

# Applications of reproducing kernels and Berezin symbols

Mubariz T. Garayev, Hocine Guediri  
and Houcine Sadraoui

ABSTRACT. Using techniques from the theory of reproducing kernels and Berezin symbols, we investigate some problems related to classes of linear operators acting on reproducing kernel Hilbert spaces (RKHS's). In particular, we establish new estimates related to the numerical radii and Berezin numbers of some operators on RKHS's. Further, in terms of the distance function, we describe invariant subspaces of isometric composition operators on a RKHS  $\mathcal{H}(\Omega)$  of complex-valued, but not necessarily analytic, functions on a set  $\Omega$ . Moreover, we consider a modification of Sarason's question about truncated Toeplitz operators. We also discuss related problems.

## CONTENTS

1. Introduction	583
2. Estimates for Berezin numbers and numerical radii	585
3. On a modification of a question of Sarason for truncated Toeplitz operators	589
4. Distance function and invariant subspaces of isometric composition operators	593
5. Submodules of the Hardy space over the bidisc	595
6. On operator norm inequalities	598
References	602

## 1. Introduction

In this paper we apply techniques from the theory of reproducing kernel Hilbert spaces (RKHS's) and Berezin symbols to study questions about linear operators on various reproducing kernel Hilbert spaces. We establish

---

Received September 1, 2015.

2010 *Mathematics Subject Classification*. 47A12.

*Key words and phrases*. Reproducing kernel, Berezin symbol, Invariant subspace, Distance function, Truncated Toeplitz operator, Module, Composition operator.

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project no. RGP-VPP-323.

estimates for numerical radii and Berezin numbers. We prove some operator norm inequalities. We study invariant subspaces and boundedness questions for truncated Toeplitz operators with bounded Berezin numbers.

Recall that a RKHS is a Hilbert space  $\mathcal{H} = \mathcal{H}(\Omega)$  of complex-valued functions on a (nonempty) set  $\Omega$  which has the property that point evaluations  $f \rightarrow f(\lambda)$  are continuous in  $\mathcal{H}$  for all  $\lambda \in \Omega$ . The classical Riesz representation theorem guarantees the existence of a unique element  $k_{\mathcal{H},\lambda} \in \mathcal{H}$  such that  $f(\lambda) = \langle f, k_{\mathcal{H},\lambda} \rangle$  for all  $f \in \mathcal{H}$ , where  $\langle \cdot, \cdot \rangle$  stands for the inner product in  $\mathcal{H}$ . The function  $k_{\mathcal{H},\lambda}(z)$  is called the reproducing kernel of  $\mathcal{H}$ . The normalized reproducing kernel  $\widehat{k}_{\mathcal{H},\lambda}$  is defined by  $\widehat{k}_{\mathcal{H},\lambda} := \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}$ . A RKHS is said to be *standard* if the underlying set  $\Omega$  is a subset of a topological space and the boundary  $\partial\Omega$  is nonempty and has the property that  $\{\widehat{k}_{\mathcal{H},\lambda_n}\}$  converges weakly to 0 whenever  $\{\lambda_n\}$  is a sequence in  $\Omega$  that converges to a point in  $\partial\Omega$ . (This concept is due to Nordgren and Rosenthal [NR94].) It is well known that the Hardy, Bergman and Fock spaces of analytic functions are examples of standard RKHS's.

For any bounded linear operator  $A$  on  $\mathcal{H}$ , its *Berezin symbol*  $\widetilde{A}$  is defined by

$$\widetilde{A}(\lambda) := \langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \rangle \quad (\lambda \in \Omega).$$

The Berezin symbol of an operator provides important information about it. For example, it is well known that on most familiar RKHS's, including the Hardy, Bergman and Fock spaces, the Berezin symbol uniquely determines the operator, i.e.,  $A_1 = A_2$  if and only if  $\widetilde{A}_1 = \widetilde{A}_2$ . (See, Zhu [Z07].) It is one of the most useful tools in the study of Toeplitz and Hankel operators on Hardy and Bergman spaces. The concept of the Berezin symbol of an operator arose in connection with quantum mechanics and non-commutative geometry (see, for instance, [Ber72, BerC86]). On the other hand, the method of reproducing kernels is actively developed by Saitoh & Castro and their collaborators, in order to solve various problems of applied mathematics (see, [CSSS12, CIS12, CS13], and the references therein). For other applications of reproducing kernels and Berezin symbols, see for instance the first author's papers [Kar08a, Kar12, Kar06, Kar08b].

Finally, recall that for a bounded operator  $A$  on a RKHS, the corresponding Berezin set and Berezin number of  $A$  are defined, respectively, as follows (see [Kar06]):

$$\begin{aligned} \text{Ber}(A) &= \text{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \\ \text{ber}(A) &= \sup \{ |\mu| : \mu \in \text{Ber}(A) \}. \end{aligned}$$

This paper is organized as follows:

In Section 2, we prove some results concerning Berezin numbers and numerical radii of operators. We demonstrate some new relations among these numerical characteristics of operators on RKHS's.

Section 3 is devoted to the solution of the following modification of a question first studied by Sarason: *Does every truncated Toeplitz operator  $A_\varphi^\theta$  with finite Berezin number  $\text{ber}(A_\varphi^\theta)$  possess an  $L^\infty$  symbol?* Here also an inequality for the Berezin number of  $A_\varphi^\theta$  is proved.

In Section 4, we describe the invariant subspaces of isometric composition operators  $C_\varphi$  on the RKHS  $\mathcal{H}(\Omega)$ .

In Section 5, submodules  $M$  of the Hardy module  $H^2(\mathbb{D}^2)$  over the bidisk  $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$  are investigated in terms of Berezin symbols  $\widetilde{P}_M$  of the orthogonal projection  $P_M$  onto  $M$ . We improve some of results of Yang [Yan04] and Guo and Yang [GY04].

In Section 6, we estimate the limits  $\lim_n \|T^n S\|$  for some appropriate operators  $T$  and  $S$  on a RKHS. Such limits have important applications, for example, in investigating the compactness of operators  $S$  (see, for instance, [ESZ, KZ09, Le09, Muh71, MusH14]).

## 2. Estimates for Berezin numbers and numerical radii

In this section, we prove some new inequalities for Berezin numbers and numerical radii of operators on the Hardy space  $H^2$ .

Let  $(\Sigma)$  denote the set of all inner functions in  $H^2$ , and

$$(H^2)_1 := \{f \in H^2 : \|f\|_2 = 1\}$$

the unit sphere of  $H^2$ .

**Proposition 1.** *Let  $A : H^2 \rightarrow H^2$  be an arbitrary bounded linear operator. Then*

$$(2.1) \quad \sup_{\theta \in (\Sigma) \cup \{1\}} \text{ber}(T_{\bar{\theta}} A T_\theta) \leq w(A) \leq \sup_{f \in H^\infty \cap (H^2)_1} \text{ber}(T_{\bar{f}} A T_f).$$

**Proof.** Since  $H^\infty$  is dense in  $H^2$ , it is easy to show that

$$\sup \{|\langle Af, f \rangle| : f \in (H^2)_1\} = \sup \{|\langle Af, f \rangle| : f \in H^\infty \cap (H^2)_1\}.$$

Then we have:

$$\begin{aligned} w(A) &= \sup \{|\langle Af, f \rangle| : f \in H^\infty \cap (H^2)_1\} \\ &= \sup \{|\langle T_{\bar{f}}^* A T_f 1, 1 \rangle| : f \in H^\infty \cap (H^2)_1\} \\ &= \sup \left\{ \left| \langle T_{\bar{f}} A T_f \widehat{k}_0, \widehat{k}_0 \rangle \right| : f \in H^\infty \cap (H^2)_1 \right\} \\ &= \sup \left\{ \left| \widetilde{T_{\bar{f}} A T_f}(0) \right| : f \in H^\infty \cap (H^2)_1 \right\} \\ &\leq \sup_{f \in H^\infty \cap (H^2)_1} \sup \left\{ \left| \widetilde{T_{\bar{f}} A T_f}(\lambda) \right| : \lambda \in \mathbb{D} \right\} \\ &= \sup_{f \in H^\infty \cap (H^2)_1} \text{ber}(T_{\bar{f}} A T_f), \end{aligned}$$

i.e.,

$$(2.2) \quad w(A) \leq \sup_{f \in H^\infty \cap (H^2)_1} \text{ber} \left( T_{\bar{f}} A T_f \right).$$

On the other hand, if  $\theta \in (\Sigma)$  is an arbitrary inner function, then, by considering that  $\theta g \in (H^2)_1$  for every  $g \in (H^2)_1$ , we have:

$$\begin{aligned} \text{ber} (T_{\bar{\theta}} A T_\theta) &= \sup_{\lambda \in \mathbb{D}} \left| \widetilde{T_{\bar{\theta}} A T_\theta}(\lambda) \right| = \sup_{\lambda \in \mathbb{D}} \left| T_{\bar{\theta}} A T_\theta \widehat{k}_\lambda, \widehat{k}_\lambda \right| \\ &= \sup_{\lambda \in \mathbb{D}} \left| \left\langle A T_\theta \widehat{k}_\lambda, T_\theta \widehat{k}_\lambda \right\rangle \right| = \sup_{\lambda \in \mathbb{D}} \left| \left\langle A \theta \widehat{k}_\lambda, \theta \widehat{k}_\lambda \right\rangle \right| \\ &\leq \sup_{\lambda \in \mathbb{D}} \sup_{h \in (H^2)_1} |\langle A h, h \rangle| \\ &= \sup_{h \in (H^2)_1} |\langle A h, h \rangle| = w(A). \end{aligned}$$

