrak{p}(N)$ denote its length. Then we put

$$l_\mathfrak{p}(M \otimes_A A_\mathfrak{p}) := \sup\{l_\mathfrak{p}(M_\alpha \otimes_A A_\mathfrak{p})\},$$

where M_α varies among all finitely generated submodules of M. Note that, since $A_\mathfrak{p}$ is flat, $M_\alpha \otimes_A A_\mathfrak{p}$ is still a submodule of $M \otimes_A A_\mathfrak{p}$; furthermore, the length $l_\mathfrak{p}$ is an increasing function on finitely generated torsion $A_\mathfrak{p}$-modules (partially ordered by inclusion).

2.1.2. Power series. In the rest of this section, A will denote a Krull domain and $B := A[[t]]$ the ring of power series in one variable over A.

Proposition 2.3. *Let A be a Krull domain and $\mathfrak{p} \subset A$ a height 1 prime. Then B is also a Krull domain and $\mathfrak{p}B$ is a height 1 prime of B.*

This is well-known (actually, one can prove the analogue even with infinitely many variables: see [11]). In order to make the paper as self-contained as possible, and for lack of a suitable reference for the second part of the proposition, we provide a quick proof.

Proof. Let Q be the fraction field of A. Since A is Krull, we have $A = \cap A_\mathfrak{q}$ as $\mathfrak{q}$ varies among all prime ideals of height 1. Furthermore, each $A_\mathfrak{q}$ is a discrete valuation ring: then [8, VII, §3.9, Proposition 8] shows that $A_\mathfrak{q}[[t]]$ is a unique factorization domain. In particular every $A_\mathfrak{q}[[t]][t^{-1}]$ is a Krull domain and we get

$$(2.3) \quad B = A[[t]] = Q[[t]] \cap \bigcap_\mathfrak{q} A_\mathfrak{q}[[t]][t^{-1}] = Q[[t]] \cap \bigcap_\mathfrak{q} \bigcap_{\mathfrak{P} \in S_\mathfrak{q}} \left(A_\mathfrak{q}[[t]][t^{-1}]\right)_\mathfrak{P}$$

(where $S_\mathfrak{q}$ denotes the set of height 1 primes in $A_\mathfrak{q}[[t]][t^{-1}]$). This shows that B is an intersection of discrete valuation rings. A power series $\lambda = t^h \sum_{i \geqslant 0} c_i t^i \in B$ (with $c_0 \neq 0$) is a unit in $A_\mathfrak{q}[[t]][t^{-1}]$ unless $c_0 \in \mathfrak{q}$ and, in the latter case, λ is still a unit in $(A_\mathfrak{q}[[t]][t^{-1}])_\mathfrak{P}$ unless it can be divided by the generator of $\mathfrak{P}$. This proves that B is a Krull domain.

Since $A_\mathfrak{p}$ is a discrete valuation ring, its maximal ideal $\mathfrak{p}A_\mathfrak{p}$ is principal: let π be a uniformizer. Then π is irreducible in $A_\mathfrak{p}[[t]]$, hence it generates a height 1 prime ideal $\mathfrak{P} := \pi A_\mathfrak{p}[[t]] = \mathfrak{p}A_\mathfrak{p}[[t]]$. By the general theory of Krull domains, $\mathfrak{P}$ corresponds to a discrete valuation $\nu_\mathfrak{P}$ on the fraction field $\mathrm{Frac}(A_\mathfrak{p}[[t]])$; the restriction of $\nu_\mathfrak{P}$ to Q is precisely the discrete valuation associated with $\mathfrak{p}$. Similarly, restricting $\nu_\mathfrak{P}$ to $\mathrm{Frac}(B)$ yields a discrete valuation, with ring of integers $D_\mathfrak{P}$ and maximal ideal $\mathfrak{m}_\mathfrak{P}$. The ring $D_\mathfrak{P}$ is the localization of B at $\mathfrak{m}_\mathfrak{P}$: hence it is flat over B and, by [8, VII, §1.10, Proposition 15], the prime ideal

$$\mathfrak{m}_\mathfrak{P} \cap B = \mathfrak{P} \cap B = \mathfrak{p}B \neq 0$$

has height 1. $\qquad\qquad\square$

2.2. Pseudo-null B-modules. Now assume that A (and hence B) is Noetherian. In this section P will be a pseudo-null B-module. We denote by P_t the kernel of multiplication by t and remark that in the exact sequence

(2.4) $$P_t \longhookrightarrow P \xrightarrow{\;t\;} P \longtwoheadrightarrow P/tP\,,$$

P_t and P/tP are finitely generated B-modules, because so is P. The former ones are also finitely generated as A-modules, because t acts as 0 on them. Moreover they are torsion A-modules: we have $P \otimes_B B_{tB} = 0$ (since the ideal tB is prime of height 1), hence there is some x in $\mathrm{Ann}_B(P) - tB$ and the projection of this x into A (via $t \mapsto 0$) kills both P_t and P/tP. Therefore the characteristic ideals $\mathrm{Ch}_A(P_t)$ and $\mathrm{Ch}_A(P/tP)$ are given by Definition 2.1 (there is no need for supernatural divisors here) and both of them are nonzero.

2.2.1. Preliminaries. For $\mathfrak{p}$ a prime of height one in A, define

$$\widehat{A_\mathfrak{p}} := \varprojlim A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}.$$

By a slight abuse of notation, we shall denote by $\mathfrak{p}$ also the maximal ideals of $A_\mathfrak{p}$ and $\widehat{A_\mathfrak{p}}$. The natural embedding of A into $\widehat{A_\mathfrak{p}}$ allows to identify B with a subring of $\widehat{A_\mathfrak{p}}[[t]]$.

Lemma 2.4. *The ring $\widehat{A_\mathfrak{p}}[[t]]$ is a flat B-algebra.*

Proof. Put $S_\mathfrak{p} := A - \mathfrak{p}$. We claim that $\widehat{A_\mathfrak{p}}[[t]]$ is the completion of $S_\mathfrak{p}^{-1}B$ with respect to the ideal generated by $\mathfrak{p}$ and t. This is enough, since formation of fractions and completion of a noetherian ring both generate flat algebras, and the composition of flat morphisms is still flat. To verify the claim consider the inclusions

$$A_\mathfrak{p}[t]/(\mathfrak{p}, t)^n \subset S_\mathfrak{p}^{-1}B/(\mathfrak{p}, t)^n \subset \widehat{A_\mathfrak{p}}[[t]]/(\mathfrak{p}, t)^n$$

and note that they are preserved by taking the inverse limit with respect to n. To conclude observe that $\varprojlim A_\mathfrak{p}[t]/(\mathfrak{p}, t)^n = \widehat{A_\mathfrak{p}}[[t]]$. $\qquad\square$

The advantage of working over $\widehat{A_\mathfrak{p}}[[t]]$ is that one can apply the Weierstrass Preparation Theorem (for a proof see, e.g., [8, VII, §3.8, Proposition 6]): given $\alpha = \sum a_i t^i \in \widehat{A_\mathfrak{p}}[[t]]$ such that not all coefficients are in $\mathfrak{p}$, there exist $u \in \widehat{A_\mathfrak{p}}[[t]]^*$ and a monic polynomial $\beta \in \widehat{A_\mathfrak{p}}[t]$ such that $\alpha = u\beta$ (the degree of β is equal to the minimum of the indices i such that $a_i \notin \mathfrak{p}$). Actually, as it is going to be clear from the proof of Lemma 2.6 below, we shall need just a weaker form of this statement.

