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Geometry of Farey–Ford polygons

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Abstract. The Farey sequence is a natural exhaustion of the set of rational numbers between 0 and 1 by finite lists. Ford Circles are a natural family of mutually tangent circles associated to Farey fractions: they are an important object of study in the geometry of numbers and hyperbolic geometry. We define two sequences of polygons associated to these objects, the Euclidean and hyperbolic *Farey–Ford polygons*. We study the asymptotic behavior of these polygons by exploring various geometric properties such as (but not limited to) areas, length and slopes of sides, and angles between sides.

Contents

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1. Introduction and main results

Ford circles were introduced by Lester R. Ford [6] in 1938 and since then have appeared in several different areas of mathematics. Their connection with *Farey fractions* is well known. We define the notion of *Farey–Ford polygons*. We study some statistics associated to both the Euclidean and hyperbolic geometry of these objects, in particular distances between vertices, angles, slopes of edges and areas. The analysis of these quantities is an interesting study in itself as distances, angles and areas are among the most common notions traditionally studied in geometry. We use dynamical methods to compute limiting distributions, and analytic number theory to compute asymptotic behavior of moments for the *hyperbolic* and *Euclidean* distance between the vertices.

Below, we define Farey–Ford polygons (§1.1), describe the statistics that we study (§1.2) and state our main theorems (§1.3 and §1.4). In §3 we prove our results on moments, in §4 we prove our distribution results, and in §5 we compute the tail behavior of the statistics we study.

1.1. Farey–Ford polygons. Recall the definition of *Ford circles*: The Ford circle

$$\mathcal{C}_{p/q} := \left\{ z \in \mathbb{C} : \left| z - \left(\frac{p}{q} + i\frac{1}{2q^2} \right) \right| = \frac{1}{2q^2} \right\}$$

is the circle in the upper-half plane tangent to the point $(p/q, 0)$ with diameter $1/q^2$ (here and in what follows we assume that p and q are relatively prime).

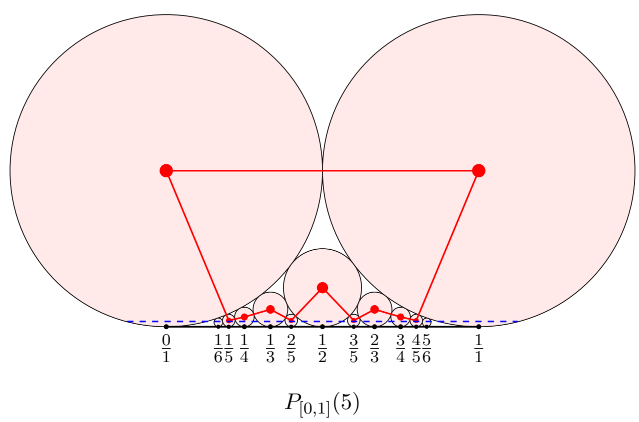


FIGURE 1. Euclidean Farey–Ford Polygons

Given $\delta > 0$ and $I = [\alpha, \beta] \subseteq [0, 1]$, we consider the subset

$$\mathcal{F}_{I,\delta} := \left\{ \mathcal{C}_{p/q} : p/q \in I, \frac{1}{2q^2} \geq \delta \right\}$$

of Ford circles with centers above the horizontal line $y = \delta$ and with tangency points in I. Writing

$$Q = \left\lfloor (2\delta)^{-1/2} \right\rfloor,$$

we have that the set of tangency points to the x-axis is

$$\mathcal{F}(Q) \cap I,$$

where

$$\mathcal{F}(Q) := \{p/q : 0 \leq p \leq q \leq Q, (p,q) = 1\}$$

denotes the Farey fractions of order Q. We write

$$\mathcal{F}(Q) \cap I := \{p_1/q_1 < p_2/q_2 < \cdots < p_{N_I(Q)}/q_{N_I(Q)}\},$$

where $N_I(Q)$ is the cardinality of the above set (which grows quadratically in Q, so linearly in δ^{-1}). We denote the circle based at p_j/q_j by C_j for $1 \leq j \leq N_I(Q)$, and define the *Farey–Ford polygon* $P_I(Q)$ of order Q by:

- connecting $(0, 1/2)$ to $(1, 1/2)$ by a geodesic;
- forming the "bottom" of the polygon by connecting the points

  $$(p_j/q_j, 1/2{q_j}^2) \quad \text{and} \quad (p_{j+1}/q_{j+1}, 1/2{q_{j+1}}^2)$$

  for $1 \leq j \leq N_I(Q) - 1$ by a geodesic.

We call these polygons Euclidean (respectively hyperbolic) Farey–Ford polygons if the geodesics connecting the vertices are Euclidean (resp. hyperbolic). Figure 1 shows the Euclidean Farey–Ford polygon $P_{[0,1]}(5)$.

Remark. For $n \in \mathbb{N}$, consider the sequence of functions $f_n : [0,1] \to [0,1]$

$$f_n(x) := \begin{cases} \dfrac{1}{2q^2} & \text{if } x = \frac{p}{q},\ (p,q) = 1, 1 \leq q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $f_n(x)$ is nonzero only when x is a Farey fraction of order n, in which case $f_n(x)$ denotes the Euclidean distance from the real axis to the vertices of Farey–Ford polygons of order n. As we let n approach infinity, the sequence $f_n(x)$ converges and its limit is given by $\frac{1}{2}(f(x))^2$, where $f(x)$ is *Thomae's function*

$$f(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ (p,q) = 1, q > 0 \\ 0 & \text{otherwise,} \end{cases}$$

This function is continuous at every irrational point in $[0,1]$ and discontinuous at all rational points in $[0,1]$.

1.2. Geometric statistics. Given a Farey–Ford polygon $P_I(Q)$, for $1 \leq j \leq N_I(Q) - 1$, let

$$x_j \coloneqq \frac{q_j}{Q}, \quad y_j \coloneqq \frac{q_{j+1}}{Q}.$$

For $1 \leq j \leq N_I(Q) - 1$, we have $0 < x_j, y_j \leq 1$. And,

$$x_j + x_{j+1} = \frac{q_j + q_{j+1}}{Q} > 1,$$

since between any two Farey fractions $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$, of order Q, there is a Farey fraction given by $\frac{p_j+p_{j+1}}{q_j+q_{j+1}}$ which is not in $\mathcal{F}(Q) \cap I$ as $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$ are neighbours in $\mathcal{F}(Q) \cap I$.

