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# Preserving positive integer images of matrices

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Abstract. We prove that whenever $u,v\in\mathbb{N}$ and $A$ is a $u\times v$ matrix with integer entries and rank $n$, there is a $u\times n$ matrix $B$ such that $\{A\vec{k}:\vec{k}\in\mathbb{Z}^v\}\cap\mathbb{N}^u=\{B\vec{x}:\vec{x}\in\mathbb{N}^n\}\cap\mathbb{N}^u$. As a consequence we obtain the following result which answers a question of Hindman, Leader, and Strauss: Let $R$ be a subring of the rationals with $1\in R$ and let $S=\{x\in R:x>0\}$. If $A$ is a finite matrix with rational entries, then there is a matrix $B$ with no more columns than $A$ such that the set of images of $B$ in $S$ via vectors with entries from $S$ is exactly the same as as the set of images of $A$ in $S$ via vectors with entries from $R$.

We also show that the notion of image partition regularity is strictly stronger than that of weak image partition regularity in terms of Ramsey Theoretic consequences. That is, we show that for each $u\geq 3$, there are no $v$ and a $u\times v$ matrix $A$ such that for any $\vec{y}\in\{A\vec{k}:\vec{k}\in\mathbb{Z}^v\}\cap\mathbb{N}^u$, the set of entries of $\vec{y}$ form (in some order) a length $u$ arithmetic progression.

Contents



## 1. Introduction

Let $\mathbb{N}$ be the set of positive integers, let $u,v\in\mathbb{N}$, and consider the following system of homogeneous linear equations where each $a_{i,j}$ is rational.

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$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,v}x_v & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,v}x_v & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{u,1}x_1 & + & a_{u,2}x_2 & + & \ldots & + & a_{u,v}x_v & = & 0 \end{array}$$

Rado in 1933 characterized [6, Satz IV] those systems of homogeneous linear equations which are *regular* in the sense that whenever $\mathbb{N}$ is divided into finitely many classes (or *finitely colored*) there is a solution $\{x_1, x_2, \ldots, x_v\}$ which is contained in one class (or is *monochromatic*). Call a subset $M$ of $\mathbb{N}$ *large* if it contains a solution set for any regular system of homogeneous linear equations. Rado conjectured that if $M$ is large and $M$ is finitely colored, then there is a monochromatic large subset of $M$. This conjecture was proved by Deuber [1] in 1973. Deuber proved Rado's conjecture by using objects that he called $(m, p, c)$-*sets*.

**Definition 1.1.** Let $m, p, c \in \mathbb{N}$. A set $D \subseteq \mathbb{N}$ is an $(m, p, c)$-*set* if and only if there exists $\vec{x} \in \mathbb{N}^m$ such that

$$D = \{\textstyle\sum_{i=1}^{m} \lambda_i x_i : \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subseteq \{0, 1, \ldots, p\} \text{ and there is some } j \in \{1, 2, \ldots, m\} \text{ such that } \lambda_j = c \text{ and } \lambda_i = 0 \text{ for } i < j\}.$$

A significant part of Deuber's proof was establishing that when $m, p, c \in \mathbb{N}$ and $\mathbb{N}$ is finitely colored, there is a monochromatic $(m, p, c)$-set. (See [2, p. 80] for a more detailed description of how Deuber's proof of Rado's conjecture proceeded.)

Notice that Rado's Theorem is a characterization of those $u \times v$ matrices $A$ such that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $\vec{x}$ are monochromatic and $A\vec{x} = \vec{0}$.

Given $m, p, c \in \mathbb{N}$, let $A$ be a matrix consisting of all possible rows of the form $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that each $\lambda_i \in \{0, 1, \ldots, p\}$ and there is some $j \in \{1, 2, \ldots, m\}$ such that $\lambda_j = c$ and $\lambda_i = 0$ for $i < j$. Call such $A$ an $(m, p, c)$-*matrix*. Then the portion of Deuber's proof mentioned above is the assertion that whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{N}^m$ such that the entries of $A\vec{x}$ are monochromatic.

Thus, in the terminology of the following definition, Rado characterized those matrices that are kernel partition regular over $\mathbb{N}$ and Deuber established that the $(m, p, c)$-matrices are image partition regular over $\mathbb{N}$.

