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Totally umbilical hemi-slant lightlike submanifolds

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Abstract. We study hemi-slant lightlike submanifolds of indefinite Kaehler manifolds and prove that there do not exist totally umbilical hemi-slant lightlike submanifolds in indefinite Kaehler manifolds other than totally geodesic hemi-slant lightlike submanifolds. We consequently prove that the induced connection of a totally umbilical hemi-slant lightlike submanifold is a metric connection. We obtain a characterization theorem on the nonexistence of totally umbilical hemi-slant lightlike submanifold of an indefinite complex space form and some characterization theorems on minimal hemi-slant lightlike submanifolds.

Contents

1. Introduction

As a generalization of holomorphic and totally real submanifolds in complex geometry, Chen [5, 6] introduced the notion of slant submanifolds. They are further studied by many authors, e.g., [4, 3, 11]. They all studied the geometry of slant submanifolds with positive definite metric. Therefore this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. The geometry of submanifolds is one of the most important topics of differential geometry. The geometry of semi-Riemannian submanifolds has many similarities with the Riemannian case but the geometry of lightlike submanifolds is different since their normal vector bundle intersects with the tangent bundle making it more difficult. Recently Sahin [13] proved that there do not exist totally umbilical slant submanifolds in Kaehler manifolds other than totally

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geodesic slant submanifolds. Haider et al. [10] introduced hemi-slant lightlike submanifolds of an indefinite Kaehler manifold. Moreover, the growing importance of lightlike submanifolds in mathematical physics and relativity makes this a topic of current interest. Therefore, we extended a result proved by Sahin in [13] to hemi-slant lightlike submanifolds of indefinite Kaehler manifolds. In particular, we prove that there do not exist totally umbilical hemi-slant lightlike submanifolds in indefinite Kaehler manifolds other than totally geodesic hemi-slant lightlike submanifolds. We also obtain a characterization theorem on the nonexistence of totally umbilical hemi-slant lightlike submanifold of an indefinite complex space form and some characterization theorems on minimal hemi-slant lightlike submanifolds.

2. Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}\, T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $\mathrm{Rad}\, TM : x \in M \longrightarrow \mathrm{Rad}\, T_xM$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $\mathrm{Rad}\, TM$ is called the radical distribution on M (for details see [7]).

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = \mathrm{Rad}\, TM \perp S(TM), \tag{1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}\, TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (\mathrm{Rad}\, TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as

$$\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\},$$

where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}\, TM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have:

Theorem 2.1 ([7]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM) \mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp} \mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, \ldots, r\}, \tag{4}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

We have four cases for a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$:

- M is an r-lightlike submanifold if $r < \min\{m, n\}$.
- M is a coisotropic lightlike submanifold if
$$r = n < m, \quad S(TM^{\perp}) = \{0\}.$$
- M is an isotropic lightlike submanifold if $r = m < n$, $S(TM) = \{0\}$.
- M is a totally lightlike submanifold if
$$r = n = m, \quad S(TM^{\perp}) = \{0\} = S(TM).$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is linear a operator on M, known as shape operator. According to (2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (5) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{6}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{7}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{8}$$

where we put

$$h^l(X, Y) = L(h(X, Y)), \qquad h^s(X, Y) = S(h(X, Y)),$$
$$\nabla^l_X N = L(\nabla^{\perp}_X N), \qquad \nabla^s_X W = S(\nabla^{\perp}_X W),$$
$$D^s(X, N) = S(\nabla^{\perp}_X N), \qquad D^l(X, W) = L(\nabla^{\perp}_X W),$$

for any $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. Using (2)–(3) and (6)–(8), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$.

Let $\bar{P}$ is a projection of TM on $S(TM)$. Now, we consider the decomposition (4), we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\,TM)$, where $\{\nabla^*_X \bar{P}Y,\ A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}\,TM)$ respectively. Here ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $\mathrm{Rad}\,TM$ respectively. By using (7), (8) and (10), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \quad \bar{g}(h^*(X, \bar{P}Y), N) = \bar{g}(A_N X, \bar{P}Y). \tag{11}$$

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is, $(\bar{\nabla}_X \bar{J})Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$.

