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Bounded height conjecture for function fields

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Abstract. We prove a function field version of the Bounded Height Conjecture formulated by Chatzidakis, Ghioca, Masser and Maurin in 2013.

Contents

1. Introduction

The Manin–Mumford Conjecture (proven by Raynaud [Ray83a, Ray83b] in the abelian case and by Hindry [Hin88] in the semiabelian case) asserts that if G is a semiabelian variety defined over the complex numbers $\mathbb{C}$, and V is an irreducible subvariety of G which is not a translate of an algebraic subgroup of G by a torsion point, then V does not contain a Zariski dense set of torsion points. If for each integer $m \geq 0$ we define $G^{[m]}$ as the union of all algebraic subgroups of G of codimension at least m, then the Manin–Mumford Conjecture states that $V \cap G^{[\dim G]}$ is not Zariski dense in V, as long as V is not a torsion translate of an algebraic subgroup of G. In [Zil02] (see also [BMZ99] in the special case $G = \mathbb{G}_m^n$), a more general conjecture was advanced. Bombieri, Masser and Zannier conjectured that if $V \subset \mathbb{G}_m^n$ is an irreducible variety of dimension d which is not contained in a translate of a proper algebraic subgroup of $\mathbb{G}_m^n$, then its intersection with $G^{[d+1]}$ is not Zariski dense in V. We also note that Pink [Pin] advanced a conjecture generalizing several known problems in arithmetic geometry: Mordell–Lang, Manin–Mumford, André–Oort, and Pink–Zilber. In [BMZ99], Bombieri, Masser and Zannier proved their conjecture for curves $V \subset \mathbb{G}_m^n$, and in [BMZ07], they formulated a possible strategy for proving

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their conjecture in general. Their proposed strategy goes through proving first the Bounded Height Conjecture (which is now a theorem due to Habegger [Hab09]). Habegger proved that once we remove from V the *anomalous locus* V^a (i.e., the union of all irreducible subvarieties W for which there exists a translate T of an algebraic subgroup of $G = \mathbb{G}_m^n$ such that $W \subseteq V \cap T$ and $\dim(W) > \max\{0, \dim(V) + \dim(T) - n\}$), then $(V \setminus V^a) \cap G^{[\dim(V)]}$ is a set of bounded height. See Zannier's recent book [Zan12] for more information on these and related topics.

Function field versions of both the Pink–Zilber Conjecture and of the Bounded Height Conjecture (see [CGMM13, Conjecture 1.8]) were formulated in [CGMM13]. While the function field version of the Pink–Zilber Conjecture was proven also in [CGMM13], on the other hand, in [CGMM13] there was proven only a partial result for plane curves of the function field version of the Bounded Height Conjecture. The main result of this paper is to prove [CGMM13, Conjecture 1.8] for all plane curves defined over a field of characteristic 0. We note that the method for our proof is significantly different than the one used in [CGMM13] for proving the special case of the Bounded Height Conjecture for plane curves of the form $f(X) = g(Y)$.

We start by stating the Bounded Height Conjecture from [CGMM13]. So, let $k \subset K$ be algebraically closed fields and let $\mathcal{X} := \mathbb{A}^n$. We assume $\text{trdeg}_k K$ is finite; let $t_1, \dots, t_\ell$ be a transcendence basis for K/k. We endow K with the valuations extending the valuations corresponding to the function field $k(t_1, \dots, t_\ell)$; we define the usual Weil height for all points in $\mathbb{A}^n(K)$. The subvarieties of $\mathcal{X}$ defined over k are the equivalent of algebraic subgroups in the Bounded Height Conjecture for $\mathbb{G}_m^n$; in particular, these subvarieties defined over k have the property (similar to the case of algebraic subgroups of $\mathbb{G}_m^n$) that contain a Zariski dense set of points of Weil height 0.

Definition 1.1. For each $m \geq 0$ we define $\mathcal{X}^{(m)}$ be the union of all subvarieties of $\mathcal{X}$ defined over k of codimension m.

We define the set of *quasi-constant* varieties, which play the role of translates of algebraic subgroups from the classical setting.

Definition 1.2. The (absolute irreducible) variety $\mathcal{Y} \subseteq \mathcal{X}$ is *quasi-constant* if it is defined over a subfield of K which has transcendence degree over k at most equal to 1.