**Definition 1.2.** Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with rational entries, let $S$ be a nontrivial subsemigroup of $(\mathbb{Q}, +)$, and let $G$ be the subgroup of $\mathbb{Q}$ generated by $S$.

(a) The matrix $A$ is *kernel partition regular over* $S$ if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that $A\vec{x} = \vec{0}$ and the entries of $\vec{x}$ are monochromatic.

(b) The matrix $A$ is *image partition regular over* $S$ if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that the entries of $A\vec{x}$ are monochromatic.

Notice that image partition regularity of matrices corresponds naturally with many classical results of Ramsey Theory. For example, Schur's Theorem [7] is the assertion that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ is image partition regular over $\mathbb{N}$ and the length 5 version of van der Waerden's Theorem [8] is the assertion that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

is image partition regular over $\mathbb{N}$.

In the early 1990's the first author of this paper was working on some Ramsey Theoretic problems with Walter Deuber, Imre Leader, and Hanno Lefmann and was surprised to learn, given the important role image partition regularity played in the proof of Rado's conjecture and the way they naturally represent important results in Ramsey Theory, that there was no known characterization of those matrices that are image partition regular over $\mathbb{N}$.

Accordingly, he and Leader got to work on an attempt to characterize matrices that are image partition regular over $\mathbb{N}$, and in reasonably short order, they almost succeeded. The "almost" refers to the fact that they had to allow the entries of $\vec{x}$ to be 0 or negative. That is, they came up with some characterizations of *weakly imge partition regular* matrices over $\mathbb{N}$.

**Definition 1.3.** Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with rational entries, let $S$ be a nontrivial subsemigroup of $(\mathbb{Q}, +)$, and let $G$ be the subgroup of $\mathbb{Q}$ generated by $S$. The matrix $A$ is *weakly image partition regular over* $S$ if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in G^v$ such that the entries of $A\vec{x}$ are monochromatic.

To see why they were not happy at this stage, consider the matrix above representing the length 5 version of van der Waerden's Theorem. The fact that it is weakly image partition regular over $\mathbb{N}$ is completely trivial. (Let $x_1 = 1$ and $x_2 = 0$.)

Eventually, they did find some characterizations of image partition regularity over $\mathbb{N}$ and published these, along with the characterizations of weak image partition regularity, in [4].

It is apparent that image partition regularity and weak image partition regularity are vastly different notions. However, recently the following surprising (and surprisingly easy) result was obtained by Hindman, Leader and

Strauss [5]. Here $\omega = \mathbb{N} \cup \{0\}$ is the first infinite ordinal, and the entries of the matrix are indexed by ordinals. The group denoted $S - S$ is the group of differences of $S$.

**Theorem 1.4.** *Let $(S, +)$ be a commutative cancellative semigroup with at least three elements and let $G = S - S$. Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}$. Define a $u \times (2 \cdot v)$ matrix $C$ by, for $i < u$ and $j < v$, $c_{i,2\cdot j} = a_{i,j}$ and $c_{i,2\cdot j+1} = -a_{i,j}$. Then*

$$\{A\vec{x} : \vec{x} \in G^v\} = \{C\vec{y} : \vec{y} \in (S \setminus \{0\})^{2\cdot v}\}.$$

**Proof.** [5, Theorem 1.5]. □

This and several other related results in [5] led to the following question being asked [5, Question 3.4].

**Question 1.5.** *Let $S$ be a nontrivial proper subsemigroup of $\mathbb{Q}^+$ and let $G$ be the subgroup of $\mathbb{Q}$ generated by $S$. Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with rational entries that is weakly image partition regular over $S$. Must there exist $w \leq v$ and a $u \times w$ matrix $C$ such that*

$$\{A\vec{x} : \vec{x} \in G^v\} \cap (S \setminus \{0\})^u = \{C\vec{y} : \vec{y} \in (S \setminus \{0\})^w\} \cap (S \setminus \{0\})^u\,?$$

In the case $S = \mathbb{N}$ and $G = \mathbb{Z}$, this asks whether given any finite matrix $A$ with rational entries (which is weakly image partition regular over $\mathbb{N}$), there must exist a matrix $B$ with no more columns than $A$ such that the set of images of $B$ in $\mathbb{N}$ via vectors with entries from $\mathbb{N}$ is exactly the same as as the set of images of $A$ in $\mathbb{N}$ via vectors with entries from $\mathbb{Z}$.