2.2.2. Characteristic ideals. Now we deal with the equality between $\mathrm{Ch}_A(P_t)$ and $\mathrm{Ch}_A(P/tP)$.

Lemma 2.5. *For any finitely generated torsion $A_{\mathfrak{p}}$-module N one has the equality of lengths*

$$l_{A_{\mathfrak{p}}}(N) = l_{\widehat{A_{\mathfrak{p}}}}(N \otimes_{A_{\mathfrak{p}}} \widehat{A_{\mathfrak{p}}}).$$

Proof. Since both $A_{\mathfrak{p}}$ and $\widehat{A_{\mathfrak{p}}}$ are discrete valuation rings and

$$A_{\mathfrak{p}}/\mathfrak{p}^n \simeq \widehat{A_{\mathfrak{p}}}/\mathfrak{p}^n \simeq (A_{\mathfrak{p}}/\mathfrak{p}^n) \otimes \widehat{A_{\mathfrak{p}}},$$

the statement follows directly from the structure theorem for finitely generated torsion modules over principal ideal domains. $\qquad\square$

Lemma 2.6. *Let P be a pseudo-null B-module. Then $P \otimes_B \widehat{A_{\mathfrak{p}}}[[t]]$ is a finitely generated $\widehat{A_{\mathfrak{p}}}$-module for any height 1 prime ideal $\mathfrak{p} \subset A$.*

Proof. We consider P as an A-module and work separately with primes $\mathfrak{p}$ belonging or not belonging to the support of P. If the prime $\mathfrak{p}$ is not in this support, there is some $r \in \mathrm{Ann}_A(P)$ which becomes a unit in $A_{\mathfrak{p}} \subset \widehat{A_{\mathfrak{p}}}[[t]]$, hence $P \otimes_B \widehat{A_{\mathfrak{p}}}[[t]] = 0$ and the statement is trivial. Thus, from now on, we assume $\mathfrak{p} \in \mathrm{Supp}_A(P)$ (i.e., $\mathrm{Ann}_A(P) \subset \mathfrak{p}$). Since $\mathfrak{p}B$ is a height 1 prime ideal in B, the hypothesis on P yields $\mathrm{Ann}_B(P) \not\subset \mathfrak{p}B$. Hence there exists $\alpha \in \mathrm{Ann}_B(P) - \mathfrak{p}B$, i.e.,

$$\alpha = \sum_{i \geqslant 0} a_i t^i \in \mathrm{Ann}_B(P) \qquad (\text{with } a_i \in A \text{ for any } i)$$

with at least one $a_i \notin \mathfrak{p}$. For such an α, let n be the smallest index such that $a_n \notin \mathfrak{p}$. Then, by [8, VII, §3.8, Proposition 5] (which is a key step in the proof of the Weierstrass Preparation Theorem), one has a decomposition

$$\widehat{A_{\mathfrak{p}}}[[t]] = \alpha\widehat{A_{\mathfrak{p}}}[[t]] \oplus \left(\bigoplus_{i=0}^{n-1} \widehat{A_{\mathfrak{p}}}t^i\right).$$

Now one just uses $P \otimes_B \alpha\widehat{A_{\mathfrak{p}}}[[t]] = \alpha \cdot (P \otimes_B \widehat{A_{\mathfrak{p}}}[[t]]) = 0$. $\qquad\square$

Proposition 2.7. *Let P be a pseudo-null B-module. Then*

$$\mathrm{Ch}_A(P_t) = \mathrm{Ch}_A(P/tP).$$

Proof. By Remark 2.2 and Lemma 2.5, we need to show that

$$l_{\widehat{A_{\mathfrak{p}}}}(P_t \otimes_A \widehat{A_{\mathfrak{p}}}) = l_{\widehat{A_{\mathfrak{p}}}}((P/tP) \otimes_A \widehat{A_{\mathfrak{p}}})$$

for any height 1 prime ideal $\mathfrak{p}$ of A. By Lemma 2.4, the functor $\otimes_B\widehat{A_{\mathfrak{p}}}[[t]]$ is exact. Applying it to (2.4), we get an exact sequence

$$(2.5) \quad P_t \otimes_B \widehat{A_{\mathfrak{p}}}[[t]] \hookrightarrow P \otimes_B \widehat{A_{\mathfrak{p}}}[[t]] \xrightarrow{t} P \otimes_B \widehat{A_{\mathfrak{p}}}[[t]] \twoheadrightarrow (P/tP) \otimes_B \widehat{A_{\mathfrak{p}}}[[t]].$$

Lemma 2.6 shows that all terms of (2.5) are finitely generated $\widehat{A_{\mathfrak{p}}}$-modules. Hence, the first and last term of the sequence have the same length. Finally, just observe that if N is a B-module annihilated by t then

$$N \otimes_B \widehat{A_{\mathfrak{p}}}[[t]] = N \otimes_A \widehat{A_{\mathfrak{p}}}.$$

$\qquad\square$

Example 2.8. If P happens to be finitely generated over A then the statement of the proposition is obvious. We give a few examples of pseudo-null $B := \mathbb{Z}_p[[s,t]]$-modules which are not finitely generated as $A := \mathbb{Z}_p[[s]]$-modules, providing nontrivial examples in which the above theorem applies. However we remark that the main consequence of Lemma 2.6 is exactly the fact that we can ignore the issue of checking whether a pseudo-null B-module is finitely generated over A or not.

1. $P = B/(p,s)$. Then $P \simeq \mathbb{F}_p[[t]]$ is not finitely generated over A. In this case $P_t = 0$ and $P/tP \simeq \mathbb{F}_p$ (both A-pseudo-null), so that

$$\mathrm{Ch}_A(P_t) = \mathrm{Ch}_A(P/tP) = A.$$

2. $P = B/(s,pt)$. Then $P \simeq \mathbb{Z}_p[[t]]/(pt)$ is not finitely generated over A and elements in P can be written as

$$m = \sum_{i \geqslant 0} a_i t^i \quad a_0 \in \mathbb{Z}_p \text{ and } a_i \in \{0, ..., p-1\} \ \forall i \geqslant 1.$$

Moreover

$$P_t = p\mathbb{Z}_p[[t]]/(pt) \simeq p\mathbb{Z}_p \simeq \mathbb{Z}_p \simeq A/(s)$$

and

$$P/tP = \mathbb{Z}_p[[s,t]]/(t,s,pt) \simeq \mathbb{Z}_p \simeq A/(s),$$

so both have characteristic ideal (s) (as A-modules).