3. Hemi-slant lightlike submanifolds

Definition 3.1. Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then the submanifold M is said to be hemi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\mathrm{Rad}\,TM$ is a distribution on M such that $\bar{J}(\mathrm{Rad}\,TM) = ltr(TM)$.

(ii) For all $x \in U \subset M$ and for each nonzero vector field X tangent to $S(TM)$, the angle $\theta(X)$, called slant angle, between $\bar{J}X$ and the vector space $S(TM)$ is constant.

A hemi-slant lightlike submanifold is said to be proper if $S(TM) \neq 0$ and $\theta \neq 0, \pi/2$.

Example 1. Let $\bar{M} = (\Re^8_2, \bar{g})$ be a semi-Riemannian manifold, where $\Re^8_2$ is a semi-Euclidean space of signature $(-,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8\}$. Let M be a submanifold of $\Re^8_2$ given by

$$X(s, t, u, v) = (s, t, \sin u, \cos u, -u \sin v, -u \cos v, t, s),$$

then the tangent bundle TM is spanned by

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_8, \\ Z_2 &= \partial x_2 + \partial x_7, \\ Z_3 &= \cos u \partial x_3 - \sin u \partial x_4 - \sin v \partial x_5 - \cos v \partial x_6, \\ Z_4 &= -u \cos v \partial x_5 + u \sin v \partial x_6. \end{aligned}$$

Then it is easy to see that M is a 2-lightlike submanifold with

$$\mathrm{Rad}\,TM = \mathrm{span}\{Z_1, Z_2\}, \quad S(TM) = \mathrm{span}\{Z_3, Z_4\}$$

and that M is Riemannian. It is easy to prove that $S(TM)$ is a slant distribution with slant angle $\theta = \pi/4$. Moreover, the screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W_1 = \sin u\partial x_3 + \cos u\partial x_4, \quad W_2 = \sec u\partial x_3 + \sin v\partial x_5 + \cos v\partial x_6.$$

The transversal lightlike bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\partial x_1 + \partial x_8), \quad N_2 = \frac{1}{2}(-\partial x_2 + \partial x_7).$$

Hence M is a hemi-slant lightlike submanifold of $\bar{M}$.

Using the decomposition in (1), we denote P and Q the projection morphisms on the distributions $S(TM)$ and $\operatorname{Rad} TM$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = PX + QX. \tag{12}$$

Applying $\bar{J}$ to (12) we obtain

$$\bar{J}X = TPX + FPX + FQX. \tag{13}$$

Then using the definition of hemi-slant lightlike submanifolds, we get

$$TPX \in \Gamma(S(TM)), \quad FPX \in \Gamma(S(TM^{\perp})), \quad FQX \in \Gamma(ltr(TM)). \tag{14}$$

Similarly, for any $U \in \Gamma(tr(TM))$, we have

$$\bar{J}U = BU + CU, \tag{15}$$

where BU and CU are the tangential and the transversal components of $\bar{J}U$, respectively. Differentiating (13) and using (6) to (8) and (15), for any $X, Y \in \Gamma(TM)$, we have

$$(\nabla_X T)PY = A_{FPY}X + A_{FQY}X + Bh^s(X,Y) + \bar{J}h^l(X,Y), \tag{16}$$

$$(\nabla_X F)PY = Ch^s(X,Y) - h^s(X,TPY) - D^s(X,FQY), \tag{17}$$

$$(\nabla_X F)QY = -h^l(X,TPY) - D^l(X,FPY). \tag{18}$$

where

$$(\nabla_X T)PY = \nabla_X TPY - TP\nabla_X Y,$$
$$(\nabla_X F)PY = \nabla^s_X FPY - FP\nabla_X Y,$$
$$(\nabla_X F)QY = \nabla^l_X FQY - FQ\nabla_X Y.$$

Theorem 3.2 ([10]). *The necessary and sufficient conditions for a $2q$-lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ of index $2q$ to be hemi-slant lightlike submanifold are:*

(a) *$\bar{J}(ltr(TM))$ is a distribution on M.*
(b) *For any vector field X tangent to M, there exists $\lambda \in [-1, 0]$ such that $(PT)^2X = \lambda X$, where $X \in \Gamma(TM)$ and $\lambda = -\cos^2\theta$.*

Corollary 3.3 ([10])**.** *Let M be a hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then we have*

$$g(TPX, TPX) = \cos^2\theta g(PX, PX), \tag{19}$$

$$\bar{g}(FPX, FPX) = \sin^2\theta g(PX, PX), \tag{20}$$

for any $X, Y \in \Gamma(TM)$.