Let

$$\Omega \coloneqq \{(x, y) \in (0, 1]^2 : x + y > 1\},$$

so we have $(x_j, y_j) \in \Omega$ for $1 \leq j \leq N_I(Q) - 1$. A *geometric statistic* F is a measurable function $F : \Omega \to \mathbb{R}$. Given a geometric statistic F, we define the *limiting distribution for Ford circles* $G_F(t)$ of F (if it exists) to be the limiting proportion of Ford circles for which F exceeds t, that is,

$$G_F(t) \coloneqq \lim_{\delta \to 0} \frac{\#\{1 \leq j \leq N_I(Q) : F(x_j, y_j) \geq t\}}{N_I(Q)}.$$

Similarly, we define the *moment* of F by

$$M_{F,I,\delta}(\mathcal{F}) \coloneqq \frac{1}{|I|} \sum_{j=1}^{N_I(Q)-1} |F(x_j, y_j)|. \tag{1.1}$$

We describe how various natural geometric properties of Farey–Ford polygons can be studied using this framework.

1.2.1. Euclidean distance. Given consecutive circles C_j and C_{j+1} in $\mathcal{F}_{I,\delta}$, we consider the *Euclidean* distance d_j between their centers O_j and O_{j+1}. Note that since consecutive circles are mutually tangent, the Euclidean geodesic connecting the centers of the circles passes through the point of tangency, and has length equal to the sum of the radii, that is,

$$d_j = \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}.$$

Therefore, we have

$$Q^2 d_j = \frac{1}{2x_j^2} + \frac{1}{2y_j^2}.$$

Thus, by letting $F(x_j, y_j) = \frac{1}{2{x_j}^2} + \frac{1}{2{y_j}^2}$, we have expressed the Euclidean distance (appropriately normalized) as a geometric statistic.

In [4], the latter three authors computed averages of first and higher moments of the Euclidean distance d_j, in essence, they computed explicit formulas for

$$\frac{1}{N_I(Q)}\int_X^{2X}\sum_{j=1}^{N_I(Q)-1} d_j^k \, d(2\delta)^{-1/2} \quad \text{for } k \geq 1,$$

where X was a large real number.

1.2.2. Slope. Let t_j denote the Euclidean slope of the Euclidean geodesic joining the centers O_j and O_{j+1} of two consecutive circles C_j and C_{j+1}. It is equal to

$$t_j = \frac{1}{2}\left(\frac{q_i}{q_{i+1}} - \frac{q_{i+1}}{q_i}\right).$$

We associate the geometric statistic $F(x_j, y_j) = \frac{1}{2}\left(\frac{x_j}{y_j} - \frac{y_j}{x_j}\right)$ to it.

1.2.3. Euclidean angles. Let θ_j be the angle between the Euclidean geodesic joining O_j with $(p_j/q_j, 0)$ and the Euclidean geodesic joining O_j with the point of tangency of the circles C_j and C_{j+1}. In order to compute θ_j, we complete the right angled triangle with vertices O_j, O_{j+1} and the point lying on the Euclidean geodesic joining O_j and $(p_j/q_j, 0)$. Then,

$$\tan\theta_j = \begin{cases} \frac{2q_jq_{j+1}}{q_{j+1}^2 - q_j^2} & \text{if } q_{j+1} > q_j, \\ \frac{2q_jq_{j+1}}{q_j^2 - q_{j+1}^2} & \text{if } q_{j+1} < q_j. \end{cases}$$

Thus, the geometric statistic associated to $\tan\theta_j$ is given by

$$F(x_j, y_j) = \begin{cases} \frac{2x_jy_j}{y_j^2 - x_j^2} & \text{if } y_j > x_j, \\ \frac{2x_jy_j}{x_j^2 - y_j^2} & \text{if } y_j < x_j. \end{cases} \tag{1.2}$$

1.2.4. Euclidean area. Let A_j denote the area of the trapezium whose sides are formed by the Euclidean geodesics joining the points $\left(\frac{p_j}{q_j}, 0\right)$ and O_j; $\left(\frac{p_{j+1}}{q_{j+1}}, 0\right)$ and O_{j+1}; O_j and O_{j+1}; $\left(\frac{p_j}{q_j}, 0\right)$ and $\left(\frac{p_{j+1}}{q_{j+1}}, 0\right)$. This gives $A_j = \frac{1}{4q_j^3q_{j+1}} + \frac{1}{4q_jq_{j+1}^3}$. The geometric statistic (normalized) in this case is

$$F(x_j, y_j) = \frac{1}{4x_j^3y_j} + \frac{1}{4x_jy_j^3}.$$

1.2.5. Hyperbolic distance. Given consecutive circles in $\mathcal{F}_{I,\delta}$ we let ρ_j denote the *hyperbolic* distance between their centers O_j and O_{j+1}, where we use the standard hyperbolic metric on the upper half plane $\mathbb{H}$ defined as

$$ds := \frac{\sqrt{dx^2 + dy^2}}{y}.$$

A direct computation shows that

$$\sinh \frac{\rho_j}{2} = \frac{q_j}{2q_{j+1}} + \frac{q_{j+1}}{2q_j} = \frac{x_j}{2y_j} + \frac{y_j}{2x_j} = \frac{1}{2}\left(\frac{x_j}{y_j} + \frac{y_j}{x_j}\right).$$

Letting $F(x_j, y_j) = \frac{1}{2}\left(\frac{x_j}{y_j} + \frac{y_j}{x_j}\right)$, we have expressed (sinh of half of) the hyperbolic distance as a geometric statistic.

