

Scaffolds and integral Hopf Galois module structure on purely inseparable extensions

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ABSTRACT. Let p be prime. Let L/K be a finite, totally ramified, purely inseparable extension of local fields, $[L : K] = p^n$, $n \geq 2$. It is known that L/K is Hopf Galois for numerous Hopf algebras H , each of which can act on the extension in numerous ways. For a certain collection of such H we construct “Hopf Galois scaffolds” which allow us to obtain a Hopf analogue to the Normal Basis Theorem for L/K . The existence of a scaffold structure depends on the chosen action of H on L . We apply the theory of scaffolds to describe when the fractional ideals of L are free over their associated orders in H .

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1. Introduction

Let L/K be a totally ramified extension of local fields of degree p^n , where the residue field of K has characteristic p . Suppose further that L/K is Galois with $G = \text{Gal}(L/K)$. Let \mathfrak{D}_K and \mathfrak{D}_L denote the valuation rings of K and L respectively. There are two natural ways to describe the elements of L , namely by using its valuation v_L or by using its Galois action. If $\pi \in L$ is a uniformizing parameter, then every element of L is a K -linear combination of powers of π ; computing its valuation is a simple process. A

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drawback of the valuation representation of L is that the Galois action is not necessarily transparent.

Alternatively, we have the Normal Basis Theorem, which asserts that there exists a $\rho \in L$ whose Galois conjugates form a K -basis for L/K ; equivalently, L is a free rank one module over the group algebra KG . Here, every element of L is a K -linear combination of $\{\sigma(\rho) : \sigma \in G\}$, which allows for a simple description of the Galois action; however, the valuation representation is not transparent, making certain Galois module theory questions difficult to answer. For example, \mathfrak{D}_L is an $\mathfrak{D}_K G$ -module, however by Noether's Theorem [13] \mathfrak{D}_L is not free of rank one if L/K is wildly ramified. The $\mathfrak{D}_K G$ -module structure of \mathfrak{D}_L when \mathfrak{D}_L does not possess a normal integral basis can be more difficult. A typical strategy, thanks to Leopoldt [11] is to replace $\mathfrak{D}_K G$ with a larger \mathfrak{D}_K -subalgebra of KG , namely

$$\mathfrak{A} = \{\alpha \in KG : \alpha(\mathfrak{D}_L) \subset \mathfrak{D}_L\},$$

which also acts on \mathfrak{D}_L ; the structure of \mathfrak{D}_L as an \mathfrak{A} -module can be simpler to describe.

In an attempt to unite these representations, G. Griffith Elder [5] first developed a theory of ‘‘Galois scaffolds’’. In that work a Galois scaffold consists of a subset $\{\theta_1, \theta_2, \dots, \theta_n\}$ of KG , together with a positive integer v , called an integer certificate, such that $\{v_L(\theta_i^j(\rho)) : 1 \leq i \leq n, 0 \leq j \leq p-1\}$ is a complete set of residues mod p^n where $\rho \in L$ is any element of valuation v . Certainly, $\{\theta_i^j(\rho)\}$ forms a K -basis for L , and this basis facilitates the study of both valuation and Galois action, particularly if $\theta_i^p = 0$ for all i . A simple example of a Galois scaffold arises when $n = 1$ and the break number b is relatively prime to p ; in this case, if $G = \langle \sigma \rangle$ then $\theta_1 = \sigma - 1$, $v = b$ is an example of a Galois scaffold. Such scaffolds do not always exist — in fact, integer certificates may not exist, for example if L/K is unramified and $\pi^p = 1$ [1]. This notion of scaffold was refined in [3], and then again in [2], the latter version being the most useful for describing the integral Galois module structure.

The version in [2] is also the most general as it does not insist that L/K be Galois, merely that there is a K -algebra A which acts on L in a very reasonable way. A classic example of such an algebra is a K -Hopf algebra. There are many more Hopf Galois extensions than Galois extensions. For example, any Galois extension is Hopf Galois for at least one Hopf algebra (namely, $H = KG$) and, if $n \geq 2$, many more: the exact determination of the number of such H is a group theory problem thanks to [7], which covers all separable extensions. At the other extreme, if the extension L/K is purely inseparable, then it is also Hopf Galois [4]; if $[L : K] \geq p^2$, then there are numerous Hopf algebras which make L/K Hopf Galois [10].

In the setting where L/K is Hopf Galois with Hopf algebra H , one can study the structure of \mathfrak{D}_L as an H -module. Given [2], a natural approach would be an attempt to construct an H -scaffold which, loosely, consists of

$\{\lambda_t : t \in \mathbb{Z}\} \subset L$ with $v_L(\lambda_t) = t$, along with $\{\Psi_i : 0 \leq i \leq n - 1\} \subset H$ such that Ψ_i acts on λ_t in a manner which makes $v_L(\Psi_i(\lambda_t))$ easy to compute.

Here, we focus on the case where L/K is a totally ramified, purely inseparable extension of local fields, $[L : K] = p^n$, $n \geq 2$. Such extensions are necessarily primitive, generated as a K -algebra by a (nonunique) element $x \in L$ with $x^{p^n} \in K, x^{p^{n-1}} \notin K$. We take a collection of Hopf algebras H which make L/K Hopf Galois and describe the generalized integral Hopf Galois module structure of \mathfrak{D}_L . The integral Hopf Galois module structure we seek is a description of all of the fractional ideals of L as H -modules. In detail, each fractional ideal of L is of the form \mathfrak{P}_L^h for $h \in \mathbb{Z}$, where \mathfrak{P}_L is the maximal ideal of \mathfrak{D}_L . In other words, $\mathfrak{P}_L^h = \{x \in L : v_L(x) \geq h\}$. For each h we let \mathfrak{A}_h be the largest subset of H which acts on \mathfrak{P}_L^h , i.e.,

$$\mathfrak{A}_h = \left\{ \alpha \in H : \alpha \mathfrak{P}_L^h \subset \mathfrak{P}_L^h \right\}.$$

We call \mathfrak{A}_h the associated order of \mathfrak{P}_L^h in H : it is clearly an \mathfrak{D}_K -subalgebra of H and $\mathfrak{A}_h \otimes_{\mathfrak{D}_K} K \cong H$. By construction, \mathfrak{A}_h acts on \mathfrak{P}_L^h ; the existence of the scaffold allows for a numerical criterion for determining whether \mathfrak{P}_L^h is a free \mathfrak{A}_h -module. The criterion itself is independent of the scaffold, provided the scaffold exists.

