New York Journal of Mathematics

New York J. Math. **22** (2016) 933–942.

Differentiating along rectangles, in lacunary directions

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Abstract. We show that, given some lacunary sequence of angles $\boldsymbol{\theta} = (\theta_j)_{j\in\mathbb{N}}$ not converging too fast to zero, it is possible to build a rare differentiation basis $\mathscr{B}$ of rectangles parallel to the axes that differentiates $L^1(\mathbb{R}^2)$ while the basis $\mathscr{B}_{\boldsymbol{\theta}}$ obtained from $\mathscr{B}$ by allowing its elements to rotate around their lower left vertex by the angles θ_j, $j \in \mathbb{N}$, fails to differentiate all Orlicz spaces lying between $L^1(\mathbb{R}^2)$ and $L\log L(\mathbb{R}^2)$.

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1. Introduction

Assume that $\boldsymbol{\theta} = (\theta_j)_{j\in\mathbb{N}} \subseteq (0, 2\pi)$ is a lacunary sequence going to zero and denote by $\mathscr{B}_{\boldsymbol{\theta}}$ the set of all rectangles in $\mathbb{R}^2$, one of whose sides makes an angle θ_j with the horizontal axis, for some $j \in \mathbb{N}$. It follows from results by Córdoba and Fefferman [2] (for $p > 2$) and Nagel, Stein and Wainger [7] (for all $p > 1$) that for every $f \in L^p(\mathbb{R}^2)$, one has:

$$f(x) = \lim_{\substack{R\in\mathscr{B}_{\boldsymbol{\theta}} \\ R\ni x \\ \operatorname{diam} R\to 0}} \frac{1}{|R|}\int_R f, \tag{1}$$

for almost every $x \in \mathbb{R}^2$ (we say, in this case, that $\mathscr{B}_{\boldsymbol{\theta}}$ *differentiates* $L^p(\mathbb{R}^n)$). This is often equivalent, according to Sawyer-Stein principles (see *e.g.* Garsia [3, Chapter 1]), to the fact that the associated maximal operator $M_{\mathscr{B}}$,

Received May 16, 2016.

2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 26B05.

Key words and phrases. Lebesgue differentiation theorem, Hardy–Littlewood maximal operator, lacunary directions.

This work was partially supported by the French ANR project "GEOMETRYA" no. ANR-12-BS01-0014.

ISSN 1076-9803/2016

defined for measurable functions f by:

$$M_{\mathscr{B}}f(x) := \sup_{\substack{R \in \mathscr{B} \\ R \ni x}} \frac{1}{|R|} \int_R |f|,$$

satisfies a weak (p,p) inequality, *i.e.*, verifies:

$$|\{M_{\mathscr{B}}f > \alpha\}| \leqslant \frac{C}{\alpha^p} \int_{\mathbb{R}^2} |f|^p,$$

for all $\alpha > 0$ and all $f \in L^p(\mathbb{R}^2)$. By interpolation, of course, such a property for all $p > 1$ implies that $M_{\mathscr{B}}$ sends boundedly $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $p > 1$.

Since then, the L^p $(p > 1)$ behaviour of the operators $M_{\mathscr{B}_{\boldsymbol{\theta}}}$ has been studied when the lacunary sequence $\boldsymbol{\theta}$ is replaced by some Cantor sets (see *e.g.* Katz [5] and Hare [4]); recently, Bateman [1] obtained necessary and sufficient (geometrical) conditions on $\boldsymbol{\theta}$ providing the L^p boundedness of $M_{\mathscr{B}_{\boldsymbol{\theta}}}$.

In this paper we explore the behaviour of some maximal operators associated to rare differentiation bases of rectangles oriented in a lacunary set of directions $\boldsymbol{\theta} = \{\theta_j : j \in \mathbb{N}\}$, provided that the sequence (θ_j) does not converge too fast to zero. More precisely, we prove the following theorem.