3. With $P = B/(p,st)$, a similar reasoning shows that

$$\mathrm{Ch}_A(P/tP) = \mathrm{Ch}_A(P_t) = (p).$$

Remark 2.9. The hypothesis that P is pseudo-null is necessary: if M is a torsion B-module then it is not true, in general, that $\mathrm{Ch}_A(M_t) = \mathrm{Ch}_A(M/tM)$. We give an easy example: let again $B = \mathbb{Z}_p[[s,t]]$ with $A = \mathbb{Z}_p[[s]]$, and consider $M = \mathbb{Z}_p[[s,t]]/(p^2 + s + t)$, which is a torsion B-module. Observe that M_t is trivial (so $\mathrm{Ch}_A(M_t) = A$) and

$$M/tM = \mathbb{Z}_p[[s,t]]/(t, p^2 + s) \simeq A/(p^2 + s)$$

has characteristic ideal over A equal to $(p^2 + s)$. Moreover $\mathrm{Ch}_A(M/tM) = (p^2+s)$ is the image of $\mathrm{Ch}_B(M) = (p^2+s+t)$ under the projection $\pi \colon B \to A$, $t \mapsto 0$. Hence, in this case,

$$\mathrm{Ch}_A(M_t) \, \pi \, (\mathrm{Ch}_B(M)) = \mathrm{Ch}_A(M/tM)$$

which anticipates the general formula of Proposition 2.10.

As mentioned in the Introduction, the following proposition will be crucial in the study of characteristic ideals for Iwasawa modules under descent.

Proposition 2.10. *Let $\pi \colon B \to A$ be the projection given by $t \mapsto 0$ and let M be a finitely generated torsion B-module. Then*

$$(2.6) \qquad \mathrm{Ch}_A(M_t) \, \pi \, (\mathrm{Ch}_B(M)) = \mathrm{Ch}_A(M/tM).$$

Moreover,

$$\mathrm{Ch}_A(M_t) = 0 \iff \pi\left(\mathrm{Ch}_B(M)\right) = 0 \iff \mathrm{Ch}_A(M/tM) = 0$$

and in this case M_t and M/tM are A-modules of the same rank.

Proof. Recall that the structure theorem (2.1) provides a pseudo-isomorphism between M and its associated elementary module $E(M)$. As noted above, being pseudo-isomorphic is an equivalence relation for torsion modules: therefore one has a (noncanonical) sequence

$$E(M) \lhook\joinrel\longrightarrow M \longrightarrow\joinrel\twoheadrightarrow P$$

where P is pseudo-null over B and the injectivity on the left comes from the fact that elementary modules have no nontrivial pseudo-null submodules (just use the valuation on $B_{\mathfrak{p}}$ to check that the annihilator of any $x \in B/\mathfrak{p}^e - \{0\}$ must be contained in $\mathfrak{p}$). The snake lemma sequence coming from the diagram

$$
\begin{array}{ccccc}
E(M) & \lhook\joinrel\longrightarrow & M & \longrightarrow\joinrel\twoheadrightarrow & P \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\
E(M) & \lhook\joinrel\longrightarrow & M & \longrightarrow\joinrel\twoheadrightarrow & P
\end{array}
$$

reads as

$$(2.7) \qquad E(M)_t \hookrightarrow M_t \longrightarrow P_t \longrightarrow E(M)/tE(M) \longrightarrow M/tM \twoheadrightarrow P/tP.$$

As we remarked at the beginning of Section 2.2, both P_t and P/tP are finitely generated torsion A-modules. It is also easy to see that all modules in the sequence (2.7) are finitely generated over A. Now observe that $(B/\mathfrak{p}^e)_t$ is zero if $\mathfrak{p} \neq (t)$ and isomorphic to A if $\mathfrak{p} = (t)$; similarly, $(B/\mathfrak{p}^e)/t(B/\mathfrak{p}^e)$ is either pseudo-null or isomorphic to A. Thus, putting $E(M) = \oplus B/\mathfrak{p}_i^{e_i}$, we find $E(M)_t \simeq A^r$ and

$$E(M)/tE(M) = \oplus B/(\mathfrak{p}_i^{e_i}, t) \simeq \oplus A/(\pi(\mathfrak{p}_i)^{e_i}) \simeq A^r \oplus \bullet,$$

where $r := \#\{i \mid \mathfrak{p}_i = tB\}$ and $\bullet$ is a pseudo-null B-module. Moreover (2.7) shows that $E(M)/tE(M)$ is A-torsion if and only if M/tM is A-torsion and $E(M)_t$ is A-torsion if and only if M_t is A-torsion. Therefore we have two cases:

1. If $r > 0$, then (t) divides $\mathrm{Ch}_B(M)$, so $\pi(\mathrm{Ch}_B(M)) = 0$ and, since M_t and M/tM are not A-torsion, $\mathrm{Ch}_A(M_t) = \mathrm{Ch}_A(M/tM) = 0$ as well (the statement on A-ranks is immediate from (2.7): e.g., apply the exact functor $\otimes_A \mathrm{Frac}(A)$).
2. If $r = 0$, then, because of the equivalent conditions above, all the characteristic ideals involved in (2.6) are nonzero; moreover we have

$$\mathrm{Ch}_A(E(M)/tE(M)) = \pi(\mathrm{Ch}_B(E(M))) = \pi(\mathrm{Ch}_B(M))$$

and (2.6) follows from the sequence (2.7), Proposition 2.7 and the multiplicativity of characteristic ideals. $\qquad\square$

Corollary 2.11. *In the above setting assume that M/tM is a finitely generated torsion A-module. Then M is a pseudo-null B-module if and only if $\mathrm{Ch}_A(M_t) = \mathrm{Ch}_A(M/tM)$. Moreover if $M/tM \sim_A 0$, then $M \sim_B 0$.*

Proof. The "only if" part is provided by Proposition 2.7. For the "if" part we assume the equality of characteristic ideals (which are nonzero by hypothesis). By (2.6) we have $\pi(\mathrm{Ch}_B(M)) = A$, hence there is some $f \in \mathrm{Ch}_B(M)$ such that $\pi(f) = 1$. But then $f = \sum_{i \geq 0} c_i t^i$ with $c_0 = 1$, which is an obvious unit in $B = A[[t]]$. Therefore $\mathrm{Ch}_B(M) = B$, i.e., M is pseudo-null over B. For the last statement just note that $\mathrm{Ch}_A(M/tM) = A$ yields $\mathrm{Ch}_A(M_t)\,\pi(\mathrm{Ch}_B(M)) = A$, so $\mathrm{Ch}_A(M_t) = A$ as well. □

Remarks 2.12.
  1. When $R \simeq \mathbb{Z}_p[[t_1, \ldots, t_d]]$ (i.e., the Iwasawa algebra for a $\mathbb{Z}_p^d$-extension of global fields), the statement of the previous corollary appears in [20, Lemme 4]. Note anyway that the proof given there relies on the choice of a $\mathbb{Z}_p^{d-1}$-subextension (i.e., on the strategy mentioned in Remark 1.4).
  2. The possibility of lifting pseudo-nullity from M/tM to M has been used to prove some instances of Greenberg's Generalized Conjecture (for statement and examples see, e.g., [2], [3] and [19]).