That is

$$\text{ber} (T_{\bar{\theta}} A T_\theta) \leq w(A)$$

for any  $\theta \in (\Sigma)$ . Thus

$$\text{ber} (T_{\bar{\eta}} A T_\eta) \leq w(A)$$

for any  $\eta \in (\Sigma) \cup \{1\}$ . Therefore, we obtain

$$(2.3) \quad \sup_{\eta \in (\Sigma) \cup \{1\}} \text{ber} (T_{\bar{\eta}} A T_\eta) \leq w(A).$$

Now, the desired inequality (2.1) follows from (2.2) and (2.3), which completes the proof. □

**Corollary 1.** *If  $\sup_{f \in H^\infty \cap (H^2)_1} \text{ber} (T_{\bar{f}} A T_f) = \text{ber} (T_{\bar{\theta}} A T_\theta)$  for some  $\theta \in (\Sigma)$ , then  $w(A) = \text{ber} (T_{\bar{\theta}} A T_\theta)$ .*

**Corollary 2.** *If*

$$\sup_{\eta \in (\Sigma) \cup \{1\}} \text{ber} (T_{\bar{\eta}} A T_\eta) = \sup_{f \in H^\infty \cap (H^2)_1} \text{ber} (T_{\bar{f}} A T_f) = \text{ber} (A),$$

then  $w(A) = \text{ber} (A)$ .

**Corollary 3.** *Let  $\varphi \in L^\infty(\mathbb{T})$  and  $T_\varphi$  be a Toeplitz operator on the Hardy space  $H^2$ . Then*

$$\text{ber} (T_\varphi) = w(T_\varphi) = \|T_\varphi\| = \|\varphi\|_\infty.$$

The proof of the latter corollary uses the well-known result that  $\widetilde{T}_\varphi = \widetilde{\varphi}$  for any Toeplitz operator  $T_\varphi$ ,  $\varphi \in L^\infty(\mathbb{T})$ , on the Hardy space  $H^2$ , where  $\widetilde{\varphi}$  denotes the harmonic extension of  $\varphi$ .

Similar arguments allow us to state the following result in the Bergman space  $L_a^2(\mathbb{D})$ .

**Proposition 2.** *Let  $\varphi \in L^\infty(\mathbb{D})$  and  $T_\varphi$  be a Toeplitz operator on the Bergman space  $L^2_a(\mathbb{D})$ . Then*

$$w(T_\varphi) \leq \sup_{f \in H^\infty \cap (L^2_a)_1} \text{ber} \left( T_{|f|^2 \varphi} \right).$$

Recall that for any inner function  $\theta$  and any symbol  $\varphi \in L^\infty(\mathbb{T})$ , the truncated Toeplitz operator  $A_\varphi^\theta : K_\theta \rightarrow K_\theta$  is defined by  $A_\varphi^\theta f = P_\theta(\varphi f)$ . Our next result estimates the numerical radius of  $A_\varphi^\theta$ .

**Proposition 3.** *Let  $\theta$  be an inner function,  $\varphi \in L^\infty(\mathbb{T})$  be a function and  $A_\varphi^\theta : K_\theta \rightarrow K_\theta$  be a truncated Toeplitz operator. Then the numerical radius of  $A_\varphi^\theta$  satisfies the following inequality*

$$w \left( A_\varphi^\theta \right) \geq \sup_{\lambda \in \mathbb{D}} \frac{\left| \left( \left| 1 - \overline{\theta(\lambda)} \theta \right|^2 \varphi \right) \sim (\lambda) \right|}{1 - |\theta(\lambda)|^2}.$$

**Proof.** Since  $\varphi \in L^\infty(\mathbb{T})$ , the truncated Toeplitz  $A_\varphi^\theta$  is bounded. For any  $f \in K_\theta$ ,  $\|f\|_2 = 1$ , we have

$$\left\langle A_\varphi^\theta f, f \right\rangle = \left\langle P_\theta T_\varphi f, f \right\rangle = \left\langle T_\varphi f, f \right\rangle.$$

By considering this and the formula

$$\widehat{k}_{\theta, \lambda}(z) = \left( \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \right)^{\frac{1}{2}} \frac{1 - \overline{\theta(\lambda)} \theta(z)}{1 - \bar{\lambda}z},$$

we have

$$\begin{aligned} & \sup_{\|f\|_2=1} \left| \left\langle A_\varphi^\theta f, f \right\rangle \right| \\ &= \sup_{\|f\|_2=1} \left\langle T_\varphi f, f \right\rangle \geq \sup_{\lambda \in \mathbb{D}} \left| \left\langle T_\varphi \widehat{k}_{\theta, \lambda}, \widehat{k}_{\theta, \lambda} \right\rangle \right| \\ &= \sup_{\lambda \in \mathbb{D}} \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left| \left\langle T_\varphi \frac{1 - \overline{\theta(\lambda)} \theta}{1 - \bar{\lambda}z}, \frac{1 - \overline{\theta(\lambda)} \theta}{1 - \bar{\lambda}z} \right\rangle \right| \\ &= \sup_{\lambda \in \mathbb{D}} \frac{1}{1 - |\theta(\lambda)|^2} \left| \left\langle T_{1 - \theta(\lambda) \bar{\theta}} T_\varphi T_{1 - \overline{\theta(\lambda)} \theta} \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}z}, \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}z} \right\rangle \right| \\ &= \sup_{\lambda \in \mathbb{D}} \frac{1}{1 - |\theta(\lambda)|^2} \left| \left\langle T_{\varphi |1 - \overline{\theta(\lambda)} \theta|^2} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &= \sup_{\lambda \in \mathbb{D}} \frac{1}{1 - |\theta(\lambda)|^2} \left| \widetilde{T}_{\varphi |1 - \overline{\theta(\lambda)} \theta|^2}(\lambda) \right| \end{aligned}$$

$$= \sup_{\lambda \in \mathbb{D}} \frac{\left( \left| 1 - \overline{\theta(\lambda)}\theta \right|^2 \varphi \right) \sim (\lambda)}{1 - |\theta(\lambda)|^2}.$$

In the last equality we again used the well known result that  $\tilde{T}_h = \tilde{h}$  for every  $h \in L^\infty(\mathbb{T})$  (see, for example, Zhu [Z07]). Thus we have

$$w\left(A_\varphi^\theta\right) \geq \sup_{\lambda \in \mathbb{D}} \frac{\left( \left| 1 - \overline{\theta(\lambda)}\theta \right|^2 \varphi \right) \sim (\lambda)}{1 - |\theta(\lambda)|^2},$$

which proves the proposition. □

Our next result establishes an inequality for the Berezin number of an abstract operator.

**Proposition 4.** *Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a reproducing kernel Hilbert space of complex-valued functions on some set  $\Omega$  with reproducing kernel*

$$k_{\mathcal{H},\lambda}(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z),$$

where  $\{e_n(z)\}_{n \geq 0}$  is any orthonormal basis of the space  $\mathcal{H}$ . Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator with matrix entries

$$a_{n,m} := \langle Ae_n(z), e_m(z) \rangle, \quad n, m = 0, 1, 2, \dots,$$

satisfying the inequality

$$(2.4) \quad |a_{n,m}| \leq C |a_n| |b_m| \leq \|A\| \quad (\forall n, m \geq 0),$$

for some sequences  $\{a_n\}_{n \geq 0} \in \ell^2$ ,  $\{b_n\}_{n \geq 0} \in \ell^2$  and  $C > 0$ . Then

$$\text{ber}(A) \leq C \|\{a_n\}\|_{\ell^2} \|\{b_n\}\|_{\ell^2}.$$

**Proof.** First, let us determine the Berezin symbol of the operator  $A$  :

$$\begin{aligned} \tilde{A}(\lambda) &= \left\langle A\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \right\rangle = \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \langle Ak_\lambda, k_\lambda \rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \left\langle A \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z), \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z) \right\rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \left\langle \sum_{n=0}^{\infty} \overline{e_n(\lambda)} Ae_n(z), \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z) \right\rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n,m=0}^{\infty} \overline{e_n(\lambda)} e_m(\lambda) \langle Ae_n(z), e_m(z) \rangle \\ &= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n,m=0}^{\infty} a_{n,m} \overline{e_n(\lambda)} e_m(\lambda), \end{aligned}$$

or

$$(2.5) \quad \tilde{A}(\lambda) = \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n,m=0}^{\infty} a_{n,m} \overline{e_n(\lambda)} e_m(\lambda),$$

for all  $\lambda \in \Omega$ .