The paper is organized as follows. After giving a definition of an H -scaffold, a simpler version than the one in [2], we consider the family of monogenic K -Hopf algebras $H_{n,r,f}$, $1 \leq r \leq n - 1$, $f \in K^\times$ introduced in [10] which make L an $H_{n,r,f}$ -Galois object. We examine the case where $2r \geq n$ and consider actions of the linear dual $H := H_{n,r,f}^*$ which give L/K the structure of a Hopf Galois extension. A subtlety that arises is that H possesses an infinite number of actions on L ; in each case, L/K is H -Galois. The different actions correspond with different choices for K -algebra generator $x \in L$; and for each choice of x we will construct H -scaffolds for infinitely many actions. As with the Galois case, the H -scaffold will allow us to consider the effect of the action on the valuation of some specially chosen elements, and using [2, Th 3.1, 3.7] we will use it to describe the integral Hopf Galois module structure. We will then focus on a specific action for which an H -scaffold exists, and explicitly describe which fractional ideals \mathfrak{P}_L^h are free over their associated orders. We conclude with some remarks concerning selecting the “best” choices of r and f , and the action on L , for answering integral Hopf Galois module theory questions.

The evident purpose of this work is to construct H -scaffolds. However, our results contribute to the bigger picture of scaffolds. The definition of a scaffold has evolved significantly since Elder’s 2009 paper, which required L/K to be a Galois extension. At this point, it is not yet clear how prevalent scaffolds are for general Hopf Galois extensions. But we will see that in the finite purely inseparable case, many scaffolds exist.

Throughout, we fix an integer $n \geq 2$ and L a totally ramified purely inseparable extension of $K = \mathbb{F}_q((T))$ of degree p^n . Let v_K be the T -adic valuation, v_L the extension of v_K to L . Write $L = K(x)$, $x^{p^n} = \beta \in K$, $v_K(\beta) = -b < 0$, $p \nmid b$. We let H and $H_{n,r,f}$ be as above, and we assume $2r \geq n$.

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2. Scaffolds

The definition of an A -scaffold in [2] is very general — more so than we need here. We will simplify this definition as much as possible, and since our acting K -algebra is a Hopf algebra we will refer to it as an H -scaffold.

Definition 2.1. Let a be an integer such that $ab \equiv -1 \pmod{p^n}$. Let $\mathfrak{T} > 1$ be an integer. An H -scaffold on L of tolerance \mathfrak{T} consists of:

- (1) A set $\{\lambda_j : j \in \mathbb{Z}, v_L(\lambda_j) = j\}$ of elements of L such that $\lambda_{j_1} \lambda_{j_2}^{-1} \in K$ when $j_1 \equiv j_2 \pmod{p^n}$.
- (2) A collection $\{\Psi_s : 0 \leq s \leq n-1\}$ of elements in H such that

$$\Psi_s(1_K) = 0$$

for all s and, mod $\lambda_{j+p^s b} \mathfrak{P}_L^{\mathfrak{T}}$,

$$\Psi_s(\lambda_j) \equiv \begin{cases} u_{s,j} \lambda_{j+p^s b} & \text{res}(aj)_s > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $u_{s,j} \in \mathfrak{O}_K^\times$, $\text{res}(aj)$ is the least nonnegative residue of aj mod p^n , and

$$\text{res}(aj) = \sum_{s=0}^{n-1} \text{res}(aj)_s p^s, \quad 0 \leq \text{res}(aj)_s \leq p-1$$

is the p -adic expansion of $\text{res}(aj)$.

Given an H -scaffold we know the effect of applying Ψ_s to λ_j , provided $\text{res}(aj)_s > 0$. For $0 < i \leq p-1$ it can be readily seen that

$$\text{res}(a(b + p^s bi))_s = p - i > 0,$$

hence $\Psi_s^i(\lambda_b) \equiv u \lambda_{b+p^s bi} \pmod{\lambda_{j+p^s b} \mathfrak{P}_L^{\mathfrak{T}}}$ for some $u \in \mathfrak{O}_K^\times$. More generally,

$$v_L(\Psi_0^{i_0} \Psi_1^{i_1} \cdots \Psi_{n-1}^{i_{n-1}}(\lambda_b)) = b + b \sum_{s=0}^{n-1} i_s p^s, \quad 0 \leq i_s \leq p-1.$$

By allowing the $\{i_s\}$ to vary, we obtain p^n elements of L , pairwise incongruent modulo p^n , hence

$$\left\{ \Psi_0^{i_0} \Psi_1^{i_1} \cdots \Psi_{n-1}^{i_{n-1}}(\lambda_b) : 0 \leq i_s \leq p-1 \right\}$$

is a K -basis for L .

We will use the result below to construct our H -scaffolds.

Lemma 2.2. *Suppose we have $\{\Psi_s : 0 \leq s \leq n-1\} \subset H$ such that, for $i \leq p^n - 1$, $i = \sum_{s=0}^{n-1} i_s p^s$,*

$$\Psi_s(x^i) \equiv i_s x^{i-p^s} \pmod{x^{i-p^s} \mathfrak{P}_L^{\mathfrak{T}}}$$

for some $\mathfrak{T} > 1$. Let

$$\lambda_j = T^{(j+b \operatorname{res}(aj))/p^n} x^{\operatorname{res}(aj)}.$$

Then $\{\lambda_j\}, \{\Psi_s\}$ form a scaffold of tolerance \mathfrak{T} .

Proof. First, since $v_L(x) = -b$,

$$v_L(\lambda_j) = j + b \operatorname{res}(aj) - b \operatorname{res}(aj) = j,$$

and clearly $v_L(\lambda_{j_1} \lambda_{j_2}^{-1}) = j_1 - j_2$, so condition (1) of the definition above is satisfied. Next, we have

$$\begin{aligned} \Psi_s(\lambda_j) &= \Psi_s(T^{(j+b \operatorname{res}(aj))/p^n} x^{\operatorname{res}(aj)}) \\ &= T^{(j+b \operatorname{res}(aj))/p^n} \Psi_s(x^{\operatorname{res}(aj)}) \\ &\equiv T^{(j+b \operatorname{res}(aj))/p^n} \operatorname{res}(aj)_s x^{\operatorname{res}(aj)-p^s} \pmod{x^{\operatorname{res}(aj)-p^s} \mathfrak{P}_L^{\mathfrak{T}}}. \end{aligned}$$

