Operator convexity in Krein spaces

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ABSTRACT. We introduce the notion of Krein-operator convexity in the setting of Krein spaces. We present an indefinite version of the Jensen operator inequality on Krein spaces by showing that if $(\mathscr{H}, J)$ is a Krein space, $\mathcal{U}$ is an open set which is symmetric with respect to the real axis such that $\mathcal{U} \cap \mathbb{R}$ consists of a segment of real axis and f is a Krein-operator convex function on $\mathcal{U}$ with $f(0) = 0$, then

$$f(C^\sharp A C) \leq^J C^\sharp f(A) C$$

for all J-positive operators A and all invertible J-contractions C such that the spectra of A, $C^\sharp A C$ and $D^\sharp A D$ are contained in $\mathcal{U}$, where D is a defect operator for $C^\sharp$. We also show that in contrast with usual operator convex functions the converse of this implication is not true, in general.

Contents

1. Introduction and preliminaries

Linear spaces with indefinite inner products were used for the first time in the quantum field theory in physics by Dirac [10] and Pauli [17]. Their first mathematical definition was provided by Pontrjagin [18] and since then they have been studied by many mathematicians. Krein spaces as an indefinite generalization of Hilbert spaces were formally defined by Ginzburg [14] and applies in the quantum field theory by Jakóbczyk [16] and others. Strohmaier [22] applied Krein spaces in the definition of semi-Riemannian spectral triples in noncommutative geometry. Bebiano et al. [7] proved an extension of the classical theory of Courant and Fisher to the case of J-Hermitian matrices. We present the standard terminology and some basic results on Krein spaces. The reader is referred to [3, 5, 9] for a complete exposition on the subject.

Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the identity operator $I_{\mathscr{H}}$. An operator $T \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathscr{H}$. If T is a positive invertible operator we write $T > 0$. For bounded selfadjoint operators T and S on $\mathscr{H}$, we say $T \leq S$ if $S - T \geq 0$.

Suppose that a nontrivial selfadjoint involution J on $\mathscr{H}$, i.e., $J = J^* = J^{-1}$, is given to produce an indefinite inner product

$$[x, y]_J := \langle Jx, y \rangle \qquad (x, y \in \mathscr{H}).$$

In this case $(\mathscr{H}, J)$ is called a Krein space. The operators $\mathrm{P}_+ = \frac{I+J}{2}$ and $\mathrm{P}_- = \frac{I-J}{2}$ are orthogonal projections onto $\mathscr{H}_+ = \mathrm{ran}(\mathrm{P}_+)$ and $\mathscr{H}_- = \mathrm{ran}(\mathrm{P}_-)$, respectively and

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-.$$

In correspondence to this orthogonal decomposition, each bounded linear operator C on $\mathscr{H}$ is uniquely represented by the matrix

$$(1.1) \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where $C_{11} = \mathrm{P}_+ C \mathrm{P}_+|_{\mathscr{H}_+}$, $C_{12} = \mathrm{P}_+ C \mathrm{P}_-|_{\mathscr{H}_-}$, $C_{21} = \mathrm{P}_- C \mathrm{P}_+|_{\mathscr{H}_+}$, $C_{22} = \mathrm{P}_- C \mathrm{P}_-|_{\mathscr{H}_-}$. The n-dimensional Minkowski space is a well-known example of a Krein space:

Example 1.1. Let $M_n(\mathbb{C})$ be the set of all complex $n \times n$ matrices and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$. For selfadjoint involution

$$J_0 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

where I_{n-1} denotes the identity of $M_{n-1}(\mathbb{C})$, one can define an indefinite inner product $[.,.]_{J_0}$ on $\mathbb{C}^n$ by

$$[x, y]_{J_0} = \langle J_0 x, y \rangle = \sum_{k=1}^{n-1} x_k \bar{y}_k - x_n \bar{y}_n$$

for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$. The Krein space $(\mathbb{C}^n, J_0)$ is called the n-dimensional Minkowski space.