2.3. Pro-characteristic ideals. We can now define an analogue of characteristic ideals for finitely generated modules over certain nonnoetherian Krull domains Λ. We need Λ to be the inverse limit of noetherian Krull domains and we limit ourselves to finitely generated modules because characteristic ideals are usually defined only for them.

Let $\{\Lambda_d\}_{d \geq 0}$ be an inverse system of noetherian Krull domains such that

$$\Lambda_d \simeq \Lambda_{d+1}/\mathfrak{p}_{d+1} \quad \text{and} \quad \Lambda_{d+1} \simeq \varprojlim_n \Lambda_{d+1}/\mathfrak{p}_{d+1}^n \quad \text{for any } d \geq 0$$

($\mathfrak{p}_{d+1}$ a principal prime ideal of Λ_{d+1} of height 1). Let $\Lambda := \varprojlim \Lambda_d$ and note that, by hypothesis, $\Lambda_{d+1} \simeq \Lambda_d[[t_{d+1}]]$, where the variable t_{d+1} corresponds to a generator of the ideal $\mathfrak{p}_{d+1}$. Take a finitely generated Λ-module M which can be written as the inverse limit of Λ_d-modules $M = \varprojlim M_d$ (all the relevant arithmetic applications to Iwasawa modules satisfy this requirement).

Theorem 2.13. *Let notations be as above. If, for every $d \gg 1$,*
  1. *$(M_d)_{t_d}$ (the $\mathfrak{p}_d$-torsion submodule of M_d) is a pseudo-null Λ_{d-1}-module,*
  2. *$\mathrm{Ch}_{\Lambda_{d-1}}(M_d/t_dM_d) \subseteq \mathrm{Ch}_{\Lambda_{d-1}}(M_{d-1})$,*

then the ideals $\mathrm{Ch}_{\Lambda_d}(M_d)$ form a projective system (with respect to the maps $\pi_{\Lambda_{d-1}}^{\Lambda_d}$ defining Λ).

Proof. We can assume that the M_d are torsion Λ_d-modules (at least for $d \gg 0$), otherwise the $\mathrm{Ch}_{\Lambda_d}(M_d)$ are zero and there is nothing to prove. By Proposition 2.10, applied to $A = \Lambda_{d-1}$, $B = \Lambda_d$ and $M = M_d$, we get

$$\mathrm{Ch}_{\Lambda_{d-1}}((M_d)_{t_d})\pi^{\Lambda_d}_{\Lambda_{d-1}}(\mathrm{Ch}_{\Lambda_d}(M_d)) = \mathrm{Ch}_{\Lambda_{d-1}}(M_d/t_d M_d).$$

For $d \gg 1$ the hypotheses yield

$$\pi^{\Lambda_d}_{\Lambda_{d-1}}(\mathrm{Ch}_{\Lambda_d}(M_d)) \subseteq \mathrm{Ch}_{\Lambda_{d-1}}(M_{d-1}),$$

which shows that the ideals $\mathrm{Ch}_{\Lambda_d}(M_d)$ form a projective system with respect to the maps defining Λ. $\square$

As mentioned in the Introduction, this shows that we can define the pro-characteristic ideal of M as

$$\widetilde{\mathrm{Ch}}_\Lambda(M) := \varprojlim_d \mathrm{Ch}_{\Lambda_d}(M_d) \subseteq \Lambda.$$

Our pro-characteristic ideal maintains two classical properties of characteristic ideals.

Corollary 2.14. *Let M, M' and M'' be finitely generated Λ-modules which verify the hypotheses of Theorem 2.13.*

1. *The pro-characteristic ideals are multiplicative, i.e., if there is an exact sequence*

(2.8) $$M' \lhook\joinrel\longrightarrow M \longrightarrow\mkern-14mu\rightarrow M'' \ ,$$

   then

   $$\widetilde{\mathrm{Ch}}_\Lambda(M) = \widetilde{\mathrm{Ch}}_\Lambda(M')\widetilde{\mathrm{Ch}}_\Lambda(M'').$$

2. *$\widetilde{\mathrm{Ch}}_\Lambda(M) \neq 0$ if and only if M_d is a finitely generated torsion Λ_d-module for $d \gg 0$.*

Proof. **1**. For any $d \geqslant 0$ we have exact sequences (arising from (2.8))

$$M'_d \lhook\joinrel\longrightarrow M_d \longrightarrow\mkern-14mu\rightarrow M''_d \ ,$$

for which the equality $\mathrm{Ch}_{\Lambda_d}(M_d) = \mathrm{Ch}_{\Lambda_d}(M'_d)\,\mathrm{Ch}_{\Lambda_d}(M''_d)$ holds. The previous theorem allows to take limits on both sides maintaining the equality.

2. Obvious. $\square$

Remark 2.15. In the previous corollary it is enough to assume that M' and M'' verify the hypotheses of Theorem 2.13. Indeed, using the snake lemma exact sequence

$$(M'_d)_{t_d} \hookrightarrow (M_d)_{t_d} \to (M''_d)_{t_d} \to M'_d/t_d M'_d \to M_d/t_d M_d \twoheadrightarrow M''_d/t_d M''_d,$$

one immediately has that

$$(M'_d)_{t_d} \text{ and } (M''_d)_{t_d} \sim_{\Lambda_{d-1}} 0 \implies (M_d)_{t_d} \sim_{\Lambda_{d-1}} 0$$

and

$$\mathrm{Ch}_{\Lambda_{d-1}}(M_d/t_d M_d) = \mathrm{Ch}_{\Lambda_{d-1}}(M'_d/t_d M'_d)\,\mathrm{Ch}_{\Lambda_{d-1}}(M''_d/t_d M''_d)$$
$$\subseteq \mathrm{Ch}_{\Lambda_{d-1}}(M'_{d-1})\,\mathrm{Ch}_{\Lambda_{d-1}}(M''_{d-1}) = \mathrm{Ch}_{\Lambda_{d-1}}(M_{d-1}).$$

3. Class groups in global fields

For the rest of the paper we adjust our notations a bit to be more consistent with the usual ones in Iwasawa theory. We fix a prime number p and a global field F (note that for now we are not making any assumption on the characteristic of F). For any finite extension E/F let $\mathcal{M}(E)$ be the p-adic completion of the group of divisor classes of E, i.e.,

$$\mathcal{M}(E) := (E^* \backslash \mathbf{I}_E / \Pi_v \mathcal{O}^*_{E_v}) \otimes \mathbb{Z}_p$$

where $\mathbf{I}_E$ is the group of finite ideles of E, v varies over all nonarchimedean places of E and $\mathcal{O}_{E_v}$ is the ring of integers of the completion of E at v. When $\mathcal{L}/F$ is an infinite extension, we put $\mathcal{M}(\mathcal{L}) := \varprojlim \mathcal{M}(E)$ as E runs among the finite subextensions of $\mathcal{L}/F$ (the limit being taken with respect to norm maps). Class field theory yields a canonical isomorphism

$$(3.1) \qquad\qquad \mathcal{M}(E) \xrightarrow{\sim} X(E) := \mathrm{Gal}(L(E)/E),$$

where $L(E)$ is the maximal unramified abelian pro-p-extension of E. Passing to the limit shows that (3.1) is still true for infinite extensions.