If $\operatorname{res}(aj)_s = 0$ then $\Psi_s(\lambda_j) = 0$. Otherwise, $a(j + bp^s) \equiv aj - p^s \pmod{p^n}$ and

$$\begin{aligned} \operatorname{res}(a(j + bp^s)) &= \operatorname{res}(aj - p^s) \\ &= \operatorname{res}(aj) - p^s, \end{aligned}$$

the latter equality since $\operatorname{res}(aj) \geq p^s$. Thus $\operatorname{res}(aj) = p^s + \operatorname{res}(a(j + bp^s))$ and so

$$\begin{aligned} j + b \operatorname{res}(aj) &= j + b(p^s + \operatorname{res}(a(j + bp^s))) \\ &= j + b \operatorname{res}(a(j + bp^s)) + bp^s, \end{aligned}$$

giving

$$\begin{aligned} \operatorname{res}(aj)_s T^{(j+b \operatorname{res}(aj))/p^n} x^{\operatorname{res}(aj)-p^s} \\ &= \operatorname{res}(aj)_s T^{(j+b \operatorname{res}(a(j+bp^s))+bp^s)/p^n} x^{\operatorname{res}(a(j+bp^s))} \\ &= \operatorname{res}(aj)_s \lambda_{j+bp^s}. \end{aligned}$$

Setting $u_{s,j} = \operatorname{res}(aj)_s$ shows that (2) is also satisfied. \square

Remark 2.3. By adjusting each λ_j by a scalar it is possible to have $u_{s,j} = 1$. This is the primary difference between the construction above and the one found in [2, Sec. 5.3].

In the work to follow, we will use the definition of H -scaffold given by the description in Lemma 2.2. As the choice of $\{\lambda_j\}$ will remain fixed (assuming a constant b), we will refer to the scaffold as $\{\Psi_s\}$.

3. The Hopf algebra structure

In this section, we introduce the class of Hopf algebras we will use to construct our H -scaffolds. To do so, we first recall a family of Hopf algebras introduced in [10]. For $0 < r < n \leq 2r$ and $f \in K^\times$, let $H_{n,r,f}$ be the K -Hopf algebra whose K -algebra structure is $H_{n,r,f} = K[t]/(t^{p^n})$; whose counit and antipodal map are $\varepsilon(t) = 0$ and $\lambda(t) = -t$ respectively; and whose comultiplication is

$$\Delta(t) = t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} t^{p^r \ell} \otimes t^{p^r(p-\ell)}.$$

Let us fix values for r, n , and f as above; and let $H = H_{n,r,f}^*$. Certainly, H has a K -basis $\{z_0 = 1, z_1, \dots, z_{p^n-1}\}$ with $z_i : H \rightarrow K$ given by

$$z_j(t^i) = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta. The algebra structure on H is induced from the coalgebra structure on $H_{n,r,f}$; explicitly,

$$(1) \quad z_{j_1} z_{j_2}(h) = \text{mult}(z_{j_1} \otimes z_{j_2})\Delta(h).$$

In this section we will show that $\{z_{p^s} : 0 \leq s \leq n-1\}$ generate H as a K -algebra. This set will be (part of) the scaffolds we develop.

We start by recalling a result which will facilitate the study of the algebra structure of H as well as the action of H on L .

Lemma 3.1. *Let*

$$S_f(u, v) = u + v + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p^r \ell} v^{p^r(p-\ell)}.$$

Then, for every positive integer i , $S_f(u, v)^i$ is an K^\times -linear combination of elements of the form

$$f^{i_3} u^{i_1+p^r \ell'} v^{i_2+p^r \ell''},$$

where

$$\begin{aligned} i &= i_1 + i_2 + i_3 \\ \ell' &= i_{3,1} + 2i_{3,2} + \dots + (p-1)i_{3,p-1} \\ \ell'' &= (p-1)i_{3,1} + (p-2)i_{3,2} + \dots + i_{3,p-1}, \end{aligned}$$

and $i_{3,1} + i_{3,2} + \dots + i_{3,p-1} = i_3$.

Proof. This is a straightforward calculation from [10, Lemma 5.1] — we recall it here for the reader's convenience.

We have

$$\begin{aligned} S_f(u, v)^i &= \left(u + v + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p^r \ell} v^{p^r(p-\ell)} \right)^i \\ &= \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} (u^{i_1} v^{i_2}) \left(f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p^r \ell} v^{p^r(p-\ell)} \right)^{i_3}. \end{aligned}$$

The last factor in each summand can be expanded as

$$f^{i_3} \sum_{i_{3,1}+\dots+i_{3,p-1}=i_3} \left(\binom{i_3}{i_{3,1}, \dots, i_{3,p-1}} \left(\prod_{j=1}^{p-1} \frac{1}{i_{3,j}!(p-i_{3,j})!} \right) u^{i_1+p^r \ell'} v^{i_2+p^r \ell''} \right).$$

The result follows. \square

Next, we consider powers of the z_{p^s} 's.

Lemma 3.2. *For $0 \leq s \leq r$, $1 \leq m \leq p-1$; or $0 \leq s \leq r-1$, $m = p$ we have $z_{p^s}^m = m!z_{mp^s}$. In particular, $z_{p^s}^p = 0$.*

Proof. See [10, Lemmas 5.2, 5.3]. While the result there was for $n = r+1$, its validity depended on the form of the comultiplication; the more general $2r \geq n$ case the comultiplication has the same form., and hence a nearly identical proof. \square

The result above does not hold for $s > r$. However, we do have:

Lemma 3.3. *For $0 \leq s \leq n-1$, $1 \leq j, m \leq p-1$, we have*

$$z_{p^s}^j(t^{mp^s}) = m!\delta_{j,m}.$$

Furthermore, if $s \geq r$ then $z_{p^s}^p(t^{p^i}) = f^{p^s-r} \delta_{i,s-r}$. In particular, $z_{p^s}^p \neq 0$.

Proof. Certainly, if $s < r$ then the result follows from the previous lemma. Thus, we will assume that $s \geq r$. The statement $z_{p^s}^j(t^{mp^s}) = m!\delta_{j,m}$ is clearly true for $j = 1$. Suppose $z_{p^s}^{j-1}(t^{(m-1)p^s}) = (m-1)!\delta_{j,m-1}$. Since $s+r \geq n$ we have that t^{p^s} is a primitive element, hence

$$\begin{aligned} z_{p^s}^j(t^{mp^s}) &= \text{mult}(z_{p^s}^{j-1} \otimes z_{p^s})(t^{p^s} \otimes 1 + 1 \otimes t^{p^s})^m \\ &= \sum_{i=0}^m \binom{m}{i} z_{p^s}^{j-1}(t^{ip^s}) z_{p^s}(t^{(m-i)p^s}). \end{aligned}$$

Recalling that $z_i(t^j) = 0$ for $i \neq j$, for this to be nonzero, we require $i = j - 1$ and $m - i = 1$. Thus, $m = j$ and

$$\begin{aligned} z_{p^s}^m(t^{mp^s}) &= \binom{m}{m-1} z_{p^s}^{m-1}(t^{p^s(m-1)}) z_{p^s}(t^{p^s}) \\ &= m(m-1)! \\ &= m!, \end{aligned}$$

proving the first statement of the lemma.