Let $(\mathscr{H}_1, J_1)$ and $(\mathscr{H}_2, J_2)$ be Krein spaces. The (J_1, J_2)-adjoint operator $A^\sharp$ of $A \in \mathbb{B}(\mathscr{H}_1, \mathscr{H}_2)$ is defined by

$$[Ax, y]_{J_2} = [x, A^\sharp y]_{J_1} \qquad (x \in \mathscr{H}_1, y \in \mathscr{H}_2),$$

which is equivalent to say that $A^\sharp = J_1 A^* J_2$. Trivially $(A^\sharp)^\sharp = A$. An operator $A \in \mathbb{B}(\mathscr{H})$ on a Krein space $(\mathscr{H}, J)$ is said to be J-selfadjoint if $A^\sharp = A$, or equivalently, $A = J A^* J$. The spectrum of a J-selfadjoint operator on a Krein space $(\mathscr{H}, J)$ is not necessarily real (it can even cover the whole plane); see [9].

For J-selfadjoint operators A and B, the J-order, denoted as $A \leq^J B$, is defined by

$$[Ax, x]_J \leq [Bx, x]_J \qquad (x \in \mathcal{H}).$$

Clearly $A \leq^J B$ if and only if $JA \leq JB$. The J-selfadjoint operator $A \in \mathbb{B}(\mathcal{H})$ is said to be J-positive if $A \geq^J 0$. Evidently, A is J-positive if and only if AJ is positive. It is easy to see that neither $A \geq 0$ implies $A \geq^J 0$ nor $A \geq^J 0$ implies $A \geq 0$.

Let $(\mathcal{H}_1, J_1)$ and $(\mathcal{H}_2, J_2)$ be Krein spaces. As usual, let $\mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ be the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$. An operator $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called a (J_1, J_2)-contraction if $C^\sharp C \leq^{J_1} I_{\mathcal{H}_1}$, that is, $C^* J_2 C \leq J_1$. The operator C is called (J_1, J_2)-bicontraction if C and $C^\sharp$ are (J_1, J_2)-contraction and (J_2, J_1)-contraction, respectively. In the case that $\mathcal{H}_1 = \mathcal{H}_2$ and $J_1 = J_2 = J$ we write J-contraction instead of (J, J)-contraction. An invertible operator $U \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $U^\sharp = U^{-1}$ is said to be (J_1, J_2)-unitary.

Note that in contrast to the setting of Hilbert spaces, not all J-contractions are J-bicontractions; see [12, Example 1.3.8]. The following theorem presenting a suitable condition for a J-contraction to being a J-bicontraction.

Theorem 1.2 ([3, Corollary 3.3.3])**.** *A J-contraction C on a Krein space $(\mathcal{H}, J)$ is a J-bicontraction if and only if the operator C_{22} in the matrix form (1.1) of C is invertible. In particular if C is an invertible J-contraction, then C is a J-bicontraction.*

Let $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$. By a defect operator for C we mean any operator $E \in \mathbb{B}(\tilde{\mathcal{H}}_2, \mathcal{H}_1)$, where $(\tilde{\mathcal{H}}_2, \tilde{J}_2)$ is a Krein space, such that E has zero kernel and $I_{\mathcal{H}_1} - C^\sharp C = EE^\sharp$. A Julia operator for C is a $(J_1 \oplus \tilde{J}_1, J_2 \oplus \tilde{J}_2)$-unitary $U : \mathcal{H}_1 \oplus \tilde{\mathcal{H}}_1 \to \mathcal{H}_2 \oplus \tilde{\mathcal{H}}_2$ of the form

$$U = \begin{pmatrix} C & D \\ E^\sharp & -L^\sharp \end{pmatrix},$$

where $(\tilde{\mathcal{H}}_1, \tilde{J}_1)$ and $(\tilde{\mathcal{H}}_2, \tilde{J}_2)$ are Krein spaces such that the operators $D \in \mathbb{B}(\tilde{\mathcal{H}}_1, \mathcal{H}_2)$ and $E \in \mathbb{B}(\tilde{\mathcal{H}}_2, \mathcal{H}_1)$ have zero kernels. In this case E is a defect operator for C, and D is a defect operator for $C^\sharp$; cf. [11].