In this paper, we answer Question 1.5 in the affirmative (without the assumption that $A$ is weakly image partition regular over $S$) whenever $S$ is the set of positive elements of a subring $R$ of $\mathbb{Q}$ with $1 \in R$. In particular, we have that for any finite matrix $A$ which is weakly image partition regular over $\mathbb{N}$, there is a matrix $B$ which is no bigger than $A$ such that the portion of the image of $B$ over $\mathbb{N}$ which lies in $\mathbb{N}$ is exactly the same as the portion of the image of $A$ over $\mathbb{Z}$ which lies in $\mathbb{N}$.

Section 2 consists of a proof that if $u, v \in \mathbb{N}$ and $A$ is a $u \times v$ matrix of rank $n$ with integer entries, then there is a $u \times n$ matrix $B$ with integer entries such that $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^u$.

The relatively short Section 3 consists of the derivation of the answer to Question 1.5 for subrings of $\mathbb{Q}$.

In Section 4, we show that image partition regular matrices are strictly stronger from a Ramsey Theoretic point of view than are weakly image partition regular matrices. Specifically, we show that for any $u \geq 3$, there are no $v$ and a $u \times v$ matrix $A$ such that for any $\vec{y} \in \{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^u$, the set of entries of $\vec{y}$ form (in some order) a length $u$ arithmetic progression. Consequently, van der Waerden's Theorem cannot be proved in the standard way, just using the weak image partition regularity of some matrix, without strengthening the conclusion of the theorem.

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## 2. Preserving integer images

Say that a matrix is in *column echelon* form if it is the transpose of a matrix in row echelon form. The first step in our construction is to convert a matrix with integer entries into a matrix with integer entries in column echelon form which has the same image over the integers and preserves the linear dependencies among the rows. The following simple lemma allows us to do this. (We follow the custom of denoting an entry of a matrix by the lower case letter corresponding to the upper case name of the matrix.)

**Lemma 2.1.** *Let $u, v \in \mathbb{N}\setminus\{1\}$ and let $A$ be a $u\times v$ matrix with entries from $\mathbb{Z}$. Let $s \in \{1, 2, \ldots, u\}$ and let $t, r \in \{1, 2, \ldots, v\}$ with $t \neq r$ and assume that $a_{s,t} \neq 0$. For $i \in \{1, 2, \ldots, u\}$ and for $j \in \{1, 2, \ldots, v\} \setminus \{t, r\}$, let $b_{i,j} = a_{i,j}$. Let $w = \gcd(a_{s,t}, a_{s,r})$ and pick $p$ and $q$ in $\mathbb{Z}$ such that $w = pa_{s,t} + qa_{s,r}$. For $i \in \{1, 2, \ldots, u\}$, let $b_{i,t} = pa_{i,t} + qa_{i,r}$ and let $b_{i,r} = (a_{s,t}a_{i,r} - a_{s,r}a_{i,t})/w$. Then:*

1. *The entries of $B$ are integers.*
2. *$b_{s,t} = w$ and $b_{s,r} = 0$.*
3. *$\{B\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{A\vec{k} : \vec{k} \in \mathbb{Z}^v\}$.*
4. *The matrix $B$ is obtainable from $A$ by at most four standard column operations.*
5. *The matrices $A$ and $B$ have the same linear dependencies among their rows.*
6. *If $u = v$, then $\det(B) = \det(A)$.*

**Proof.** Conclusions (1) and (2) are immediate.

(3) ($\subseteq$) Let $\vec{x} \in \mathbb{Z}^v$. For $j \in \{1, 2, \ldots, v\} \setminus \{t, r\}$ (if any) let $k_j = x_j$. Let $k_t = px_t - (a_{s,r}x_r/w)$ and let $k_r = qx_t + (a_{s,t}x_r/w)$. Then $\vec{k} \in \mathbb{Z}^v$ and for each $i \in \{1, 2, \ldots, u\}$, a simple calculation establishes that $\sum_{j=1}^{v} a_{i,j}k_j = \sum_{j=1}^{v} b_{i,j}x_j$.