For the second, we have

$$\begin{aligned} z_{p^s}^p(t^{p^i}) &= \text{mult}(z_{p^s}^{p-1} \otimes z_{p^s}) \left(t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} t^{p^r \ell} \otimes t^{p^r(p-\ell)} \right)^{p^i} \\ &= z_{p^s}^{p-1}(t^{p^i}) z_{p^s}(1) + z_{p^s}^{p-1}(1) z_{p^s}(t^{p^i}) \\ &\quad + f^{p^i} \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} z_{p^s}^{p-1}(t^{p^{r+i}\ell}) z_{p^s}(t^{p^{r+i}(p-\ell)}). \end{aligned}$$

Since $z_{p^s}(1) = 0$ we may ignore the first two terms, and so

$$z_{p^s}^p(t^{p^i}) = f^{p^i} \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} z_{p^s}^{p-1}(t^{p^{r+i}\ell}) z_{p^s}(t^{p^{r+i}(p-\ell)}).$$

In order that a summand be nonzero we require $p^{r+i}(p-\ell) = p^s$, i.e. $\ell = p-1$, and hence $i = s - r$. We have, since $z_{p^s}^{p-1}(t^{mp^s}) = (p-1)! \delta_{p-1, m}$,

$$\begin{aligned} z_{p^s}^p(t^{p^i}) &= f^{p^{s-r}} \frac{1}{(p-1)!} z_{p^s}^{p-1}(t^{p^s(p-1)}) \\ &= f^{p^{s-r}} \frac{1}{(p-1)!} (p-1)! \\ &= f^{p^{s-r}}. \end{aligned}$$

For $i \neq s - r$ we have $z_{p^s}^p(t^{p^i}) = 0$. □

It can be shown that the set

$$\left\{ \prod_{s=0}^{n-1} z_{p^s}^{j_s} : 0 \leq j_s \leq p-1 \right\}$$

is a K -basis for H . A formal proof will be given in section 5. By counting dimensions, it is clear that $z_{p^s}^{p^2} = 0$ for $r \leq s \leq n-1$.

The coalgebra structure on H is induced from the multiplication on $H_{n,r,f}$ and is simply

$$\Delta(z_j) = \sum_{i=0}^j z_{j-i} \otimes z_i.$$

4. The Hopf Galois action

In [10] we describe how L can be viewed as an $H_{n,r,f}$ -Galois object. Since $2r \geq n$ the K -algebra map $\alpha : L \rightarrow L \otimes H_{n,r,f}$ given by

$$(2) \quad \alpha(x) = x \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} x^{p^r \ell} \otimes t^{p^r(p-\ell)}$$

provides an $H_{n,r,f}$ -comodule structure on L ; furthermore, the map

$$\gamma : L \otimes L \rightarrow L \otimes H_{n,r,f}$$

given by $\gamma(x^i \otimes x^j) = x^i \alpha(x^j)$ is an isomorphism, hence L is an $H_{n,r,f}$ -Galois object. In this section, we describe the induced action of $H = H_{n,r,f}^*$ on L which makes L/K an H -Galois extension.

Before proceeding, notice that this action depends on two choices: the choice of x , the K -algebra generator for L , and the choice of t , the K -algebra generator for $H_{n,r,f}$. By replacing x with x' , $p \nmid v_L(x')$ we may define

$$\alpha^{x'}(x') = x' \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} (x')^{p^r \ell} \otimes t^{p^r(p-\ell)}$$

and obtain a different coalgebra structure. Alternatively, if we replace t with, say, $t_g := gt$, $g \in K^\times$ we may define

$$\alpha_g(x) = x \otimes 1 + 1 \otimes t_g + fg^{1-p^{r+1}} \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} x^{p^r \ell} \otimes t_g^{p^r(p-\ell)}$$

which also results in a different coalgebra structure. Furthermore, each of the coalgebra structures here give L the structure of an $H_{n,r,f}$ -Galois object. Combined, we have coactions given by

$$\alpha_h^y(y) = y \otimes 1 + 1 \otimes t_h + fh^{1-p^{r+1}} \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} y^{p^r \ell} \otimes t_h^{p^r(p-\ell)},$$

$$y \in L^\times, p \nmid v_L(y), h \in K^\times$$

although some choices of h, y produce the same actions, e.g. $\alpha_1^x = \alpha_{T^{-1}}^x$. By fixing $x \in L$ we eliminate some of the ambiguity as to which coaction is being used. For the rest, notice that $K[t_g]/(t_g^{p^n}) = H_{n,r,fg^{1-p^{r+1}}}$, and so $H_{n,r,f} = H_{n,r,fg^{1-p^{r+1}}}$ for any choice of $g \in K^\times$, hence choosing the K -algebra generator for the Hopf algebra is equivalent to choosing a representative of a coset in $K^\times/(K^\times)^{p^{r+1}-1}$; once such a choice f is made, it is assumed that the coaction of $H_{n,r,f}$ follows the coaction given in Equation (2). In other words, we will always use the action α_1^x .

Generally, if A is a K -Hopf algebra such that L is an A -Galois object, then A^* acts on L by

$$(3) \quad h(y) = \text{mult}(1 \otimes h)\alpha(y), \quad h \in A^*, y \in L.$$

As H is generated by $\{z_{p^s} : 0 \leq s \leq n-1\}$, it suffices to compute $z_{p^s}(x^i)$ for $0 \leq s \leq n-1$, $1 \leq i \leq p^n - 1$.

Proposition 4.1. *For $0 \leq i \leq p^n - 1$, write*

$$i = \sum_{s=0}^{n-1} i_s p^s.$$

Then, for $0 \leq s \leq r-1$ we have

$$z_{p^s}(x^i) = i_s x^{i-p^s}.$$

Additionally,

$$z_{p^r}(x^i) = i_r x^{i-p^r} - i f x^{p^r(p-1)+i-1}.$$

Remark 4.2. Note that if $i < p^s$ then $z_{p^s}(x^i) = 0$, and if $i < p^r$ then $z_{p^r}(x^i) = -i f x^{p^r(p-1)+i-1}$.