Definition 3.4 ([12])**.** Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is called a semi-transversal lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\mathrm{Rad}\,TM$ is a distribution on M such that $\bar{J}(\mathrm{Rad}\,TM) = ltr(TM)$.

(ii) There exists a real nonnull distribution $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad \bar{J}D = D, \quad \bar{J}(D^{\perp}) \subset S(TM^{\perp}),$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

Lemma 3.5 ([12])**.** *Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with constant index $2q$ such that $2q < \dim(\bar{M})$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

Theorem 3.6. *Let M be a $2q$-dimensional lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then any coisotropic semi-transversal lightlike submanifold is a hemi-slant lightlike submanifold with $\theta = 0$. Moreover, any semi-transversal lightlike submanifold of $\bar{M}$ with $D = \{0\}$ is a hemi-slant submanifold with $\theta = \pi/2$.*

Proof. Let M be a $2q$-dimensional semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then by the definition of semi-transversal lightlike submanifold, $\mathrm{Rad}\,TM$ is a distribution on M such that $\bar{J}(\mathrm{Rad}\,TM) = ltr(TM)$. If M is coisotropic, then $S(TM^{\perp}) = \{0\}$ thus $D^{\perp} = \{0\}$. Using the Lemma (3.5), the screen distribution $S(TM)$ is Riemannian. Since D is invariant with respect to $\bar{J}$ therefore it follows that $\theta = 0$. The second assertion is clear. □

Theorem 3.7. *There exist no proper hemi-slant totally lightlike or isotropic submanifold in indefinite Kaehler manifolds.*

Proof. Let M be a totally lightlike submanifold of an indefinite Kaehler manifold then $TM = \mathrm{Rad}(TM)$ and hence $S(TM) = \{0\}$. The other assertion follows similarly. □

Lemma 3.8. *Let M be a hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $FPX \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$.*

Proof. Using (2) and (4) it is clear that $FPX \in \Gamma(S(TM^{\perp}))$ if and only if $\bar{g}(FPX, \xi) = 0$, for any $\xi \in \Gamma(\mathrm{Rad}\,TM)$. Therefore

$$\bar{g}(FPX, \xi) = \bar{g}(\bar{J}PX - TPX, \xi) = \bar{g}(\bar{J}PX, \xi) = -\bar{g}(PX, \bar{J}\xi) = 0$$

gives the result. □

Thus from the (14) it follows that $F(S(TM))$ is a subspace of $S(TM^{\perp})$. Therefore there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^{\perp}) = F(S(T_pM))\perp\mu_p. \tag{21}$$

Thus

$$T_p\bar{M} = S(T_pM)\perp\{\mathrm{Rad}(T_pM)\oplus ltr(T_pM)\}\perp\{F(S(T_pM))\perp\mu_p\}.$$

Definition 3.9 ([8])**.** A lightlike submanifold M of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H\in\Gamma(tr(TM))$ on M, called transversal curvature vector field of M, such that, for all $X,Y\in\Gamma(TM)$

$$h(X,Y) = Hg(X,Y). \tag{22}$$

It is clear that M is totally umbilical if and only if on each coordinate neighborhood u there exists smooth vector fields $H^l\in\Gamma(ltr(TM))$ and $H^s\in\Gamma(S(TM^{\perp}))$, such that

$$h^l(X,Y) = H^lg(X,Y),\quad D^l(X,W)=0,\quad h^s(X,Y)=H^sg(X,Y). \tag{23}$$

A lightlike submanifold is said to be totally geodesic if $h(X,Y)=0$, for any $X,Y\in\Gamma(TM)$, that is, a lightlike submanifold is totally geodesic if $H^l=0$ and $H^s=0$.