Next we define the quasi-anomalous locus that we need to remove from any subvariety $\mathcal{Y} \subseteq \mathcal{X}$ in order to obtain a set of bounded Weil height when we intersect $\mathcal{Y}$ with $\mathcal{X}^{(\dim(\mathcal{Y}))}$.

Definition 1.3. The anomalous part $\mathcal{Y}^a$ of a variety $\mathcal{Y}$ in $\mathcal{X}$ is the union of all irreducible subvarieties W in $\mathcal{Y}$ such that W is contained in some quasi-constant subvariety $\mathcal{Z}$ of X satisfying

$$\dim W > \max\{0, \dim \mathcal{Y} + \dim \mathcal{Z} - n\}.$$

In [CGMM13, Conjecture 1.8], it was conjectured that for any subvariey $\mathcal{Y} \subset \mathcal{X}$, the points in $(\mathcal{Y} \setminus \mathcal{Y}^a) \cap \mathcal{X}^{(\dim \mathcal{Y})}$ over K have Weil height bounded above. The first interesting case of [CGMM13, Conjecture 1.8] is the case of plane curves $\mathcal{Y}$ (i.e., when $\mathcal{X} = \mathbb{A}^2$); this is [CGMM13, Conjecture 1.6]. As mentioned above, in [CGMM13], only a partial result was obtained for plane curves of the form $f(X) = g(Y)$. In this paper we prove [CGMM13, Conjecture 1.6] for all plane curves $\mathcal{Y}$ defined over a field of characteristic 0. In this case, an irreducible curve $\mathcal{Y}$ is either itself quasi-constant, in which case $\mathcal{Y}^a = \mathcal{Y}$ and so, [CGMM13, Conjecture 1.6] holds trivially, or $\mathcal{Y}$ is not quasi-constant, i.e. the minimal field of $\mathcal{Y}$ has transcendence degree at least equal to 2 and then $\mathcal{Y}^a$ is empty. So, in all that follows we assume $\operatorname{trdeg}_k K \geq 2$, and also that k has characteristic 0. We also note that (as pointed out by the referee) we use in one essential point of our proof the hypothesis that k has characteristic 0. So, our main result is the following:

Theorem 1.4. *Let k be an algebraically closed field of characteristic 0, and let K be an algebraically closed field containing k such that $2 \leq \operatorname{trdeg}_k K < \infty$. Let $\mathcal{Y} \subset \mathcal{X} := \mathbb{A}^2$ be an absolutely irreducible curve defined over K which is not defined over a subfield of K of transcendence degree 1. Then the points of $\mathcal{Y} \cap \mathcal{X}^{(1)}$ over K have height bounded from above.*

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2. Preliminaries

In this Section we start by introducing the Weil height for a function field, and then we prove a couple of useful results which will be used later in Section 3 in the proof of Theorem 1.4.

Since the proof of Theorem 1.4 in the case when $\operatorname{trdeg}_k K > 2$ follows by the same argument as the case when $\operatorname{trdeg}_k K = 2$, then for the sake of simplifying the notation we restrict our attention to the case $\operatorname{trdeg}_k K = 2$. So, we let k be an algebraically closed field, and we let K be a fixed algebraic closure of $k(s,t)$. We define the Weil height $h(x)$ of each point x in the function field K/k following either [Ser89, Chapter 2], or [BG06]. Alternatively, we can define the Weil height of $u \in K$ as follows. We let $d := [k(s,t,u) : k(s,t)]$ and we let $b_0, b_1, \dots, b_d \in k[s,t]$ relatively prime such that

$$b_d u^d + \cdots + b_1 u + b_0 = 0.$$

Then we define the height $h(u)$ as $\frac{\max_i \deg(b_i)}{d}$; for more details, see [DM12, Lemma 2.1]. Finally, for a point $(x,y) \in \mathbb{A}^2(K)$, its height is defined to be $h(x) + h(y)$.

We note the following property for computing the Weil height.