1.2.6. Hyperbolic angles. Let α_j denote the angle between the hyperbolic geodesic parallel to the y-axis, passing through the center O_j, and the hyperbolic geodesic joining the centers O_j and O_{j+1}. This angle can be calculated by first computing the center of the hyperbolic geodesic joining O_j and O_{j+1}. Note that the center of this geodesic lies on the x-axis. Let us denote it by $(c_j, 0)$. Now, α_j is also the acute angle formed between the line joining $(c_j, 0)$ and O_j, and the x-axis. Thus $\tan \alpha_j$ is given by

$$\tan \alpha_j = \frac{4q_j^5}{q_{j+1}^3} - 4q_j q_{j+1} + \frac{16q_j^3}{q_{j+1}}.$$

Consequently the geometric statistic (normalized) associated to it is

$$F(x_j, y_j) = \frac{4x_j^5}{y_j^3} - 4x_j y_j + \frac{16x_j^3}{y_j}.$$

1.3. Moments. The following results concern the growth of the moments $M_{F,I,\delta}$ for the geometric statistics expressed in terms of Euclidean distances, hyperbolic distances and angles.

Theorem 1.1. *For any real number $\delta > 0$, and any interval $I = [\alpha, \beta] \subset [0, 1]$, with $\alpha, \beta \in \mathbb{Q}$ and $F(x, y) = \frac{1}{2x^2} + \frac{1}{2y^2}$,*

$$M_{F,I,\delta}(\mathcal{F}) = \frac{6}{\pi^2} Q^2 \log Q + D_I Q^2 + \mathrm{O}_I\left(Q \log Q\right),$$

where $M_{F,I,\delta}(\mathcal{F})$ is defined as in (1.1), Q is the integer part of $\frac{1}{\sqrt{2\delta}}$ and D_I is a constant depending only on the interval I. Here the implied constant in the Big O-term depends on the interval I.

Theorem 1.2. *For any real number $\delta > 0$, any interval $I = [\alpha, \beta] \subset [0, 1]$ and $F(x, y) = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right)$,*

$$M_{F,I,\delta}(\mathcal{F}) = \frac{9}{2\pi^2} Q^2 + \mathrm{O}_{I,\epsilon}\left(Q^{7/4+\epsilon}\right),$$

where $M_{F,I,\delta}(\mathcal{F})$ *is defined as in* (1.1)*,* C_I *is a constant depending only on the interval* I*,* ϵ *is any positive real number, and* Q *is the integer part of* $\frac{1}{\sqrt{2\delta}}$*. The implied constant in the Big O-term depends on the interval* I *and* ϵ*.*

Theorem 1.3. *For any real number* $\delta > 0$*, any interval* $I = [\alpha, \beta] \subset [0, 1]$ *and* $F(x, y)$ *as in* (1.2)*, we have,*

$$M_{F,I,\delta}(\mathcal{F}) = \frac{12}{\pi} Q^2 + \mathrm{O}_I \left(Q \log Q \right),$$

where $M_{F,I,\delta}(\mathcal{F})$ *is defined as in* (1.1)*, and* Q *is the integer part of* $\frac{1}{\sqrt{2\delta}}$*. The implied constant in the Big O-term depends on the interval* I*.*

1.4. Distributions. The equidistribution of a certain family of measures on Ω yields information on the distribution of individual geometric statistics and also on the *joint* distribution of any finite family of *shifted* geometric statistics. In particular, we can understand how all of the above statistics correlate with each other, and can even understand how the statistics observed at shifted indices correlate with each other. First, we record the result on equidistribution of measures.

Let

$$\rho_{Q,I} := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_j, y_j)}$$

be the probability measure supported on the set

$$\left\{ (x_j, y_j) = \left(\frac{q_j}{Q}, \frac{q_{j+1}}{Q} \right) : 1 \le j \le N \right\}.$$

We have [2, Theorem 1.3] (see also [10], [11]):

Theorem 1.4. *The measures* $\rho_{Q,I}$ *equidistribute as* $Q \to \infty$*. That is,*

$$\lim_{Q \to \infty} \rho_{Q,I} = m,$$

where $dm = 2dxdy$ *is the Lebesgue probability measure on* Ω *and the convergence is in the weak-$*$ topology.*

As consequences, we obtain results on individual and joint distributions of geometric statistics.

Corollary 1. *Let* $F : \Omega \to \mathbb{R}^+$ *be a geometric statistic and let* m *denote the Lebesgue probability measure on* Ω*. Then for any interval* $I \subset [0, 1]$*, the limiting distribution* $G_F(t)$ *exists, and is given by*

$$G_F(t) = m \left(F^{-1}(t, \infty) \right).$$

In order to obtain results on joint distributions, we will need the following notations. Let $T : \Omega \to \Omega$ be the *BCZ map*,

$$T(x, y) := \left(y, -x + \left\lfloor \frac{1 + x}{y} \right\rfloor y \right).$$

A crucial observation, due to Boca, Cobeli and one of the authors [3] is that

$$T(x_j, y_j) = (x_{j+1}, y_{j+1}).$$

Let $\mathbf{F} = \{F_1, \ldots, F_k\}$ denote a finite collection of geometric statistics, and let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ and $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$. Define $G_{\mathbf{F},\mathbf{n}}(\mathbf{t})$, the $(\mathbf{F}, \mathbf{n})$-*limiting distribution* for Ford circles by the limit (if it exists),

$$G_{\mathbf{F},\mathbf{n}}(\mathbf{t}) := \lim_{\delta \to 0} \frac{\#\{1 \leq j \leq N_I(Q) : F_i(x_{j+n_i}, y_{j+n_i}) \geq t_i, 1 \leq i \leq k\}}{N_I(Q)},$$

where the subscripts are viewed modulo $N_I(Q)$. This quantity reflects the proportion of circles with specified behavior of the statistics F_i at the indices $j + n_i$. Corollary 1 is in fact a special $(k = 1, \mathbf{n} = 0)$ case of the following.