Lemma 3.10. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $\nabla_XX\in\Gamma(S(TM))$ for any $X\in\Gamma(S(TM))$.*

Proof. Let $X\in\Gamma(S(TM))$ and $N\in\Gamma(ltr(TM))$ such that $\bar{J}N=\xi$ then using (6), (8), (13) and (23), we obtain

$$\begin{aligned}\bar{g}(\nabla_XX,N) &= \bar{g}(\nabla_X\bar{J}X,\bar{J}N) = \bar{g}(\bar{\nabla}_XTX,\xi)+\bar{g}(\bar{\nabla}_XFX,\xi)\\ &= \bar{g}(h^l(X,TX),\xi)+\bar{g}(D^l(X,FX),\xi)\\ &= g(X,TX)g(H^l,\xi).\end{aligned}$$

Since $g(X,TX)=g(X,JX)=0$, we have $\bar{g}(\nabla_XX,N)=0$. Hence using (1) and (4), the assertion follows. □

Theorem 3.11. *Every totally umbilical proper hemi-slant lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is totally geodesic.*

Proof. Let $X,Y\in\Gamma(S(TM))$ then by adding (17) and (18) we obtain

$$\begin{aligned}&\nabla^s_XFPY - FP\nabla_XY - FQ\nabla_XY\\ &\qquad = Ch^s(X,Y)-h^s(X,TPY)-h^l(X,TPY)-D^l(X,FPY).\end{aligned}$$

Replace X by TX and Y by X, we get

$$\begin{aligned}FP\nabla_{TX}X+FQ\nabla_{TX}X &= \nabla^s_{TX}FPX - Ch^s(TX,X)+h^s(TX,TPX)\\ &\quad + h^l(TX,TPX)+D^l(TX,FPX).\end{aligned}$$

Since M is totally umbilical hemi-slant lightlike submanifold therefore using (19), (23) and the fact $h^s(TX,X) = g(TX,X)H^s = \bar{g}(\bar{J}X,X)H^s = 0$, we get $\cos^2\theta g(X,X)H = FP\nabla_{TX}X + FQ\nabla_{TX}X - \nabla^s_{TX}FPX$. Now taking the scalar product both sides with respect to $FPX \in (S(TM^\perp))$, we obtain

$$\cos^2\theta g(X,X)\bar{g}(H^s,FPX) = \bar{g}(FP\nabla_{TX}X,FPX) - \bar{g}(\nabla^s_{TX}FPX,FPX).$$

Further using (20), we get

$$\begin{aligned}(24)\quad & \cos^2\theta g(X,X)\bar{g}(H^s,FPX)\\ &= \sin^2\theta\bar{g}(P\nabla_{TX}X,PX) - \bar{g}(\nabla^s_{TX}FPX,FPX).\end{aligned}$$

Next, taking covariant derivative of (20) with respect to $\bar{\nabla}_{TX}$, we get

$$\bar{g}(\nabla^s_{TX}FPX,FPX) = \sin^2\theta g(\nabla_{TX}PX,PX)$$

and then using this in (24), we obtain $\cos^2\theta g(X,X)\bar{g}(H^s,FPX) = 0$. Since M is a proper hemi-slant lightlike submanifold and g is a Riemannian metric on $S(TM)$ therefore we have $\bar{g}(H^s,FPX) = 0$. Thus using (14) and (21), we obtain

$$(25)\qquad H^s \in \Gamma(\mu).$$

Now, using the Kaehlerian property of $\bar{M}$ for any $X,Y \in \Gamma(S(TM))$ with (6), (8) and (22), we have

$$\begin{aligned}&\nabla_X TPY + g(X,TPY)H - A_{FPY}X + \nabla^s_X FPY + D^l(X,FPY)\\ &\qquad = T\nabla_X Y + F\nabla_X Y + g(X,Y)\bar{J}H.\end{aligned}$$

Taking the scalar product of both sides with respect to $\bar{J}H^s$ and then using invariant property of μ with (25), we obtain

$$(26)\qquad \bar{g}(\nabla^s_X FPY,\bar{J}H^s) = g(X,Y)\bar{g}(H^s,H^s).$$

Since μ is an invariant subspace therefore using the Kaehlerian character of $\bar{M}$ for any $H^s \in \Gamma(\mu)$, we get

$$\begin{aligned}&-A_{\bar{J}H^s}X + \nabla^s_X\bar{J}H^s + D^l(X,\bar{J}H^s)\\ &\qquad = -TA_{H^s}X - FA_{H^s}X + B\nabla^s_X H^s + C\nabla^s_X H^s + \bar{J}D^l(X,H^s).\end{aligned}$$

