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# CAT(0) cubical complexes for graph products of finitely generated abelian groups

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Abstract. We construct for every graph product of finitely generated abelian groups a CAT(0) cubical complex on which it acts properly and cocompactly. The complex generalizes (up to subdivision) the Salvetti complex of a right-angled Artin group and the Coxeter complex of a right-angled Coxeter group.

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Let $\Gamma$ be a (simplicial) graph in which every vertex is labeled by a finitely generated abelian group. The graph product $G(\Gamma)$ is the free product of all these vertex groups modulo the relations that the elements of two of them commute if they are connected by an edge. The purpose of this article is to show:

**Main Theorem.** *Let $\Gamma$ be a finite graph with vertices labeled by finitely generated abelian groups. There is a* $\mathrm{CAT}(0)$ *cubical complex $X(\Gamma)$ on which $G(\Gamma)$ acts faithfully, properly, cocompactly and specially.*

Every graph product of finitely generated abelian groups can be written as a graph product of cyclic groups by inflating each vertex to a complete graph whose vertices correspond to direct summands. Therefore we can restrict ourselves to graph products of cyclic groups. That $G(\Gamma)$ acts properly and cocompactly on a $\mathrm{CAT}(0)$ cubical complex can then be deduced by putting

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together results from [JŚ01], [NR03] and [CM05] as we will in the proof of Theorem 1.4. That $G(\Gamma)$ is virtually special was shown previously by Kim [Kim12, Theorem 3(2)].

For these reasons we think that the main use of the article lies not in the result as such, but in the concrete construction which is explicit, easy and natural and generalizes complexes that are widely used in the literature. One extreme is the case where all the vertex groups are infinite cyclic and so the graph product is a right-angled Artin group. In this case, the natural complex for the group to act on is the Salvetti complex, and this is what our construction produces. The other extreme is the case where all vertex groups have order two so that the graph product is a right-angled Coxeter group. Here, the natural complex to act on is the right angled Coxeter complex and again it is recovered by our construction. In fact, the construction would work in the same way for graph products of finite groups and infinite cyclic groups at the expense of making the notation more cumbersome.

Davis and Januszkiewicz [DJ00] have shown that the Salvetti complex of a right-angled Artin group is (up to subdivision) also the Coxeter complex of a right-angled Coxeter group by embedding the right-angled Artin group with finite index into a right-angled Coxeter group and then identifying an appropriate sub-Coxeter group of the same index. We will show that these inclusions also work with our construction and that the complex associated to an arbitrary graph product of finitely generated abelian groups can also be seen as the complex associated to a graph product of finite abelian groups.

The original motivation that led to the present article is the following:

**Corollary 1.** *If the graph $\Gamma$ satisfies the No SILs condition, the subgroup $\mathrm{Aut}^{PC}(G(\Gamma))$ of the automorphism group of $G(\Gamma)$ generated by partial conjugations is* CAT(0).

**Proof.** A graph $\Gamma$ contains a SIL if there are vertices $v, w$ in $\Gamma$ of distance at least 2 such that the following holds: removing the intersection of the links of these two vertices results in a connected component that does not contain either $v$ or $w$. By [CRSV10, Theorem 3.6], if $\Gamma$ does not contain any SILs then $\mathrm{Aut}^{PC}(G(\Gamma))$ is again a graph product of cyclic groups. □

When trying to prove that a supergroup $G_1$ of a CAT(0) group $G_0$ is CAT(0), it is a natural idea to see whether the action extends. For this it is of course necessary to have an explicit description of the CAT(0) space and the $G_0$-action. The group $\mathrm{Aut}^{PC}(G(\Gamma))$ $(= G_1)$ is in general a semi-direct product extension of $G(\Gamma)$ $(= G_0)$ by [GPR12, Theorem 1.2]. Under the hypotheses of the main theorem there, the extension is by a finitely generated abelian group which is finite if all vertex groups are finite. Thus one attempt at showing this group is CAT(0) is to extend the action of $G(\Gamma)$ on $X(\Gamma)$. In the end, this could be avoided by using the group theoretic statement that $\mathrm{Aut}^{PC}(G(\Gamma))$ is again a graph product of cyclic groups.

A straightforward application of the strategy to extend the action is:

**Corollary 2.** *For any graph* $\Gamma$*, the action of* $G(\Gamma)$ *on* $X(\Gamma)$ *can be extended by the automorphism group of the (labelled) graph* $\Gamma$*. In particular,*

$$G(\Gamma) \rtimes \mathrm{Aut}(\Gamma)$$

*is* CAT(0) *and so is* $\mathrm{Aut}^{PC}(G(\Gamma)) \rtimes \mathrm{Aut}(\Gamma)$ *provided* $\Gamma$ *contains no SILs.*

**Proof.** The construction of the cube complex is functorial by Observation 2.4. □

The paper is organized as follows. In Section 1 we recall some facts about graph products of groups and set notation. The cubical complex is constructed in Section 2 and shown to be contractible and CAT(0) in Section 3. In Section 4 we make the connection to the Davis–Januszkiewicz construction.