Finally, for any infinite Galois extension $\mathcal{L}/F$, let $\Lambda(\mathcal{L}) := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{L}/F)]]$ be the associated Iwasawa algebra. We shall be interested in the situation where $\mathrm{Gal}(\mathcal{L}/F)$ is an abelian p-adic Lie group: in this case, both $\mathcal{M}(\mathcal{L})$ and $X(\mathcal{L})$ are $\Lambda(\mathcal{L})$-modules (the action of $\mathrm{Gal}(\mathcal{L}/F)$ on $X(\mathcal{L})$ is the natural one via inner automorphisms of $\mathrm{Gal}(L(\mathcal{L})/F)$) and these structures are compatible with the isomorphism (3.1). Furthermore, if $\mathrm{Gal}(\mathcal{L}/F) \simeq \mathbb{Z}_p^d$ then $\Lambda(\mathcal{L}) \simeq \mathbb{Z}_p[[t_1,..,t_d]]$ is a Krull domain.

Lemma 3.1. *Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^d$-extension, ramified only at finitely many places. If $d \geqslant 2$, one can always find a $\mathbb{Z}_p$-subextension $\mathcal{F}_1/F$ such that none of the ramified places splits completely in $\mathcal{F}_1$.*

Proof. Let S denote the set of primes of F which ramify in $\mathcal{F}$ and, for any place v in S let $D_v \subset \mathrm{Gal}(\mathcal{F}/F) =: \Gamma$ be the corresponding decomposition group. Getting $\mathcal{F}_1$ amounts to finding $\alpha \in \mathrm{Hom}(\Gamma, \mathbb{Z}_p)$ such that $\alpha(D_v) \neq 0$ for all $v \in S$. By hypothesis, for such v's the vector spaces $D_v \otimes \mathbb{Q}_p$ are nonzero, hence their annihilators are proper subspaces of $\mathrm{Hom}(\Gamma \otimes \mathbb{Q}_p, \mathbb{Q}_p)$ and since a $\mathbb{Q}_p$-vector space cannot be union of a finite number of proper subspaces, we deduce that the required α exists. $\qquad\square$

The following lemma is mostly a restatement of [13, Theorem 1].

Lemma 3.2. *Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^d$-extension, ramified only at finitely many places, and $\mathcal{F}' \subset \mathcal{F}$ a $\mathbb{Z}_p^{d-1}$-subextension, with $d > 2$. Let I be the kernel*

of the natural projection $\Lambda(\mathcal{F}) \to \Lambda(\mathcal{F}')$. *Then* $X(\mathcal{F})/IX(\mathcal{F})$ *is a finitely generated torsion* $\Lambda(\mathcal{F}')$*-module and* $X(\mathcal{F})$ *is a finitely generated torsion* $\Lambda(\mathcal{F})$*-module. This holds also for* $d = 2$, *provided that no ramified place in* $\mathcal{F}/F$ *is totally split in* $\mathcal{F}'$.

Proof. The idea is to proceed by induction on d. Choose a filtration

$$F =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{d-1} := \mathcal{F}' \subset \mathcal{F}_d := \mathcal{F}$$

so that $\mathrm{Gal}(\mathcal{F}_i/\mathcal{F}_{i-1}) \simeq \mathbb{Z}_p$ for all i and no ramified place in $\mathcal{F}/F$ is totally split in $\mathcal{F}_1$ (by Lemma 3.1, this can always be achieved when $d > 2$).

Now one proceeds as in [13, Theorem 1]. Namely, a standard argument yields that a $\Lambda(\mathcal{F}_i)$-module M is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_i)}$ if $M/I_{i-1}^i M$ is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$ (where I_{i-1}^i is the kernel of the projection $\Lambda(\mathcal{F}_i) \to \Lambda(\mathcal{F}_{i-1})$) and Greenberg's proof shows that $X(\mathcal{F}_{i-1}) \in \mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$ implies $X(\mathcal{F}_i)/I_{i-1}^i X(\mathcal{F}_i) \in \mathbf{Fgt}_{\Lambda(\mathcal{F}_{i-1})}$. So it is enough to prove that $X(\mathcal{F}_1)$ is a finitely generated torsion $\Lambda(\mathcal{F}_1)$-module. This follows from Iwasawa's classical proof ([15], exposed, e.g., in [22]; the function field version can be found in [17]). $\square$

Remarks 3.3.

1. In a $\mathbb{Z}_p$-extension of a global field, only places with residual characteristic p can ramify: thus the finiteness hypothesis on the ramification locus is automatically satisfied unless $\mathrm{char}(F) = p$. Note, however, that in the latter case this hypothesis is needed (see, e.g., [12, Remark 4]).

2. Among all $\mathbb{Z}_p$-extensions of F there is a distinguished one, namely, the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} if F is a number field and the arithmetic $\mathbb{Z}_p$-extension F_{arit} (arising from the unique $\mathbb{Z}_p$-extension of the constant field) if F is a function field. The condition on $\mathcal{F}'$ (when $d = 2$) is satisfied if it contains either F_{cyc} or F_{arit}.

3. For $d = 1$, we have $\mathcal{F}' = F$ and $\Lambda(\mathcal{F}') = \mathbb{Z}_p$. Thus the analogue of Lemma 3.2 would state that $X(\mathcal{F})/IX(\mathcal{F})$ is finite. This holds quite trivially if F is a global function field and $\mathcal{F} = F_{\mathrm{arit}}$ (note also that if $char(F) = \ell \neq p$ then F_{arit} is the only $\mathbb{Z}_p$-extension of F, see, e.g., [5, Proposition 4.3]). In this case the maximal abelian extension of F contained in $L(\mathcal{F})$ is exactly $L(F)$, hence $X(\mathcal{F})/IX(\mathcal{F}) \simeq \mathrm{Gal}(L(F)/F_{\mathrm{arit}})$ which is known (e.g., by class field theory) to be finite.

3.1. Iwasawa theory for class groups in function fields. In this section F will be a global function field of characteristic p and F_{arit} its arithmetic $\mathbb{Z}_p$-extension as defined above. Let $\mathcal{F}/F$ be a $\mathbb{Z}_p^{\infty}$-extension unramified outside a finite set of places S, with $\Gamma := \mathrm{Gal}(\mathcal{F}/F)$ and associated Iwasawa algebra $\Lambda := \Lambda(\mathcal{F})$. We fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i \in \mathbb{N}}$ for Γ and for any $d \geqslant 0$ we let $\mathcal{F}_d \subset \mathcal{F}$ be the fixed field of $\{\gamma_i\}_{i>d}$. Also, we assume that our basis is such that no place in S splits completely in $\mathcal{F}_1$ (Lemma 3.1 shows that there is no loss of generality in this assumption).