Taking the scalar product of both sides with respect to FPY, we get

$$\bar{g}(\nabla^s_X\bar{J}H^s,FPY) = -\bar{g}(FA_{H^s}X,FPY) + \bar{g}(C\nabla^s_X H^s,FPY).$$

From (15), we know that for any $U \in \Gamma(tr(TM))$, BU and CU are tangential and transversal components of $\bar{J}U$, respectively. Thus if $U \in \Gamma(ltr(TM))$ then $\bar{J}U = BU \in \Gamma(\mathrm{Rad}(TM))$ and $CU = 0$. Moreover, since

$$S(TM^\perp) = F(S(TM))\perp\mu,$$

therefore for any $U \in \Gamma(S(TM^\perp))$, $BU \in \Gamma(S(TM))$ and $CU \in \Gamma(\mu)$. Since $\nabla^s_X\alpha_S \in \Gamma(S(TM^\perp))$ therefore $C\nabla^s_X\alpha_S \in \Gamma(\mu)$. Hence, we have

$$(27)\qquad \bar{g}(\nabla^s_X\bar{J}H^s,FPY) = -\bar{g}(FA_{H^s}X,FPY) = -\sin^2\theta g(A_{H^s}X,PY).$$

Since $\bar{\nabla}$ is a metric connection therefore $(\bar{\nabla}_X g)(FPY, \bar{J}H^s) = 0$ this further implies that $\bar{g}(\nabla^s_X FPY, \bar{J}H^s) = \bar{g}(\nabla^s_X \bar{J}H^s, FPY)$, therefore using (27), we obtain

$$\bar{g}(\nabla^s_X FPY, \bar{J}H^s) = -\sin^2\theta g(A_{H^s}X, PY). \tag{28}$$

From (26) and (28), we have $g(X,Y)g(H^s,H^s) = -\sin^2\theta g(A_{H^s}X, PY)$ and using (9) here, we obtain

$$g(X,Y)g(H^s,H^s) = -\sin^2\theta\bar{g}(h^s(X,PY),H^s) = -\sin^2\theta g(X,Y)g(H^s,H^s).$$

This implies that $(1+\sin)^2\theta g(X,Y)g(H^s,H^s) = 0$. Since M is a proper hemi-slant lightlike submanifold and g is a Riemannian metric on $S(TM)$ therefore we obtain

$$H^s = 0. \tag{29}$$

Furthermore, using the Kaehler character of $\bar{M}$ for any $X \in \Gamma(S(TM))$, we have

$$\begin{aligned}\nabla_X TX + h(X,TX) - A_{FX}X + \nabla^s_X FX + D^l(X,FX)\\ = T\nabla_X X + F\nabla_X X + Bh(X,X) + Ch^s(X,X).\end{aligned}$$

Since M is totally umbilical hemi-slant lightlike submanifold therefore using $h(X,TX) = 0$ and then comparing the tangential components, we obtain $\nabla_X TX - A_{FX}X = T\nabla_X X + Bh(X,X)$. Taking the scalar product both sides with respect to $N \in \Gamma(ltr(TM))$ such that $\bar{J}N = \xi \in \Gamma(\mathrm{Rad}(TM))$ and using the Lemma (3.10), we get

$$\bar{g}(A_{FX}X, N) = \bar{g}(\bar{J}h^l(X,X), N). \tag{30}$$

Now using (6), (8), (23) and the Lemma (3.10), we have

$$\begin{aligned}\bar{g}(A_{FX}X,N) &= \bar{g}(\bar{J}X, \bar{\nabla}_X N) - \bar{g}(TX, \bar{\nabla}_X N) = \bar{g}(\bar{\nabla}_X X, \xi) + \bar{g}(\bar{\nabla}_X TX, N)\\ &= \bar{g}(h^l(X,X),\xi).\end{aligned}$$

Hence using this in (30), we have $2\bar{g}(h^l(X,X),\xi) = 0$, since M is a totally umbilical proper hemi-slant lightlike submanifold therefore we have $g(X,X)\bar{g}(H^l,\xi) = 0$. Since g is a Riemannian metric on $S(TM)$ therefore $\bar{g}(H^l,\xi) = 0$ then using (4), we obtain that

$$H^l = 0. \tag{31}$$

Thus from (29) and (31), the proof is complete. □

Theorem 3.12. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the induced connection ∇ is a metric connection on M.*