($\supseteq$) Let $\vec{k} \in \mathbb{Z}^v$ and let $D = \{1, 2, \ldots, v\} \setminus \{t, r\}$. For $j \in D$, if any, let $x_j = k_j$. Let $x_t = (a_{s,t}k_t + a_{s,r}k_r)/w$ and let $x_r = k_rp - k_tq$. Then $\vec{x} \in \mathbb{Z}^v$. Let $i \in \{1, 2, \ldots, u\}$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{v} b_{i,j}x_j &= b_{i,t}(a_{s,t}k_t + a_{s,r}k_r)/w + b_{i,r}(k_rp - k_tq) + \textstyle\sum_{j\in D} a_{i,j}k_j \\ &= \big((a_{s,t}p + a_{s,r}q)a_{i,t}k_t + (a_{s,t}p + a_{s,r}q)a_{i,r}k_r\big)/w + \textstyle\sum_{j\in D} a_{i,j}k_j \\ &= \textstyle\sum_{j=1}^{v} a_{i,j}k_j.\end{aligned}$$

(4) Assume first that $p \neq 0$. Then $B$ is the result of succesively applying the following four column operations to $A$.

- (i) Multiply column $t$ by $p$.
- (ii) Add $q$ times column $r$ to column $t$, with the result replacing column $t$.

(iii) Multiply column $r$ by $(\frac{a_{s,t}}{w}+\frac{qa_{s,r}}{pw})$.
(iv) Add $-\frac{a_{s,r}}{pw}$ times column $t$ to column $r$, with the result replacing column $r$.

Now assume that $p=0$. Then $qa_{s,r}=w$ and, since $w$ divides $a_{s,r}$, $q=1$ and $a_{s,r}=w$. Then $B$ is the result of succesively applying the following three column operations to $A$.

(i) Interchange columns $t$ and $r$.
(ii) Multiply column $r$ by $-1$.
(iii) Add $\frac{a_{s,t}}{w}$ times column $t$ to column $r$, with the result replacing column $r$.

Conclusion (5) is an immediate consequence of conclusion (4). To verify conclusion (6), we consider first the case that $p\neq 0$. Then operation (i) multiplies the determinant by $p$ and operation (iii) multiplies the determinant by $(\frac{a_{s,t}}{w}+\frac{qa_{s,r}}{pw})$ while operations (ii) and (iv) leave the determinant unchanged. Since $p(\frac{a_{s,t}}{w}+\frac{qa_{s,r}}{pw})=1$, we have $\det(B)=\det(A)$.

Now assume that $p=0$. Then operations (i) and (ii) each multiply the determinant by $-1$, while operation (iii) leaves it unchanged. □

The following theorem is our major tool. Unfortunately, part of the proof is quite complicated, namely the verification that when $\vec{x}$ is produced with $B\vec{x}=A\vec{k}$, the entries of $\vec{x}$ are positive.

**Theorem 2.2.** *Let $n\in\mathbb{N}\setminus\{1\}$ and let $A$ be a lower triangular $n\times n$ matrix with integer entries such that $a_{m,m}>0$ for each $m\in\{1,2,\ldots,n\}$. There is a lower triangular $n\times n$ matrix $B$ with integer entries such that $b_{m,m}=a_{m,m}$ for each $m\in\{1,2,\ldots,n\}$ and $\{A\vec{k}:\vec{k}\in\mathbb{Z}^n\}\cap\mathbb{N}^n=\{B\vec{x}:\vec{x}\in\mathbb{N}^n\}\cap\mathbb{N}^n$.*

**Proof.** For $m\in\{1,2,\ldots,n\}$, let $b_{m,m}=a_{m,m}$ and let $b_{m,j}=0$ for $m<j\leq n$.

Let $m\in\{2,3,\ldots,n\}$ and assume that we have chosen $b_{j,i}$ for $1\leq i\leq j\leq m-1$ and $t_{j,i}$ for $1\leq i<j\leq m-1$. Pick $t_{m,m-1}\in\mathbb{N}$ such that

$$a_{m,m-1}-t_{m,m-1}a_{m,m}<0$$

and let $b_{m,m-1}=a_{m,m-1}-t_{m,m-1}a_{m,m}$. If $m=2$ this completes the definition of row $m$ of $B$. Otherwise, let $j\in\{1,2,\ldots,m-2\}$ and assume we have chosen $b_{m,s}$ and $t_{m,s}$ for $j+1\leq s\leq m-1$. Pick $t_{m,j}\in\mathbb{N}$ such that $a_{m,j}-t_{m,j}b_{m,m}-\sum_{l=j+1}^{m-1}t_{l,j}b_{m,l}<0$ (which can be done since $b_{m,m}=a_{m,m}>0$). Let $b_{m,j}=a_{m,j}-\sum_{l=j+1}^{m}t_{l,j}b_{m,l}$. This completes the definition of $B$.