Theorem 1. *Given a lacunary sequence* $\boldsymbol{\theta} = (\theta_j)_{j \in \mathbb{N}} \subseteq (0, 2\pi)$ *satisfying:*

$$0 < \varliminf_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} \leqslant \varlimsup_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} < 1,$$

there exists a differentiation basis $\mathscr{B}$ of rectangles parallel to the axes satisfying the two following properties:

(i) $M_{\mathscr{B}}$ *has weak type* $(1,1)$ *(in particular $\mathscr{B}$ differentiates $L^1(\mathbb{R}^2)$).*

(ii) *If we denote by $\mathscr{B}_{\boldsymbol{\theta}}$ the differentiation basis obtained from $\mathscr{B}$ by allowing its elements to rotate around their lower left corner by any angle θ_j, $j \in \mathbb{N}$, then for any Orlicz function Φ (see below for a definition) satisfying $\Phi = o(t \log_+ t)$ at ∞, the maximal operator$M_{\mathscr{B}_{\boldsymbol{\theta}}}$* fails to have *weak type* (Φ, Φ) *(in particular $\mathscr{B}_{\boldsymbol{\theta}}$ fails to differentiate $L^\Phi(\mathbb{R}^n)$).*

Remark 2. The differentiation basis $\mathscr{B}$ we shall construct in the proof of Theorem 1 is *rare*: it will be obtained as the smallest translation-invariant basis containing a *countable* family of rectangles with lower left corner at the origin (see Section 3 for a more precise statement).

Our paper is organized as follows: we first discuss some easy geometrical facts concerning rectangles and rotations along lacunary sequences, following with a proof of Theorem 1.

2. Some basic geometrical facts

In the sequel we always call *standard rectangle* in $\mathbb{R}^2$ a set of the form $Q = [0, L] \times [0, \ell]$ where $L > 0$ and $\ell > 0$ are real numbers; we then let $Q_+ := [L/2, L] \times [0, \ell]$. For $\theta \in [0, 2\pi)$ we also denote by r_θ the (counterclockwise) rotation of angle θ around the origin.

Lemma 3. *Fix real numbers $0 \leqslant \vartheta < \theta < \frac{\pi}{2}$ and $0 < 2\ell < L$ and let $Q := [0, L] \times [0, \ell]$. If moreover one has $\tan(\theta - \vartheta) \geqslant 1/\sqrt{\frac{1}{4}\left(\frac{L}{\ell}\right)^2 - 1}$, then $r_\vartheta Q_+$ and $r_\theta Q_+$ are disjoint.*

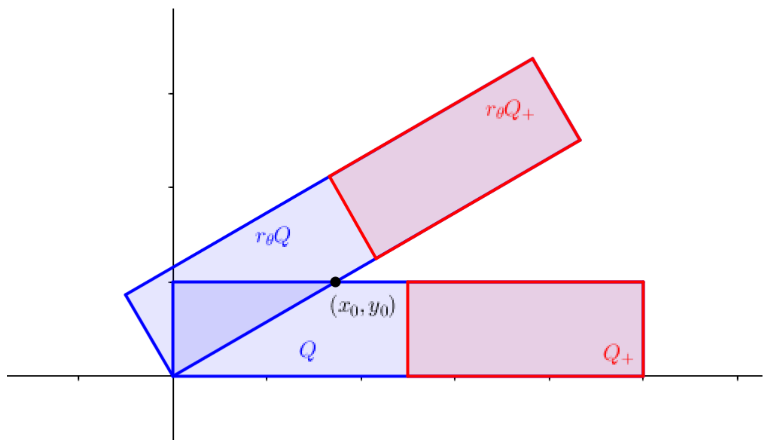

FIGURE 1. The rectangles $Q, Q_+, r_\theta Q$ and $r_\theta Q_+$

Proof. To prove this lemma, we can assume, without loss of generality, that one has $\vartheta = 0$ (for otherwise, apply $r_{-\vartheta}$ to $r_\vartheta Q_+$ and $r_\theta Q_+$). Let $m := \tan\theta$. Observe then that the lines $y = \ell$ and $y = mx$ intersect at $x_0 = \ell/m \leqslant \ell\sqrt{\frac{1}{4}\left(\frac{L}{\ell}\right)^2 - 1} \leqslant \frac{L}{2}$ and $y_0 = \ell$. Since we also have:

$$|(x_0, y_0)| \leqslant \ell\sqrt{\frac{1}{4}\left(\frac{L}{\ell}\right)^2} = \frac{L}{2},$$

this shows indeed that Q_+ and $r_\theta Q_+$ are disjoint (see Figure 1). □

Lemma 4. *Assume that the sequence $(\theta_j)_{k\in\mathbb{N}} \subseteq (0, \pi/2)$ is such that one has:*

$$0 < \lambda < \varliminf_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} \leqslant \varlimsup_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} < \mu < 1. \tag{2}$$

Let $\boldsymbol{\theta} := \{\theta_j : j \in \mathbb{N}\}$.