Remark 3.4. If $\mathcal{F}$ contains F_{arit} we can take the latter as $\mathcal{F}_1$. The additional hypothesis on $\mathcal{F}_1$ appears also in [16, Theorem 1.1]: the authors enlarge the set S and the extension $\mathcal{F}_d$ in order to get a $\mathbb{Z}_p$-extension verifying that hypothesis and use this to get a monomial Stickelberger element. This is a crucial step in the proof of the Main Conjecture provided in [10].

For notational convenience, let $t_i := \gamma_i - 1$. Then the subring

$$\mathbb{Z}_p[[t_1, \ldots, t_d]] \subset \Lambda$$

is canonically isomorphic to $\Lambda(\mathcal{F}_d)$ and, by a slight abuse of notation, the two shall be identified in the following. In particular, for any $d \geqslant 1$ we have $\Lambda(\mathcal{F}_d) = \Lambda(\mathcal{F}_{d-1})[[t_d]]$ and we can apply the results of Section 2. We shall denote by π_{d-1}^d the canonical projection $\Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_{d-1})$ with kernel $I_{d-1}^d = (t_d)$ (the augmentation ideal of $\mathcal{F}_d/\mathcal{F}_{d-1}$) and by Γ_{d-1}^d the group $\mathrm{Gal}(\mathcal{F}_d/\mathcal{F}_{d-1})$.

For two finite extensions $L \supset L' \supset F$, the degree maps $\deg_L$ and $\deg_{L'}$ fit into the commutative diagram (with exact rows)

$$(3.2) \qquad \begin{array}{ccccc} \mathcal{A}(L) & \lhook\joinrel\longrightarrow & \mathcal{M}(L) & \xrightarrow{\ \deg_L\ } & \mathbb{Z}_p \\ \downarrow{\scriptstyle N_{L'}^L} & & \downarrow{\scriptstyle N_{L'}^L} & & \downarrow \\ \mathcal{A}(L') & \lhook\joinrel\longrightarrow & \mathcal{M}(L') & \xrightarrow{\ \deg_{L'}\ } & \mathbb{Z}_p \, , \end{array}$$

where $N_{L'}^L$ denotes the norm and the vertical map on the right is multiplication by $[\mathbb{F}_L : \mathbb{F}_{L'}]$ (the degree of the extension between the fields of constants). For an infinite extension $\mathcal{L}/F$ contained in $\mathcal{F}$, taking projective limits one gets an exact sequence

$$(3.3) \qquad \mathcal{A}(\mathcal{L}) \lhook\joinrel\longrightarrow \mathcal{M}(\mathcal{L}) \xrightarrow{\ \deg_{\mathcal{L}}\ } \mathbb{Z}_p \, .$$

Remark 3.5. The map $\deg_{\mathcal{L}}$ above becomes zero exactly when $\mathcal{L}$ contains the unramified $\mathbb{Z}_p$-subextension F_{arit}.

By (3.1), Lemma 3.2 shows that $\mathcal{M}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module, so the same holds for $\mathcal{A}(\mathcal{F}_d)$. Hence, by Proposition 2.10, one has, for all $d \geqslant 1$,

$$(3.4)$$
$$\mathrm{Ch}_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)_{t_d}) \, \pi_{d-1}^d(\mathrm{Ch}_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = \mathrm{Ch}_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/t_d\mathcal{A}(\mathcal{F}_d))$$

and note that

$$\mathcal{A}(\mathcal{F}_d)_{t_d} = \mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \, , \quad \mathcal{A}(\mathcal{F}_d)/t_d\mathcal{A}(\mathcal{F}_d) = \mathcal{A}(\mathcal{F}_d)/I_{d-1}^d\mathcal{A}(\mathcal{F}_d).$$

Consider the following diagram

$$(3.5) \qquad \begin{array}{ccc} \mathcal{A}(\mathcal{F}_d) \lhook\joinrel\longrightarrow \mathcal{M}(\mathcal{F}_d) \xrightarrow{\deg} \mathbb{Z}_p \\ \downarrow{\scriptstyle t_d} \qquad \downarrow{\scriptstyle t_d} \qquad \downarrow{\scriptstyle t_d} \\ \mathcal{A}(\mathcal{F}_d) \lhook\joinrel\longrightarrow \mathcal{M}(\mathcal{F}_d) \xrightarrow{\deg} \mathbb{Z}_p \end{array}$$

(note that the vertical map on the right is 0) and its snake lemma sequence

$$(3.6) \qquad \mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \lhook\joinrel\longrightarrow \mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d} \xrightarrow{\quad\deg\quad} \mathbb{Z}_p$$

$$\mathbb{Z}_p \xleftarrow{\quad\deg\quad} \mathcal{M}(\mathcal{F}_d)/I_{d-1}^d \mathcal{M}(\mathcal{F}_d) \longleftarrow \mathcal{A}(\mathcal{F}_d)/I_{d-1}^d \mathcal{A}(\mathcal{F}_d)$$

For $d \geqslant 2$ (which implies that $\mathbb{Z}_p$ is a torsion $\Lambda(\mathcal{F}_{d-1})$-module), (3.6) and Lemma 3.2 show that $\mathcal{A}(\mathcal{F}_d)/I_{d-1}^d \mathcal{A}(\mathcal{F}_d)$ is in $\mathbf{Fgt}_{\Lambda(\mathcal{F}_{d-1})}$ as well. By Proposition 2.10 it follows that no term in (3.4) is trivial.

3.1.1. Totally ramified extensions and the Main Conjecture. The main examples we have in mind are extensions satisfying the following:

Assumption 3.6. The (finitely many) ramified places of $\mathcal{F}/F$ are totally ramified.

In what follows an extension satisfying this assumption will be called a *totally ramified extension*. A prototypical example is the $\mathfrak{a}$-cyclotomic extension of $\mathbb{F}_q(T)$ generated by the $\mathfrak{a}$-torsion of the Carlitz module ($\mathfrak{a}$ an ideal of $\mathbb{F}_q[T]$, see, e.g., [21, Chapter 12]). As usual in Iwasawa theory over number fields, most of the proofs will work (or can be adapted) simply assuming that ramified primes are totally ramified in $\mathcal{F}/\mathcal{F}_e$ for some $e \geqslant 0$, but, in the function field setting, one would need some extra hypothesis on the behaviour of these places in $\mathcal{F}_e/F$.