Proof. We have

$$\begin{aligned} z_{p^s}(x^i) &= \text{mult}(1 \otimes z_{p^s})\alpha(x^i) \\ &= \text{mult}(1 \otimes z_{p^s})S_f(x \otimes 1, 1 \otimes t)^i \\ &= \text{mult}(1 \otimes z_{p^s}) \left(x \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} x^{p^r \ell} \otimes t^{p^r(p-\ell)} \right)^i \\ &= \text{mult}(1 \otimes z_{p^s}) \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} (x^{i_1} \otimes t^{i_2}) \\ &\quad \cdot \left(f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} x^{p^r \ell} \otimes t^{p^r(p-\ell)} \right)^{i_3}. \end{aligned}$$

When simplified, the tensors are of the form $x^{i_1+i_3 p^r \ell'} \otimes t^{i_2+i_3 p^r \ell''}$, ℓ', ℓ'' as before. Applying $1 \otimes z_{p^s}$ to each tensor will give 0 unless

$$(4) \quad p^s = i_2 + i_3 p^r \ell''.$$

Assume first that $s < r$. Since $p^r > p^s$ we see that $\ell'' = 0$. This can only occur if $i_3 = 0$. Thus $i_2 = p^s$ and $i_1 = i - p^s$, giving

$$\begin{aligned} z_{p^s}(x^i) &= \binom{i}{i-p^s, p^s, 0} x^{i-p^s} z_{p^s}(t^{p^s}) \\ &= \binom{i}{p^s} x^{i-p^s} \\ &= i_s x^{i-p^s}, \end{aligned}$$

the last equality following from Lucas' Theorem (see [6]). Thus

$$z_{p^s}(x^i) = i_s x^{i-p^s},$$

as desired.

Now we consider the case $s = r$. Then $i_3 = 0$, $i_2 = p^r$, $i_1 = i - p^r$ certainly satisfies Equation (4). However, we get an additional solution to this equation, namely $i_3 = 1$, $\ell' = p - 1$, $\ell'' = 1$, $i_2 = 0$, $i_1 = i - 1$ — as

$$i_2 + i_3 p^r \ell'' = p^r(p - \ell),$$

with this solution we have the left-hand side equal $0 + p^r(1) = p^r$, hence $\ell = p - 1$. Thus

$$\begin{aligned} z_{p^r}(x^i) &= \binom{i}{i - p^r, p^r, 0} x^{i-p^r} z_{p^r}(t^{p^r}) \\ &\quad + \binom{i}{i - 1, 0, 1} x^{i-1} f \frac{1}{(p-1)!(p-(p-1))!} x^{p^r(p-1)} z_{p^r}(t^{p^r}) \\ &= i_r x^{i-p^r} - i f x^{p^r(p-1)+i-1}. \end{aligned} \quad \square$$

Much like it was for the algebra structure, describing the action for $s > r$ is more complicated as Equation (4) can have numerous solutions. However, in the sequel we will be able to effectively study how the valuation of an element of L changes when z_{p^s} is applied.

5. A scaffold on H

Recall that $L = K(x)$, $x^{p^n} = \beta$, $v_L(x) = v_K(\beta) = -b$, $p \nmid b$. In this section we build an H -scaffold for L using the action above. Initially, we will insist on a restriction on f , however this restriction will ultimately not be necessary.

We start by determining the effect of applying z_{p^s} to powers of x . The first result is fundamental.

Proposition 5.1. *Let $0 \leq s \leq n - 1$, $1 \leq i \leq p^n - 1$. Write $i = \sum_{s=0}^{n-1} i_s p^s$. If $v_K(f) \geq b p^{r+1-n}$ then*

$$z_{p^s}(x^i) \equiv i_s x^{i-p^s} \pmod{x^{i-p^s} \mathfrak{P}_L^\mathfrak{T}}$$

where $\mathfrak{T} = p^n v_K(f) - b(p^{r+1} - 1)$.

Proof. Since $z_{p^s}(x^i) = (1 \otimes z_{p^s})(\alpha(x))^i$ we have

$$z_{p^s}(x^i) = \sum_{i_1+i_2+i_3=i} f^{i_3} \sum_{i_{3,1}+\dots+i_{3,p-1}=i_3} c_{i_1,i_2,i_3} x^{i_1+p^r \ell'} z_{p^s}(t^{i_2+p^r \ell''})$$

where $c_{i_1,i_2,i_3} \in K^\times$ and, as before,

$$\begin{aligned} \ell' &= i_{3,1} + 2i_{3,2} + \dots + (p-1)i_{3,p-1} \\ \ell'' &= (p-1)i_{3,1} + (p-2)i_{3,2} \dots + i_{3,p-1}. \end{aligned}$$

For a summand to be nontrivial we require $i_2 + p^r \ell'' = p^s$, in which case the summand is a K^\times -multiple of $f^{i_3} x^{i_1 + p^r \ell''}$.

If $s < r$ then Lemma 4.1 gives

$$z_{p^s}(x^i) = i_s x^{i-p^s},$$

and clearly the desired congruence holds.

Now suppose $s \geq r$. Then $z_{p^s}(x^i)$ will again contain the summand $i_{(s)} x^{i-p^s}$ arising from $i_3 = 0$, however there may be positive choices of i_3 which make $i_2 + p^r \ell'' = p^s$. Since $i_3 \leq \ell'' \leq (p-1)i_3$ it follows that $i_3 \leq p^{s-r}$. For an ℓ'' in this interval we have $i_2 = p^s - p^r \ell''$ and $i_1 = i - (p^s - p^r \ell'') - i_3$. Since $\ell' + \ell'' = pi_3$, the $i_3 > 0$ terms in the summand are all of the form

$$\begin{aligned} c_{i_1, i_2, i_3} f^{i_3} x^{i - (p^s - p^r \ell'') - i_3 + p^r (pi_3 - \ell'')} &= c_{i_1, i_2, i_3} f^{i_3} x^{i - p^s + i_3(p^{r+1} - 1)} \\ &= (c_{i_1, i_2, i_3} f^{i_3} x^{i_3(p^{r+1} - 1)}) x^{i - p^s}. \end{aligned}$$

Thus

$$z_{p^s}(x^i) = i_s x^{i-p^s} + \sum (c_{i_1, i_2, i_3} f^{i_3} x^{i_3(p^{r+1} - 1)}) x^{i-p^s},$$

where the sum is taken over all i_1, i_2, i_3 with $i_3 > 0$. Now for $i_3 \geq 1$,

$$\begin{aligned} v_L(f^{i_3} x^{i_3(p^{r+1} - 1)}) &= p^n i_3 v_K(f) - b(i_3(p^{r+1} - 1)) \\ &= i_3(p^n v_K(f) - bp^{r+1} + b), \end{aligned}$$

and since $v_K(f) \geq bp^{r+1-n}$ this expression is minimized when i_3 is minimized, i.e., $i_3 = 1$. Thus

$$v_L(f^{i_3} x^{i_3(p^{r+1} - 1)}) \geq p^n v_K(f) - b(p^{r+1} - 1),$$

so $f^{i_3} x^{i_3(p^{r+1} - 1)} \in \mathfrak{P}_L^{\mathfrak{T}}$, $\mathfrak{T} = p^n v_K(f) - b(p^{r+1} - 1)$. Hence,

$$z_{p^s}(x^i) = i_s x^{i-p^s} \left(1 + \sum c_{i_1, i_2, i_3} f^{i_3} x^{i_3(p^{r+1} - 1)} \right)$$

and so

$$z_{p^s}(x^i) \equiv i_s x^{i-p^s} \pmod{x^{i-p^s} \mathfrak{P}_L^{\mathfrak{T}}}. \quad \square$$