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## 1. Graph products of cyclic groups

As mentioned in the introduction we may restrict ourselves to graph products of cyclic groups and we introduce notation accordingly. Let $\Gamma$ be a simplicial graph with vertex set $V$ and edge set $E$. Let $c\colon V \to \mathbb{N} \cup \{\infty\}$ be a vertex-labeling. We define the graph product of $\Gamma$ to be the group $G(\Gamma)$ with presentation

$$G(\Gamma) := \left\langle s \in V \mid s^{c(s)} = 1 \text{ for } s \in V, [s,t] = 1 \text{ for } \{s,t\} \in E \right\rangle .$$

Thus every vertex $s \in V$ is an element of order $c(s)$ in $G(\Gamma)$.

We will need a fact about general graph products which provides a solution to the word problem (with the elements of the vertex groups as generators). It was first proved by Green [Gre90] and later reproved using geometric methods by Hsu and Wise [HW99]. We only describe it in our case of cyclic vertex groups.

Every element $g \in G(\Gamma)$ can be written as a word $(s_1^{e_1}, \dots, s_k^{e_k})$ with each $s_i$ one of the vertices of $\Gamma$ (not necessarily distinct) and $e_i \in \mathbb{Z}$. By that we mean that the product $s_1^{e_1} \cdots s_k^{e_k}$ in $G(\Gamma)$ equals $g$. The following operations on words clearly do not change the element of $G(\Gamma)$ that the word describes:

(i) Remove the letter $1 (= s_i^0)$.
(ii) Replace two consecutive letters $s_i^{e_i}$ and $s_{i+1}^{e_{i+1}}$ which are powers of the same generator $s_i = s_{i+1}$ by $s_i^{e_i + e_{i+1}}$.
(iii) Replace two consecutive letters $s_i^{e_i}, s_{i+1}^{e_{i+1}}$ such that $s_i$ and $s_{i+1}$ are connected by an edge in $\Gamma$ by $s_{i+1}^{e_{i+1}}, s_i^{e_i}$.

A word that cannot be shortened using these operations is called *reduced*. The following is [HW99, Theorem 2.5].

**Theorem 1.1.** *A reduced word describes the identity element if and only if it is empty.*

By induction on the word length one obtains the seemingly stronger version formulated by Green [Gre90, Theorem 3.9].

**Corollary 1.2.** *Two words describe the same element if and only if they can be transformed into a common word using only the operations* (i) *to* (iii)*.*

In particular a word has minimal length (among those representing the corresponding element) if and only if it is reduced. For our purposes, a slightly different measurement of length will be useful. It corresponds to the generating set of $G(\Gamma)$ consisting of all the elements of the finite vertex groups, but only one generator for each infinite cyclic subgroup.

So if we partition the vertex set of $\Gamma$ into

$$V_{\text{fin}} = \{s \in V \mid c(s) < \infty\} \quad \text{and} \quad V_{\text{inf}} = \{s \in V \mid c(s) = \infty\}$$

the length of a word is given by

$$\ell(s_1^{e_1}, \dots, s_k^{e_k}) = \sum_{s_i \in V_{\text{inf}}} |e_i| + \sum_{s_i \in V_{\text{fin}}} 1.$$

It follows from Corollary 1.2 that every reduced word has minimal length with respect to this length function though the converse is not true. The length of an element of $G(\Gamma)$ is defined to be the minimal length of a word representing it (for example a reduced word).

We say that an element $g \in G(\Gamma)$ *ends with* $s \in V$ if there is a reduced word $(s_1^{e_1}, \dots, s_k^{e_k})$ representing $g$ with $s = s_k$. Another consequence of Corollary 1.2 is:

**Corollary 1.3.** *If $g$ ends with $s$ and $g$ ends with $t$ then $s$ and $t$ commute (that is, are connected by an edge).*

We end this section by giving an inexplicit proof of most of our main result by collecting references.

**Theorem 1.4.** *For $\Gamma$ as above the group $G(\Gamma)$ acts properly and cocompactly on a* CAT(0) *cubical complex.*

**Proof.** By Corollary 5.11 in [JŚ01] there is a Coxeter group $W$ that contains $G(\Gamma)$ as a finite index subgroup. In fact, tracing the proof back one finds that the Coxeter group has presentation

$$\begin{aligned} W = \langle s_v, t_v, v \in V \mid s_v^2 = t_v^2 = 1, (s_v t_v)^{c(v)} = 1 \text{ for } v \in V,& \\ [s_v, t_w] = [t_v, t_w] = 1 \text{ for } v, w \in V,& \\ [s_v, s_w] = 1 \text{ if } \{v, w\} \in E\rangle.& \end{aligned}$$

This Coxeter system has no irreducible affine Coxeter subsystem of rank $\geq 3$ because the only Coxeter relations with exponents other than 2 and $\infty$ are between generators named $s$ and generators named $t$. Using Theorem 1.3 of