Now using condition (2.4) and formula (2.5), we have

$$\begin{aligned} & \left| \tilde{A}(\lambda) \right| \\ & \leq C \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_n| |b_m| |e_n(\lambda)| |e_m(\lambda)| \\ & = C \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n=0}^{\infty} |a_n| |e_n(\lambda)| \sum_{m=0}^{\infty} |b_m| |e_m(\lambda)| \\ & \leq C \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} |e_n(\lambda)|^2 \right)^{\frac{1}{2}} \left( \sum_{m=0}^{\infty} |b_m|^2 \right)^{\frac{1}{2}} \left( \sum_{m=0}^{\infty} |e_m(\lambda)|^2 \right)^{\frac{1}{2}} \\ & = C \|\{a_n\}\|_{\ell^2} \|\{b_n\}\|_{\ell^2} \end{aligned}$$

for all  $\lambda \in \Omega$ , which implies that

$$\text{ber}(A) = \sup_{\lambda \in \Omega} \left| \tilde{A}(\lambda) \right| \leq C \|\{a_n\}\|_{\ell^2} \|\{b_n\}\|_{\ell^2}.$$

The proposition is proved. □

### 3. On a modification of a question of Sarason for truncated Toeplitz operators

Let  $S^*$  denote the backward shift on the Hardy space  $H^2$ , which is given by  $S^*f(z) = \frac{f(z)-f(0)}{z}$ , and let  $\theta$  be a nonconstant inner function. The  $S^*$ -invariant subspace  $H^2 \ominus \theta H^2$  will be denoted by  $K_\theta$ . The kernel function in  $K_\theta$  for the evaluation functional at the point  $\lambda$  of  $\mathbb{D}$  is, as mentioned above, the function

$$k_{\theta,\lambda}(z) := \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda}z} \quad (z \in \mathbb{D}).$$

Now let  $\varphi \in L^\infty(\mathbb{T})$ , and recall that the truncated Toeplitz operator  $A_\varphi^\theta$  acting on bounded functions from  $K_\theta$  is defined by the formula

$$A_\varphi^\theta f = P_\theta(\varphi f),$$

$f \in K_\theta \cap L^\infty(\mathbb{T})$ ; here  $P_\theta f = P_+ f - P_+(\bar{\theta}f)$  is the orthogonal projection onto  $K_\theta$ . In contrast with the Toeplitz operators on  $H^2$  (which satisfy  $\|T_\varphi\| = \|\varphi\|_\infty$ ), the operator  $A_\varphi^\theta$  may be extended to a bounded operator on  $K_\theta$  even for some unbounded symbol  $\varphi$ .

The symbol of a truncated Toeplitz operator is highly nonunique. In fact, Sarason proved in [Sar07, Theorem 3.1] that  $A_\varphi^\theta = 0$  if and only if  $\varphi$

belongs to  $\theta H^2 + \overline{\theta H^2}$ . A truncated Toeplitz operator obviously is bounded if it has a symbol in  $L^\infty$ . The following natural question is due to Sarason [Sar07, Sar08]: does every bounded truncated Toeplitz operator possess an  $L^\infty$  symbol?

It is shown in [BaCFMT10] that in general the answer to this question is negative. Namely, a class of inner functions  $\theta$  for which there exist rank one truncated Toeplitz operators  $K_\theta$  without bounded symbols has been constructed.

Among the results of that paper, a negative answer to a weaker question has been given, namely: *Does every truncated Toeplitz operator  $A_\varphi^\theta$  with finite Berezin number  $\text{ber}(A_\varphi^\theta)$  possess an  $L^\infty$  symbol?*

However, it is still interesting to find necessary and sufficient conditions under which a given truncated Toeplitz operator with finite Berezin number possesses an  $L^\infty$  symbol.

Here, in terms of the harmonic extension, we give the necessary and sufficient conditions ensuring the existence of a bounded symbol for any truncated Toeplitz operator  $A_\varphi^\theta$  with finite Berezin number  $\text{ber}(A_\varphi^\theta)$ .

Recall that for any function  $\psi \in L^1$ , its harmonic extension into  $\mathbb{D}$  will be denoted by  $\tilde{\psi}$ .

**Theorem 1.** *Let  $\theta$  be an inner function. For  $\varphi$  in  $L^2$ , let  $A_\varphi^\theta$  be a truncated Toeplitz operator on  $K_\theta$  defined by  $A_\varphi^\theta f = P_\theta(\varphi f)$  with finite Berezin number  $\text{ber}(A_\varphi^\theta)$ . Then  $A_\varphi^\theta$  possesses a bounded symbol if and only if there exist functions  $h_1, h_2 \in H^2$  such that*

$$\sup_{\lambda \in \mathbb{D}} \left| \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re} \left( \overline{\theta(\lambda)} \theta \right) \right)^\sim (\lambda) \right| < +\infty.$$

**Proof.** For the “only if” part, suppose that  $A_\varphi^\theta$  possesses a bounded symbol  $\psi \in L^\infty$ . This means that there exist functions  $h_1, h_2 \in H^2$  such that  $\psi = \varphi + \theta h_1 + \overline{\theta h_2}$ . Then, using standard arguments for bounded harmonic functions, we obtain:

$$\begin{aligned} & \left| \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re} \left( \overline{\theta(\lambda)} \theta \right) \right)^\sim (\lambda) \right| \\ &= \left| \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} \left( (\varphi(\xi) + \theta(\xi) h_1(\xi) + \overline{\theta(\xi) h_2(\xi)}) \operatorname{Re} \left( \overline{\theta(\lambda)} \theta(\xi) \right) \right) dm(\xi) \right| \\ &\leq \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |\varphi(\xi) + \theta(\xi) h_1(\xi) + \overline{\theta(\xi) h_2(\xi)}| \left| \operatorname{Re} \left( \overline{\theta(\lambda)} \theta(\xi) \right) \right| dm(\xi) \\ &\leq \|\varphi + \theta h_1 + \overline{\theta h_2}\|_{L^\infty(\mathbb{T})} \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |\theta(\lambda) \theta(\xi)| dm(\xi) \\ &\leq \|\varphi + \theta h_1 + \overline{\theta h_2}\|_{L^\infty(\mathbb{T})}, \end{aligned}$$



for all  $\lambda \in \mathbb{D}$ , which shows that

$$\sup_{\lambda \in \mathbb{D}} \left| \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re} \left( \overline{\theta(\lambda)\theta} \right) \right)^\sim (\lambda) \right| \leq \|\varphi + \theta h_1 + \overline{\theta h_2}\|_{L^\infty(\mathbb{T})} < +\infty.$$

Now, we prove the “if” part. We first calculate the Berezin symbol of the bounded operator  $A_\varphi^\theta$ , which is the function  $\widetilde{A}_\varphi^\theta$  defined on  $\mathbb{D}$  by

$$\widetilde{A}_\varphi^\theta (\lambda) := \left\langle A_\varphi^\theta \widehat{k}_{\theta,\lambda}, \widehat{k}_{\theta,\lambda} \right\rangle,$$

where

$$\widehat{k}_{\theta,\lambda} (z) := \left( \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \right)^{1/2} \frac{1 - \overline{\theta(\lambda)\theta} (z)}{1 - \bar{\lambda}z}$$

is the normalized reproducing kernel of the subspace  $K_\theta$ . We have:

$$\begin{aligned} \widetilde{A}_\varphi^\theta (\lambda) &= \left\langle A_\varphi^\theta \widehat{k}_{\theta,\lambda}, \widehat{k}_{\theta,\lambda} \right\rangle = \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left\langle P_\theta (\varphi k_{\theta,\lambda}), k_{\theta,\lambda} \right\rangle \\ &= \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left\langle \varphi \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z}, \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z} \right\rangle \\ &= \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left\langle P_+ \left( \varphi \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z} \right), \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z} \right\rangle \\ &= \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left\langle T_\varphi \left( \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z} \right), \frac{1 - \overline{\theta(\lambda)\theta}}{1 - \bar{\lambda}z} \right\rangle \\ &= \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \left\langle T_\varphi T_{1-\overline{\theta(\lambda)\theta}} \frac{1}{1 - \bar{\lambda}z}, T_{1-\overline{\theta(\lambda)\theta}} \frac{1}{1 - \bar{\lambda}z} \right\rangle \\ &= \frac{1}{1 - |\theta(\lambda)|^2} \left\langle T_{1-\overline{\theta(\lambda)\theta}}^* T_\varphi T_{1-\overline{\theta(\lambda)\theta}} \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda}z}, \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda}z} \right\rangle. \end{aligned}$$

Since for  $\varphi \in L^2$  one has  $S^* T_\varphi S = T_\varphi$  (see Sarason [Sar07]), where  $S$  is the shift operator on  $H^2$  defined by  $Sf(z) = zf(z)$ , it is easy to show that

$$T_{1-\overline{\theta(\lambda)\theta}}^* T_\varphi T_{1-\overline{\theta(\lambda)\theta}} = T_{(1-\overline{\theta(\lambda)\theta})\varphi(1-\overline{\theta(\lambda)\theta})}.$$

On the other hand, by considering that the function  $\widehat{k}_\lambda(z) := \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda}z}$  is the normalized reproducing kernel for the Hardy space  $H^2$  and  $\widetilde{T}_\psi = \widetilde{\psi}$  for every function  $\psi \in L^2(\mathbb{T})$  (see, for example, Zhu [Z07]), we obtain:

$$\begin{aligned} (3.1) \quad \widetilde{A}_\varphi^\theta (\lambda) &= \frac{1}{1 - |\theta(\lambda)|^2} \left\langle T_{(1-\overline{\theta(\lambda)\theta})(1-\overline{\theta(\lambda)\theta})\varphi} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \end{aligned}$$

$$\begin{aligned} &= \frac{1}{1 - |\theta(\lambda)|^2} \left\langle T_{(1-2\operatorname{Re}(\overline{\theta(\lambda)\theta})+|\theta(\lambda)|^2)\varphi} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &= \frac{1}{1 - |\theta(\lambda)|^2} \left[ \left(1 + |\theta(\lambda)|^2 - 2\operatorname{Re}(\overline{\theta(\lambda)\theta})\right) \varphi \right]^\sim(\lambda) \\ &= \frac{1}{1 - |\theta(\lambda)|^2} \left( \left(1 + |\theta(\lambda)|^2\right) \widetilde{\varphi}(\lambda) - 2 \left( \operatorname{Re}(\overline{\theta(\lambda)\theta}) \right) \varphi \right)^\sim(\lambda). \end{aligned}$$