Corollary 2. *For any finite collection of geometric statistics*

$$\mathbf{F} = \{F_1, \ldots, F_k\}, \quad \mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k, \quad \mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k,$$

the limiting distribution $G_{\mathbf{F},\mathbf{n}}(\mathbf{t})$ *exists, and is given by*

$$G_{\mathbf{F},\mathbf{n}}(\mathbf{t}) = m\left((\mathbf{F} \circ T)^{-\mathbf{n}}(t_1, \ldots, t_k)\right),$$

where

$$(\mathbf{F} \circ T)^{-\mathbf{n}}(t_1, \ldots, t_k) := \bigcap_{i=1}^{k} (F_i \circ T^{n_i})^{-1}(t_i, \infty).$$

1.4.1. Tails. For the geometric statistics listed in Section 1.2, one can also explicitly compute *tail* behavior of the limiting distributions, that is, the behavior of $G_F(t)$ as $t \to \infty$. We use the notation $f(x) \sim g(x)$ as $x \to \infty$ to mean that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,$$

and the notation $f(x) \asymp g(x)$ means that there exist positive constants c_1, c_2 and a constant x_0 such that for all $x > x_0$,

$$c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|.$$

We have the following result for the tail behavior of limiting distributions for various geometric statistics.

Theorem 1.5. *Consider the geometric statistics associated to the Euclidean distance, hyperbolic distance, Euclidean slope, Euclidean area and hyperbolic angle. The limiting distribution functions associated to these have the following tail behavior:*

(1) (Euclidean distance) *for* $F(x, y) = \frac{1}{2x^2} + \frac{1}{2y^2}, G_F(t) \sim \frac{1}{t}$.
(2) (Hyperbolic distance) *for* $F(x, y) = \frac{1}{2}\left(\frac{y}{x} + \frac{x}{y}\right)$, $G_F(t) \sim \frac{1}{2t^2}$.
(3) (Euclidean slope) *for* $F(x, y) = \frac{1}{2}\left(\frac{x}{y} - \frac{y}{x}\right)$, $G_F(t) \asymp \frac{1}{t^2}$.
(4) (Euclidean area) *for* $F(x, y) = \frac{1}{4}\left(\frac{1}{x^3 y} + \frac{1}{x y^3}\right)$, $G_F(t) \asymp \frac{1}{t^{2/3}}$.
(5) (Hyperbolic angle) *for* $F(x, y) = \frac{4x^5}{y^3} - 4xy + \frac{16x^3}{y}$, $G_F(t) \asymp \frac{1}{t^{2/3}}$.

2. Moments for Euclidean distance

In this section, we prove Theorem 1.1. Recall that the Euclidean distance d_j between the centers of two consecutive Ford circles C_j and C_{j+1} is given by

$$d_j = \frac{1}{2{q_j}^2} + \frac{1}{2{q_{j+1}}^2}.$$

As mentioned in section 1.2.1, the geometric statistic F expressed in terms of the Euclidean distance d_j is

$$F(x_j, y_j) = Q^2 d_j.$$

Thus, the moment of F is given by

$$M_{F,I,\delta}(\mathcal{F}) = \frac{Q^2}{|I|} \sum_{j=1}^{N_I(Q)-1} \left(\frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2} \right).$$

Proof of Theorem 1.1.

(2.1)

$$\begin{aligned}
&\frac{|I|}{Q^2} M_{E,I,\delta}(\mathcal{F}) \\
&= \sum_{j=2}^{N_I(Q)} \frac{1}{q_j^2} + \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} \\
&= \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} + \sum_{1\le q\le Q} \frac{1}{q^2} \sum_{\substack{\alpha q < a \le \beta q \\ (a,q)=1}} 1 \\
&= \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} + \sum_{q\le Q} \frac{1}{q^2} \sum_{\alpha q < a \le \beta q} 1 \sum_{d|(a,q)} \mu(d) \\
&= \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} + \sum_{q\le Q} \frac{1}{q^2} \sum_{d|q} \mu(d) \sum_{\frac{\alpha q}{d} < l \le \frac{\beta q}{d}} 1 \\
&= \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} \\
&\quad + \sum_{q\le Q} \frac{1}{q^2} \left(\sum_{d|q} \mu(d) \frac{(\beta-\alpha)q}{d} - \sum_{d|q} \mu(d) \left(\left\{ \frac{\beta q}{d} \right\} - \left\{ \frac{\alpha q}{d} \right\} \right) \right) \\
&= |I| \sum_{q\le Q} \frac{\phi(q)}{q^2} - \sum_{d\le Q} \frac{\mu(d)}{d^2} \sum_{m\le \frac{Q}{d}} \left(\frac{\{\beta m\} - \{\alpha m\}}{m^2} \right) + \left(\frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} \right).
\end{aligned}$$

Note that since $\alpha, \beta \in \mathbb{Q}$, for large Q one can assume that α and β are Farey fractions of order Q. Therefore, the quantity $\dfrac{1}{2q_1^2} - \dfrac{1}{2q_{N_I(Q)}^2}$ is a constant.

Now,

$$\sum_{m\le \frac{Q}{d}}\left(\frac{\{\beta m\}-\{\alpha m\}}{m^2}\right)=\sum_{m=1}^{\infty}\left(\frac{\{\beta m\}-\{\alpha m\}}{m^2}\right)-\sum_{m>\frac{Q}{d}}\left(\frac{\{\beta m\}-\{\alpha m\}}{m^2}\right)$$
$$=C_I+\mathrm{O}_I\left(\frac{d}{Q}\right).$$

This gives

$$\sum_{d\le Q}\frac{\mu(d)}{d^2}\left(\frac{\{\beta m\}-\{\alpha m\}}{m^2}\right)=\frac{C_I}{\zeta(2)}+\mathrm{O}_I\left(\frac{\log Q}{Q}\right).$$

Combining this in (2.1) and the fact that

$$\sum_{q\le Q}\frac{\phi(q)}{q^2}=\frac{6}{\pi^2}\log Q+A+\mathrm{O}\left(\frac{\log Q}{Q}\right), \tag{2.2}$$

we obtain,

$$M_{E,I,\delta}(\mathcal{F})=\frac{6\log Q}{\pi^2}+D_I+\mathrm{O}_I\left(\frac{\log Q}{Q}\right).$$

It only remains to prove (2.2).