Under this assumption any $\mathbb{Z}_p$-subextension can play the role of $\mathcal{F}_1$. Moreover $\mathcal{M}(\mathcal{F})$ is defined using norm maps and norms are surjective on class groups in totally ramified extensions, so

$$\mathcal{M}(\mathcal{F}_d) = N_{\mathcal{F}_d}^{\mathcal{F}}(\mathcal{M}(\mathcal{F})) := \mathcal{M}(\mathcal{F})_d \text{ and } \mathcal{M}(\mathcal{F}) = \varprojlim_d \mathcal{M}(\mathcal{F})_d = \varprojlim_d \mathcal{M}(\mathcal{F}_d)$$

(in the notations of Theorem 2.13). The same holds for the modules $\mathcal{A}(\mathcal{F})$ and $\mathcal{A}(\mathcal{F}_d)$.

Let $L^0(\mathcal{F}_{d-1})$ be the maximal abelian extension of $\mathcal{F}_{d-1}$ contained in $L(\mathcal{F}_d)$, we have

$$\mathcal{F}_d L(\mathcal{F}_{d-1}) \subseteq L^0(\mathcal{F}_{d-1}) \text{ and } \operatorname{Gal}(L(\mathcal{F}_d)/L^0(\mathcal{F}_{d-1})) = I_{d-1}^d \mathcal{M}(\mathcal{F}_d)$$

(see [24, Lemma 13.14]). Galois theory provides a surjection

$$\operatorname{Gal}(L^0(\mathcal{F}_{d-1})/\mathcal{F}_d) \twoheadrightarrow \operatorname{Gal}(\mathcal{F}_d L(\mathcal{F}_{d-1})/\mathcal{F}_d),$$

i.e.,

$$\mathcal{M}(\mathcal{F}_d)/I_{d-1}^d \mathcal{M}(\mathcal{F}_d) \twoheadrightarrow \mathcal{M}(\mathcal{F}_{d-1}),$$

which yields

(3.7) $\qquad \mathrm{Ch}_{\Lambda_{d-1}}(\mathcal{M}(\mathcal{F}_d)/I_{d-1}^d \mathcal{M}(\mathcal{F}_d)) \subseteq \mathrm{Ch}_{\Lambda_{d-1}}(\mathcal{M}(\mathcal{F}_{d-1})).$

The same relation holds for the characteristic ideals of the $\mathcal{A}(\mathcal{F}_d)$ for $d \geqslant 3$, because of (3.6). In particular if we have only one ramified prime, the surjection above is an isomorphism (just adapt the proof of [24, Lemma 13.15]) and (3.7) is an equality. This takes care of hypothesis **2** in Theorem 2.13.

A little modification of the proof of [6, Lemma 5.7] (note that [6, Lemmas 5.4 and 5.6] still hold in the present setting), shows that elements of $\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$ are represented by divisors supported on ramified primes. Hence $\mathcal{M}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$ (and $\mathcal{A}(\mathcal{F}_d)^{\Gamma_{d-1}^d}$) are finitely generated $\mathbb{Z}_p$-modules, i.e., pseudo-null $\Lambda(\mathcal{F}_{d-1})$-modules for $d \geqslant 3$. From (3.4) we obtain:

Corollary 3.7. *Let $\mathcal{F}_d$ be a $\mathbb{Z}_p^d$-extension of F contained in a totally ramified extension. Then, for any $\mathbb{Z}_p^{d-1}$-extension $\mathcal{F}_{d-1}$ contained in $\mathcal{F}_d$, one has*

(3.8) $\qquad \pi_{d-1}^d(\mathrm{Ch}_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d))) = \mathrm{Ch}_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{A}(\mathcal{F}_d)/I_{d-1}^d \mathcal{A}(\mathcal{F}_d))$
$$\subseteq \mathrm{Ch}_{\Lambda_{d-1}}(\mathcal{A}(\mathcal{F}_{d-1})).$$

Hence the modules $\mathcal{A}(\mathcal{F}_d)$ verify the hypotheses of Theorem 2.13 and we can define:

Definition 3.8. Let $\mathcal{F}/F$ be a totally ramified $\mathbb{Z}_p^\infty$-extension. The *pro-characteristic ideal* of $\mathcal{A}(\mathcal{F})$ is

$$\widetilde{\mathrm{Ch}}_\Lambda(\mathcal{A}(\mathcal{F})) := \varprojlim_{\mathcal{F}_d} \mathrm{Ch}_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)).$$

Remark 3.9. Definition 3.8 only depends on the extension $\mathcal{F}/F$ and not on the filtration of $\mathbb{Z}_p^d$-extension we choose inside it. Indeed take two different filtrations $\{\mathcal{F}_d\}$ and $\{\mathcal{F}_d'\}$ and define a new filtration containing both by putting

$$\mathcal{F}_0'' := F \quad \text{and} \quad \mathcal{F}_n'' = \mathcal{F}_n \mathcal{F}_n' \quad \forall n \geqslant 1$$

(note that $\mathcal{F}_n''$ is not, in general, a $\mathbb{Z}_p^n$-extension and $\mathcal{F}_n''/\mathcal{F}_{n-1}''$ is a $\mathbb{Z}_p^i$-extension with $0 \leqslant i \leqslant 2$, but these details are irrelevant for the limit process we need here). By Corollary 3.7, the limits of the characteristic ideals of the filtrations we started with coincide with the limit on the filtration $\{\mathcal{F}_n''\}$. This answers questions a and b of [6, Remark 5.11]: we had a similar definition there but it was based on the particular choice of the filtration.

We recall that, in [9, Theorem 3.1] (and [10]), the authors prove an Iwasawa Main Conjecture (IMC) at "finite level", which (in our simplified setting and notations) reads as

(3.9) $\qquad\qquad \mathrm{Ch}_{\Lambda(\mathcal{F}_d)}(\mathcal{A}(\mathcal{F}_d)) = (\theta_{\mathcal{F}_d/F,S}),$

where $\theta_{\mathcal{F}_d/F,S}$ is the classical Stickelberger element (defined, e.g., in [6, Section 5.3]). By [23, Proposition IV.1.8], the elements $\theta_{\mathcal{F}_d/F,S}$ form a coherent sequence with respect to the maps π_e^d, so, taking inverse limits in (3.9), one obtains:

Corollary 3.10 (IMC in nonnoetherian algebras)**.** *In the previous setting we have*

$$\widetilde{\mathrm{Ch}}_\Lambda(\mathcal{A}(\mathcal{F})) = \varprojlim_{\mathcal{F}_d} (\theta_{\mathcal{F}_d/F,S}) := (\theta_{\mathcal{F}/F,S}),$$

as ideals in Λ.

More details on the statement and its proof (now independent from the filtration $\{\mathcal{F}_d\}_{d\geqslant 0}$) can be found in [6, Section 5].

Remark 3.11. A different approach, using a more natural filtration of global function fields for the Carlitz $\mathfrak{p}$-cyclotomic extension of $\mathbb{F}_q(T)$ and Fitting ideals of class groups, will be carried out in [1]. It leads to a similar version of the Iwasawa Main Conjecture in the algebra Λ, but it has the advantage of having more direct and relevant arithmetic applications (see [1, Section 6]).