Then we obtain

$$(3.2) \quad \widetilde{\varphi}(\lambda) = \frac{1 - |\theta(\lambda)|^2}{1 + |\theta(\lambda)|^2} \widetilde{A}_\varphi^\theta(\lambda) + \frac{2}{1 + |\theta(\lambda)|^2} \left( \operatorname{Re}(\overline{\theta(\lambda)\theta}) \right) \varphi^\sim(\lambda),$$

for all  $\lambda \in \mathbb{D}$ . Analogously we have that

$$\begin{aligned} (3.3) \quad &(\varphi + \theta h_1 + \overline{\theta h_2})^\sim(\lambda) \\ &= (\widetilde{\varphi}(\lambda) + \theta(\lambda) h_1(\lambda) + \overline{\theta}(\lambda) \overline{h_2}(\lambda)) \\ &= \frac{1 - |\theta(\lambda)|^2}{1 + |\theta(\lambda)|^2} \widetilde{A}_{\varphi + \theta h_1 + \overline{\theta h_2}}^\theta(\lambda) \\ &\quad + \frac{2}{1 + |\theta(\lambda)|^2} \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re}(\overline{\theta(\lambda)\theta}) \right)^\sim(\lambda) \\ &= \frac{1 - |\theta(\lambda)|^2}{1 + |\theta(\lambda)|^2} \widetilde{A}_\varphi^\theta(\lambda) \\ &\quad + \frac{2}{1 + |\theta(\lambda)|^2} \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re}(\overline{\theta(\lambda)\theta}) \right)^\sim(\lambda). \end{aligned}$$

Since

$$\frac{1 - |\theta(\lambda)|^2}{1 + |\theta(\lambda)|^2} \leq 1$$

and  $|\widetilde{A}_\varphi^\theta(\lambda)| \leq C$  for all  $\lambda \in \mathbb{D}$  and some  $C > 0$ , it follows from equality (3) that  $(\varphi + \theta h_1 + \overline{\theta h_2})^\sim \in L^\infty(\mathbb{D})$  if and only if

$$(3.4) \quad \sup_{\lambda \in \mathbb{D}} \left| \left( (\varphi + \theta h_1 + \overline{\theta h_2}) \operatorname{Re}(\overline{\theta(\lambda)\theta}) \right)^\sim(\lambda) \right| < +\infty.$$

Thus, by considering that  $\Psi \in L^\infty(\mathbb{T})$  if and only if  $\widetilde{\Psi} \in L^\infty(\mathbb{D})$ , we deduce that  $\varphi + \theta h_1 + \overline{\theta h_2} \in L^\infty$  if and only if (3.4) is satisfied. The theorem is proved.  $\square$

Let  $Z_\theta := \{\lambda_n\}_{n \geq 1} \subset \mathbb{D}$  be a sequence of zeros of the inner function  $\theta$ , and let

$$\theta = B \exp \left( - \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu_\theta^s(\xi) \right)$$

be its canonical factorization. Let  $\sigma(\theta)$  denote the spectrum of the function  $\theta$ . It is well-known [Nik86] that

$$\sigma(\theta) = \operatorname{clos}(Z_\theta) \cup \operatorname{supp}(\mu_\theta^s) = \{ \lambda \in \overline{\mathbb{D}} : \lim_{z \rightarrow \lambda, z \in \mathbb{D}} |\theta(z)| = 0 \}.$$

The following is an immediate corollary of formulas (3.1) and (3.2).

**Corollary 4.** *Let  $A_\varphi^\theta$  ( $\varphi \in L^2$ ) be any truncated Toeplitz operator on  $K_\theta$  with  $\text{ber}(A_\varphi^\theta) < +\infty$ . Then we have:*

- (i)  $\text{ber}(A_\varphi^\theta) = \sup_{\lambda \in \mathbb{D}} \left| \frac{1+|\theta(\lambda)|^2}{1-|\theta(\lambda)|^2} \tilde{\varphi}(\lambda) - \frac{2}{1-|\theta(\lambda)|^2} \left( \varphi \text{Re} \left( \overline{\theta(\lambda)} \theta \right) \right)^\sim(\lambda) \right|$ .
- (ii)  $\sup_{\lambda \in \sigma(\theta)} |\tilde{\varphi}(\lambda)| \leq \text{ber}(A_\varphi^\theta)$ .

It follows from Sarason’s description that any two symbols of the same operator differ by an element of the set  $\theta H^2 + \overline{\theta} \overline{H}^2$ . Then we immediately get the following criterion: a bounded truncated Toeplitz operator  $A_\varphi^\theta$  has a bounded symbol if and only if there exists two functions  $h_1, h_2 \in H^2$  such that  $\varphi + \theta h_1 + \overline{\theta} h_2 \in L^\infty(\mathbb{T})$ , which is obviously equivalent to

$$(3.5) \quad \sup_{\lambda \in \mathbb{D}} |(\varphi + \theta h_1 + \overline{\theta} h_2)^\sim(\lambda)| < +\infty.$$

The next corollary to Theorem 1 somewhat improves Sarason’s criterion (3.5), because our criterion (3.6) below is weaker than (3.5) due to the presence of the factor  $\text{Re} \left( \overline{\theta(\lambda)} \theta \right)$ .

**Corollary 5.** *Let  $\theta$  be an inner function. For  $\varphi$  in  $L^2$ , let  $A_\varphi^\theta$  be a bounded truncated Toeplitz operator on  $K_\theta$ . Then  $A_\varphi^\theta$  possesses a bounded symbol if and only if there exist functions  $h_1, h_2 \in H^2$  such that*

$$(3.6) \quad \sup_{\lambda \in \mathbb{D}} \left| \left( (\varphi + \theta h_1 + \overline{\theta} h_2) \text{Re} \left( \overline{\theta(\lambda)} \theta \right) \right)^\sim(\lambda) \right| < +\infty.$$

#### 4. Distance function and invariant subspaces of isometric composition operators

Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a RKHS of complex-valued functions on some set  $\Omega$  with reproducing kernel  $k_{\mathcal{H},\lambda}(z)$ ; that is  $\langle f, k_{\mathcal{H},\lambda} \rangle = f(\lambda)$  for every  $f \in \mathcal{H}$ . For any suitable function  $\varphi : \Omega \rightarrow \Omega$ , the associated composition operator  $C_\varphi$  on  $\mathcal{H}$  is defined by  $C_\varphi f = f \circ \varphi$ . In this section, we describe the invariant subspaces of the composition operator  $C_\varphi$  in terms of the so-called distance function.

Recall that Nikolski introduces in [Nik95] the concept of the distance function defined in  $\Omega$ , for a closed subspace  $E \subset \mathcal{H}$  by

$$\theta_E(\lambda) := \sup \{ |f(\lambda)| : f \in E, \|f\| \leq 1 \}, \lambda \in \Omega.$$

In other words,  $\theta_E(\lambda) = \|\Phi_\lambda|_E\|$ ,  $\lambda \in \Omega$ , where  $\Phi_\lambda$  is the point evaluation functional at  $\lambda \in \Omega$  on  $\mathcal{H}$ :  $\Phi_\lambda(f) = f(\lambda)$ ,  $f \in \mathcal{H}$ . It is well known that the distance function uniquely determines the subspace (see Nikolski [Nik95]).

Note that the description of invariant subspaces of composition operators seems not to be well studied. Moreover, most of known results concern mostly RKHS’s of analytic functions on the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ ,

(For instance, see Mahvidi [Mah01] and Nordgren, Rosenthal and Wintrobe [NRW87]).