$$\sum_{n\le x}\frac{\phi(n)}{n^2}=\sum_{n\le x}\frac{1}{n^2}\sum_{d|n}\mu(d)\frac{n}{d}=\sum_{\substack{q,d\\ qd\le x}}\frac{\mu(d)}{qd^2}$$
$$=\sum_{d\le x}\frac{\mu(d)}{d^2}\sum_{q\le x/d}\frac{1}{q}=\sum_{d\le x}\frac{\mu(d)}{d^2}\left(\log x-\log d+A+\mathrm{O}\left(\frac{1}{x}\right)\right)$$
$$=\log x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>x}\frac{\mu(d)}{d^2}\right)-\left(\sum_{d=1}^{\infty}\frac{\mu(d)\log d}{d^2}-\sum_{d>x}\frac{\mu(d)\log d}{d^2}\right)$$
$$+A\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d>x}\frac{\mu(d)}{d^2}\right)+\mathrm{O}\left(\frac{1}{x}\right)$$
$$=\frac{\log x}{\zeta(2)}+B+\mathrm{O}\left(\frac{\log x}{x}\right).$$

This completes the proof of Theorem 1.1. □

3. Moments for hyperbolic distance and angles

3.1. Hyperbolic distance. In this section, we prove Theorem 1.2. Recall that ρ_j denotes the hyperbolic distance between consecutive centers of Ford circles O_j and O_{j+1}, and

$$\sinh\frac{\rho_j}{2}=\frac{q_j}{2q_{j+1}}+\frac{q_{j+1}}{2q_j}. \tag{3.1}$$

In order to compute the total sum of the distances $\sinh \frac{\rho_j}{2}$,

$$M_{H,I,\delta}(\mathcal{F}) = \frac{1}{|I|}\sum_j \sinh\frac{\rho_j}{2} = \frac{1}{|I|}\sum_{\substack{q,q' \text{ neighbours}\\ \text{in } \mathcal{F}_I(Q)}} \left(\frac{q}{2q'}+\frac{q'}{2q}\right), \tag{3.2}$$

we use the following key analytic lemma.

Lemma 3.1. [3, Lemma 2.3] *Suppose that $0 < a < b$ are two real numbers, and q is a positive integer. Let f be a piecewise C^1 function on $[a, b]$. Then*

$$\sum_{a<k\le b} \frac{\phi(k)}{k} f(k) = \frac{6}{\pi^2}\int_a^b f(x)\ dx + \mathrm{O}_I\left(\log b\left(||f||_\infty + \int_a^b |f'(x)|\ dx\right)\right).$$

We also use the Abel summation formula,

$$\sum_{x<n\le y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)\ dt, \tag{3.3}$$

where $a : \mathbb{N} \to \mathbb{C}$ is an arithmetic function, $0 < x < y$ are real numbers, $f : [x, y] \to \mathbb{C}$ is a function with continuous derivative on $[x, y]$, and

$$A(t) := \sum_{1\le n\le t} a(n).$$

Proof of Theorem 1.2. Let $\bar{q'}$ denote the unique multiplicative inverse of q' modulo q in the interval $[1, q]$. Define

$$a_q(q') = \begin{cases} 1 & \text{if } (q, q') = 1 \text{ and } \ \bar{q'} \in [q\alpha, q\beta], \\ 0 & \text{otherwise.} \end{cases}$$

Using Weil type estimates ([5], [9], [14]) for Kloosterman sums, the partial sum of $a_q(q')$ was estimated in [3]. More precisely, by [3, Lemma 1.7], for any $\epsilon > 0$,

$$A_q(t) := \sum_{1\le q'\le t} a_q(q') = \frac{\phi(q)}{q}(t-1)|I| + \mathrm{O}_{I,\epsilon}\left(q^{1/2+\epsilon}\right). \tag{3.4}$$

In order to estimate the sum in (3.2), we write it as

$$2|I|M_{H,I,\delta}(\mathcal{F}) = \sum_{\substack{q,q' \text{ neighbours}\\ \text{in } \mathcal{F}_I(Q)\\ q\ge q'}} \left(\frac{q}{q'}+\frac{q'}{q}\right) + \sum_{\substack{q,q' \text{ neighbours}\\ \text{in } \mathcal{F}_I(Q)\\ q<q'}} \left(\frac{q}{q'}+\frac{q'}{q}\right). \tag{3.5}$$

Note that it suffices to estimate the first sum above because of the symmetry in q and q'. And for the first sum, one has:

$$\sum_{\substack{q,q' \text{ neighbours} \\ \text{in } \mathcal{F}_I(Q) \\ q \geq q'}} \left(\frac{q}{q'} + \frac{q'}{q}\right) = \sum_{\substack{Q/2<q\leq Q-L \\ Q-q<q'\leq q}} a_q(q') \left(\frac{q}{q'} + \frac{q'}{q}\right) + \sum_{\substack{Q-L<q\leq Q \\ L<q'\leq q}} a_q(q') \left(\frac{q}{q'} + \frac{q'}{q}\right) + \sum_{\substack{Q-L<q\leq Q \\ Q-q<q'\leq L}} a_q(q') \left(\frac{q}{q'} + \frac{q'}{q}\right) =: S_1 + S_2 + S_3. \tag{3.6}$$

We begin with estimating the sum S_1.

$$S_1 = \sum_{Q/2<q\leq Q-L} \sum_{Q-q<q'\leq q} a_q(q') \left(\frac{q}{q'} + \frac{q'}{q}\right) =: \sum_{Q/2<q\leq Q-L} \Sigma_1.$$

Employing (3.4), (3.3) and the fact that $q > Q-q$, for the inner sum Σ_1 in S_1, we have