There exists constants $d(\mu) > c(\mu) > 0$ depending only on μ such that, for each $\varepsilon > 0$ and each integer $k \in \mathbb{N}^*$, one can find a standard rectangle

$Q_k = [0, L_k] \times [0, \ell_k]$ and a subset $\boldsymbol{\theta}_k \subset \boldsymbol{\theta}$ satisfying $\#\boldsymbol{\theta}_k = k$ such that the following hold:

(i) $0 \leqslant 2\ell_k \leqslant L_k \leqslant \varepsilon$.
(ii) $c(\mu)\lambda^{-k} \leqslant \frac{L_k}{\ell_k} \leqslant d(\mu)\lambda^{-k}$.
(iii) $\left|\bigcup_{\theta\in\boldsymbol{\theta}_k} r_\theta Q_k\right| \geqslant \frac{k}{2}|Q_k|$.

Proof. To prove this lemma, observe first that letting $m_j := \tan\theta_j$ for all $j \in \mathbb{N}$, one clearly has:

$$\lim_{j\to\infty} \frac{m_j}{\theta_j} = 1,$$

so that (2) also holds for the sequence $(m_j)_{j\in\mathbb{N}}$. There hence exists an index $j_0 \in \mathbb{N}$ such that, for all $j \geqslant j_0$, one has $\lambda \leqslant \frac{m_{j+1}}{m_j} \leqslant \mu$ (we may also and will assume that one has $m_{j_0} \leqslant 1$). For the sake of clarity, we shall now consider that $j_0 = 0$ and compute, for an integer $0 \leqslant j < k$:

$$\tan(\theta_j - \theta_k) = \frac{m_j - m_k}{1 + m_j m_k} \geqslant \frac{1}{2}(m_j - m_k).$$

Since we also have, for every integer $0 \leqslant j < k$:

$$\lambda^{k-j} m_j \leqslant m_k \leqslant \mu^{k-j} m_j,$$

we obtain under the same assumptions on j:

$$\tan(\theta_j - \theta_k) \geqslant \frac{1}{2}(m_j - m_k) \geqslant \frac{1}{2}(\mu^{j-k} - 1)m_k \geqslant \frac{1}{2}\lambda^k(\mu^{-1} - 1)m_0.$$

Now choose real numbers $0 \leqslant 2\ell \leqslant L \leqslant \varepsilon$ (we write L and ℓ instead of L_k and ℓ_k here, for the index k remains constant all through the proof) satisfying:

$$\left(\frac{L}{\ell}\right)^2 = 4 + \lambda^{-2k}[(\mu^{-1} - 1)m_0]^{-2}.$$

It is clear that one has:

$$\frac{L}{\ell} = \lambda^{-k}\sqrt{4\lambda^{2k} + [(\mu^{-1} - 1)m_0]^{-2}},$$

so that (ii) holds if we take, for example, $c(\mu) := \sqrt{[(\mu^{-1} - 1)m_0]^{-2}}$ and $d(\mu) := \sqrt{4 + [(\mu^{-1} - 1)m_0]^{-2}}$. On the other hand, (i) is clearly satisfied by assumption.

In order to show (iii), define $Q := [0, L] \times [0, \ell]$ and observe that one has

$$\tan(\theta_j - \theta_k) \geqslant \frac{1}{\sqrt{\frac{1}{4}\left(\frac{L}{\ell}\right)^2 - 1}},$$

for all integers j satisfying $j < k$. According to Lemma 3, this ensures that the family $\{r_{\theta_j} Q_+ : j \in \mathbb{N}, j < k\}$ consists of pairwise disjoints sets; in particular we get:

$$\left|\bigcup_{j=0}^{k-1} r_{\theta_j} Q\right| \geqslant \left|\bigsqcup_{j=0}^{k-1} r_{\theta_j} Q_+\right| = k \cdot \frac{|Q|}{2},$$

(we used $\sqcup$ to indicate a disjoint union) and the lemma is proved. □

We now turn to studying maximal operators associated to families of standard rectangles.