**Theorem 2.** *Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a RKHS of complex-valued functions on some set  $\Omega$ , and let  $\varphi : \Omega \rightarrow \Omega$ , be a function such that  $C_\varphi$  is an isometric operator on  $\mathcal{H}$ . If  $E \subset \mathcal{H}$  is a closed subspace such that  $C_\varphi E \subseteq E$ , then  $\theta_E(\varphi(\lambda)) \leq \theta_E(\lambda)$  for all  $\lambda \in \Omega$ .*

**Proof.** First, note that if  $k_{\mathcal{H},\lambda}(z)$  is the reproducing kernel of  $\mathcal{H}$ , then it is easy to verify that  $C_\varphi^* k_{\mathcal{H},\lambda} = k_{\mathcal{H},\varphi(\lambda)}$  for all  $\lambda \in \Omega$  (see, for instance, [MarR07, Lemma 5.1.9]). Let  $P_E : \mathcal{H} \rightarrow E$  be the orthogonal projector, and let  $k_{\mathcal{H},\lambda}^E$  denote the reproducing kernel of the subspace  $E$ . Since  $C_\varphi$  is an isometry on  $\mathcal{H}$ , then it is easy to see that

$$P_{E \ominus C_\varphi E} = P_E - C_\varphi P_E C_\varphi^*,$$

which implies that

$$(P_E - C_\varphi P_E C_\varphi^*) k_{\mathcal{H},\lambda}(z) = k_{\mathcal{H},\lambda}^{E \ominus C_\varphi E}(z).$$

Then, we obtain:

$$\begin{aligned} & \langle P_E k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle - \langle C_\varphi P_E C_\varphi^* k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle \\ &= \langle P_E k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle - \langle P_E k_{\mathcal{H},\varphi(\lambda)}(z), C_\varphi^* k_{\mathcal{H},\lambda}(z) \rangle \\ &= \langle P_E k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle - \langle k_{\mathcal{H},\varphi(\lambda)}^E(z), k_{\mathcal{H},\varphi(\lambda)}(z) \rangle \\ &= \langle P_E k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle - k_{\mathcal{H},\varphi(\lambda)}^E(\varphi(\lambda)) \\ &= \langle k_{\mathcal{H},\lambda}^{E \ominus C_\varphi E}(z), k_{\mathcal{H},\lambda}(z) \rangle = \langle P_{E \ominus C_\varphi E} k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle \end{aligned}$$

or

$$\begin{aligned} & \langle P_E k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle - \langle P_E k_{\mathcal{H},\varphi(\lambda)}(z), k_{\mathcal{H},\varphi(\lambda)}(z) \rangle \\ &= \langle P_{E \ominus C_\varphi E} k_{\mathcal{H},\lambda}(z), k_{\mathcal{H},\lambda}(z) \rangle \end{aligned}$$

for all  $\lambda \in \mathbb{D}$ . From this

$$\begin{aligned} & \left\langle P_E \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}, \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|} \right\rangle \|k_{\mathcal{H},\lambda}\|^2 - \left\langle P_E \frac{k_{\mathcal{H},\varphi(\lambda)}}{\|k_{\mathcal{H},\varphi(\lambda)}\|}, \frac{k_{\mathcal{H},\varphi(\lambda)}}{\|k_{\mathcal{H},\varphi(\lambda)}\|} \right\rangle \|k_{\mathcal{H},\varphi(\lambda)}\|^2 \\ &= \left\langle P_{E \ominus C_\varphi E} \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}, \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|} \right\rangle \|k_{\mathcal{H},\lambda}\|^2, \end{aligned}$$

or

$$(4.1) \quad \tilde{P}_E(\lambda) \|k_{\mathcal{H},\lambda}\|^2 - \tilde{P}_E(\varphi(\lambda)) \|k_{\mathcal{H},\varphi(\lambda)}\|^2 = \tilde{P}_{E \ominus C_\varphi E}(\lambda) \|k_{\mathcal{H},\lambda}\|^2,$$

for all  $\lambda \in \mathbb{D}$ .

On the other hand, it is easy to show that

$$(4.2) \quad \tilde{P}_E(\lambda) \|k_{\mathcal{H},\lambda}\|^2 = \theta_E^2(\lambda) \quad (\forall \lambda \in \Omega).$$

Indeed,

$$\begin{aligned} \theta_E^2(\lambda) &= \|\Phi_\lambda|_E\|^2 = \|P_E k_{\mathcal{H},\lambda}\|^2 = \|k_{\mathcal{H},\lambda}^E\|^2 = \langle k_{\mathcal{H},\lambda}^E, k_{\mathcal{H},\lambda}^E \rangle \\ &= \langle P_E k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle = \|k_{\mathcal{H},\lambda}\|^2 \left\langle P_E \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}, \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|} \right\rangle \\ &= \|k_{\mathcal{H},\lambda}\|^2 \tilde{P}_E(\lambda) \quad (\forall \lambda \in \Omega). \end{aligned}$$

Now, it follows from the formulae (4.1) and (4.2) that

$$(4.3) \quad \theta_E^2(\lambda) - \theta_E^2(\varphi(\lambda)) = \theta_{E \ominus C_\varphi E}^2(\lambda) \quad (\forall \lambda \in \Omega).$$

Since  $C_\varphi E \subset E$ ,  $\theta_{E \ominus C_\varphi E}(\lambda) \geq 0$ ,  $\forall \lambda \in \Omega$ , so, it follows from (4.3) that

$$\theta_E(\varphi(\lambda)) \leq \theta_E(\lambda) \quad (\forall \lambda \in \Omega),$$

as desired. This proves the theorem. □

### 5. Submodules of the Hardy space over the bidisc

Let  $\mathbb{T}$  denote the boundary of the unit disc  $\mathbb{D}$ . Let  $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$  be the Cartesian product of two copies of  $\mathbb{D}$  and  $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$  is its distinguished boundary. The points in  $\mathbb{D}^2$  are thus ordered pairs  $z = (z_1, z_2)$ . As usual,  $H^2(\mathbb{D}^2)$ , which is  $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ , is the Hardy space over the bidisc  $\mathbb{D}^2$ . The bidisc algebra  $C_A(\mathbb{D}^2)$  acts on  $H^2(\mathbb{D}^2)$  by pointwise multiplication, which makes  $H^2(\mathbb{D}^2)$  into a  $C_A(\mathbb{D}^2)$ -module. A closed subspace  $M$  of  $H^2(\mathbb{D}^2)$  is called a submodule if  $M$  is invariant under the module action, or equivalently,  $M$  is invariant under multiplications by both  $z_1$  and  $z_2$ . For more details about the Hardy space  $H^2(\mathbb{D}^2)$  and its submodules, see for instance, Rudin [R69], Yang [Yan04] and the references therein.

In the classical Hardy space  $H^2(\mathbb{D})$  over the unit disc  $\mathbb{D}$ , every  $z$ -invariant (i.e., shift-invariant) subspace  $M$  is of the form  $\theta H^2(\mathbb{D})$  for some inner function  $\theta$  by the celebrated Beurling theorem [H62], and the reproducing kernel of  $M$  is  $k_{M,\lambda}(z) = \frac{\overline{\theta(\lambda)}\theta(z)}{1-\lambda\bar{z}}$ . The fact that  $\theta$  is inner implies that  $(1 - |\lambda|^2) k_{M,\lambda}(z)$  has boundary value 1 almost everywhere on  $\mathbb{T}$ . In the two variable space  $H^2(\mathbb{D}^2)$ , these questions are far more complicated and there is no similar characterization of invariant subspaces  $M$  in terms of inner functions. However, Yang [Yan04] showed an analogous phenomenon in terms of reproducing kernels, namely,  $(1 - |\lambda_1|^2)(1 - |\lambda_2|^2) k_{M,(\lambda_1,\lambda_2)}(\lambda_1, \lambda_2)$  has boundary value 1 almost everywhere on  $\mathbb{T}^2$ . Yang’s paper [Yan04] sticks to the idea of Beurling’s theorem and shows, in terms of their reproducing kernels, that submodules in  $H^2(\mathbb{D}^2)$  do exhibit a Beurling-type phenomenon. For a submodule  $M$  in  $H^2(\mathbb{D}^2)$ , a natural analogue of  $(1 - |\lambda|^2) k_{M,\lambda}(\lambda)$  is so-called core function (see Yang [Yan04])

$$G^M(\lambda_1, \lambda_2) := (1 - |\lambda_1|^2)(1 - |\lambda_2|^2) k_{M,(\lambda_1,\lambda_2)}(\lambda_1, \lambda_2),$$

where  $k_{M,\lambda}(z)$  is the reproducing kernel of  $M$ . As is proved in [Yan04],  $G^M$  completely determines the submodule  $M$ . Moreover, Yang proved [Yan04] under a mild condition that  $G^M(\lambda_1, \lambda_2) = 1$  almost everywhere on  $\mathbb{T}^2$  (see [Yan04, Theorem 4.5 and Corollary 4.6]). In [Yan04], it is also conjectured that  $G^M(z) = 1$  almost everywhere on the distinguished boundary  $\mathbb{T}^2$  for every submodule  $M$ . This conjecture was affirmatively solved by Guo and Yang in [GY04, Theorem 2.1]. For the more general case see also the Corollaries 2.3 and 2.4 in [Kar08b].

In the present section, we characterize submodules  $M$  of  $H^2(\mathbb{D}^2)$  in terms of Berezin symbols  $\tilde{P}_M$  of the orthogonal projection  $P_M$  onto  $M$ . We also prove that

$$\lim_{z \rightarrow \xi} G^M(z) = 1$$

for every submodule  $M$  with finite co-dimension and  $\xi \in \partial\mathbb{D}^2$ . Since  $\partial\mathbb{D}^2 \neq \mathbb{T}^2$ , in case of submodules with finite co-dimensions, our result is stronger than the results of Yang [Yan04] and Guo and Yang [GY04].

**Theorem 3.** *For any nontrivial submodule  $M$  of the Hardy space  $H^2(\mathbb{D}^2)$ , the Berezin symbol  $\tilde{P}_M$  of the operator  $P_M$  has the representation*

$$(5.1) \quad \tilde{P}_M(\lambda_1, \lambda_2) = \left(1 - |\lambda_1|^2\right) \sum_{i=1}^{\infty} |\eta_i(\lambda_1, \lambda_2)|^2, \quad (\lambda_1, \lambda_2) \in \mathbb{D}^2,$$

for some orthonormal basis  $\{\eta_i\}_{i \geq 1}$  of the subspace  $M \ominus z_2M$ .