Defect and Julia operators have played important roles in the Krein space operator theory. The first constructions of these operators in the Krein space setting are due to Arsene et al. [2]. An abstract theory of Julia operators in Krein spaces and its applications appear in a number of sources; see e.g., [11, 12, 13]. We need the following theorem as a result of Corollary 1.4.3, Theorem 2.4.5 of [12] and Theorem 13 of [13].

Theorem 1.3. *Suppose that $(\mathcal{H}_1, J_1)$ and $(\mathcal{H}_2, J_2)$ are Krein spaces and $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an injective (J_1, J_2)-bicontraction. Then C has a unique (up to unitary) Julia operator of the form*

$$U = \begin{pmatrix} C & D \\ E^\sharp & -L^* \end{pmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \tilde{\mathcal{H}}_1, \mathcal{H}_2 \oplus \tilde{\mathcal{H}}_2),$$

for which $\tilde{\mathscr{H}}_1$ and $\tilde{\mathscr{H}}_2$ are Hilbert spaces and $-L^*$ is a Hilbert space contraction.

A real valued continuous function f on an interval $\mathcal{I} \subseteq \mathbb{R}$ is said to be operator convex if

$$f\Big((1-\lambda)A + \lambda B\Big) \leq (1-\lambda)f(A) + \lambda f(B),$$

for all $\lambda \in [0,1]$ and all selfadjoint operators A and B on a Hilbert space $\mathscr{H}$, whose spectra are contained in $\mathcal{I}$. The notion of operator/matrix convex functions, as well as that of operator/matrix monotone functions, have played a vital role in operator/matrix analysis and its applications, e.g., to quantum information. On the other hand, the operator theory in spaces with an indefinite inner product (notably in Pontryagin spaces and Krein spaces) has been investigated for a long time with the aim to establish mathematical formalism of quantum field theory. An indefinite analogue of the concept of monotone matrix function was studied in [1].

In this paper we introduce the notion of Krein-operator convexity in the setting of Krein spaces. We also present an indefinite version of Jensen's operator inequality based on the ideas due to Hansen and Pedersen [15].

2. Main results

Let A be a J-selfadjoint operator on a Krein space $(\mathscr{H}, J)$ and let $\mathcal{U}$ be an open set which is not necessarily connected such that $\sigma(A) \subseteq \mathcal{U}$. Suppose that $f : \mathcal{U} \to \mathbb{C}$ is an analytic function. Then the operator $f(A)$ is defined by the usual Dunford-Riesz integral

$$(2.1) \qquad f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I_{\mathscr{H}} - A)^{-1} d\lambda,$$

where Γ is a suitable finite family of closed rectifiable contours with positive direction surrounding $\sigma(A)$ in its interior; see [20].

From now on assume that $\mathcal{U}$ is an open set (not necessarily connected) in the plane, which is symmetric with respect to real axis and $\mathcal{U} \cap \mathbb{R}$ consists of a segment of real axis. The following proposition provide some conditions for $f(A)$ to be J-selfadjoint whenever A is J-selfadjoint.