$$\begin{aligned}
\Sigma_1 &= A_q(q)\left(\frac{q}{q}+\frac{q}{q}\right) - A_q(Q-q)\left(\frac{Q-q}{q}+\frac{q}{Q-q}\right) \\
&\quad - \int_{Q-q}^{q} A_q(t)\left(\frac{1}{q}-\frac{q}{t^2}\right)\, dt \\
&= 2\left(\frac{\phi(q)}{q}(q-1)|I| + \mathrm{O}_{I,\epsilon}\left(q^{1/2+\epsilon}\right)\right) \\
&\quad - \left(\frac{\phi(q)}{q}(Q-q-1)|I| + \mathrm{O}_{I,\epsilon}\left(q^{1/2+\epsilon}\right)\right)\left(\frac{Q-q}{q}+\frac{q}{Q-q}\right) \\
&\quad - \int_{Q-q}^{q} \left(\frac{\phi(q)}{q}(t-1)|I| + \mathrm{O}_{I,\epsilon}\left(q^{1/2+\epsilon}\right)\right)\left(\frac{1}{q}-\frac{q}{t^2}\right)\, dt \\
&= |I|\frac{\phi(q)}{q}\Bigg(2(q-1) - (Q-q-1)\left(\frac{Q-q}{q}+\frac{q}{Q-q}\right) \\
&\quad - \int_{Q-q}^{q}(t-1)\left(\frac{1}{q}-\frac{q}{t^2}\right)\, dt\Bigg) + \mathrm{O}_{I,\epsilon}\left(q^{1/2+\epsilon}\right)\left(\frac{q}{Q-q}\right) \\
&= |I|\frac{\phi(q)}{q}\Bigg(2(q-1) - (Q-q-1)\left(\frac{Q-q}{q}+\frac{q}{Q-q}\right) + \frac{Q^2}{2q} - \frac{Q}{q} \\
&\quad - \frac{q}{Q-q} - q\log(Q-q) + q\log q - Q + 3\Bigg) + \mathrm{O}_{I,\epsilon}\left(\frac{q^{3/2+\epsilon}}{Q-q}\right) \\
&= -\frac{Q^2}{2q} - q\log(Q-q) + q\log q + Q + \mathrm{O}_{I,\epsilon}\left(\frac{q^{3/2+\epsilon}}{Q-q}\right).
\end{aligned} \tag{3.7}$$

Therefore,

$$S_1 = |I| \sum_{\frac{Q}{2}<q\le Q-L} \frac{\phi(q)}{q} g(q) + \mathrm{O}_{I,\epsilon}\left(\frac{Q^{5/2+\epsilon}}{L}\right),$$

where $g(q) := -\frac{Q^2}{2q} - q\log(Q-q) + q\log q + Q$. Also, from (3.7) and Lemma 3.1, we obtain

$$S_1 = \frac{6|I|}{\pi^2}\int_{Q/2}^{Q-L} g(x)\ dx + \mathrm{O}_{I,\epsilon}\left(\log Q(||g||_\infty + \int_{Q/2}^{Q-L} |g'(x)|\ dx)\right) + \mathrm{O}_{I,\epsilon}\left(\frac{Q^{5/2+\epsilon}}{L}\right). \tag{3.8}$$

And,

$$\int_{Q/2}^{Q-L} g(q) = \frac{Q^2-(Q-L)^2}{2}\log\left(\frac{L}{Q-L}\right) + \frac{3(Q-L)Q}{2} - \frac{3Q^2}{4}; \tag{3.9}$$

$$||g||_\infty = \mathrm{O}_I\left(Q\log Q\right) = \int_{Q/2}^{Q-L} |g'(x)|\ dx. \tag{3.10}$$

Therefore, from (3.8), (3.9), (3.10) and Lemma 3.1,

$$S_1 = \frac{6|I|}{\pi^2}\left(\frac{Q^2-(Q-L)^2}{2}\log\left(\frac{L}{Q-L}\right) + \frac{3(Q-L)Q}{2} - \frac{3Q^2}{4}\right) + \mathrm{O}_{I,\epsilon}\left(\frac{Q^{5/2+\epsilon}}{L}\right) + \mathrm{O}_I\left(Q\log Q\right). \tag{3.11}$$

Next, we estimate the sum S_2 in (3.6). Recall that since $q' \le q$, we have

$$S_2 = \sum_{Q-L<q\le Q}\sum_{L<q'\le q} a_q(q')\left(\frac{q}{q'}+\frac{q'}{q}\right) = \mathrm{O}_I\left(\sum_{Q-L<q\le Q}\sum_{L<q'\le q}\frac{q}{q'}\right) = \mathrm{O}_I\left(\sum_{Q-L<q\le Q} q \sum_{L<q'\le q}\frac{1}{q'}\right) = \mathrm{O}_I\left(QL\log Q\right). \tag{3.12}$$

Lastly,

$$S_3 = \sum_{Q-L<q\le Q}\sum_{Q-q<q'\le L} a_q(q')\left(\frac{q}{q'}+\frac{q'}{q}\right) = \mathrm{O}_I\left(\sum_{1\le q'\le L}\sum_{Q-q'<q\le Q}\frac{q}{q'}\right) = \mathrm{O}_I\left(QL\right), \tag{3.13}$$

Setting $L = Q^{3/4}$, and from (3.11), (3.12), (3.13), and (3.6), we obtain the first moment for hyperbolic distances as

$$M_{H,I,\delta}(\mathcal{F}) = \frac{9}{2\pi^2}Q^2 + \mathrm{O}_{I,\epsilon}\left(Q^{7/4+\epsilon}\right). \qquad \square$$

3.2. Angles. First we consider the case when q_{j+1} is smaller than q_j. Let θ_j denote the angle for the circle C_j. Then, the sum of the angles for both circles is given by $\theta_j + \frac{\pi}{2} - \theta_{j+1} + \frac{\pi}{2} = \pi$. Similarly, in the case $q_j < q_{j+1}$, the sum of the angles is π. This gives

$$M_{\theta,I,\delta}(\mathcal{F}) = \frac{1}{|I|}\sum_{j=1}^{N_I(Q)}(\theta_j + \theta_{j+1}) = \frac{2\pi}{|I|}N_I(Q) = \frac{12}{\pi}Q^2 + \mathrm{O}_I(\log Q).$$

4. Dynamical methods

In this section, we prove Corollaries 1 and 2. As we stated before, they are both consequences of the equidistribution of a certain family of measures on the region

$$\Omega = \{(x,y) \in (0,1]^2 : x + y > 1\}.$$

To see this, we write (with notation as in §1.4)

$$\frac{\#\{1 \le j \le N_I(Q) : F(x_j, y_j) \ge t\}}{N_I(Q)} = \rho_{Q,I}\left(F^{-1}(t,\infty)\right)$$

$$\frac{\#\{1 \le j \le N_I(Q) : F_i(x_{j+n_i}, y_{j+n_i}) \ge t_i, 1 \le i \le k\}}{N_I(Q)} = \rho_{Q,I}\left((\mathbf{F} \circ T)^{-\mathbf{n}}(t_1, \dots, t_k)\right)$$

Now the results follow from applying Theorem 1.4 to the above expressions. $\square$

Remark (Shrinking Intervals)**.** Versions of Corollaries 1 and 2 for *shrinking intervals* $I_Q = [\alpha_Q, \beta_Q]$ where the difference $\beta_Q - \alpha_Q$ is permitted to tend to zero with Q can be obtained by replacing Theorem 1.4 with appropriate discretizations of results of Hejhal [8] and Strömbergsson [13], see [2, Remark 1].