Proof. Using (23) and (31), we have $h^l = 0$ then using the Theorem 2.2 in [7], at page 159, the induced connection ∇ becomes a metric connection on M. □

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively then by straightforward calculations ([7]), we have

$$\begin{aligned}\bar{R}(X,Y)Z = {}& R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ & + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ & - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)),\end{aligned} \tag{32}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{33}$$

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z). \tag{34}$$

An indefinite complex space form is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c, denoted by $\bar{M}(c)$, whose curvature tensor $\bar{R}$ is given by [1]

$$\begin{aligned}\bar{R}(X,Y)Z = \frac{c}{4}\{&\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\ & - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},\end{aligned} \tag{35}$$

for X, Y, Z vector fields on $\bar{M}$.

Theorem 3.13. *There exists no totally umbilical proper hemi-slant lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $c \neq 0$.*

Proof. Suppose M be a totally umbilical proper hemi-slant lightlike submanifold of $\bar{M}(c)$ such that $c \neq 0$. Then using (35), for any $X \in \Gamma(S(TM))$ and $\xi, \xi' \in \Gamma(\mathrm{Rad}(TM))$, we obtain

$$\bar{g}(\bar{R}(X, \bar{J}X)\xi', \xi) = -\frac{c}{2} g(X,X) g(\bar{J}\xi', \xi). \tag{36}$$

On the other hand using (23) and (32), we get

$$\bar{g}(\bar{R}(X, \bar{J}X)\xi', \xi) = \bar{g}((\nabla_X h^l)(\bar{J}X, \xi'), \xi) - \bar{g}((\nabla_{\bar{J}X} h^l)(X, \xi'), \xi). \tag{37}$$

Now, using (23) and (34), we have

$$\begin{aligned}(\nabla_X h^l)(\bar{J}X, \xi') &= -g(\nabla_X \bar{J}X, \xi')H^l - g(\bar{J}X, \nabla_X \xi')H^l \\ &= \bar{g}(\bar{\nabla}_X TX, \xi') = \bar{g}(h^l(X, TX), \xi') \\ &= g(X, \bar{J}X)\bar{g}(H^l, \xi') = 0.\end{aligned} \tag{38}$$

Similarly

$$\begin{aligned}(\nabla_{\bar{J}X} h^l)(X, \xi') &= -g(\nabla_{\bar{J}X} X, \xi')H^l - g(X, \nabla_{\bar{J}X}\xi')H^l = \bar{g}(\bar{\nabla}_{\bar{J}X} X, \xi') \\ &= \bar{g}(h^l(\bar{J}X, X), \xi') = g(\bar{J}X, X)\bar{g}(H^l, \xi') = 0.\end{aligned} \tag{39}$$

Thus from (36) to (40), we obtain

$$\frac{c}{2} g(X,X) g(\bar{J}\xi', \xi) = 0.$$

Since g is a Riemannian metric on $S(TM)$ and (4) implies that $g(\bar{J}\xi', \xi) \neq 0$, therefore $c = 0$. This contradiction completes the proof. □

In [7], a minimal lightlike submanifold M is defined when M is a hypersurface of a 4-dimensional Minkowski space. Then in [2], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is introduced as follows:

Definition 3.14. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:

(i) $h^s = 0$ on $\mathrm{Rad}(TM)$.
(ii) $\mathrm{trace}\, h = 0$, where trace is written with respect to g restricted to $S(TM)$.

We use the quasi orthonormal basis of M given by

$$\{\xi_1, \ldots, \xi_r, e_1, \ldots, e_k\},$$

such that $\{\xi_1, \ldots, \xi_r\}$ and $\{e_1, \ldots, e_k\}$ form a basis of $\mathrm{Rad}(TM)$ and $S(TM)$ respectively.

Theorem 3.15. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then M is minimal.*

Proof. The assertion follows directly using the Theorem (3.11). □

Lemma 3.16. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $\dim(S(TM)) = \dim(S(TM^\perp))$. If $\{e_1, \ldots, e_k\}$ is a local orthonormal basis of $\Gamma(S(TM))$ then*

$$\{\csc\theta F e_1, \ldots, \csc\theta F e_k\}$$

is an orthonormal basis of $S(TM^\perp)$).