Lemma 2.1. *Let Σ be a surface with function field $k(s,t,u)$, with u algebraic over $k(s,t)$, of degree m. Suppose that for all but finitely many $c \in k$ there is a polynomial $P_c \in k[s,t]$, of degree D such that $P_c(s,t)$ vanishes for all points of Σ where $u = c$. Then $h(u) \leq \frac{D}{m}$.*

Proof. Note that since c is varying it does not matter which birational model of Σ we are considering, and we may refer to the affine surface in $\mathbb{A}^3$ with equation

$$b_m u^m + b_{m-1} u^{m-1} + \cdots + b_1 u + b_0 = 0.$$

Without loss of generality, we may assume each $b_i \in k[s,t]$ and moreover that the polynomials b_i share no common factor. In this case the points in question are the points (s_0, t_0, c) with

$$b_m(s_0,t_0)c^m + b_{m-1}(s_0,t_0)c^{m-1} + \cdots + b_0(s_0,t_0) = 0.$$

The coordinate u is a root of the irreducible polynomial

$$b_m U^m + b_{m-1} U^{m-1} + \cdots + b_0,$$

and so, using the irreducibility of the above polynomial, then for almost all specialisations $U \mapsto c \in k$ the resulting polynomial in $k[s,t]$ has no repeated factors. Indeed, we can consider the discriminant of the above polynomial with respect to the variable s; then we obtain a polynomial in t and u which is not identically 0. So, the specialization $U \mapsto c$ will not make this resultant equal to 0 for all but finitely many $c \in k$. This yields that the specialised polynomial at such c is square-free and so, it must divide P_c. Since this is true for almost all $c \in k$, then $\max \deg(b_i) \leq D$, as required. An application of [DM12, Lemma 2.1] finishes the proof. □

We will also use the following general result regarding the gonality of curves. Before proving our result, we note that for a field extension L_2/L_1 and for a place v of L_1, our convention for a place w of L_2 lying above v is that $w|_{L_1} = e(w|v) \cdot v$, where $e(w|v)$ is the corresponding ramification index.

Lemma 2.2. *Let ℓ be an algebraically closed field, and let $L_1 \subseteq L_2$ be a finite extension of function fields over ℓ of transcendence degree 1. Let $t \in L_2$ be a primitive element of the extension L_2/L_1 and let*

$$f(x) := x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in L_1[x]$$

be the minimal polynomial of t. Let v be a place of L_1/ℓ, and let

$$m := \max\{0, -v(a_0), \ldots, -v(a_{d-1})\}.$$

We let

$$M := \sum_{\substack{w \text{ is a place of } L_2 \\ \text{lying over } v}} \max\{0, -w(t)\}.$$

Then $m \leq M \leq dm$.

Proof. By using the Puiseux series of t at all places w lying above the place v (they are series in a fractional power of a given uniformizer z of v, with coefficients in ℓ) and comparing this with the Laurent series of the coefficients a_i, we immediately derive the desired result; of course we have taken here into account ramification indices, which are at most equal to d, explaining the factor d in the upper bound. □

This implies the following

Corollary 2.3. *In the notation of the preceding lemma, and setting*

$$h_{L_1}(f) := \sum_v \max\{0, -v(a_0), \ldots, -v(a_{d-1})\},$$

we have

$$h_{L_1}(f) \leq \deg(t) \leq d h_{L_1}(f).$$

A proof follows immediately from the lemma on summing over all places of L_1/ℓ.

Remark 2.4. Corollary 2.3 yields in particular that the gonality of a curve is a non increasing function under a rational map, and the left inequality immediately proves e.g. Luroth's theorem (without invoking the notion of genus and even differential forms): indeed, if $L_2 = \ell(t)$ is a rational function field, the degree of t is 1, whence $h_{L_1}(f) = 1$, which implies that any non constant coefficient of f has degree 1, and thus generates L_1 over ℓ.

If the field ℓ is not algebraically closed then Lemma 2.2 still holds once we take into account the degree of each place.