Proposition 2.1. *Let A be a J-selfadjoint operator on a Krein space $(\mathscr{H}, J)$ such that $\sigma(A) \subseteq \mathcal{U}$. If $f : \mathcal{U} \to \mathbb{C}$ is an analytic function such that $f(x)$ is real for all $x \in \mathcal{U} \cap \mathbb{R}$, then $f(A)$ is J-selfadjoint.*

Proof. By the definition of a J-selfadjoint operator, $AJ = JA^*$. Hence

$$(2.2) \qquad Jf(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) J(\lambda I_{\mathscr{H}} - A)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda J - AJ)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma f(\lambda) \Big(J(\lambda I_{\mathscr{H}} - A^*) \Big)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I_{\mathscr{H}} - A^*)^{-1} J d\lambda$$

$$= f(A^*)J.$$

The spectrum of a J-selfadjoint operator on a Krein space $(\mathscr{H}, J)$ is symmetric with respect to the real axis (see [9, Corollary 6.3]). Suppose that Γ is a closed rectifiable contour with positive direction surrounding $\sigma(A)$ in its interior such that it is symmetric with respect to the real axis. By the change of variable λ to $\bar{\lambda}$, we have

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma^{-1}} f(\bar{\lambda})(\bar{\lambda} I_{\mathscr{H}} - A)^{-1} d\bar{\lambda} = -\frac{1}{2\pi i} \int_\Gamma f(\bar{\lambda})(\bar{\lambda} I_{\mathscr{H}} - A)^{-1} d\bar{\lambda}.$$

By the reflection principle, we have $\overline{f(\bar{z})} = f(z)$ for all $z \in \Gamma$. Therefore

$$(f(A))^* = \Big(-\frac{1}{2\pi i} \int_\Gamma f(\bar{\lambda})(\bar{\lambda} I_{\mathscr{H}} - A)^{-1} d\bar{\lambda} \Big)^*$$

$$= \frac{1}{2\pi i} \int_\Gamma \overline{f(\bar{\lambda})}(\lambda I_{\mathscr{H}} - A^*)^{-1} d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I_{\mathscr{H}} - A^*)^{-1} d\lambda$$

$$= f(A^*).$$

It follows from (2.2) that $Jf(A) = f(A^*)J = f(A)^*J$. Hence $f(A)$ is J-selfadjoint. □

The notion of operator convexity is a generalization of that of usual convexity. This is based on the fact that the selfadjoint operators (Hermitian matrices) can be regarded as a generalization of the real numbers. We aim to introduce the notion of Krein-operator convexity as a generalization of the operator convexity.

One may immediately say that a real valued function f being analytic on an interval $\mathcal{I}$ is Krein-operator convex if

$$f\Big((1-\lambda)A + \lambda B\Big) \leq^J (1-\lambda)f(A) + \lambda f(B),$$

for all $\lambda \in [0,1]$ and all J-selfadjoint operators A and B on any Krein space $(\mathscr{H}, J)$ with spectra contained in $\mathcal{I}$. As noticed by Ando [4], this definition is vain. In fact if $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ and

$$J = \begin{pmatrix} I_{\mathscr{H}_+} & 0 \\ 0 & -I_{\mathscr{H}_-} \end{pmatrix}$$

with respect to this decomposition, $\alpha, \beta \in \mathcal{I}$ and consider $A = \alpha I_{\mathscr{H}}$ and $B = \beta I_{\mathscr{H}}$, then A, B are J-selfadjoint (selfadjoint) and $\sigma(A) = \{\alpha\} \subseteq \mathcal{I}$

and $\sigma(B) = \{\beta\} \subseteq \mathcal{I}$. Hence

$$f\Big((1-\lambda)A + \lambda B\Big) = f\Big((1-\lambda)\alpha + \lambda\beta\Big)I_{\mathscr{H}}.$$

and

$$(1-\lambda)f(A) + \lambda f(B) = \Big((1-\lambda)f(\alpha) + \lambda f(\beta)\Big)I_{\mathscr{H}},$$

for all $\lambda \in [0,1]$. Therefore

$$f\Big((1-\lambda)A + \lambda B\Big) \leq^J (1-\lambda)f(A) + \lambda f(B)$$

whence

$$\Big((1-\lambda)f(\alpha) + \lambda f(\beta)\Big) - f\Big((1-\lambda)\alpha + \lambda\beta\Big)\Big) \geq 0$$

and

$$\Big((1-\lambda)f(\alpha) + \lambda f(\beta)\Big) - f\Big((1-\lambda)\alpha + \lambda\beta\Big)\Big) \leq 0.$$