[CM05] we deduce that $W$ contains only finitely many conjugacy classes of triangle groups. Thus by Theorem 1 and Theorem 4 of [NR03] there exists a CAT(0) cubical complex on which $W$ acts properly and cocompactly and thus the same is true of $G(\Gamma)$. □

## 2. The complex

From now on fix a finite graph $\Gamma$ with labeling $c$ and let $G := G(\Gamma)$ be the associated group. Let $V^{\pm} = V \cup V^{-1}$ denote the set of generators and their inverses of $G(\Gamma)$ and define $V^{\pm}_{\mathrm{inf}}$ accordingly. We define a new graph $\Delta$ whose vertex set is $\hat{V} := V_{\mathrm{fin}} \cup V^{\pm}_{\mathrm{inf}}$ and whose edges are given by pulling back the edges of $\Gamma$ via the obvious projection $\hat{V} \to V$ (note that this means that $s$ and $s^{-1}$ are *not* connected for $s \in V_{\mathrm{inf}}$).

For $s \in V^{\pm}$ we define the expression

$$[s] := \begin{cases} \langle s \rangle\,, & s \in V_{\mathrm{fin}},\\ \{1, s\}, & s \in V^{\pm}_{\mathrm{inf}}. \end{cases}$$

We extend this expression to cliques (vertices spanning complete subgraphs) of $\Delta$ by setting $[C] = [s_1] \cdots [s_k]$ (element-wise product) if $s_1, \ldots, s_k$ are the elements of $C$. Note that the order of the product does not matter since all the $s_i$ commute. Moreover the clique $C$ can be recovered from $[C]$, being just $[C] \cap \hat{V}$. Note that $[\emptyset] = \{1\}$.

We omit the straightforward proof of the following lemma.

**Lemma 2.1.** *Let $g \in G(\Gamma)$ be arbitrary and let $C \subseteq V(\Gamma)$ be a clique, i.e., any two elements of $C$ commute. For $s \in C$ we have $\max \ell(g[s]) > \ell(g)$ if and only if $\max \ell(g[C]) > \max \ell(g[C \setminus \{s\}])$.*

Let $\mathcal{S}$ be the poset of the $[C]$ where $C$ ranges over cliques of $\Delta$ and let $K := |\mathcal{S}|$ be its realization. That is, the vertices of $K$ are the sets $[C]$ and the simplices are flags of those, ordered by inclusion. The poset $\mathcal{S}$ is covered by the intervals $[[\emptyset], [C]]$ which are boolean lattices. Therefore $K$ naturally carries a cubical structure in which intervals are cubes (see for example [AB08, Proposition A.38]). The link of $[\emptyset]$ in this cubical structure is the flag complex of $\Delta$ (see Observation 3.5 below). This can be used to see that $K$ is CAT(0) cubical, which also follows from the fact that $\mathcal{S}$ is a discrete median semilattice which is equivalent to the structure of a discrete median algebra or of a median graph (see Theorem 3.1 and Theorem 4.3 of [BH83]) and gives rise to a CAT(0) cube structure by [Rol98, Theorem 10.3] or [Che00, Theorem 6.1]. However, neither that $K$ is CAT(0) nor the median structures will be used in what follows.

Let $\mathcal{T} := G\mathcal{S}$ be the set of cosets of elements of $\mathcal{S}$ (see below for pictures). This set is again ordered by inclusion and its realization $X = X(\Gamma) := |\mathcal{T}|$ is the space we are looking for. For the same reason as before $X$ can be regarded as a cubical complex. Note that if $\Gamma$ has no vertices labeled $\infty$, then $\Gamma = \Delta$ and $X(\Gamma)$ is just the usual coset-complex (the right-angled building from [Dav98, Section 5]). Every edge of $X$ is of the form

$$gs^k[C] \leq gs^k[C \cup \{s\}]$$

with $k \in \mathbb{Z}$, $g$ not ending with $s$, and $s \in \hat{V} \setminus C$.

**Observation 2.2.** *The subspace $K$ is a weak fundamental domain for the action of $G$ on $X$.*

An alternative description of $X$ is as a quotient $G \times K/\sim$ where $\sim$ is the equivalence relation generated by (the reflexive, symmetric, transitive hull of) the following two kinds of relations: firstly $(g, x) \sim (gt, x)$ if $t \in \langle s \rangle$ and $x \in [\langle t \rangle, \langle C \rangle]$ for $t \in C \subseteq V_{\text{fin}}$; and secondly $(gs, x) \sim (gs^{-1}, y)$ for $s \in V_{\text{inf}}$ if the barycentric coordinates of $y$ can be obtained from those of $x$ by replacing $s^{-1}$ by $s$.

**Observation 2.3.** *The complex $X$ is locally finite and the action of $G$ is proper.*

**Proof.** This essentially follows from the finiteness of the sets $[C]$. □

We close the section by discussing functoriality of the assignments $\Gamma \mapsto G(\Gamma)$ and $\Gamma \mapsto X(\Gamma)$. We describe a category **VGph** whose objects are simple, vertex labelled graphs. A morphism $\Gamma \to \Gamma'$ is a map $\varphi \colon V(\Gamma) \to V(\Gamma')$ that takes incident vertices to incident vertices such that $c(\varphi(s))$ divides $c(s)$ for every vertex $s$. The convention here is that every natural number divides $\infty$. Such a morphism gives rise to a homomorphism $G(\Gamma) \to G(\Gamma')$ that takes $s$ to $\varphi(s)$. Similarly, taking $[s]$ to $[\varphi s]$ induces a morphism of cube complexes.