5. Computations of the tail behavior of geometric statistics

In this section, we compute the tails of our geometric statistics (§1.4.1).

5.1. Euclidean distance. For $F(x,y) = \dfrac{1}{2x^2} + \dfrac{1}{2y^2}$, by Corollary 1,

$$G_F(t) = m(F^{-1}(t,\infty)) = m\left(\left\{(x,y) \in \Omega, \frac{1}{x^2} + \frac{1}{y^2} \geq 2t\right\}\right).$$

Define

$$B_t := \left\{(x,y) \in \Omega, \frac{1}{x^2} + \frac{1}{y^2} \geq 2t\right\}.$$

Therefore,

$$G_F(t) = \int_{B_t} dm = 2\int\int_{B_t} dx\ dy.$$

Note that if $y < \frac{1}{\sqrt{2t}}$, the inequality $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq 2t$ automatically holds true. The region bounded by lines $0 < y < 1/\sqrt{2t}$ and $1 - y < x < 1$ is exactly an isosceles right triangle with area $\dfrac{1}{4t}$. Let us assume in what follows that $y > \frac{1}{\sqrt{2t}}$. Then,

$$x \leq \frac{1}{\sqrt{2t - \frac{1}{y^2}}}.$$

This yields

$$1 - y < x < \min\left\{1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}}\right\}.$$

Therefore, $1 - y < \frac{1}{\sqrt{2t - \frac{1}{y^2}}}$, which implies either $\alpha_1 < y < \alpha_2$ or $\alpha_3 < y < \alpha_4$, where

$$\alpha_1 := \frac{t - \sqrt{t^2 + 2t + 2t(\sqrt{1+2t})}}{2t}; \quad \alpha_2 := \frac{t - \sqrt{t^2 + 2t - 2t(\sqrt{1+2t})}}{2t},$$

and

$$\alpha_3 := \frac{t + \sqrt{t^2 + 2t - 2t(\sqrt{1+2t})}}{2t}; \quad \alpha_4 := \frac{t + \sqrt{t^2 + 2t + 2t(\sqrt{1+2t})}}{2t}.$$

Also,

$$\alpha_1 < 0 < \frac{1}{\sqrt{2t}} < \frac{1}{\sqrt{2t-1}} < \alpha_2 < \alpha_3 < 1 < \alpha_4. \tag{5.1}$$

Hence a point (x,y) with $y > 1/\sqrt{2t}$ belongs to B_t if and only if

$$1 - y < x < \min\left\{1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}}\right\} \text{ and } y \in \left(\frac{1}{\sqrt{2t}}, \alpha_2\right) \cup (\alpha_3, 1). \tag{5.2}$$

Now for the measure of the set B_t minus the isosceles right triangle already discussed above, we consider the following two cases.

Case I. $\min\left\{1, \frac{1}{\sqrt{2t-\frac{1}{y^2}}}\right\} = 1$, i.e., $y \le \frac{1}{\sqrt{2t-1}}$. Employing (5.1) and (5.2), we have that $1-y < x < 1$ and $\frac{1}{\sqrt{2t}} < y \le \frac{1}{\sqrt{2t-1}}$.

Case II. $\min\left\{1, \frac{1}{\sqrt{2t-\frac{1}{y^2}}}\right\} = \frac{1}{\sqrt{2t-\frac{1}{y^2}}}$, i.e., $y \ge \frac{1}{\sqrt{2t-1}}$. Then, using (5.1) and (5.2), we have that $1-y < x < \frac{1}{\sqrt{2t-\frac{1}{y^2}}}$ and either $\frac{1}{\sqrt{2t-1}} < y < \alpha_2$ or $\alpha_3 < y < 1$.

Combining the two cases above and adding the contribution from the isosceles right triangle, we obtain

$$
\begin{aligned}
G_F(t) &= \frac{1}{2t} + 2\int_{\frac{1}{\sqrt{2t}}}^{\frac{1}{\sqrt{2t-1}}} \int_{1-y}^{1} dx\, dy \\
&\quad + 2\int_{\frac{1}{\sqrt{2t-1}}}^{\alpha_2} \int_{1-y}^{\frac{1}{\sqrt{2t-\frac{1}{y^2}}}} dx\, dy + 2\int_{\alpha_3}^{1} \int_{1-y}^{\frac{1}{\sqrt{2t-\frac{1}{y^2}}}} dx\, dy \\
&= \frac{4t-1}{\sqrt{2t-1}} - 1 - \frac{1}{t}\sqrt{\frac{1}{2t-1}} + \frac{1}{t}\sqrt{t\left(t-2\sqrt{2t+1}+2\right)} \\
&\quad + \frac{1}{t}\sqrt{t-\sqrt{2t+1}-\sqrt{t\left(t-2\sqrt{2t+1}+2\right)}} \\
&\quad - \frac{1}{t}\sqrt{t-\sqrt{2t+1}+\sqrt{t\left(t-2\sqrt{2t+1}+2\right)}}.
\end{aligned} \tag{5.3}
$$

This gives

$$G_F(t) \sim \frac{1}{t}.$$

5.2. Hyperbolic distance. In this case $F(x,y) = \frac{1}{2}\left(\frac{y}{x}+\frac{x}{y}\right)$ and

$$G_F(t) = m(C_t), \text{ where } C_t := \left\{(x,y)\in\Omega, \frac{1}{2}\left(\frac{y}{x}+\frac{x}{y}\right) \ge t\right\}.$$

$$\frac{1}{2}\left(\frac{y}{x}+\frac{x}{y}\right) \ge t \Rightarrow \text{ either } x \ge y(t+\sqrt{t^2-1}) \text{ or } x \le y(t-\sqrt{t^2-1}).$$