Therefore

$$(1-\lambda)f(\alpha) + \lambda f(\beta) = f\Big((1-\lambda)\alpha + \lambda\beta\Big),$$

for all $\alpha, \beta \in \mathcal{I}$ an all $\lambda \in [0,1]$. Thus f is linear on $\mathcal{I}$, that is, $f(t) = at + b$ for some $a, b \in \mathbb{R}$.

To avoid such trivialities, we restrict ourselves to the J-positive operators instead of J-selfadjoint operators. It is well known that the spectrum of a J-positive operator on a Krein space $(\mathscr{H}, J)$ is real (see [3, Theorem 2.1]). From this point of view, the J-positive operators seem to behave like the selfadjoint operators on Hilbert spaces. The following definition seems to be satisfactory.

Definition 2.2. Suppose that $f : \mathcal{U} \to \mathbb{C}$ is an analytic function such that $f(x)$ is real for all $x \in \mathcal{U} \cap \mathbb{R}$. Then f is said to be Krein-operator convex if

(2.3) $$f\Big((1-\lambda)A + \lambda B\Big) \leq^J (1-\lambda)f(A) + \lambda f(B),$$

for all $\lambda \in [0,1]$ and all J-positive operators A and B on any Krein space $(\mathscr{H}, J)$, such that spectra of A, B and $(1-\lambda)A + \lambda B$ are contained in $\mathcal{U}$.

Also, the condition (2.3) can be obviously replaced by

$$f\left(\frac{A+B}{2}\right) \leq^J \frac{f(A) + f(B)}{2}.$$

The following example shows that operator convex functions are not necessarily Krein-operator convex.

Example 2.3. Consider the 2-dimensional Minkowski space $(\mathbb{C}^2, J_0)$ with $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$. Since the J_0-selfadjoitness of A is equivalent to the usual selfadjointness of $J_0 A$, we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & a_{22} \end{pmatrix}$$

in which a_{11} and a_{22} are real. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}$. It is easy to see that A and B are J_0-positive and they have real eigenvalues. Also we have

$$J_0 \left(\frac{A^2 + B^2}{2} \right) - J_0 \left(\frac{A + B}{2} \right)^2 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \ngeq 0.$$

It shows that

$$\left(\frac{A + B}{2} \right)^2 \nleq^{J_0} \frac{A^2 + B^2}{2}.$$

Therefore $f(t) = t^2$ is not Krein-operator convex, while it is well known that, this function is operator convex on $\mathbb{R}$ (see [19, p. 8]).

The following lemma characterize the invertible J-positive operators on a Krein space $(\mathscr{H}, J)$. For more information of J-positivite operators see [3, Chapter 2].

Proposition 2.4. *Let $(\mathscr{H}, J)$ be a Krein space and $A \in \mathbb{B}(\mathscr{H})$. Then A is J-positive and invertible if and only if it is of the form $A = J\tilde{A}$ for some $\tilde{A} > 0$.*

Proof. Let $A = J\tilde{A}$ with $\tilde{A} > 0$. Then A is invertible and $JA = J(J\tilde{A}) = \tilde{A} \geq 0$. It follows that A is J-positive and invertible.