This can be summarized as follows. Denote by **Grp** the category of groups, by **Cub** the category of cube complexes and by **CubAct** the category of groups acting on cube complexes:

**Observation 2.4.** *The above data define functors*

$$G\colon \textbf{\textit{VGph}} \to \textbf{\textit{Grp}} \quad \textit{and} \quad X\colon \textbf{\textit{VGph}} \to \textbf{\textit{Cub}}$$

*that are compatible in the sense that they define a functor*

$$(G \curvearrowright X)\colon \textbf{\textit{VGph}} \to \textbf{\textit{CubAct}}.$$

It seems likely that the construction in the proof of Theorem 1.4 has the same functoriality property, but it is not as easily verified.

## 3. Contractibility and CAT(0)-ness

One reason for the popularity of cubical complexes is Gromov's link condition [Gro87, Proposition 4.2.C], see also [BH99, Theorem II.5.20]:

**Theorem 3.1.** *A finite dimensional cubical complex has nonpositive curvature if and only if the link of each of its vertices is a flag complex.*

To get a criterion for a complex to be CAT(0) (rather than nonpositively curved) we also need the Cartan–Hadamard theorem [BH99, Theorem II.4.1]:

**Theorem 3.2.** *A complete connected metric space that is nonpositively curved and simply connected is* CAT(0)*.*

Using both results, we only have to show that $X$ is simply connected and has flag complexes as links to conclude that $X$ is CAT(0). However, in our case showing that $X$ is simply connected is not significantly easier than showing it to be contractible. We will therefore directly show:

**Theorem 3.3.** *$X$ is contractible.*

We will prove Theorem 3.3 by building up from a vertex to the whole complex while taking care that contractibility is preserved at each step. The technical tool to do this is combinatorial Morse theory.

A *Morse function* on an affine cell complex $X$ is a map $f\colon X^{(0)} \to \mathbb{Z}$ such that every cell $\sigma$ of $X$ has a unique vertex in which $f$ attains its maximum. The *descending link* $\operatorname{lk}^{\downarrow} v$ of a vertex $v$ consists of those cofaces $\sigma$ for which $v$ is that vertex. We denote the sublevel set, the complex supported on $f^{-1}((-\infty, n])$, by $X^{\le n}$. The Morse lemma in its most basic form — which is good enough for us — can be stated as follows.

**Lemma 3.4.** *If $X^{\le n-1}$ is contractible and for every vertex $v$ with $f(v) = n$ the descending link $\operatorname{lk}^{\downarrow} v$ is contractible then $X^{\le n}$ is contractible.*

Returning to our concrete setting, we first study the links of vertices in $X$. Let $L$ denote the flag complex of $\Delta$.

**Observation 3.5.** *The link of a vertex of the form $g[\emptyset] = \{g\}$ is isomorphic to $L$. Indeed, the correspondence $C \mapsto g[C]$ is a bijection between the faces of $L$ and the cofaces of $g[\emptyset]$.*

The link of a general vertex is not much more complicated:

**Observation 3.6.** *Let $g[C]$ be a vertex of $X$. The link decomposes as*

$$\operatorname{lk} g[C] = \operatorname{ulk} g[C] * \operatorname{dlk} g[C]$$

*into an up-link $\operatorname{ulk} g[C]$ with simplices $g[C']$ with $C' \supsetneq C$, and a down-link $\operatorname{dlk} g[C]$ with simplices $gh[C']$ with $C' \subsetneq C$, $h \in [C \setminus C']$. The up-link is isomorphic to the link of $C$ in $L$. The down-link is isomorphic to the join of the sets $[s], s \in C$.*

Note that the *simplices* in the link of a vertex correspond to *vertices* in the ambient complex. This is a feature of cubical complexes, see Figure 1. Observation 3.6 implies in particular:

**Corollary 3.7.** *The link of every vertex of $X$ is a flag complex.*

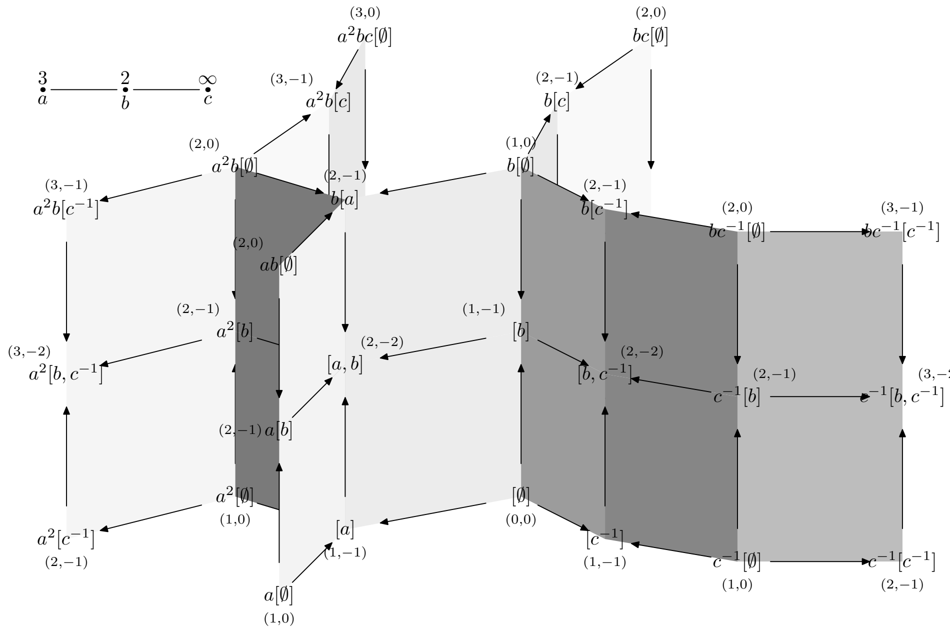


FIGURE 1. Part of the complex $X(\Gamma)$ for the graph $\Gamma$ indicated in the upper left corner. The smaller ordered pair of numbers next to a vertex indicates the value of $f$ on that vertex. The arrows indicate the poset relation. The "bottom layer" of the complex is the complex of the graph obtained from $\Gamma$ by deleting the vertex $b$, see Figure 2.

We are now ready to start the proof of Theorem 3.3. The function we will be using is

$$\begin{aligned} f\colon X &\to \mathbb{Z} \times \mathbb{Z} \\ g[C] &\mapsto (\max \ell(g[C]), -\#C) \end{aligned}$$

where $\ell$ is length of an element as defined in Section 1 and the codomain is ordered lexicographically. This is not technically a Morse function as defined before since the codomain is not $\mathbb{Z}$. However, since $\#C$ is uniformly bounded by $\#V$, one could work with the map $v \mapsto (\#V + 1)f_1(v) + f_2(v)$ in order to match the definition literally. The values of $f$ are indicated in Figure 1.

**Lemma 3.8.** *Let $\sigma$ be a cube in $X$ and let $v$ be a vertex of $\sigma$. If $v$ is a local $f$-maximum on $\sigma$ then it is a global $f$-maximum. That is if $f(v) > f(w)$ for all $w \in \sigma$ adjacent to $v$ then $f(v) > f(w)$ for all $w \in \sigma$, $w \neq v$.*

**Proof.** Let the cube $\sigma$ be $g[[C_1], [C_2]]$ and assume that the vertex $v = g[C]$ with $C_1 \subseteq C \subseteq C_2$ is a local maximum for $f$. We claim that $C$ consists of $C_1$ and those elements $s \in C_2$ with the property that $g[s]$ contains an element

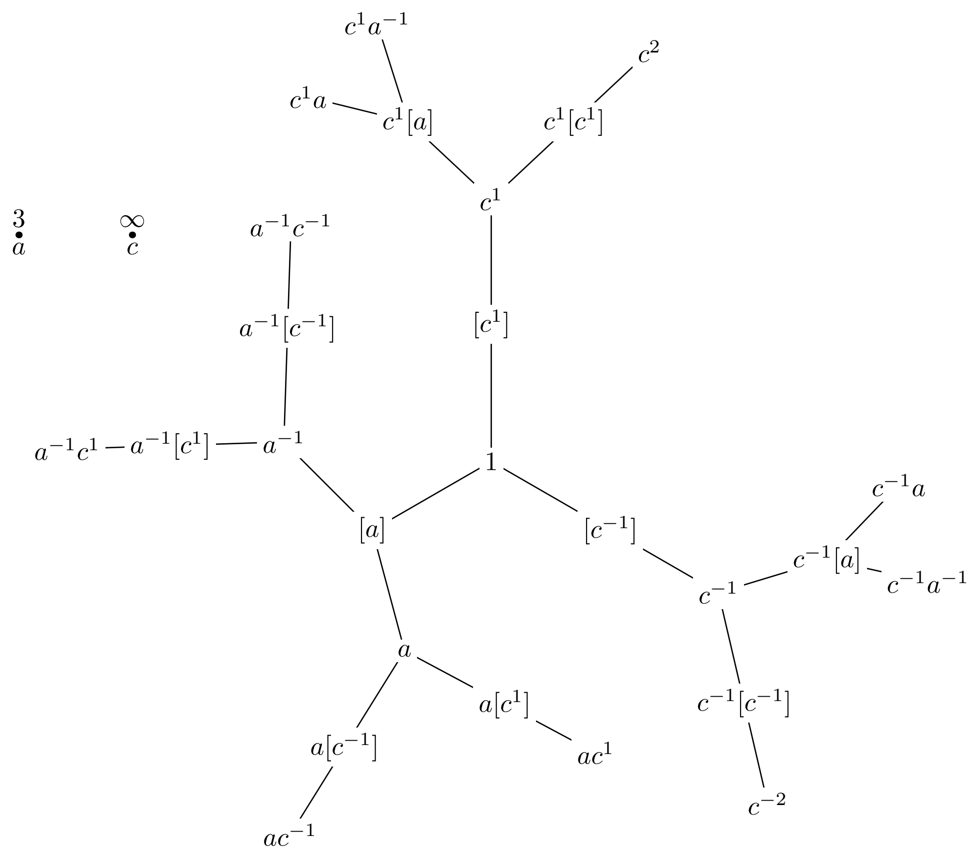


FIGURE 2. Part of the complex $X(\Gamma)$ for the graph $\Gamma$ indicated in the upper left corner.