3. Proof of our main result

We continue with the notation as in Theorem 1.4; in particular, k has characteristic 0. Since the case when $\operatorname{trdeg}_k K > 2$ follows by the exact same argument, then for the sake of simplifying the notation we restrict to the case $\operatorname{trdeg}_k K = 2$. Also, $\mathcal{Y} \subset \mathcal{X} = \mathbb{A}^2$ is a curve defined over K which is not quasi-constant. Then $\mathcal{Y}$ is defined over a finite extension L of $k(s,t)$; at the expense of replacing $\mathcal{Y}$ by the finite union

$$\bigcup_{\substack{\sigma: L \longrightarrow K \\ \sigma|_{k(s,t)} = \mathrm{id}}} \mathcal{Y}^\sigma$$

(where $\mathcal{Y}^\sigma$ is the curve obtained by applying σ to each coefficient of the equation defining $\mathcal{Y}$), we may assume $\mathcal{Y}$ is defined over $k(s,t)$. Furthermore, it is sufficient to assume $\mathcal{Y}$ is irreducible over $k(s,t)$. Hence, $\mathcal{Y} \subset \mathcal{X}$ is the zero locus of an irreducible polynomial $f(X,Y)$ whose coefficients are in $k[s,t]$; we may also assume these polynomials in $k[s,t]$ share no common factor. Now, since $\mathcal{Y}$ is not quasi-constant, the ratio of the coefficients of f generate a field of transcendence degree 2 over k. Sometimes, by abuse of notation, we will write $f(s,t,X,Y)=0$ to denote the corresponding 3-fold defined over k (contained in $\mathcal{X}$ seen now as $\mathbb{A}^4_k$).

We view now $f(s,t,X,Y)$ as a polynomial in s and t over $k(X,Y)$ and we replace f by an absolutely irreducible factor of it; because we assumed before that the coefficients of f as a polynomial in X and Y are coprime polynomials in $k[s,t]$, we conclude that each such absolute irreducible factor of f is not of the form $A\cdot g$ where $A\in\overline{k(X,Y)}$ and $g\in k[s,t]$. At the expense of replacing (s,t) by the corresponding variables after using an automorphism of $k(s,t)$, we may assume that the leading coefficient of f as a polynomial in t does not depend on s. Then dividing $f(s,t,X,Y)$ (seen as a polynomial in t) by its leading coefficient (which, by our assumption lives in $\overline{k(X,Y)}$) we obtain a polynomial of degree d in t of the form

$$t^d+A_{d-1}t^{d-1}+\cdots+A_0\in\overline{k(X,Y)}[s][t],$$

i.e., each A_i is a polynomial in s with coefficients in $\overline{k(X,Y)}$. Then we write each A_i as a finite sum $A_i=\sum_j A_{i,j}s^j$ with $A_{i,j}\in\overline{k(X,Y)}$. There are two cases: the functions $A_{i,j}\in\overline{k(X,Y)}$ either generate a field E_f of transcendence degree 2 over k, or not. We see first that the latter case is impossible.

Indeed, assume the field E_f defined above has transcendence degree less than 2. Since $\operatorname{trdeg}_k(E_f)>0$ (because f is not of the form $A\cdot g$, where $A\in\overline{k(X,Y)}$ and $g\in k[s,t]$), then it must be that $\operatorname{trdeg}_k(E_f)=1$. So, let $A\in k(\mathcal{X})$ such that E_f is algebraic over $k(A)$. Then, letting $\mathcal{Y}_1$ be an absolutely irreducible component of $\mathcal{Y}$, we have that A is constant on $\mathcal{Y}_1$; hence $\mathcal{Y}_1$ is quasi-constant, which is a contradiction.

So, from now on we assume that $\operatorname{trdeg}_k(E_f)=2$. Then we can view the functions $A_{i,j}$ also as $\tilde{A}_{i,j}\circ\varphi^{-1}$ for some rational functions $\tilde{A}_{i,j}$ defined on a given surface S_0 which is endowed with a finite morphism $\varphi:S_0\longrightarrow\mathbb{A}^2$. Then each time when we *evaluate* $A_{i,j}$ at some point $P\in\mathbb{A}^2(K)$ we mean $\tilde{A}_{i,j}(\varphi^{-1}(P))$. In particular, we say that $A_{i,j}$ is *well-defined* at $P\in\mathbb{A}^2(K)$ if $\varphi^{-1}(P)$ is not contained in the pole-divisor of $\tilde{A}_{i,j}$. Even though $\varphi^{-1}(P)$ is not uniquely defined, because φ is a finite map, for the purpose of bounding the height of $\tilde{A}_{i,j}(\varphi^{-1}(P))$ this ambiguity is not relevant.