Conversely, if A is J-positive and invertible, then $A = J\tilde{A}$ with $\tilde{A} := JA$ which is positive and invertible. $\square$

Example 2.5 ([4]). Define the function $f(z) = \frac{1}{z}$ $(z \neq 0)$ and $f(0) = 0$. Then f is Krein-operator convex on $\mathcal{U} = \mathbb{C} \setminus \{0\}$. To see this, assume that A and B are invertible J-positive operators such that the spectra of A, B and $(1 - \lambda)A + \lambda B$ are contained in $\mathcal{U} \cap \mathbb{R} = \mathbb{R} \setminus \{0\}$. Using the Dunford–Riesz representation (2.1), $\left((1 - \lambda)A + \lambda B \right)^{-1}$ exists. By Proposition 2.4, $A = J\tilde{A}$ and $B = J\tilde{B}$ for some $\tilde{A} > 0$ and $\tilde{B} > 0$. The restriction of f to $(0, \infty)$ is operator convex [19, p. 8]. Therefore

$$\left((1 - \lambda)A + \lambda B \right)^{-1} J = \left((1 - \lambda)\tilde{A} + \lambda \tilde{B} \right)^{-1}$$
$$\leq (1 - \lambda)\tilde{A}^{-1} + \lambda \tilde{B}^{-1}$$
$$= \left((1 - \lambda)A^{-1} + \lambda B^{-1} \right) J.$$

Hence

$$\left((1 - \lambda)A + \lambda B \right)^{-1} \leq^J (1 - \lambda)A^{-1} + \lambda B^{-1}.$$

Generally, suppose that $\mathcal{U}$ contains an interval $\mathcal{I} \subseteq [0, \infty)$ and f is a Krein-operator convex function on $\mathcal{U}$. Let $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$,

for which A and B are positive operators on a Hilbert space $\mathscr{H}$. Then

$$f\Big((1-\lambda)\tilde{A} + \lambda\tilde{B}\Big) \leq^{J} (1-\lambda)f(\tilde{A}) + \lambda f(\tilde{B})$$

with $J = \begin{pmatrix} I_{\mathscr{H}} & 0 \\ 0 & -I_{\mathscr{H}} \end{pmatrix}$ for all $\lambda \in [0,1]$. It follows that

$$f\big((1-\lambda)A + \lambda B\big) \leq (1-\lambda)f(A) + \lambda f(B)$$

for all positive operators A and B with spectra in $\mathcal{I}$ and all $\lambda \in [0,1]$. Hence the restriction of f to $\mathcal{I}$ is operator convex. Thus under mild condition, when the Krein space happens to be a Hilbert space, the definition of a Krein-operator convex function then coincids with the traditional definition, when we restricted ourself to the class of positive operators.

Before we present our main result, we give a lemma needed later.

Lemma 2.6. *Let $(\mathscr{H}, J)$ be a Krein space and A be a J-selfadjoint operator with $\sigma(A) \subseteq \mathcal{U}$. Then*

$$f(U^{\sharp}AU) = U^{\sharp}f(A)U$$

for any analytic function $f : \mathcal{U} \to \mathbb{C}$ and any J-unitary U.

Proof. The operator $U^{\sharp}AU$ is J-selfadjoint, since

$$(U^{\sharp}AU)^{\sharp} = J(U^{\sharp}AU)^{*}J = J(U^{*}A^{*}JUJ)J = JU^{*}A^{*}JU = JU^{*}JAU$$
$$= U^{\sharp}AU,$$

where we use the J-selfadjointness of A at the fourth equality. By the elementary operator theory, if A is some invertible operator, then

$$\sigma(U^{\sharp}AU) \cup \{0\} = \sigma(AUU^{\sharp}) \cup \{0\} = \sigma(A) \cup \{0\},$$

and since $U^{\sharp}AU$ is invertible if A is, it follows that these operators have the same spectra in this case. Finally

$$f(U^{\sharp}AU) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda I_{\mathscr{H}} - U^{\sharp}AU)^{-1}d\lambda$$
$$= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)U^{\sharp}(\lambda I_{\mathscr{H}} - A)^{-1}Ud\lambda$$
$$= U^{\sharp}f(A)U. \qquad \square$$

In the following theorem we present a Jensen type inequality for Krein-operator convex functions in the setting of Krein spaces.