longer than $g$. By Lemma 2.1 for this purpose it does not matter whether we look at $g$ or at $g[C']$ for some $C' \subseteq C_2$ not containing $s$.

For $s \in C_2 \backslash C$ the vertex $w = g[C \cup \{s\}]$ is adjacent to $v$. So by assumption $f(w) < f(v)$. Looking at the first component of $f$ this means that $\max \ell(C \cup \{s\}) \leq \max \ell(C)$ and since $C \cup \{s\} \supseteq C$ we see that actually

$$\max \ell(g[C \cup \{s\}]) = \max \ell(g[C]). \tag{3.1}$$

That is, multiplying on the right by any element of $[C_2 \setminus C]$ does not make $g$ longer.

Similarly if $s \in C \setminus C_1$, then the vertex $x = g[C \setminus \{s\}]$ is adjacent to $v$. Now the second component of $f$ has $-\#(C \setminus \{s\}) > -\#C$ so in order for $f(x)$ to be smaller than $f(v)$, it is necessary that the first component is smaller, that is,

$$\max \ell(g[C \setminus \{s\}]) < \max \ell(g[C]). \tag{3.2}$$

That is, multiplying on the right by some element of $C \setminus C_1$ makes $g$ longer.

Now let $y \in \sigma$ be an arbitrary vertex and write $y = g[C']$ with $C_1 \subseteq C' \subseteq C_2$. If $C'$ does not contain $C$ then (3.1) and (3.2) imply that $\max \ell(g[C']) < \max \ell(g[C])$. So $f(y) < f(v)$ by the first component. On the other hand if $C'$ strictly contains $C$ then (3.1) implies $\max \ell(g[C']) = \max \ell(g[C])$. Then $f(y) < f(v)$ by the second component: $-\#C' < -\#C$. The only remaining case is $C' = C$ where there is nothing to show. □

The first consequence of this is that $f$ is indeed a Morse function.

**Corollary 3.9.** *The function $f$ attains its maximum over the vertices of a cube of $X$ in a unique vertex.*

**Proof.** Note that two adjacent vertices cannot have same height: if $C \subsetneq C'$ then $-\#C' < -\#C$. So following an ascending edge path one finds that every cube has a strict local maximum. Now Lemma 3.8 shows that this is in fact a strict global maximum and in particular is unique. □

To apply Lemma 3.4 we have to show that descending links are contractible. For this purpose we separately look at the up- and down-link.

**Corollary 3.10.** *For every vertex the descending link is a full subcomplex of the link. In particular it decomposes as $\mathrm{lk}^{\downarrow} v = \mathrm{ulk}^{\downarrow} v * \mathrm{dlk}^{\downarrow} v$ into the descending up-link $\mathrm{ulk}^{\downarrow} v = \mathrm{lk}^{\downarrow} v \cap \mathrm{ulk}\, v$ and the descending down-link $\mathrm{dlk}^{\downarrow} v = \mathrm{lk}^{\downarrow} v \cap \mathrm{dlk}\, v$.*

**Proof.** This is immediate from Lemma 3.8: a simplex in the link of $v$ whose vertices lie in the descending link corresponds to a cube $\sigma$ above $v$ such that the neighbors of $v$ in $\sigma$ are descending. The Lemma now implies that the whole simplex lies in the descending link. □

Since the join of a contractible complex with any complex is contractible, it suffices to show that one of the two join factors is contractible. For vertices of the form $g[\emptyset]$ we prove it for the up-link, for all others for the down-link.

**Lemma 3.11.** *Let $g[C]$ be a vertex of $X$. If $C \neq \emptyset$ then the descending down-link is contractible.*

**Proof.** Let $s \in C$. The vertices corresponding to the join factor $[s]$ in Observation 3.6 point to the vertices $gh[C \setminus \{s\}], h \in [s]$. Such a vertex is descending only for the unique $h \in [s]$ such that $g$ ends with $h^{-1}$ or, if no such $h$ exists, for $h = 1$. In either case it is a single point. Thus the descending down-link is a join of singleton sets, that is, a simplex. □