As we have seen, the restriction on $v_K(f)$ is not a restriction on the Hopf algebra, merely on the ways in which this Hopf algebra can act on L . We must write $H = H_{n,r,f}^*$, $v_K(f) \geq bp^{r+1-n}$ for the action (induced from the coaction in Equation (2) for this choice of f) to provide an H -scaffold. As $H_{n,r,f} = H_{n,r, T^{p^{r+1}-1}f}$, it is clear that there will be an infinite number of actions of H on L which produce scaffolds. To ensure a scaffold of tolerance $\mathfrak{T} > 1$ we require a slight increase in the lower bound for $v_K(f)$. For the rest of the section, we shall assume $v_K(f) > bp^{r+1-n}$.

Theorem 5.2. *For $v_K(f) > bp^{r+1-n}$, the set*

$$\left\{ z_1^{j_0} z_p^{j_1} \cdots z_{p^{n-1}}^{j_{n-1}} : 0 \leq j_s \leq p-1 \right\}$$

constructed above is an H -scaffold on L with tolerance

$$\mathfrak{T} = p^n v_K(f) - b(p^{r+1} - 1) > 1.$$

The presentation of the scaffold above follows the form given in Lemma 2.2. To obtain a scaffold which follows Definition 2.1, we pick an integer a such that $ab \equiv -1 \pmod{p^n}$ and set

$$\lambda_j = T^{(j+b\text{res}(aj))/p^n} x^{\text{res}(aj)}, \quad j \in \mathbb{Z}.$$

This set, together, with $\{\Psi_s = z_{p^s} : 0 \leq s \leq n-1\}$, forms the scaffold on L of tolerance \mathfrak{T} as in the sense of Definition 2.1. In particular,

$$\begin{aligned} (5) \quad \lambda_b &= T^{(b+b\text{res}(ab))/p^n} x^{\text{res}(ab)} \\ &= T^{(b+b(p^n-1))/p^n} x^{p^n-1} \\ &= T^b x^{p^n-1}. \end{aligned}$$

As an immediate consequence, we get:

Corollary 5.3. *The set*

$$\left\{ \prod_{s=0}^{n-1} z_{p^s}^{j_s}(\lambda_b) : 0 \leq j_s \leq p-1 \right\}$$

is a K -basis for L .

Proof. This follows from the discussion between Definition 2.1 and Lemma 2.2. In particular, note that

$$\left\{ v_L \left(\prod_{s=0}^{n-1} z_{p^s}^{j_s}(\lambda_b) \right) : 0 \leq s \leq n-1, 0 \leq j_s \leq p-1 \right\}$$

forms a complete set of residues mod p^n . \square

We devote the remainder of this section to showing that the action of H on L has an “integer certificate”. In classical Galois module theory, a number $c \in \mathbb{Z}$ is called an integer certificate if, for all $\rho \in L$ with $v_L(\rho) = c$, the set $\{\sigma(\rho) : \sigma \in \text{Gal}(L/K)\}$ is a K -basis for L . We modify that here: a number $c \in \mathbb{Z}$ is an integer certificate if whenever $v_L(\rho) = c$ the set

$$\left\{ z_1^{j_0} z_p^{j_1} \cdots z_{p^{n-1}}^{j_{n-1}}(\rho) : 0 \leq j_s \leq p-1 \right\}$$

is a K -basis for L .

As an immediate consequence to Proposition 5.1 we get

Corollary 5.4. *Let $0 \leq s \leq n-1$, $1 \leq i \leq p^n-1$. Suppose $z_{p^s}(x^i) \neq 0$. Then $v_L(z_{p^s}(x^i)) = b(p^s - i) = v_L(x^i) + bp^s$.*

As each application of z_{p^s} increases valuation by bp^s , the above result allows us to determine the effect, on valuation, of applying our basis elements of H to the standard K -basis of L .

Corollary 5.5. *Let $1 \leq i \leq p^n - 1$, and let $0 \leq j_s \leq p - 1$ for all $0 \leq s \leq n - 1$. If $z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^i) \neq 0$ then*

$$v_L(z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^i)) = v_L(x^i) + b \sum_{s=0}^{n-1} j_s p^s.$$

To set some notation, given $0 \leq j \leq p^n - 1$, we define $0 \leq j_0, \dots, j_{n-1} \leq p - 1$ to be the unique integers such that

$$j = \sum_{s=0}^{n-1} j_s p^s.$$

Conversely, given a collection $\{j_0, \dots, j_{n-1}\}$ with $0 \leq j_s \leq p - 1$ for all $0 \leq s \leq n - 1$ we define j using the summation above.

We claim that if $v_L(\rho) = b$ then

$$\left\{ z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho) : 0 \leq j_\ell \leq p - 1 \right\}$$

forms a basis for L/K . The crucial step to establishing this is the following.

Proposition 5.6. *Pick $\rho \in L$ with $v_L(\rho) = b$. Then*

$$v_L(z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho)) = b(1 + j).$$

Proof. Any $\rho \in L$ with $v_L(\rho) = b$ has the form

$$\rho = g \left(x^{-1} + \sum_{\ell=1}^{p^n} a_\ell x^{-1-\ell} \right)$$

with $g \in K$, $a_\ell \in K$, $v_K(g) = 0$, and $v_L(a_\ell) > -b\ell$ for all $1 \leq \ell \leq p^n$. Let us write $g = g_0 T^b x^{p^n}$, and for simplicity we assume $g_0 = 1$. Then

$$\rho = T^b x^{p^n-1} + T^b \sum_{\ell=1}^{p^n} a_\ell x^{p^n-1-\ell}$$

(note that $T^b x^{p^n-1}$ is the element λ_b from Equation (5), and thus is part of the scaffold in the Definition 2.1 sense) and

$$\begin{aligned} & z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho) \\ &= T^b z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1}) + T^b \sum_{\ell=1}^{p^n} a_\ell z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1-\ell}). \end{aligned}$$

Applying Corollary 5.5 to the case where $i = p^n - 1 - \ell$, either

$$z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1-\ell}) = 0$$

or

$$v_L(z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1-\ell})) = -b(p^n - 1 - \ell) + bj.$$