Theorem 2.7. *Let f be a Krein-operator convex function on $\mathcal{U}$ and $f(0) = 0$. Then*

(2.4) $$f(C^{\sharp}AC) \leq^{J} C^{\sharp}f(A)C$$

for all J-positive operators A and all invertible J-contractions C on a Krein space $(\mathscr{H}, J)$ such that the spectra of A, $C^{\sharp}AC$ and $D^{\sharp}AD$ are contained in $\mathcal{U}$, where D is a defect operator for $C^{\sharp}$.

Proof. Let $f : \mathcal{U} \to \mathbb{C}$ be a Krein-operator convex function and let A be a J-positive operator with $\sigma(A) \subseteq \mathcal{U}$. Also assume that C is an invertible J-contraction. By Theorem 1.2, C is a J-bicontraction. Therefore Theorem 1.3 implies that C has a unique Julia operator

$$U = \begin{pmatrix} C & D \\ E^\sharp & -L^* \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \tilde{\mathcal{H}}_1, \mathcal{H} \oplus \tilde{\mathcal{H}}_2),$$

where $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ are Hilbert spaces such that the operators $D \in \mathbb{B}(\tilde{\mathcal{H}}_1, \mathcal{H})$ and $E \in \mathbb{B}(\tilde{\mathcal{H}}_2, \mathcal{H})$ have zero kernels and $-L^*$ is a Hilbert space contraction. It is well known that $\mathcal{H} \oplus \tilde{\mathcal{H}}_i$ are Krein spaces with fundamental symmetries $\tilde{\mathbf{J}}_{\mathbf{i}} = J \oplus I_{\tilde{\mathcal{H}}_i} = \begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathcal{H}}_i} \end{pmatrix}$ for $i = 1, 2$. Then U is a $(\tilde{\mathbf{J}}_{\mathbf{1}}, \tilde{\mathbf{J}}_{\mathbf{2}})$-unitary i.e., $U^\sharp U = I_{\mathcal{H} \oplus \tilde{\mathcal{H}}_1}$ and $UU^\sharp = I_{\mathcal{H} \oplus \tilde{\mathcal{H}}_2}$. Let

$$V = \begin{pmatrix} C & -D \\ E^\sharp & L^* \end{pmatrix}.$$

Note that V is simply U multiplied by fundamental symmetry $I_{\mathcal{H}} \oplus -I_{\tilde{\mathcal{H}}_1}$. So V is a $(\tilde{\mathbf{J}}_{\mathbf{1}}, \tilde{\mathbf{J}}_{\mathbf{2}})$-unitary. Let

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H} \oplus \tilde{\mathcal{H}}_2 \to \mathcal{H} \oplus \tilde{\mathcal{H}}_2.$$

Since

$$X^\sharp = \tilde{\mathbf{J}}_{\mathbf{2}} X^* \tilde{\mathbf{J}}_{\mathbf{2}} = \begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathcal{H}}_2} \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathcal{H}}_2} \end{pmatrix} = \begin{pmatrix} A^\sharp & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = X,$$

we conclude that X is $\tilde{\mathbf{J}}_{\mathbf{2}}$-selfadjoint. Moreover,

$$\tilde{\mathbf{J}}_{\mathbf{2}} X = \begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathcal{H}}_2} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} JA & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

Hence X is $\tilde{\mathbf{J}}_{\mathbf{2}}$-positive. It is clear that $U^\sharp X U$ and $V^\sharp X V$ are $\tilde{\mathbf{J}}_{\mathbf{1}}$-positive operators. If $\lambda \notin \sigma(A) \cup \{0\}$, then we have

$$(\lambda I_{\mathcal{H} \oplus \tilde{\mathcal{H}}_2} - X)^{-1} = \begin{pmatrix} (\lambda I_{\mathcal{H}} - A)^{-1} & 0 \\ 0 & \lambda^{-1} I_{\tilde{\mathcal{H}}_2} \end{pmatrix},$$