3. Maximal operators associated to lacunary sequences of directions

From now on, given a family $\mathscr{R}$ of standard rectangles and a set $\boldsymbol{\theta} \subseteq [0, 2\pi)$, we let $r_{\boldsymbol{\theta}}\mathscr{R} := \{r_\theta Q : Q \in \mathscr{R}, \theta \in \boldsymbol{\theta}\}$, and we define, for $f : \mathbb{R}^2 \to \mathbb{R}$ measurable:

$$M_{\mathscr{R}} f(x) := \sup \left\{ \frac{1}{|Q|} \int_{\tau(Q)} |f| : Q \in \mathscr{R}, \tau \text{ translation}, x \in \tau(Q) \right\},$$

and:

$$M_{r_{\boldsymbol{\theta}}\mathscr{R}} f(x) := \sup \left\{ \frac{1}{|R|} \int_{\tau(R)} |f| : R \in r_{\boldsymbol{\theta}}\mathscr{R}, \tau \text{ translation}, x \in \tau(R) \right\}.$$

Notice, in particular, that in case one has $\inf\{\operatorname{diam} R : R \in \mathscr{R}\} = 0$, $M_{\mathscr{R}}$ and $M_{r_{\boldsymbol{\theta}}\mathscr{R}}$ are the maximal operators associated to the translation-invariant differentiation bases $\mathscr{B}$ and $\mathscr{B}_{\boldsymbol{\theta}}$ defined respectively by:

$$\mathscr{B} := \{\tau(Q) : Q \in \mathscr{R}, \tau \text{ translation}\}$$

and

$$\mathscr{B}_{\boldsymbol{\theta}} := \{\tau(r_\theta Q) : Q \in \mathscr{R}, \theta \in \boldsymbol{\theta}, \tau \text{ translation}\}.$$

The next proposition will be useful in order to study the maximal operator $M_{r_{\boldsymbol{\theta}}\mathscr{R}}$. Observe that it has the flavour of Stokolos' [8, Lemma 1].

Proposition 5. *Assume that* $(\theta_j)_{j\in\mathbb{N}} \subseteq (0, 2\pi)$ *satisfies:*

$$0 < \lambda < \varliminf_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} \leqslant \varlimsup_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} < \mu < 1,$$

and let $\boldsymbol{\theta} := \{\theta_j : j \in \mathbb{N}\}$. *There exists a (countable) family* $\mathscr{R}$ *of standard rectangles in* $\mathbb{R}^2$ *which is totally ordered by inclusion, verifies* $\inf\{\operatorname{diam} R : R \in \mathscr{R}\} = 0$ *and satisfies the following property: for any* $k \in \mathbb{N}^*$, *there exists sets* $\Theta_k \subseteq \mathbb{R}^2$ *and* $Y_k \subseteq \mathbb{R}^2$ *satisfying the following conditions:*

(i) $|Y_k| \geqslant \kappa(\mu) \cdot k\lambda^{-k}|\Theta_k|$.

(ii) *For any* $x \in Y_k$, *one has* $M_{r_{\boldsymbol{\theta}}\mathscr{R}}\chi_{\Theta_k} f(x) \geqslant \kappa'(\mu)\lambda^k$.

Here, $\kappa(\mu) > 0$ *and* $\kappa'(\mu) > 0$ *are two constants depending only on* μ.