Since $x, y \in \Omega$, we note that either

$$1-y < x < \min\left\{1, y(t-\sqrt{t^2-1})\right\}$$

or

$$\max\left\{1-y, y(t+\sqrt{t^2-1})\right\} < x < 1.$$

Depending on the bounds of x and y, we have the following cases:

Case I. $1-y<x<\min\left\{1, y(t-\sqrt{t^2-1})\right\}$. As $0<y<1$ and

$$t-\sqrt{t^2-1}=\frac{1}{t+\sqrt{t^2+1}}<1,$$

we note that $y(t-\sqrt{t^2-1})<1$ and therefore

$$\min\left\{1, y(t-\sqrt{t^2-1})\right\}=y(t-\sqrt{t^2-1}).$$

This implies $y>\frac{1}{t+1-\sqrt{t^2-1}}$. Also, $y(t-\sqrt{t^2-1})<1$ implies

$$y<\left(t-\sqrt{t^2-1}\right)^{-1}=t+\sqrt{t^2-1},$$

which always holds since $y<1$ and $t\gg 1$. And so in this case the range for x and y is given by,

$$\frac{1}{t+1-\sqrt{t^2-1}}<y<1 \quad\text{and}\quad 1-y<x<y(t-\sqrt{t^2-1}). \tag{5.4}$$

Case II. $\max\left\{1-y, y(t+\sqrt{t^2-1})\right\}<x<1$. Here, we consider two cases.

Subcase I. $\max\left\{1-y, y(t+\sqrt{t^2-1})\right\}=1-y$. Then

$$y<\frac{1}{t+\sqrt{t^2-1}+1}$$

and the range for x and y is given by

$$0<y<\frac{1}{t+\sqrt{t^2-1}+1} \quad\text{and}\quad 1-y<x<1. \tag{5.5}$$

Subcase II. When $\max\left\{1-y, y(t+\sqrt{t^2-1})\right\}=y(t+\sqrt{t^2-1})$. This implies

$$y>\frac{1}{t+\sqrt{t^2-1}+1} \quad\text{and}\quad y(t+\sqrt{t^2-1})<1,$$

and so we get,

$$\frac{1}{t+\sqrt{t^2-1}+1}<y<\frac{1}{(t+\sqrt{t^2-1})} \quad\text{and}\quad y(t+\sqrt{t^2-1})<x<1. \tag{5.6}$$

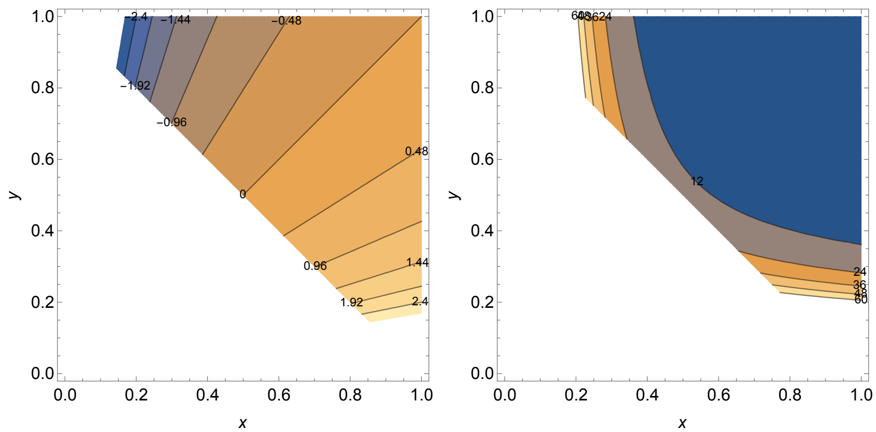
Combining all the above cases, from (5.4), (5.5) and (5.6), we have

$$\begin{aligned}
G_F(t)&=2\int_{\frac{1}{t-\sqrt{t^2-1}+1}}^{1}\int_{1-y}^{y(t-\sqrt{t^2-1})}dx\,dy+2\int_0^{\frac{1}{t+\sqrt{t^2-1}+1}}\int_{1-y}^{1}dx\,dy\\
&\quad+2\int_{\frac{1}{t+\sqrt{t^2-1}+1}}^{\frac{1}{t+\sqrt{t^2-1}}}\int_{y(t+\sqrt{t^2-1})}^{1}dx\,dy\\
&=\frac{1}{(t+\sqrt{t^2-1})(t+1+\sqrt{t^2-1})}+\frac{1}{(t+\sqrt{t^2-1}+1)^2}\\
&\quad+\frac{1}{(t+\sqrt{t^2-1})(t+\sqrt{t^2-1}+1)^2}
\end{aligned}$$

$$= \frac{2}{(t+\sqrt{t^2-1})(t+\sqrt{t^2-1}+1)}.$$

Thus,

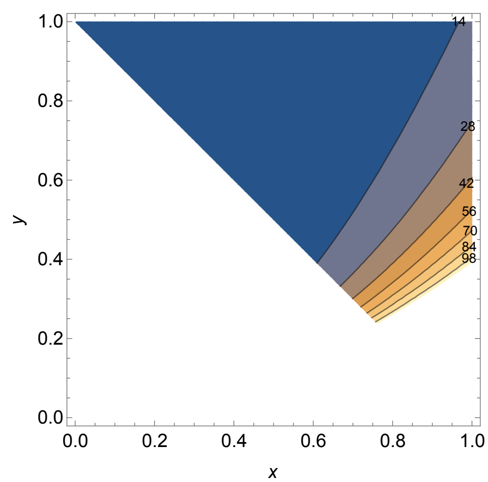
$$G_F(t) \sim \frac{1}{2t^2}.$$



(A) Level curves for geometric statistic for Euclidean slope in the region Ω.

(B) Level curves for geometric statistic for Euclidean area in the region Ω.

The tails for the geometric statistics for Euclidean slope, Euclidean area and hyperbolic angle can be computed in a similar manner.



(A) Level curves for geometric statistic for hyperbolic angle.

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