Furthermore, observe that

$$z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1}) \neq 0, \quad 0 \leq j_\ell \leq p-1$$

since $p^n - 1 = (p-1) + (p-1)p + \dots + (p-1)p^{n-1}$. Thus,

$$\begin{aligned} v_L(T^b z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1})) &= bp^n - b(p^n - 1) + bj \\ &= b(1 + j) \end{aligned}$$

since

$$v_L(T^b a_\ell z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1-\ell})) \geq p^n b + v_L(a_\ell) - b(p^n - 1 - \ell) + bj$$

and, since $v_L(a_\ell) > -b\ell$,

$$\begin{aligned} p^n b + v_L(a_\ell) - b(p^n - 1 - \ell) + bj &= p^n b + v_L(a_\ell) - bp^n + b + b\ell + bj \\ &= v_L(a_\ell) + b\ell + b(1 + j) \\ &> b(1 + j) \\ &= v_L(T^b z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(x^{p^n-1})), \end{aligned}$$

hence

$$v_L(z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho)) = \min \{b(1 + j), v_L(a_\ell) + b\ell + b(1 + j)\} = b(1 + j)$$

since the minimum is uniquely achieved. \square

Remark 5.7. Generally, it is not the case that if $z_{p^s}(y) \neq 0$ then

$$v_L(z_{p^s}(y)) = v_L(y) + bp^s,$$

i.e., that an application of z_{p^s} universally increases valuation by bp^s . For example, $v_L(x^{p-1} + Tx^p) = -(p-1)b$ but $v_L(z_p(x^{p-1} + Tx^p)) = v_L(T) = p^n$. However, it is always true that $v_L(z_{p^s}(y)) \geq v_L(y) + bp^s$.

Corollary 5.8. *The set $\{z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho) : 0 \leq j_s \leq p-1\}$ forms a K -basis for L , i.e., b is an integer certificate.*

Proof. Observe that

$$\left\{ v_L(z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho)) : 0 \leq j_s \leq p-1 \right\} = \{b(1 + j) : 0 \leq j \leq p^n - 1\}.$$

Now $\{b(1 + j) : 0 \leq j \leq p^n - 1\}$ is a complete set of residues mod p^n since $p \nmid b$. Thus, $\{z_1^{j_0} z_p^{j_1} \dots z_{p^{n-1}}^{j_{n-1}}(\rho) : 0 \leq j_s \leq p-1\}$ is K -linearly independent, and hence a basis for L . \square

6. Integral Hopf Galois module structure

In this section we describe the Hopf Galois module structure of \mathfrak{D}_L and of all of the fractional ideals \mathfrak{P}_L^h of L . Given a high enough tolerance level, the results of [2] enable us to describe the H -module structure of \mathfrak{P}_L^h . We apply their work below, and then we will take a look at a specific action of H on L .

Let $h \in \mathbb{Z}$. Since $\mathfrak{P}_L^{h+p^n} = T\mathfrak{P}_L^h$ and $\mathfrak{A}_{h+p^n} = \mathfrak{A}_h$ it suffices to consider the Hopf Galois module structure on a complete set of residues mod p^n . We will pick the set of residues h such that $0 \leq b - h \leq p^n - 1$.

We start with:

Lemma 6.1. *There exists actions of H on L which produce H -scaffold structures on L/K with arbitrarily high tolerance.*

Proof. As we can write $H = H_{n,r,f}^*$ with $v_K(f)$ of arbitrarily high valuation, this is clear since $\mathfrak{T} = p^n v_K(f) - b(p^{r+1} - 1)$ for $v_K(f) \geq bp^{r+1-n}$. \square

For the remainder of this section, pick f such that

$$v_K(f) \geq \frac{2p^n - 1 + b(p^{r+1} - 1)}{p^n},$$

so $\mathfrak{T} \geq 2p^n - 1$. This level of tolerance allows us to determine integral Hopf Galois module structure.

Remark 6.2. This new bound on $v_K(f)$ is larger than the one we imposed in Section 5. While we could have simply assumed

$$v_K(f) \geq (2p^n - 1 + b(p^{r+1} - 1))p^{-n}$$

throughout, we wanted to also provide examples of H -scaffolds for which Hopf Galois module structure could not be completely determined.

We will now introduce numerical data from [2]. For each $0 \leq j \leq p^n - 1$, let

$$d_h(j) = \left\lfloor \frac{bj + b - h}{p^n} \right\rfloor$$

$$w_h(j) = \min \{d_h(i + j) - d_h(i) : 0 \leq i \leq p^n - 1, i_s + j_s \leq p - 1 \text{ for all } s\},$$

using our convention that $j = \sum j_s p^s$, $i = \sum i_s p^s$ as before. Then, using Theorem 3.1, Theorem 3.7, and Corollary 3.2 of [2] we get all of the following.

Proposition 6.3. *With the notation as above:*

- (1) \mathfrak{A}_h has \mathfrak{D}_K -basis $\left\{ T^{-w_h(j)} z_1^{j_0} z_p^{j_1} \cdots z_{p^{n-1}}^{j_{n-1}} : 0 \leq j \leq p^n - 1 \right\}$.
- (2) \mathfrak{D}_K is a free \mathfrak{A} -module of rank one — explicitly,

$$\mathfrak{D}_L = \mathfrak{A} \cdot \rho, \quad v_L(\rho) = b$$

— if $\text{res}(b) \mid (p^m - 1)$ for some $1 \leq m \leq n$.

- (3) \mathfrak{P}_L^h is a free \mathfrak{A}_h -module if and only if $w_h(j) = d_h(j)$ for all $0 \leq j \leq p^n - 1$; furthermore if this equality holds then $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot \rho$, $v_L(\rho) = b$.
- (4) If $w_h(j) \neq d_h(j)$, then \mathfrak{P}_L^h can be generated over \mathfrak{A}_h using ℓ generators, where

$$\ell = \# \{i : d_h(i) > d_h(i - j) + w_h(j) \text{ for all } 0 \leq j \leq p^{n-1} \text{ with } j_s \leq i_s\}.$$

Remark 6.4. It is important to note that the determination as to whether \mathfrak{P}_L^h is free over \mathfrak{A}_h does not depend on the H -scaffold itself, merely on the behavior of d_h and w_h .

Remark 6.5. Note that if $\text{res}(b) \mid (p^m - 1)$ then \mathfrak{D}_K is free over \mathfrak{A} , but in general the converse does not hold. But since (2) is a special case of (3) where $h = 0$ we do have necessary and sufficient conditions for when \mathfrak{D}_K is free over \mathfrak{A} .