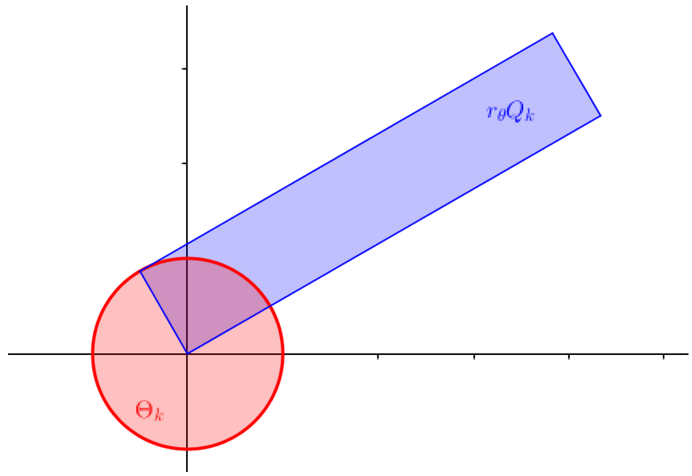

FIGURE 2. The intersection $\Theta_k \cap r_\theta Q_k$

Proof. Define $\mathscr{R} = \{Q_k : k \in \mathbb{N}^*\}$ where the sequence $(Q_k)_{k\in\mathbb{N}^*}$ is defined inductively as follows. We choose $Q_1 = [0, L_1] \times [0, \ell_1]$ and $\boldsymbol{\theta}_1 \subseteq \boldsymbol{\theta}$ associated to $k = 1$ and $\varepsilon = 1$ according to Lemma 4. Assuming that $Q_1, \ldots, Q_k$ have been constructed, for some integer $k \in \mathbb{N}^*$, we choose $Q_{k+1} = [0, L_{k+1}] \times [0, \ell_{k+1}]$ and $\boldsymbol{\theta}_{k+1}$ associated to $k+1$ and $\varepsilon = \min(\ell_k, 1/k)$ according to Lemma 4. Since the sequence $(Q_k)_{k\in\mathbb{N}^*}$ is a nonincreasing sequence of rectangles, it is clear that $\mathscr{R}$ is totally ordered by inclusion. It is also clear by construction that one has $\inf\{\operatorname{diam} R : R \in \mathscr{R}\} = 0$.

Now fix $k \in \mathbb{N}^*$ and define $\Theta_k := B(0, \ell_k)$ and $Y_k := \bigcup_{\theta\in\boldsymbol{\theta}_k} r_\theta Q_k$. Compute hence, using [Lemma 4, (ii) and (iii)]:

$$|Y_k| \geqslant \frac{1}{2} k L_k \ell_k = \frac{1}{2\pi} k \frac{L_k}{\ell_k} \cdot \pi \ell_k^2 \geqslant \frac{c(\mu)}{2\pi} \cdot k \lambda^{-k} |\Theta_k|,$$

so that (i) is proved in case one lets $\kappa(\mu) := \frac{c(\mu)}{2\pi}$.

For $x \in Y_k$, choose $\theta \in \boldsymbol{\theta}_k$ for which one has $x \in r_\theta Q_k$ and observe that one has (see Figure 2):

$$M_{r_{\boldsymbol{\theta}}\mathscr{R}} \chi_{\Theta_k}(x) \geqslant \frac{|\Theta_k \cap r_\theta Q_k|}{|Q_k|} = \frac{\frac{1}{4} \cdot \pi \ell_k^2}{L_k \ell_k} = \frac{\pi}{4} \cdot \frac{\ell_k}{L_k} \geqslant \frac{\pi}{4d(\mu)} \lambda^k,$$

which finishes the proof of (ii) if we let $\kappa'(\mu) := \frac{\pi}{4d(\mu)}$. □

For our purposes, an *Orlicz function* is a convex and increasing function $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(0) = 0$; we then let $L^\Phi(\mathbb{R}^2)$ denote the set of all measurable functions f in $\mathbb{R}^2$ for which $\Phi(|f|)$ is integrable (for $\Phi(t) = t^p$, $p \geqslant 1$ this yields the usual Lebesgue space $L^p(\mathbb{R}^2)$, while for

$$\Phi(t) = \Phi_0(t) := t(1 + \log_+ t)$$

we get the Orlicz space $L\log_+ L(\mathbb{R}^2) := L^{\Phi_0}(\mathbb{R}^2))$. Recall that a sublinear operator T is said to be *of weak type* (Φ, Φ) in case there exists a constant $C > 0$ such that, for all $f \in L^{\Phi}(\mathbb{R}^2)$ and all $\alpha > 0$, one has:

$$|\{x \in \mathbb{R}^2 : Tf(x) > \alpha\}| \leqslant \int_{\mathbb{R}^2} \Phi\left(\frac{|f|}{\alpha}\right).$$

Whenever $\Phi(t) = t^p$ for $p \geqslant 1$, we shall say that T has *weak type* (p,p).

The next result specifies the announced Theorem 1. It is mainly a consequence of the preceding proposition and some standard techniques as developed in Moonens and Rosenblatt [6].

Theorem 6. *Assume that* $(\theta_j)_{j\in\mathbb{N}} \subseteq (0, 2\pi)$ *satisfies:*

$$0 < \varliminf_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} \leqslant \varlimsup_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} < 1,$$

and let $\boldsymbol{\theta} := \{\theta_j : j \in \mathbb{N}\}$. *There exists a (countable) family* $\mathscr{R}$ *of standard rectangles in* $\mathbb{R}^2$ *with* $\inf\{\operatorname{diam} R : R \in \mathscr{R}\} = 0$, *satisfying the following conditions:*

(i) $M_{\mathscr{R}}$ *has weak type* $(1,1)$, *and hence the associated differentiation basis* $\mathscr{B}$ *differentiates* $L^1(\mathbb{R}^2)$.