**Proof.** Let  $M$  be any nontrivial submodule of  $H^2(\mathbb{D}^2)$ , i.e.,  $z_1M \subset M$  and  $z_2M \subset M$ . Let

$$k_\lambda(z) = \frac{1}{(1 - \bar{\lambda}_1 z_1)(1 - \bar{\lambda}_2 z_2)}$$

be a reproducing kernel of the space  $H^2(\mathbb{D}^2)$ . Then

$$k_{M,\lambda}(z) = P_M k_\lambda(z) \quad (\lambda, z \in \mathbb{D}^2)$$

is the reproducing kernel of the submodule  $M$  and

$$P_{M \ominus z_2M} k_\lambda = (1 - \bar{\lambda}_2 z_2) k_{M,\lambda}$$

is the reproducing kernel of  $M \ominus z_2M$ . Therefore,

$$k_{M \ominus z_2M,\lambda}(z) = \sum_{i=1}^{\infty} \overline{\eta_i(\lambda)} \eta_i(z)$$

for some orthonormal basis  $\{\eta_i\}_{i \geq 1}$  of the subspace  $M \ominus z_2M$ . Then we have

$$\begin{aligned} (1 - \bar{\lambda}_1 z_1)(1 - \bar{\lambda}_2 z_2) k_{M,\lambda}(z) &= (1 - \bar{\lambda}_1 z_1) k_{M \ominus z_2M,\lambda}(z) \\ &= (1 - \bar{\lambda}_1 z_1) \sum_{i=1}^{\infty} \overline{\eta_i(\lambda_1, \lambda_2)} \eta_i(z_1, z_2), \end{aligned}$$

which implies that

$$\begin{aligned}
 & P_M k_\lambda(z) \\
 &= \frac{\sum_{i=1}^{\infty} \overline{\eta_i(\lambda_1, \lambda_2)} \eta_i(z_1, z_2)}{(1 - \bar{\lambda}_2 z_2)} \\
 &= (1 - \bar{\lambda}_1 z_1) \sum_{i=1}^{\infty} \overline{\eta_i(\lambda_1, \lambda_2)} \eta_i(z_1, z_2) k_\lambda(z) \\
 &= \left( \sum_{i=1}^{\infty} \overline{\eta_i(\lambda_1, \lambda_2)} \eta_i(z_1, z_2) - \sum_{i=1}^{\infty} \bar{\lambda}_1 \overline{\eta_i(\lambda_1, \lambda_2)} z_1 \eta_i(z_1, z_2) \right) k_\lambda(z) \\
 &= \sum_{i=1}^{\infty} \overline{\eta_i(\lambda_1, \lambda_2)} \eta_i(z_1, z_2) k_\lambda(z) - \sum_{i=1}^{\infty} \bar{\lambda}_1 \overline{\eta_i(\lambda_1, \lambda_2)} z_1 \eta_i(z_1, z_2) k_\lambda(z) \\
 &= \sum_{i=1}^{\infty} T_{\eta_i(z_1, z_2)} T_{\eta_i(z_1, z_2)}^* k_{(\lambda_1, \lambda_2)}(z_1, z_2) \\
 &\quad - \sum_{i=1}^{\infty} T_{z_1 \eta_i(z_1, z_2)} T_{z_1 \eta_i(z_1, z_2)}^* k_{(\lambda_1, \lambda_2)}(z_1, z_2) \\
 &= \sum_{i=1}^{\infty} \left( T_{\eta_i(z_1, z_2)} T_{\eta_i(z_1, z_2)}^* - T_{z_1 \eta_i(z_1, z_2)} T_{z_1 \eta_i(z_1, z_2)}^* \right) k_{(\lambda_1, \lambda_2)}(z_1, z_2).
 \end{aligned}$$

Since  $\text{span} \{k_{(\lambda_1, \lambda_2)}(z_1, z_2) : (\lambda_1, \lambda_2) \in \mathbb{D}^2\} = H^2(\mathbb{D}^2)$ , the latter equality shows that

$$P_M = \sum_{i=1}^{\infty} (T_{\eta_i} T_{\eta_i}^* - T_{z_1 \eta_i} T_{z_1 \eta_i}^*),$$

which implies that

$$\begin{aligned}
 \tilde{P}_M(\lambda_1, \lambda_2) &= \sum_{i=1}^{\infty} (|\eta_i(\lambda)|^2 - |\lambda_1|^2 |\eta_i(\lambda)|^2) \\
 &= (1 - |\lambda_1|^2) \sum_{i=1}^{\infty} |\eta_i(\lambda)|^2,
 \end{aligned}$$

or

$$\tilde{P}_M(\lambda_1, \lambda_2) = (1 - |\lambda_1|^2) \sum_{i=1}^{\infty} |\eta_i(\lambda_1, \lambda_2)|^2$$

for all  $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ , as desired. The theorem is proved.  $\square$

**Corollary 6.** *For any nontrivial finite co-dimensional submodule  $M$  of the Hardy space  $H^2(\mathbb{D}^2)$  we have*

$$\lim_{(\lambda_1, \lambda_2) \rightarrow \partial\mathbb{D}^2} G^M(\lambda_1, \lambda_2) = 1.$$

**Proof.** By using Theorem 3, the definition of the function  $G^M(\lambda_1, \lambda_2)$ , and formula (5.1), we obtain:

$$\begin{aligned} G^M(\lambda_1, \lambda_2) &= (1 - |\lambda_1|^2) (1 - |\lambda_2|^2) k_{M,(\lambda_1, \lambda_2)}(\lambda_1, \lambda_2) \\ &= (1 - |\lambda_1|^2) k_{M \ominus z_2 M, (\lambda_1, \lambda_2)}(\lambda_1, \lambda_2) \\ &= (1 - |\lambda_1|^2) \sum_{i=1}^{\infty} |\eta_i(\lambda_1, \lambda_2)|^2 \\ &= \tilde{P}_M(\lambda_1, \lambda_2). \end{aligned}$$

In other words

$$(5.2) \quad G^M(\lambda_1, \lambda_2) = \tilde{P}_M(\lambda_1, \lambda_2) \quad (\forall (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

It is proved by the first author in [Kar12] that the closed subspace  $E$  of the reproducing kernel Hilbert space  $\mathcal{H}(\Omega)$  over some set  $\Omega$  has finite codimension if and only if  $\lim_{\lambda \rightarrow \partial\Omega} \tilde{P}_{UE}(\lambda) = 1$  for any unitary operator  $U : \mathcal{H}(\Omega) \rightarrow \mathcal{H}(\Omega)$ . Since  $\text{co-dim } M < +\infty$ , by applying this result we conclude from (5.2) that  $\lim_{(\lambda_1, \lambda_2) \rightarrow \xi \in \partial\mathbb{D}^2} G^M(\lambda_1, \lambda_2) = 1$  for any  $\xi \in \partial\mathbb{D}^2$ . This proves the corollary.  $\square$

Note that since  $\mathbb{T}^2 \subsetneq \partial\mathbb{D}^2$ , this corollary somewhat improves the result of Guo and Yang [GY04] in case of submodules with finite codimensions.

### 6. On operator norm inequalities

In this section, we prove some operator norm inequalities. Namely, we will estimate  $\lim_{n \rightarrow \infty} \|T^n S\|$  for some appropriate operators  $T$  and  $S$  on a Hilbert space. Such type of limits is important, in particular, for the study of the compactness property of the operator  $S$ . For, it is enough to remember, for instance, the well known Hartman-Sarason [Nik86] theorem for compact model operators  $\varphi(M_\theta)$  ( $\varphi \in H^\infty$ ) defined on the model space  $K_\theta = H^2 \ominus \theta H^2$  by  $\varphi(M_\theta) f = P_\theta \varphi f$ ,  $f \in K_\theta$ . There are also many other recent papers where the limit  $\lim_{n \rightarrow \infty} \|T^n S\|$  is investigated for different goals, see, for instance, [ESZ, KZ09, Le09, Muh71, MusH14].

Before giving the results of this section, first let us introduce some additional notation.

Let  $H$  be a complex Hilbert space. If  $\{x_n\}_{n \geq 1} \subset H$ , we denote by  $\text{span}\{x_n : n = 1, 2, \dots\}$  the closure of the linear hull generated by  $\{x_n\}_{n \geq 1}$ . The sequence  $\mathbb{X} = \{x_n\}_{n \geq 1}$  is called:



- complete if  $\text{span} \{x_n : n = 1, 2, \dots\} = H$ ;
- a Riesz basis if there exists an isomorphism  $U : H \rightarrow H$  such that  $\{Ux_n\}_{n \geq 1}$  is an orthonormal basis in  $H$ . The operator  $U$  will be called the orthogonalizer of  $\mathbb{X}$ .

It is well known that (see, for example, Nikolski [Nik86])  $\mathbb{X}$  is a Riesz basis in its closed linear span if there are constants  $C_1 > 0, C_2 > 0$  such that

$$(6.1) \quad C_1 \left( \sum_{n \geq 1}^{\infty} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \geq 1}^{\infty} a_n x_n \right\| \leq C_2 \left( \sum_{n \geq 1}^{\infty} |a_n|^2 \right)^{1/2},$$

for all finite complex sequences  $\{a_n\}_{n \geq 1}$ . Note that  $\|U\|^{-1}$  and  $\|U^{-1}\|$  are the best constants in inequality (6.1).