so that $\lambda \notin \sigma(X)$. Hence $\sigma(X) \subseteq \mathcal{U}$. Since $\sigma(U^\sharp X U) = \sigma(V^\sharp X V) = \sigma(X)$, the spectra of $\tilde{\mathbf{J}}_{\mathbf{1}}$-positive operators $U^\sharp X U$ and $V^\sharp X V$ are contained in $\mathcal{U}$. Consequently, by the Krein-operator convexity of f and Lemma 2.6, we infer that

$$\begin{pmatrix} f(C^\sharp A C) & 0 \\ 0 & f(D^\sharp A D) \end{pmatrix} = f \begin{pmatrix} C^\sharp A C & 0 \\ 0 & D^\sharp A D \end{pmatrix}$$

$$= f \left(\frac{U^\sharp X U + V^\sharp X V}{2} \right)$$

$$\leq^{\mathbf{\tilde{J}_1}} \frac{f(U^\sharp X U) + f(V^\sharp X V)}{2}$$

$$= \frac{U^\sharp f(X) U + V^\sharp f(X) V}{2}$$

$$= \frac{1}{2} U^\sharp \begin{pmatrix} f(A) & 0 \\ 0 & f(0) \end{pmatrix} U + \frac{1}{2} V^\sharp \begin{pmatrix} f(A) & 0 \\ 0 & f(0) \end{pmatrix} V$$

$$= \frac{1}{2} U^\sharp \begin{pmatrix} f(A) & 0 \\ 0 & 0 \end{pmatrix} U + \frac{1}{2} V^\sharp \begin{pmatrix} f(A) & 0 \\ 0 & 0 \end{pmatrix} V$$

$$= \begin{pmatrix} C^\sharp f(A) C & 0 \\ 0 & D^\sharp f(A) D \end{pmatrix}.$$

Hence

$$\begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathscr{H}}_1} \end{pmatrix} \begin{pmatrix} f(C^\sharp A C) & 0 \\ 0 & f(D^\sharp A D) \end{pmatrix} \leq \begin{pmatrix} J & 0 \\ 0 & I_{\tilde{\mathscr{H}}_1} \end{pmatrix} \begin{pmatrix} C^\sharp f(A) C & 0 \\ 0 & D^\sharp f(A) D \end{pmatrix}.$$

It follows that $Jf(C^\sharp A C) \leq JC^\sharp f(A) C$. Therefore $f(C^\sharp A C) \leq^J C^\sharp f(A) C$.
$\square$

Remark 2.8. In the classical case the usual operator convexity is equivalent to

$$f(C^* A C) \leq C^* f(A) C,$$

where A is selfadjoint and C is an isometry; see [19, Theorem 1.9] or [8, Section V]. Validity of an analogous relation as $f(C^\sharp A C) \leq^J C^\sharp f(A) C$ for all J-selfadjoint A and all invertible J-isometries C (i.e., $C^\sharp C = I$) on a Krein space $(\mathscr{H}, J)$ is vain, since if $C^\sharp C = I$, then $CC^\sharp = I$ and so $C^\sharp = C^{-1}$ and we enter into a trivial situation.

In the following example we show that in contrast with usual operator convex functions, the condition (2.4) in Theorem 2.7 is not equivalent to the Krein-operator convexity of f.

Example 2.9. Let A be a J-positive operator and C be an invertible J-contraction on a Krein space $(\mathscr{H}, J)$. Since A is J-selfadjoint and

$$CC^\sharp \leq^J I_{\mathscr{H}},$$

we have

$$(C^\sharp A C)^2 = (C^\sharp A^\sharp) CC^\sharp (AC) = (AC)^\sharp CC^\sharp AC \leq^J (AC)^\sharp I_{\mathscr{H}} AC = C^\sharp A A C.$$

It follows that $(C^\sharp A C)^2 \leq^J C^\sharp A^2 C$.

Therefore the function $f(t) = t^2$ satisfies (2.4). By Example 2.3, this function, however, is not Krein-operator convex.

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