(ii) *For any Orlicz function* Φ *satisfying* $\Phi = o(\Phi_0)$ *at* ∞, $M_{r_{\boldsymbol{\theta}}\mathscr{R}}$ *fails to be of weak type* (Φ, Φ). *In particular,* $M_{r_{\boldsymbol{\theta}}\mathscr{R}}$ *fails to have weak type* $(1,1)$, *and hence the associated differentiation basis* $\mathscr{B}_{\boldsymbol{\theta}}$ *fails to differentiate* $L^1(\mathbb{R}^2)$.

Proof. Begin by choosing real numbers $0 < \lambda < \mu < 1$ such that one has:

$$0 < \lambda < \varliminf_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} \leqslant \varlimsup_{j\to\infty} \frac{\theta_{j+1}}{\theta_j} < \mu < 1,$$

and keep the notations of Proposition 5.

Let now $\mathscr{R}$ be the family of rectangles given by Proposition 5. Observe first that, since $\mathscr{R}$ is totally ordered by inclusion, it follows *e.g.* from [9, Claim 1] that $M_{\mathscr{R}}$ satisfies a weak $(1,1)$ inequality.

In order to show (ii), define, for k sufficiently large, $f_k := [1/\kappa'(\mu)] \cdot \lambda^{-k}\chi_{\Theta_k}$, where Θ_k and Y_k are associated to k and $\mathscr{R}$ according to Proposition 5.

Claim 1. For each sufficiently large k, we have:

$$|\{x \in \mathbb{R}^2 : M_{\mathscr{R}} f_k(x) \geqslant 1\}| \geqslant c_1(\lambda, \mu) \int_{\mathbb{R}^2} \Phi_0(f_k),$$

where $c_1(\lambda,\mu) := \frac{2\log\frac{1}{\lambda}}{\kappa(\mu)\cdot\kappa'(\mu)}$ is a constant depending only on λ and μ.

Proof of the claim. To prove this claim, one observes that for $x \in Y_k$ we have $M_{\mathscr{R}} f_k(x) \geqslant 1$ according to [Proposition 5, (ii)]. Yet, on the other hand, one computes, for k sufficiently large:

$$\int_{\mathbb{R}^2} \Phi_0(f_k) \leqslant \frac{1}{\kappa'(\mu)} \cdot \lambda^{-k} |\Theta_k| \left[1 - \log_+ \kappa'(\mu) + k \log \frac{1}{\lambda}\right] \leqslant \frac{2 \log \frac{1}{\lambda}}{\kappa'(\mu)} \cdot k \lambda^{-k} |\Theta_k| \leqslant c_1(\lambda, \mu) \cdot |Y_k|,$$

and the claim follows. □

Claim 2. For any Φ satisfying $\Phi = o(\Phi_0)$ at ∞ and for each $C > 0$, we have:

$$\lim_{k \to \infty} \frac{\int_{\mathbb{R}^2} \Phi_0(|f_k|)}{\int_{\mathbb{R}^2} \Phi(C|f_k|)} = \infty.$$

Proof of the claim. Compute for any k:

$$\frac{\int_{\mathbb{R}^2} \Phi(C|f_k|)}{\int_{\mathbb{R}^2} \Phi_0(|f_k|)} = \frac{\Phi(\lambda^{-k} C / \kappa'(\mu))}{\Phi_0(\lambda^{-k} / \kappa'(\mu))} = \frac{\Phi(\lambda^{-k} C / \kappa'(\mu))}{\Phi_0(\lambda^{-k} C / \kappa'(\mu))} \frac{\Phi_0(\lambda^{-k} C / \kappa'(\mu))}{\Phi_0(\lambda^{-k} / \kappa'(\mu))},$$

observe that the quotient $\frac{\Phi_0(\lambda^{-k} C / \kappa'(\mu))}{\Phi_0(\lambda^{-k} / \kappa'(\mu))}$ is bounded as $k \to \infty$ by a constant independent of k, while by assumption the quotient $\frac{\Phi(\lambda^{-k} C / \kappa'(\mu))}{\Phi_0(\lambda^{-k} C / \kappa'(\mu))}$ tends to zero as $k \to \infty$. The claim is proved. □