We let F_1 and F_2 be two algebraically independent functions $A_{i,j}\in\overline{k(X,Y)}$ from the above set. Hence there exist integers $d,e\geq 1$ and there exist $B_i,C_j\in k[F_1,F_2]$ for $0\leq i<d$ and $0\leq j<e$ such that

$$X^d+B_{d-1}X^{d-1}+\cdots+B_1X+B_0=0$$

and

$$Y^e+C_{e-1}Y^{e-1}+\cdots+C_1Y+C_0=0.$$

The following result will be used in our proof.

Lemma 3.1. *Let $x,y\in K$ and assume that the functions B_i and C_j are well-defined when evaluated for $X=x$ and $Y=y$. Then for each positive* *real number H_0 there exists a positive real number H_1 (depending only on H_0 and on F_1 and F_2) such that if $h(F_i(x,y)) \le H_0$ for each $i = 1, 2$, then $h((x,y)) \le H_1$.*

Proof of Lemma 3.1. This follows immediately since our hypothesis yields that x and y satisfy equations of bounded degree and with coefficients of bounded height. □

Lemma 3.1 yields that it suffices to bound uniformly the heights of all $A_{i,j}$ evaluated at the points (x,y) which lie in the intersection $\mathcal{Y} \cap \mathcal{X}^{(1)}$.

Let $g \in k[X,Y]$ such that the zero locus of $g = 0$ is an irreducible curve $\mathcal{C}$ contained in $\mathbb{A}^2$. We first note that if there is some B_i or some C_j which is not well-defined along the curve $g = 0$, then this curve belongs to a finite set of absolutely irreducible curves defined over k. On the other hand, the intersection of each one of these finitely many curves with $\mathcal{Y}$ is a finite set of points (because $\mathcal{Y}$ is irreducible and it is not defined over k). Hence the heights of the coordinates of these points in the intersection are uniformly bounded independent of the polynomial g (and depending only on $\mathcal{Y}$).

So, from now on, we may assume that each function B_i and each function C_j is well-defined when specialized along the curve $\mathcal{C}$. We let C be a nonsingular model of an irreducible component of $\varphi^{-1}(\mathcal{C})$. We view $\varphi^* X$ and $\varphi^* Y$ as rational functions on C and we denote them by x and y. So, we assume that x, y are elements of a field extension of $k(s,t)$ such that $\varphi^* f = 0$ and $\varphi^* g = 0$. Hence we obtain a surface Σ defined over k endowed with a dominant map to $\mathbb{P}^2$ given by composing φ with the projection map on the first two coordinates of $\mathcal{X} = \mathbb{A}^2_K = \mathbb{A}^4_k$. Also, this surface is endowed with a natural projection map to C. Also note that x, y may be viewed as algebraic functions of s, t; this follows from the fact that $\mathcal{Y}$ is not a constant curve. Then, by Lemma 3.1, it suffices to bound the heights of the algebraic functions $A_{i,j}$ evaluated at (x,y). We denote by $a_{i,j} := A_{i,j}$ evaluated at (x,y), and similarly, we let a_i be the evaluation of A_i at (x,y). We let $L := k\left(x, y, (a_{i,j})_{i,j}\right)$, which is a finite extension of $k(x,y)$; moreover, $[L : k(x,y)]$ is uniformly bounded independent of $\mathcal{C}$.

By a linear invertible map on s, t we may assume that L and $k(s)$ are independent over k.

Since we assumed f is absolutely irreducible as a polynomial in s and t, there is a proper (closed) subset $\mathcal{Z}$ of $\mathcal{X} = \mathbb{A}^2$ defined over k such that if the curve C is not contained in $\mathcal{Z}$, specializing the functions $A_{i,j}$ to $a_{i,j}$ along the curve C (and therefore specializing f along C) yields an irreducible polynomial in s and t. This fact follows from a theorem of Noether (see [Sch00, Theorem 32]), or equivalently by viewing $f(s,t) = 0$ as a 1-dimensional scheme over the surface S_0 and applying [DS84, Theorem 2.10 (i)] to find a proper closed subset Z_0 of S_0 such that specializing $\tilde{A}_{i,j}$ at points away from Z_0 yields irreducible polynomials; then $\mathcal{Z} = \varphi(Z_0)$. Now, if the curve $\mathcal{C}$ is an irreducible component of $\mathcal{Z}$, then again we have a finite set of points in the intersection with $\mathcal{Y}$ whose heights are bounded uniformly. So, from now on, assume the curve $\mathcal{C}$ is not contained in $\mathcal{Z}$. Hence the minimal polynomial of t over the field $M := k(s)\,(x, y, (a_{i,j})_{i,j}) = L(s)$ is the polynomial