Let us interpret these results in the case where $b = 1$, which requires that $v_K(f) \geq 3$. (Note that scaffolds exist for $v_K(f) = 2$, as well as for $v_K(f) = 1$ unless $n = r + 1$.) Then $2 - p^n \leq h \leq 1$ and

$$d_h(j) = \left\lfloor \frac{j + 1 - h}{p^n} \right\rfloor = \begin{cases} 1 & j \geq p^n - 1 + h \\ 0 & j < p^n - 1 + h. \end{cases}$$

Since $w_h(j) \leq d_h(j)$, which is readily seen by setting $i = 0$ in the definition of $w_h(j)$, the statement $w_h(j) = d_h(j)$ for all $0 \leq j \leq p^n - 1$ is true if and only if $w_h(j) = 1$ whenever $j \geq p^n - 1 + h$. Suppose $h > (1 - p^n)/2$ and $d_h(j) = 1$. Then $j > p^n - 1 + (1 - p^n)/2 = (p^n - 1)/2$. Now assume there exists an i such that $d_h(i + j) - d_h(i) = 0$ and $i_s + j_s \leq p - 1$ for all s . Then $d_h(i + j) \geq d_h(j) = 1$ so $d_h(i) = 1$ as well. Thus $i > (p^n - 1)/2$. But then $i + j \geq p^n$, contradicting the fact that $i_s + j_s \leq p - 1$ for all s . Therefore, no such i can occur, hence $w_h(j) = d_h(j)$ for all j and $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot \rho$.

Now suppose that $h \leq (1 - p^n)/2$ and let $j = p^n + h - 1$. Then $d_h(j) = 1$. Let

$$i = p^n - 1 - j = p^n - 1 - (p^n + h - 1) = -h.$$

Then $i_s + j_s = p - 1$ for all s . As above, $d_h(i + j) = 1$. But $i = -h < p^n - 1 - h$ so $d_h(i) = 0$. Thus $w_h(j) = w_h(p^n + h - 1) = 0$ and \mathfrak{P}_L^h is not free over \mathfrak{A}_h .

We summarize, generalizing to all $h \in \mathbb{Z}$.

Theorem 6.6. Let $H = H_{n,r,f}^*$, $0 < r < n \leq 2r$, $f \in K^\times$. Suppose $v_L(x) = -1$ and $v_L(\rho) = 1$. Let $h \in \mathbb{Z}$, and let $m = \lfloor h/p^n \rfloor$. Then \mathfrak{P}_L^h is free over \mathfrak{A}_h if and only if $\text{res}(h - 2) > (p^n - 3)/2$; under this restriction, $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot (T^m \rho)$.

Remark 6.7. Notice that we do not need $v_K(f) \geq 2(1 - p^{-n}) + p^{r+1-n}$ in the statement above since, for any $f \in K^\times$, an $H_{n,r,f}^*$ of suitably high tolerance exists.

Proof. Consider first the case $2 - p^n \leq h \leq 1$. Then, $0 \leq h - 2 + p^n \leq p^n - 1$. We have seen that \mathfrak{P}_L^h is \mathfrak{A}_h -free if and only if $h > (1 - p^n)/2$, and since $h \leq 1$ this inequality holds if and only if

$$\frac{p^n - 3}{2} < h - 2 + p^n \leq p^n - 1.$$

Thus, \mathfrak{P}_L^h is \mathfrak{A}_h -free if and only if $\text{res}(h - 2) > (p^n - 3)/2$.

Now for more general h , \mathfrak{P}_L^h is free over \mathfrak{A}_h if and only if $\mathfrak{P}_L^{\text{res}(h)}$ is free over $\mathfrak{A}_{\text{res}(h)} = \mathfrak{A}_h$, so we have freeness if and only if

$$\text{res}(\text{res}(h) - 2) > (p^n - 3)/2,$$

and since the left-hand side reduces to $\text{res}(h - 2)$ we get the inequality desired. That $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot (T^m \rho)$ is immediate since $\mathfrak{P}_L^h = T^m \mathfrak{P}_L^{\text{res}(h)}$. \square

In particular, notice that \mathfrak{D}_L is free over \mathfrak{A} when $b = 1$.

7. Picking the best Hopf algebra and action

In the examples provided here — with $L = K(x)$, $v_L(x) = b$, $p \nmid b$ — questions concerning the Hopf Galois module structure of \mathfrak{D}_L have little to do with the exact Hopf algebra chosen. For any choice of $0 < r < n \leq 2r$ and $v_K(f) \geq 2 - p^n(1 - b(p^{r+1} - 1))$ we have scaffolds of sufficiently high tolerance, and their existence allows us to apply the numerical data of Proposition 6.3. So, if \mathfrak{P}_L^h is free over \mathfrak{A}_h for $H = H_{n,r,f}^*$, then \mathfrak{P}_L^h is free over \mathfrak{A}_h for any $H = H_{n,r',f'}^*$, $0 < r' < n \leq 2r$ and $v_K(f') \geq 2 - p^n(1 - b(p^{r'+1} - 1))$. Additionally, the description of \mathfrak{A}_h given in Proposition 6.3 is independent of which Hopf algebra H is chosen since the value of $T^{-w_h(j)}$ is independent of H ; of course, the actual elements z_{p^s} depend on the chosen H .

In addition to the family constructed here, the divided power K -Hopf algebra A of rank p^n found in ([12, Ex. 5.6.8], where it is denoted H) acts on L : in terms of its dual, A^* represents the n^{th} Frobenius kernel of the additive group scheme, and its simple coaction is given by Chase in [4]. In [2, Sec. 5.2] a scaffold of infinite tolerance (so the congruences are replaced by equalities) is constructed for A . Their scaffold is similar to our constructions — indeed, for large values of $v_K(f)$, A^* and $H_{n,r,f}$ act very similarly on L , and we can view $H_{n,r,f}$ as a deformation of A^* .

Thus, it is natural to ask: which Hopf algebra is “best”? As the determination of integral Hopf Galois module structure does not depend on the choice of H , there would need to be further properties of interest to make a distinction.

For a single choice of $H_{n,r,f}$, different actions lead to scaffolds of different tolerances, though we can always make \mathfrak{T} arbitrarily large. So here, we may ask: which action is the “best”? If one is primarily interested in describing \mathfrak{D}_L as an \mathfrak{A} -module then the action where $L = K(x)$, $v_L(x) = -1$ appears to be a good choice since \mathfrak{D}_L is free over \mathfrak{A} whenever $v_K(f) \geq 3$. If, on the other hand, one is primarily interested in describing \mathfrak{P}_L^h for a specific value of h

there may be better choices. For example, in an unpublished work by Jelena Sundukova, she states that \mathfrak{P}_L^h is a free \mathfrak{A}_h -module if $v_L(x) = -h$. Her work also describes choices of $v_L(x)$ which make \mathfrak{P}_L^h free over \mathfrak{A}_h reasonably rare, for example $v_L(x) = p^n - 2$. As with choosing the Hopf algebra, we would need to have more properties of this action which we deem “desirable” in order to pick one action over another.

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