Now we state our main results of this section:

**Theorem 4.** *Let  $H$  be an infinite dimensional separable Hilbert space with Riesz basis  $\{R_i\}_{i=1}^{\infty}$ , and let  $T : H \rightarrow H$  be a bounded linear operator. Then:*

- (i)  $\|T^n S\| = O \left( \sup_{m \geq 1} \left( \sum_{i=1}^m \|T^{*n} R_i\|^2 \right)^{1/2} \right)$ , as  $n \rightarrow \infty$ , for every  $S \in \mathcal{B}(H)$ .
- (ii) If  $\lim_{n \rightarrow \infty} \sup_{m \geq 1} \left( \sum_{i=1}^m \|T^{*n} R_i\|^2 \right)^{1/2} = 0$ , then  $\lim_{n \rightarrow \infty} \|T^n S\| = 0$  for every  $S \in \mathcal{B}(H)$ .

**Proof.** (i) Since  $\{R_i\}_{i=1}^{\infty}$  is a Riesz basis in  $H$  (and hence is complete in  $H$ ), for every  $x \in H$  with  $\|x\| = 1$  and  $\varepsilon > 0$ , there exists an integer  $N := N(x, \varepsilon) > 0$  and scalars  $c_i = c_i(x, \varepsilon) \in \mathbb{C}$  ( $i = 1, 2, \dots, N$ ) such that

$$\left\| x - \sum_{i=1}^N c_i R_i \right\| < \varepsilon,$$

which implies that

$$(6.2) \quad \left\| \sum_{i=1}^N c_i R_i \right\| < 1 + \varepsilon.$$

Taking into account the fact that  $\{R_i\}_{i=1}^{\infty}$  is a Riesz basis, and using (6.1) and (6.2), we obtain for any  $S \in \mathcal{B}(H)$  and  $n \geq 1$  that

$$\begin{aligned} \|T^n S\| &= \|(T^n S)^*\| = \|S^* T^{*n}\| = \sup_{\|x\|=1} \|S^* T^{*n} x\| \\ &= \sup_{\|x\|=1} \left\| \left\| S^* T^{*n} \left( x - \sum_{i=1}^N c_i R_i \right) + S^* T^{*n} \sum_{i=1}^N c_i R_i \right\| \right\| \\ &\leq \sup_{\|x\|=1} \left[ \|T^n S\| \varepsilon + \|S\| \sum_{i=1}^N |c_i| \|T^{*n} R_i\| \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \sup_{\|x\|=1} \left[ \|T^n S\| \varepsilon + \|S\| \left( \sum_{i=1}^N |c_i|^2 \right)^{1/2} \left( \sum_{i=1}^N \|T^{*n} R_i\|^2 \right)^{1/2} \right] \\
 &\leq \sup_{\|x\|=1} \left[ \|T^n S\| \varepsilon + \|S\| \|U\| \left\| \sum_{i=1}^N c_i R_i \right\| \left( \sum_{i=1}^N \|T^{*n} R_i\|^2 \right)^{1/2} \right] \\
 &\leq \sup_{\|x\|=1} \left[ \|T^n S\| \varepsilon + \|S\| \|U\| (\varepsilon + 1) \left( \sum_{i=1}^N \|T^{*n} R_i\|^2 \right)^{1/2} \right] \\
 &\leq \|T^n S\| \varepsilon + \|S\| \|U\| (\varepsilon + 1) \sup_{\|x\|=1} \left( \sum_{i=1}^N \|T^{*n} R_i\|^2 \right)^{1/2} \\
 &\leq \|T^n S\| \varepsilon + \|S\| \|U\| (\varepsilon + 1) \sup_{m \geq 1} \left( \sum_{i=1}^m \|T^{*n} R_i\|^2 \right)^{1/2},
 \end{aligned}$$

where  $U$  is the orthogonalizer of  $\{R_i\}_{i \geq 1}$ .

Since  $n \geq 1$  is an arbitrary fixed number and  $\varepsilon$  is arbitrary, by letting  $\varepsilon$  tend to zero, from the latter we deduce for all  $S \in \mathcal{B}(H)$  that

$$(6.3) \quad \|T^n S\| \leq \|S\| \|U\| \sup_{m \geq 1} \left( \sum_{i=1}^m \|T^{*n} R_i\|^2 \right)^{1/2},$$

for all  $n \geq 1$ , which proves assertion (i) of the theorem.

Assertion (ii) is immediate from inequality (6.3). □

**Corollary 7.** *If  $\sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 < +\infty$  for all  $n \geq 1$  and*

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 \right) = 0,$$

*then  $\lim_{n \rightarrow \infty} \|T^n S\| = 0$  for all  $S \in \mathcal{B}(H)$ .*

A particular case of this corollary is the following.

**Corollary 8.** *If  $\lim_{n \rightarrow \infty} \|T^{*n} R_i\| = 0$  for each  $i \geq 1$  and  $\sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 < +\infty$  for each  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} \|T^n S\| = 0$  for all  $S \in \mathcal{B}(H)$ .*

**Remark 1.** Observe that since  $R_i = U^{-1}e_i$  for some orthonormal basis  $\{e_i\}_{i \geq 1} \subset H$ , the condition

$$\sup_{m \geq 1} \left( \sum_{i=1}^m \|T^{*n} R_i\|^2 \right)^{1/2} =: M_n < +\infty$$

is satisfied, for example, for Hilbert-Schmidt operator  $T \in \mathcal{B}(H)$ .

**Proposition 5.** *Let  $H$  be any infinite dimensional separable Hilbert space with Riesz basis  $\{R_i\}_{i \geq 1}$ , and let  $T, S \in \mathcal{B}(H)$  be two operators such that:*

- (i) *There exists an isometry  $V : H \rightarrow H$  such that  $s\text{-}\lim_n T^n = V$ .*
- (ii)  $\sum_{i=1}^{\infty} \|SR_i\|^2 < +\infty$ .

Then

$$(6.4) \quad \lim_{n \rightarrow \infty} \|T^n S\| \leq \|U\| \left( \sum_{i=1}^{\infty} \|SR_i\|^2 \right)^{1/2}.$$

**Proof.** The proof is similar to the proof of Theorem 4. Indeed, since  $\lim_{n \rightarrow \infty} T^n x = Vx$  for all  $x \in H$ , where  $V$  is an isometry. Then, it is standard to show that  $T$  is power bounded, i.e.,  $\|T^n\| \leq C$  for all  $n \geq 0$  and some constant  $C > 0$ . Then, as in the proof of Theorem 4, we have for any  $\varepsilon \in (0, 1)$  that

$$\begin{aligned} \|T^n S\| &\leq \sup_{\|x\|=1} \left[ C \|S\| \varepsilon + \|U\| (\varepsilon + 1) \left( \sum_{i=1}^N \|T^n SR_i\|^2 \right)^{1/2} \right] \\ &\leq \|S\| C\varepsilon + \|U\| (\varepsilon + 1) \left( \sum_{i=1}^{\infty} \|T^n SR_i\|^2 \right)^{1/2} \\ &\rightarrow \|U\| \left( \sum_{i=1}^{\infty} \|T^n SR_i\|^2 \right)^{1/2}, \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T^n S\| &\leq \|U\| \left( \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} \|T^n SR_i\|^2 \right)^{1/2} \\ &= \|U\| \left( \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} \|V SR_i\|^2 \right)^{1/2} \\ &= \|U\| \left( \sum_{i=1}^{\infty} \|SR_i\|^2 \right)^{1/2}, \end{aligned}$$

which proves inequality (6.4), as desired. □

**Remark 2.** If  $\Lambda := \{\lambda_n\}_{n \geq 1}$  is a sequence of distinct points in  $\mathbb{D}$  and  $B = B_\Lambda = \bigcap_{n \geq 1} b_{\lambda_n}$ , where  $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \bar{\lambda}_n z}$  is the corresponding Blaschke product, then it is well known that (see, for instance, Nikolski [Nik86]):

(i) If  $\Lambda := \{\lambda_n\}_{n \geq 1}$  is a Blaschke sequence, i.e.,

$$\sum_{n=1}^{\infty} (1 - |\lambda_n|^2) < \infty,$$

then the system

$$\{k_{H^2, \lambda_n}(z)\}_{n \geq 1} := \left\{ \frac{1}{1 - \overline{\lambda_n} z} \right\}_{n \geq 1}$$

is complete in the model space  $K_B := H^2 \ominus BH^2$ .

(ii) The system

$$\{\tilde{k}_{H^2, \lambda_n}(z)\}_{n \geq 1} := \left\{ \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda_n} z} \right\}_{n \geq 1}$$

is a Riesz basis of  $K_B$  if and only if  $\{\lambda_n\}_{n \geq 1}$  satisfies Carleson's condition

$$\inf_{n \geq 1} |B_n(\lambda_n)| > 0,$$

where  $B_n = \frac{B}{b_{\lambda_n}}$ .

Thus, all results obtained above can be stated for the operators acting in  $K_B$ .

**Acknowledgements.** We thank to the referee for his useful remarks and suggestions which improved the presentation of the paper.