**Lemma 3.12.** *The descending up-link of a vertex of the form $g[\emptyset]$ is contractible provided $g \neq 1$.*

**Proof.** A vertex of the form $g[C]$ is descending for $g[\emptyset]$ unless $\max \ell(g[C]) > \ell(g)$, that is, unless $C$ contains a letter that $g$ does not end with. In other words $g[C]$ is descending if $C$ contains only letters that $g$ ends with. The poset of such $g[C]$ is a barycentrically subdivided simplex by Corollary 1.3 (unless $g = 1$ in which case it is empty). □

**Proof of Theorem 3.3.** The sublevel set $X^{\leq(0,0)}$ contains vertices $g[C]$ whose longest elements have length 0. The only possibility for this is $\{1\}$, so $X^{\leq(0,0)}$ consists of a single vertex and in particular is contractible. Using that the descending link of every vertex is contractible by Lemmas 3.11 and 3.12, an inductive application of Lemma 3.4 shows that every sublevel set of $X$ is contractible. But the whole complex is the limit of these, and thus is contractible as well. □

From Theorems 3.1, 3.2 and 3.3 and Corollary 3.7 we get:

**Corollary 3.13.** *$X$ is* CAT(0)*.*

That $X$ is CAT(0) implies that it is in particular *special* in the sense of Haglund and Wise [HW08]. We would like to say that the quotient $G\backslash X$ (and in fact $H\backslash X$ for every $H \leq G$) is special as well. This follows from [HW10, Theorem 3.5] if the action of $G$ on $X$ is special in the sense of [HW10, Definition 3.4]. In view of the known cases for right-angled Artin and Coxeter groups [HW08, Example 3.3 (ii)] it is not surprising that this condition is indeed satisfied:

**Observation 3.14.** *The action of $G$ on $X$ is special.*

**Proof.** Since every edge is of the form $gs^k[C] \leq gs^k[C \cup \{s\}]$, it is naturally oriented and labeled by some $s^k$. Orientation and label are preserved along walls and under the action of $G$. Walls with the same label cannot cross and cannot osculate. Walls with label $s^k$ and $t^\ell$ can cross only if $s$ and $t$ commute in which case no two such walls can osculate. The conditions in [HW10, Definition 3.4] now follow. □

**Corollary 3.15.** *$G$ is virtually cocompact special.*

**Proof.** By Corollary 3.13 $X$ is CAT(0) and hence special (see [HW08, Example 3.3 (ii)]). Every torsion element in $G$ fixes a cell of $X$ [BH99, Corollary II.2.8(1)] and hence is conjugate to an element of $\langle C \rangle$ for some clique $C \subseteq V_{\text{fin}}$. Thus any nontrivial torsion element is mapped to a nontrivial element under the projection $G \to \prod_{s \in V_{\text{fin}}} \langle s \rangle$. So the kernel $H$ of this map acts freely and cocompactly on $X$. The quotient $H\backslash X$ is special by Observation 3.14 and [HW10, Theorem 3.5]. □

## 4. Comparison to Davis–Januszkiewicz

When all vertices of $\Gamma$ are labeled $\infty$, then $G(\Gamma)$ is a right-angled Artin group. For that case Davis and Januszkiewicz constructed graphs $\Gamma'$ and $\Gamma''$, which in our notation would have every vertex labeled 2. They showed that $G(\Gamma)$ embeds as a finite index subgroup into $G(\Gamma'')$ which in turn acts on the Coxeter complex of $G(\Gamma')$. We want to explain how their construction carries over to graph products of general cyclic groups and how it relates to the construction from Section 2. The contents of this section could be used to reduce the proof of the Main Theorem to the (known) case of graph products of finite cyclic groups. However, having the self-contained proof from Section 3, we will skip the technical verifications needed.

The graph $\Gamma'$ has vertices $V_{\text{fin}} \cup (V_{\text{inf}} \times \{-1,1\})$ and edges are given by pulling back the edges from $\Gamma$ via the obvious projection to $V$. The graph $\Gamma''$ has vertices $V_{\text{fin}} \cup (V_{\text{inf}} \times \{0,1\})$ and the following edges: the subgraph on $V_{\text{fin}} \cup (V_{\text{inf}} \times \{1\})$ is canonically isomorphic to $\Gamma$; and a vertex $(v,0)$ is connected to every other vertex except for $(v,1)$. Each of the vertices in $V_{\text{fin}}$ keeps its label and the vertices in $(V_{\text{inf}} \times \{-1,0,1\})$ are labeled by 2. For a vertex $i \in V_{\text{inf}}$ Davis–Januzskiewicz denote these elements $g_i = i$, $s_i = (i,1)$, $t_i = (i,-1)$, and $r_i = (i,0)$ respectively.