We now finish the proof of Theorem 6. To this purpose, fix Φ an Orlicz function satisfying $\Phi = o(\Phi_0)$ at ∞ and assume that there exists a constant $C > 0$ such that, for any $\alpha > 0$, one has:

$$|\{x \in \mathbb{R}^2 : M_{\mathscr{R}} f(x) > \alpha\}| \leqslant \int_{\mathbb{R}^2} \Phi\left(\frac{C|f|}{\alpha}\right).$$

Using Claim 1, we would then get, for each k sufficiently large:

$$0 < c_1(\lambda, \mu) \int_{\mathbb{R}^2} \Phi_0(f_k) \leqslant \left|\left\{x \in \mathbb{R}^2 : M_{\mathscr{R}} f_k(x) > \frac{1}{2}\right\}\right| \leqslant \int_{\mathbb{R}^n} \Phi(2C f_k),$$

contradicting the previous claim and proving the theorem. □

Remark 7. If we are solely interested in the weak $(1,1)$ behaviour of the maximal operators $M_{\mathscr{R}}$ and $M_{r_{\boldsymbol{\theta}} \mathscr{R}}$, observe that Theorem 6 in particular applies to $\Phi(t) = t$, ensuring that the maximal operator $M_{r_{\boldsymbol{\theta}} \mathscr{R}}$ also fails to have weak type $(1,1)$.

Moreover, as pointed out by the referee, the construction, given a sequence of distinct angles $\boldsymbol{\theta} = (\theta_j)_j \subseteq (0, \pi/2)$, of a countable family $\mathscr{R}$ of rectangles for which $M_{\mathscr{R}}$ is of weak type $(1,1)$ while $M_{r_{\boldsymbol{\theta}} \mathscr{R}}$ is not, can be done almost immediately from Lemma 3 — and does not require a growth condition on the sequence $\boldsymbol{\theta}$.

To see this, observe that for each k, it is easy, according to Lemma 3 and making $L_k/\ell_k \gg 1$ large enough, to construct a rectangle $Q_k = [0, L_k] \times [0, \ell_k]$ such that the rectangles $r_{\theta_j} Q_{k,+}$, $0 \leqslant j \leqslant k$ are pairwise disjoint. We can also inductively construct (Q_k) such that one has $Q_{k+1} \subseteq Q_k$ for all $k \in \mathbb{N}$. Hence $\mathscr{R} := \{Q_k : k \in \mathbb{N}\}$ is totally ordered by inclusion, ensuring that $M_{\mathscr{R}}$ has weak type $(1,1)$.

On the other hand, define for $k \in \mathbb{N}$ a function $f_k := |Q_k| \frac{\chi_{B(0,\ell_k)}}{|B(0,\ell_k)|}$. For all $x \in Y_k := \bigcup_{j=0}^{k} r_{\theta_j} Q_k$, choose an integer $0 \leqslant j \leqslant k$ for which one has $x \in r_{\theta_j} Q_k$ and compute (see Figure 2 again):

$$M_{r_{\boldsymbol{\theta}}\mathscr{R}} f_k(x) \geqslant \frac{|Q_k|}{|B(0,\ell_k)|} \frac{|B(0,\ell_k) \cap r_{\theta_j} Q_k|}{|Q_k|} = \frac{1}{4}.$$

It hence follows that one has:

$$\begin{aligned}(k+1)\|f_k\|_1 &= (k+1)|Q_k| = 2(k+1)|Q_{k,+}| \\ &\leqslant 2|Y_k| \leqslant 2\left|\left\{x \in \mathbb{R}^2 : M_{r_{\boldsymbol{\theta}}\mathscr{R}} f_k(x) \geqslant \frac{1}{4}\right\}\right|,\end{aligned}$$

so that $M_{r_{\boldsymbol{\theta}}\mathscr{R}}$ cannot have weak type $(1,1)$.

Remark 8. In [Theorem 6, (ii)], it is not clear to us whether or not the space $L\log L(\mathbb{R}^2)$ is sharp; we don't know, for example, whether or not $\mathscr{B}_{\boldsymbol{\theta}}$ differentiates $L\log^{1+\varepsilon} L(\mathbb{R}^2)$ for $\varepsilon \geqslant 0$.

Acknowledgements. I would like to thank my colleague and friend Emma D'Aniello for her careful reading of the first manuscript of this paper. I also express my gratitude to the referee for his/her careful reading of the paper and his/her nice suggestions which were of a great help to improve it.

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This paper is available via http://nyjm.albany.edu/j/2016/22-44.html.