Proof. Since $e_1, \ldots, e_k$ is a local orthonormal basis of $S(TM)$ and $S(TM)$ is Riemannian therefore using (20), we obtain

$$\bar{g}(\csc\theta F e_i, \csc\theta F e_j) = \csc^2\theta \sin^2\theta g(e_i, e_j) = \delta_{ij},$$

this proves the assertion. □

Definition 3.17 ([9])**.** A lightlike submanifold is called irrotational if and only if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$.

Theorem 3.18. *Let M be an irrotational hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if*

$$\mathrm{trace}\, A_{W_q}|_{S(TM)} = 0, \mathrm{trace}\, A^*_{\xi_j}|_{S(TM)} = 0,$$

where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $\mathrm{Rad}(TM)$.

Proof. Since M is irrotational so this implies that $h^s(X,\xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. Thus h^s vanishes on $\mathrm{Rad}(TM)$. Hence M is minimal if and only if trace $h = 0$ on $S(TM)$, that is, M is minimal if and only $\sum_{i=1}^{k} h(e_i, e_i) = 0$. Using (9) and (11) we obtain

$$\sum_{i=1}^{k} h(e_i,e_i) = \sum_{i=1}^{k} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_i, e_i) N_j + \frac{1}{l} \sum_{q=1}^{l} g(A_{W_q} e_i, e_i) W_q \right\}. \tag{40}$$

Thus the assertion follows from (40). □

Theorem 3.19. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if*

$$\mathrm{trace}\, A_{W_q}|_{S(TM)} = 0, \mathrm{trace}\, A^*_{\xi_j}|_{S(TM)} = 0,$$

and

$$\bar{g}(D^l(X,W),Y) = 0, \forall X, Y \in \Gamma(\mathrm{Rad}(TM)),$$

where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $\mathrm{Rad}(TM)$.

Proof. Let $X, Y \in \Gamma(\mathrm{Rad}(TM))$ then using (9), it is clear that $h^s = 0$ on $\mathrm{Rad}(TM)$ if and only if $\bar{g}(D^l(X,W),Y) = 0$. Moreover using the Proposition 3.1 of [2], $h^l = 0$ on $\mathrm{Rad}(TM)$. Therefore M is minimal if and only if $\sum_{i=1}^{k} h(e_i,e_i) = 0$. Using (9) and (11) we obtain $\sum_{i=1}^{k} h(e_i,e_i) = \sum_{i=1}^{k}\{\frac{1}{r}\sum_{j=1}^{r} g(A^*_{\xi_j}e_i,e_i)N_j + \frac{1}{l}\sum_{q=1}^{l} g(A_{W_q}e_i,e_i)W_q\}$. Thus the assertion follows. □

Theorem 3.20. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that* $\dim(S(TM)) = \dim(S(TM^\perp))$. *Then M is minimal if and only if*

$$\mathrm{trace}\, A_{\csc\theta F e_i}|_{S(TM)} = 0, \mathrm{trace}\, A^*_{\xi_j}|_{S(TM)} = 0,$$

and

$$\bar{g}(D^l(X,Fe_i),Y) = 0, \forall X, Y \in \Gamma(\mathrm{Rad}(TM)),$$

where $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$.

Proof. Let $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$. By Lemma (3.16) $\{\csc\theta F e_i\}_{i=1}^{k}$ is a basis of $S(TM^\perp)$. Therefore we can write

$$h^s(X,X) = \sum_{i=1}^{k} \lambda_i \csc\theta F e_i, \tag{41}$$

for any $X \in \Gamma(TM)$ and for some functions $\lambda_i, i \in \{1,\ldots,k\}$. Using (9) we have $\bar{g}(h^s(X,X),W) = \bar{g}(A_W X, X)$ for any $X \in \Gamma(S(TM))$. Then using (20) and (41), we obtain $\lambda_i = \bar{g}(A_{\csc\theta F e_i} X, X)$ and hence we get

$$h^s(X,X) = \sum_{i=1}^{k} \csc\theta F e_i \bar{g}(A_{\csc\theta F e_i} X, X),$$

for any $X \in \Gamma(S(TM))$. Then the assertion comes from Theorem (3.20). □

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