We now show that if $\vec{x}\in\mathbb{Z}^n$, $\vec{k}\in\mathbb{Z}^n$, $x_1=k_1$, and for $l\in\{2,3,\ldots,n\}$, $x_l=k_l+\sum_{j=1}^{l-1}k_jt_{l,j}$, then for each $m\in\{1,2,\ldots,n\}$,

$$\textstyle\sum_{l=1}^{m}b_{m,l}x_l=\sum_{l=1}^{m}a_{m,l}k_l.$$

If $m = 1$ this holds because $x_1 = k_1$ and $b_{1,1} = a_{1,1}$, so assume that $m > 1$. Then

$$\begin{aligned}\sum_{l=1}^{m} b_{m,l}x_l &= b_{m,1}k_1 + \sum_{l=2}^{m} b_{m,l}(k_l + \sum_{j=1}^{l-1} k_j t_{l,j}) \\ &= b_{m,1}k_1 + \sum_{l=2}^{m} b_{m,l}k_l + \sum_{j=1}^{m-1} k_j(\sum_{l=j+1}^{m} b_{m,l}t_{l,j}) \\ &= \sum_{j=1}^{m} b_{m,j}k_j + \sum_{j=1}^{m-1} k_j(\sum_{l=j+1}^{m} b_{m,l}t_{l,j}) \\ &= b_{m,m}k_m + \sum_{j=1}^{m-1} k_j(b_{m,j} + \sum_{l=j+1}^{m} b_{m,l}t_{l,j}) \\ &= a_{m,m}k_m + \sum_{j=1}^{m-1} a_{m,j}k_j.\end{aligned}$$

Now to see that $\{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^n \subseteq \{A\vec{k} : \vec{k} \in \mathbb{Z}^n\} \cap \mathbb{N}^n$, let $\vec{x} \in \mathbb{N}^n$ such that all entries of $B\vec{x}$ are positive. Let $k_1 = x_1$ and inductively for $l \in \{2, 3, \ldots, n\}$, let $k_l = x_l - \sum_{j=1}^{l-1} k_j t_{l,j}$. Then as established above, for each $m \in \{1, 2, \ldots, n\}$, $\sum_{l=1}^{m} b_{m,l}x_l = \sum_{l=1}^{m} a_{m,l}k_l$.

To see that $\{A\vec{k} : \vec{k} \in \mathbb{Z}^n\} \cap \mathbb{N}^n \subseteq \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^n$, let $\vec{k} \in \mathbb{Z}^n$ such that all entries of $A\vec{k}$ are positive. Let $x_1 = k_1$ and for $l \in \{2, 3, \ldots, n\}$, let $x_l = k_l + \sum_{j=1}^{l-1} k_j t_{l,j}$. Then as established above, for each $m \in \{1, 2, \ldots, n\}$, $\sum_{l=1}^{m} b_{m,l}x_l = \sum_{l=1}^{m} a_{m,l}k_l$.

To complete the proof, we need to show that for each $m \in \{1, 2, \ldots, n\}$, $x_m > 0$. As we remarked before stating the theorem, this will be a lengthy process.

If $m = 1$ we have $a_{1,1}k_1 > 0$ and $a_{1,1} > 0$, so $x_1 = k_1 > 0$. We could at this stage proceed to induction on $m$. But we will verify separately the cases $m = 2$ and $m = 3$. There are two reasons for this. The first is that it will be convenient to assume that $m \geq 4$. The more important reason is that the $m = 3$ case illustrates the main ideas of the proof while still being relatively uncomplicated.

We have that $x_2 = k_2 + k_1t_{2,1}$ and $a_{2,2} > 0$ so to see that $x_2 > 0$, it suffices to show that $k_2a_{2,2} + k_1t_{2,1}a_{2,2} > 0$. We have that $t_{2,1}a_{2,2} > a_{2,1}$ so $t_{2,1}a_{2,2}k_1 > a_{2,1}k_1$. Also $a_{2,1}k_1 + a_{2,2}k_2 > 0$ so $a_{2,2}k_2 > -a_{2,1}k_1$. Adding these two inequalities we get that $k_2a_{2,2} + k_1t_{2,1}a_{2,2} > 0$ as required.