References

[1] ANGLÈS, B.; BANDINI, A.; BARS, F.; LONGHI, I. Iwasawa main conjecture for the Carlitz cyclotomic extension and applications. In progress.
[2] BANDINI, ANDREA. Greenberg's conjecture for $\mathbb{Z}_p^d$-extensions. *Acta Arith.* **108** (2003), no. 4, 357–368. MR1979904 (2004c:11201), Zbl 1091.11037, doi: 10.4064/aa108-4-5.
[3] BANDINI, ANDREA. Greenberg's conjecture and capitulation in $\mathbb{Z}_p^d$-extensions. *J. Number Theory* **122** (2007), no. 1, 121–134. MR2287114 (2007j:11150), Zbl 1106.11037, doi: 10.1016/j.jnt.2006.04.004.
[4] BANDINI, A.; LONGHI, I. Control theorems for elliptic curves over function fields. *Int. J. Number Theory* **5** (2009), no. 2, 229–256. MR2502807 (2010a:11100), Zbl 1234.11071, arXiv:/math/0604249, doi: 10.1142/S1793042109002067.
[5] BANDINI, ANDREA; LONGHI, IGNAZIO. Selmer groups for elliptic curves in $\mathbb{Z}_l^d$-extensions of function fields of characteristic p. *Ann. Inst. Fourier* **59** (2009), no. 6, 2301–2327. MR2640921 (2011j:11101), Zbl 1207.11061, arXiv:0707.1143.
[6] BANDINI, ANDREA; BARS, FRANCESC; LONGHI, IGNAZIO. Aspects of Iwasawa theory over function fields. To appear in the EMS Congress Reports, 2011. arXiv:1005.2289.
[7] BANDINI, ANDREA; BARS, FRANCESC; LONGHI, IGNAZIO. Characteristic ideals and Selmer groups. 2014. arXiv:1404.2788.
[8] BOURBAKI, NICHOLAS. Commutative algebra. Chapters 1–7. Elements of Mathematics. *Springer-Verlag, Berlin*, 1998. xxiv+625 pp. ISBN: 3-540-64239-0. MR1727221 (2001g:13001), Zbl 0902.13001.
[9] BURNS, DAVID. Congruences between derivatives of geometric L-functions. *Invent. Math.* **184** (2011), no. 2, 221–256. MR2793857 (2012e:11116), Zbl 1234.11080, doi: 10.1007/s00222-010-0286-3.
[10] BURNS, D.; LAI, K. F.; TAN, K.-S. On geometric main conjectures. Appendix to [9].
[11] GILMER, ROBERT. Power series rings over a Krull domain. *Pacific J. Math.* **29** (1969), 543–549. MR0245571 (39 #6877), Zbl 0179.34502, doi: 10.2140/pjm.1969.29.543.

[12] GOLD, R.; KISILEVSKY, H. On geometric $\mathbb{Z}_p$-extensions of function fields. *Manuscripta Math.* **62** (1988), no. 2, 145–161. MR0963002 (90e:11160), Zbl 0662.12016 doi: 10.1007/BF01278975.

[13] GREENBERG, RALPH. The Iwasawa invariants of Γ-extensions of a fixed number field. *Amer. J. Math.* **95** (1973), 204–214. http://www.jstor.org/stable/2373652. MR0332712 (48 #11038), Zbl 0268.12005.

[14] GREENBERG, RALPH. On the structure of certain Galois groups. *Invent. Math.* **47** (1978), no. 1, 85–99. MR0504453 (80b:12007), Zbl 0403.12004, doi: 10.1007/BF01609481.

[15] IWASAWA, KENKICHI. On Γ-extensions of algebraic number fields. *Bull. Amer. Math. Soc.* **65** (1959), 183–226. MR0124316 (23 #A1630), Zbl 0089.02402, doi: 10.1090/S0002-9904-1959-10317-7.

[16] KUEH, KA-LAM; LAI, KING FAI; TAN, KI-SENG. Stickelberger elements for $\mathbb{Z}_p^d$-extensions of function fields. *J. Number Theory* **128** (2008), no. 10, 2776–2783. MR2441076 (2009g:11158), Zbl 1225.11156, doi: 10.1016/j.jnt.2008.04.006.

[17] LI, CHAOQUN; ZHAO, JIANQIANG. Iwasawa theory of $\mathbb{Z}_p^d$-extensions over global function fields. *Exposition. Math.* **15** (1997), no. 4, 315–337. MR1486400 (98h:11150), Zbl 0917.11058.

[18] OCHIAI, TADASHI. Euler system for Galois deformations. *Ann. Inst. Fourier* **55** (2005), no. 1, 113–146. MR2141691 (2006a:11068), Zbl 1112.11031, doi: 10.5802/aif.2091.

[19] OZAKI, MANABU. Iwasawa invariants of $\mathbb{Z}_p$-extensions over an imaginary quadratic field. *Class field theoryits centenary and prospect (Tokyo, 1998)*, 387–399, Adv. Stud. Pure Math., 30. *Math. Soc. Japan, Tokyo*, 2001. MR1846467 (2002e:11147), Zbl 1002.11078.

[20] PERRIN-RIOU, BERNADETTE. Arithmétique des courbes elliptiques et théorie d'Iwasawa. *Mém. Soc. Math. Fr. (N.S.)* **17**, (1984), 130 pp. MR0799673 (87h:11058), Zbl 0599.14020.

[21] ROSEN, MICHAEL. Number theory in function fields. Graduate Texts in Mathematics, 210. *Springer-Verlag, New York*, 2002. xii+358 pp. ISBN: 0-387-95335-3. MR1876657 (2003d:11171), Zbl 1043.11079.

[22] SERRE, JEAN-PIERRE. Classes des corps cyclotomiques (d'après K. Iwasawa), *Séminaire Bourbaki, Vol. 5*, 74, 83–93. *Soc. Math. France, Paris*, 1959. MR1603459, Zbl 0119.27603.

[23] TATE, JOHN. Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$. Progress in Mathematics 47. *Birkhäuser Boston, Inc., Boston, MA*, 1984. 143 pp. ISBN: 0-8176-3188-7. MR0782485 (86e:11112), Zbl 0545.12009.

[24] WASHINGTON, LAWRENCE C. Introduction to cyclotomic fields. Second edition. Graduate Texts in Mathematics, 83. *Springer-Verlag, New York*, 1997. xiv+487 pp. ISBN: 0-387-94762-0. MR1421575 (97h:11130), Zbl 0966.11047, doi: 10.1007/978-1-4612-1934-7.

(Andrea Bandini) Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze, 53/A - 43124 Parma (PR), Italy
andrea.bandini@unipr.it

(Francesc Bars) Departament Matemàtiques, Edif. C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia
francesc@mat.uab.cat

(Ignazio Longhi) Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren Ai Road, Dushu Lake Higher Education Town, Suzhou Industrial Park, Suzhou, Jiangsu, 215123, China
Ignazio.Longhi@xjtlu.edu.cn

This paper is available via http://nyjm.albany.edu/j/2014/20-38.html.