## References

- [BaCFMT10] BARANOV, ANTON; CHALENDAR, ISABELLE; FRICAIN, EMMANUEL; MASHREGHI, JAVAD; TIMOTIN, DAN. Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators. *J. Funct. Anal.* **259** (2010), no. 10, 2673–2701. MR2679022 (2011h:47051), Zbl 1204.47036, arXiv:0909.0131, doi: 10.1016/j.jfa.2010.05.005.
- [Ber72] BEREZIN, F. A. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1134–1167. MR0350504 (50 #2996), Zbl 0247.47019, doi: 10.1070/IM1972v006n05ABEH001913.
- [BerC86] BERGER, C. A.; COBURN, L. A. Toeplitz operators and quantum mechanics. *J. Funct. Anal.* **68** (1986), no. 3, 273–299. MR0859136 (88b:46098), Zbl 0629.47022. doi: 10.1016/0022-1236(86)90099-6.
- [CIS12] CASTRO, L. P.; ITOU, H.; SAITOH, S. Numerical solutions of linear singular integral equations by means of Tikhonov regularization and reproducing kernels. *Houston J. Math.* **38** (2012), no. 4, 1261–1276. MR3019034, Zbl 1262.65196.
- [CS13] CASTRO, L. P.; SAITOH, S. Fractional functions and their representations. *Complex Anal. Oper. Theory* **7** (2013), no. 4, 1049–1063. MR3079842, Zbl 1285.46024, doi: 10.1007/s11785-011-0154-1.
- [CSSS12] CASTRO, L. P.; SAITOH, S.; SAWANO, Y.; SIMÕES, A. M. General inhomogeneous discrete linear partial differential equations with constant coefficients on the whole spaces. *Complex Anal. Oper. Theory* **6** (2012), no. 1, 307–324. MR2886621, Zbl 1273.35017, doi: 10.1007/s11785-010-0083-4.

- [Doug73] DOUGLAS, R. G. Banach algebra techniques in the theory of Toeplitz operators. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, 15. *American Mathematical Society, Providence, R. I.*, 1973. v+53 pp. MR0361894 (50 #14336), Zbl 0252.47025.
- [Eng95] ENGLIŠ, MIROSLAV. Toeplitz operators and the Berezin transform on  $H^2$ . *Linear Algebra Appl.* **223/224** (1995), 171–204. MR1340692 (96k:47047), Zbl 0827.47017, doi: 10.1016/0024-3795(94)00056-J.
- [ESZ] ESTERLE, J.; STROUSE, E.; ZOUAKIA, F. Theorems of Katznelson–Tzafriri type for contractions. *J. Funct. Anal.* **94** (1990), no. 2, 273–287. MR1081645 (92c:47016), Zbl 0723.47013, doi: 10.1016/0022-1236(90)90014-C.
- [GY04] GUO, KUNYU; YANG, RONGWEI. The core function of submodules over the bidisk. *Indiana Univ. Math. J.* **53** (2004), no. 1, 205–222. MR2048190 (2005m:46048), Zbl 1062.47009, doi: 10.1512/iumj.2004.53.2327.
- [H62] HOFFMAN, KENNETH. Banach spaces of analytic functions. Prentice-Hall Series in Modern Analysis. *Prentice-Hall, Inc., Englewood Cliffs, N.J.* 1962. xiii+217 pp. MR0133008 (24 #A2844), Zbl 0117.34001.
- [Kar06] KARAĖV, MUBARIZ T. Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238** (2006), no. 1, 181–192. MR2253012 (2007d:47027), Zbl 1102.47018. doi: 10.1016/j.jfa.2006.04.030.
- [Kar08a] KARAĖV, M. T. On the Riccati equations. *Monatsh. Math.* **155** (2008), no. 2, 161–166. MR2447999 (2009i:47060), Zbl 1148.47013. doi: 10.1007/s00605-008-0542-4.
- [Kar08b] KARAĖV, M. T. On a Beurling–Arveson type theorem for some functional Hilbert spaces and related questions. *Integral Equations Operator Theory* **62** (2008), no. 1, 77–84. MR2442904 (2010c:46070), Zbl 1218.47014, doi: 10.1007/s00020-008-1607-4.
- [Kar12] KARAĖV, MUBARIZ T. Use of reproducing kernels and Berezin symbols technique in some questions of operator theory. *Forum Math.* **24** (2012), no. 3, 553–564. MR2926635, Zbl 1302.47043, doi: 10.1515/form.2011.073.
- [KZ09] KELLAY, K.; ZARRABI, M. Compact operators that commute with a contraction. *Integral Equations Operator Theory* **65** (2009), no. 4, 543–550. MR2576308 (2011c:47038), Zbl 1179.47021, doi: 10.1007/s00020-009-1724-8.
- [Le09] LÉKA, ZOLTÁN. A Katznelson–Tzafriri type theorem in Hilbert spaces. *Proc. Amer. Math. Soc.* **137** (2009), no. 11, 3763–3768. MR2529885 (2011e:47025), Zbl 1180.47011, doi: 10.1090/S0002-9939-09-09939-0.
- [Mah01] MAHVIDI, ALI. Invariant subspaces of composition operators. *J. Operator Theory* **46** (2001), no. 3, 453–476. MR1897149 (2003c:47044), Zbl 0994.47027, doi: 10.7900/jot.2013nov14.2041.
- [MarR07] MARTÍNEZ–AVENDAÑO RUBÉN A.; ROSENTHAL PETER. An introduction to operators on the Hardy–Hilbert space. Graduate Texts in Mathematics, 237. *Springer, New York*, 2007. xii+220 pp. ISBN: 978-0-387-35418-7; 0-387-35418-2. MR2270722 (2007k:47051), Zbl 1116.47001, doi: 10.1007/978-0-387-48578-2.
- [Muh71] MUHLY, PAUL A. Compact operators in the commutant of a contraction. *J. Functional Analysis* **8** (1971), 197–224. MR0315486 (47 #4035), Zbl 0225.47007, doi: 10.1016/0022-1236(71)90010-3.
- [MusH14] MUSTAFAYEV, H. S.; HÜSEYNOV, F. B. Compact operators in the commutant of essentially normal operators. *Banach J. Math. Anal.* **8** (2014), no. 2, 1–15. MR3189534, Zbl 1320.47020, doi: 10.15352/bjma/1396640047.
- [Nik86] NIKOLSKIĪ, N. K. Treatise on the shift operator. Spectral function theory. Grundlehren der Mathematischen Wissenschaften, 273. *Springer-Verlag*,

- Berlin*, 1986. xii+491 pp. ISBN: 3-540-15021-8. MR0827223 (87i:47042), Zbl 0587.47036, doi: 10.1007/978-3-642-70151-1.
- [Nik95] NIKOLSKI, NIKOLAI. Distance formulae and invariant subspaces, with an application to localization of zeros of the Riemann  $\zeta$ -function. *Ann. Inst. Fourier, Grenoble* **45** (1995), no. 1, 143–159. MR1324128 (96c:47005), Zbl 0816.30026, doi: 10.5802/aif.1451.
- [NR94] NORDGREN, ERIC; ROSENTHAL, PETER. Boundary values of Berezin symbols. *Nonselfadjoint operators and related topics* (Beer Sheva, 1992), 362–368, *Oper. Theory Adv. Appl.*, 73. *Birkhäuser, Basel*, 1994. MR1320554 (96b:46036), Zbl 0874.47013, doi: 10.1007/978-3-0348-8522-5\_14.
- [NRW87] NORDGREN, ERIC; ROSENTHAL, PETER; WINTROBE, F. S. Invertible composition operators on  $H^p$ . *J. Funct. Anal.* **73** (1987), no. 2, 324–344. MR0899654 (89c:47044), Zbl 0643.47034, doi: 10.1016/0022-1236(87)90071-1.
- [R69] RUDIN, WALTER. Function theory in polydiscs. *W. A. Benjamin, Inc., New York-Amsterdam*, 1969. vii+188 pp. MR0255841 (41 #501), Zbl 0294.32001, doi: 10.1002/zamm.19730530624.
- [Sar07] SARASON, DONALD. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices* **1** (2007), no. 4, 491–526. MR2363975 (2008i:47060), Zbl 1144.47026, doi: 10.7153/oam-01-29.
- [Sar08] SARASON, DONALD. Unbounded Toeplitz operators. *Integral Equations Operator Theory* **61** (2008), no. 2, 281–298. MR2418122 (2010c:47073), Zbl 1155.47033, doi: 10.1007/s00020-008-1588-3.
- [Yan04] YANG, RONGWEI. Beurling’s phenomenon in two variables. *Integral Equations Operator Theory* **48** (2004), no. 3, 411–423. MR2038510 (2004j:46038), Zbl 1061.46023, doi: 10.1007/s00020-002-1195-7.
- [Z07] ZHU, KEHE. Operator theory in function spaces, Second edition. *Mathematical Surveys and Monographs*, 138. *American Mathematical Society, Providence R.I.*, 2007. xvi+348 pp. ISBN: 978-0-8218-3965-2. MR2311536 (2008i:47064), Zbl 1123.47001, doi: 10.1016/0377-0427(91)90049-P.

(Mubariz T. Garayev) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, RIYADH 11451, SAUDI ARABIA  
mgarayev@ksu.edu.sa

(Hocine Guediri) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, RIYADH 11451, SAUDI ARABIA  
hguediri@ksu.edu.sa

(Houcine Sadraoui) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, RIYADH 11451, SAUDI ARABIA.  
sadrawi@ksu.edu.sa

This paper is available via <http://nyjm.albany.edu/j/2016/22-25.html>.