$$T^d + a_{d-1}T^{d-1} + \cdots + a_0 \in L[s][T]. \tag{3.1.1}$$

Now, the field M is the function field of C when we view it as a curve defined over $k(s)$. In this view, the field $L(s,t)$ is the function field of a smooth curve S defined over $k(s)$, endowed with a map $\pi : S \longrightarrow C$. This curve over $k(s)$ is the surface Σ over k.

Let δ be the degree of t as a rational function on S (as a curve); then δ is the number of poles of t counted with multiplicity. So,

$$\delta = [k(s)(S) : k(s)(t)] = [L(s,t) : k(s,t)].$$

Let $u := \sum_{i,j} \gamma_{i,j} a_{i,j}$ be a *generic* linear combination of the $a_{i,j}$ with coefficients in k. Then u is a rational function on C; and the poles of u are precisely the poles of the $a_{i,j}$. Furthermore, since u is a generic linear combination of the $a_{i,j}$'s, and $a_i = \sum_j a_{i,j}s^j$, then for each place v of the function field $k(s)(C)$, the poles of u are the poles of the a_i's with the same multiplicity. So, we have

$$\max\{0, -v(u)\} = \max\{0, \max_i\{-v(a_i)\}\}. \tag{3.1.2}$$

Summing the left hand-side of (3.1.2) over all places v and also taking into account the degree of each place, we obtain the degree of u as a rational function on C, which we denote by μ. Then using (3.1.2) and Corollary 2.3 we obtain the inequality

$$\mu \leq \delta \leq d\mu. \tag{3.1.3}$$

We also note that in the conclusion of our proof we only employ the left-hand side of inequality (3.1.3).

Now, u is a map $u : C \longrightarrow \mathbb{P}^1$ and above a generic point $c \in \mathbb{P}^1(k)$ we have $\mu = \deg u$ points of C, which in turn correspond to points $(x_0, y_0) \in \mathbb{A}^2(k)$ such that $g(x_0, y_0) = 0$. Note that it suffices to bound uniformly the height of the points in $\varphi^{-1}((x_0, y_0))$ when $(x_0, y_0) \in \mathcal{Y} \cap \mathcal{C}$.

We now view S as the surface Σ above the (s,t)-plane. This S maps to C (and in turn to $\mathcal{C}$) and the curve above $(x_0, y_0) \in \mathcal{C}$ is defined by

$$f(s, t, x_0, y_0) = 0.$$

We are in position to apply Lemma 2.1. Taking then the product over all (x_0, y_0) above $u = c$ we see that

$$P_c(s,t) := \prod_{u(x_0,y_0)=c} f(s, t, x_0, y_0)$$

vanishes on the curve determined by $u = c$ on the surface Σ defined above. But then

$$\deg(P_c) = O(\mu) = O(\delta), \tag{3.1.4}$$

by inequality (3.1.3). Now, since k has charactersitic 0, then by the theorem of primitive element, for general $\gamma_{i,j}$ we have $k(s,t)\,(x,y,(a_{i,j})_{i,j}) = k(s,t,u)$ and also $k(s,t)\,(x,y,(a_{i,j})_{i,j}) = L(s,t)$. Moreover we recall that $\delta = [k(s,t,u) : k(s,t)]$ and so, by Lemma 2.1 and (3.1.4), we conclude that $h(u) = O(1)$. We remark that it is precisely this point where we use the hypothesis that k has characteristic 0; we thank the referee for pointing this to our attention.

So, for all such functions u, namely, for general coefficients $\gamma_i \in k$, we have $h(u) = O(1)$. We conclude that the heights of all $a_{i,j}$ are $O(1)$. In particular, $h(F_1(x,y))$ and $h(F_2(x,y))$ are both bounded independently of $\mathcal{C}$, and thus Lemma 3.1 yields the desired conclusion.

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