The graph products associated to these groups are related via the maps

$$\begin{aligned} \beta\colon G(\Gamma) &\to G(\Gamma'') \\ s &\mapsto s\ , \quad s \in V_{\text{fin}} \\ s &\mapsto (s,1)\cdot(s,0)\ , \quad s \in V_{\text{inf}} \end{aligned}$$

and

$$\begin{aligned} \alpha\colon G(\Gamma') &\to G(\Gamma'') \\ s &\mapsto s\ , \quad s \in V_{\text{fin}} \\ (s,1) &\mapsto (s,1)\ , \quad s \in V_{\text{inf}} \\ (s,-1) &\mapsto (s,0)\cdot(s,1)\cdot(s,0)\ , \quad s \in V_{\text{inf}} \end{aligned}$$

which are easily seen to be injective. In fact, letting $E$ denote the subgroup of $G(\Gamma'')$ generated by the elements $(s,0), s \in V_{\text{inf}}$ we see that $G(\Gamma'')$ can be written as semidirect products

$$G(\Gamma) \rtimes E = G(\Gamma'') = G(\Gamma') \rtimes E \tag{4.1}$$

where the action is always trivial on $V_{\text{fin}}$ and on the remaining generators is given by

$$s^{(t,0)} = \begin{cases} s & s \neq t \\ s^{-1} & s = t \end{cases} \qquad \text{respectively} \qquad (s,\pm 1)^{(t,0)} = \begin{cases} (s,\pm 1) & s \neq t \\ (s,\mp 1) & s = t\ . \end{cases}$$

This can be seen by writing

$$\begin{aligned} G(\Gamma'') = \langle V \cup (V_{\text{inf}} \times \{-1,0,1\}) \mid & \text{ all the previous relations,} \\ & s = (s,1)\cdot(s,0),\ \text{for } s \in V_{\text{inf}}, \\ & (s,-1) = (s,0)\cdot(s,1)\cdot(s,0) \text{ for } s \in V_{\text{inf}} \rangle \end{aligned}$$

and then applying Tietze transformations to remove generators. These algebraic considerations play the role of the geometric arguments in [DJ00]. One of the main ingredients here is that the groups $G(\Gamma)$ and $G(\Gamma')$ do indeed admit the described actions of the group $E$. The basic example to keep in mind is the following.

**Example 4.1.** If $\Gamma$ has just one vertex labeled $\infty$, then $X(\Gamma)$ can be thought of as the real line. Then $G(\Gamma)$ is the group generated by an element $s$ which is translation by 2. The group $G(\Gamma')$ is generated by elements $(s,1)$ and $(s,-1)$ which are reflection at 1 and $-1$ respectively. Finally, $G(\Gamma'')$ is generated by elements $(s,1)$ and $(s,0)$ which are reflection at 1 and 0.

It is clear from the description that $\Gamma'$ is isomorphic to the graph $\Delta$ from Section 2. Therefore $X(\Gamma')$ is isomorphic to $X(\Delta)$. We will show below that they are also isomorphic to $X(\Gamma)$. Note that $X(\Gamma'')$ is not typically homeomorphic to these complexes. Indeed if $\Gamma$ consists of two isolated vertices at least one of which is labeled $\infty$, then $X(\Gamma'')$ is 2-dimensional while $X(\Gamma')$ and $X(\Gamma')$ are 1-dimensional. The importance of $\Gamma''$ lies not so much in the complex $X(\Gamma'')$ but rather in the group $G(\Gamma'')$.

To show that $G(\Gamma'')$ acts on $X(\Gamma)$ and $X(\Gamma')$ and that both are equivariantly isomorphic, we define a third complex $Y$, that is a coset complex of $G(\Gamma'')$. Recall that $X(\Gamma)$ is the coset complex of sets of the form $\langle s\rangle, s \in V_{\text{fin}}$ and $[s], [s^{-1}], s \in V_{\text{inf}}$, while $X(\Gamma')$ is the coset complex of subgroups of the form $\langle s\rangle, s \in V_{\text{fin}}$ and $\langle (s,1)\rangle, \langle (s,-1)\rangle, s \in V_{\text{inf}}$. The construction of $Y$ is based on the observation that

$$[s]E = \langle (s,1)\rangle E \quad \text{and} \quad [s^{-1}]E = \langle (s,-1)\rangle E \tag{4.2}$$

in $G(\Gamma'')$ for $s \in V_{\text{inf}}$ (this follows from the formulas $s = (s,1)\cdot(s,0)$ and $s^{-1} = (s,-1)\cdot(s,0)$ for $s \in V_{\text{inf}}$). We therefore define $\mathcal{P}$ to be the poset of sets $\langle s\rangle E, s \in V_{\text{fin}}$ as well as those in (4.2). Further, $\mathcal{Q}$ is defined to be the poset $G(\Gamma'')\mathcal{P}$ of cosets of these sets and $Y$ to be the realization of $\mathcal{Q}$.

**Proposition 4.2.** *The maps*

$$g[s] \mapsto \alpha(g[s])E,$$
$$g\langle (s,\pm 1)\rangle \mapsto \beta(g\langle s,\pm 1\rangle)E,$$

*induce ($G(\Gamma)$- respectively $G(\Gamma')$-) equivariant isomorphisms $X(\Gamma)\to Y$ respectively $X(\Gamma')\to Y$. In particular $G(\Gamma'')$ acts on $X(\Gamma)$ and $X(\Gamma')$ and they are equivariantly isomorphic.*

**Proof.** Bijectivity of both maps follows from the semidirect product decompositions (4.1). Equivariance is clear by construction. The order is preserved since it is just inclusion. □

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