Now we have that $x_3 = k_3 + k_2t_{3,2} + k_1t_{3,1}$ and $a_{3,3} > 0$ so to see that $x_3 > 0$ it suffices to show that $k_3a_{3,3} + k_2t_{3,2}a_{3,3} + k_1t_{3,1}a_{3,3} > 0$. Now $t_{3,1}a_{3,3} > a_{3,1} - t_{2,1}b_{3,2}$ and $k_1 > 0$ so $t_{3,1}a_{3,3}k_1 > a_{3,1}k_1 - t_{2,1}b_{3,2}k_1$. Also, $b_{3,2} = a_{3,2} - t_{3,2}a_{3,3}$ so we have

$$t_{3,1}a_{3,3}k_1 > a_{3,1}k_1 - a_{3,2}t_{2,1}k_1 + t_{3,2}t_{2,1}a_{3,3}k_1. \tag{1}$$

Next $x_2 = k_2 + k_1t_{2,1} > 0$ and $t_{3,2}a_{3,3} - a_{3,2} > 0$ so the product is positive and consequently we have

$$t_{3,2}a_{3,3}k_2 > a_{3,2}k_2 + a_{3,2}k_1t_{2,1} - t_{3,2}t_{2,1}a_{3,3}k_1. \tag{2}$$

Since the third entry of $A\vec{k}$ is positive we have that

$$k_3a_{3,3} > -k_1a_{3,1} - k_2a_{3,2}. \tag{3}$$

Since the right hand sides of inequalities (1), (2), and (3) sum to 0, we have that $x_3 > 0$ as claimed.

Note that the reasons for inequalities (1), (2), and (3) were all different. In the case $m \geq 4$, the reasons for inequalities $(2), (3), \ldots, (m-1)$ all correspond to the reason for inequality (2) above.

Now let $m \in \{4, 5, \ldots, n\}$ and assume that $x_j > 0$ for $j \in \{1, 2, \ldots, m-1\}$.

Now $x_m = k_m + \sum_{j=1}^{m-1} k_jt_{m,j}$ so $x_ma_{m,m} = k_ma_{m,m} + \sum_{j=1}^{m-1} k_jt_{m,j}a_{m,m}$. Since $a_{m,m} > 0$, it suffices to show that $k_ma_{m,m} + \sum_{j=1}^{m-1} k_jt_{m,j}a_{m,m} > 0$.

We have by the choice of $t_{m,1}$ that $t_{m,1}a_{m,m} > a_{m,1} - \sum_{l=2}^{m-1} t_{l,1}b_{m,l}$ and we know $k_1 > 0$ so

$$t_{m,1}a_{m,m}k_1 > k_1a_{m,1} - \textstyle\sum_{l=2}^{m-1} t_{l,1}b_{m,l}k_1. \tag{1}$$

Now let $2 \leq j \leq m-1$. Then $k_j + \sum_{l=1}^{j-1} k_lt_{j,l} = x_j > 0$ and

$$t_{m,j}a_{m,m} + \textstyle\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j} - a_{m,j} > 0$$

so the product is positive and thus

$$\begin{aligned} t_{m,j}a_{m,m}k_j > {} & a_{m,j}k_j - k_j \textstyle\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j} + a_{m,j}\sum_{l=1}^{j-1} k_lt_{j,l} \\ & - t_{m,j}a_{m,m}\textstyle\sum_{l=1}^{j-1} k_lt_{j,l} - (\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j})(\sum_{l=1}^{j-1} k_lt_{j,l}). \end{aligned} \tag{$j$}$$

Note that if $j = m-1$, then $\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j} = 0$ so the inequality ($j$) takes the simpler form

$$\begin{aligned} t_{m,m-1}a_{m,m}k_{m-1} > {} & a_{m,m-1}k_{m-1} + a_{m,m-1}\textstyle\sum_{l=1}^{m-2} k_lt_{m-1,l} \\ & - t_{m,m-1}a_{m,m}\textstyle\sum_{l=1}^{m-2} k_lt_{m-1,l}. \end{aligned} \tag{$m-1$}$$

We also have that $\sum_{l=1}^{m} k_la_{m,l} > 0$ so

$$k_ma_{m,m} > -\textstyle\sum_{l=1}^{m-1} k_la_{m,l} \tag{$m$}$$

It suffices to show that the sum of the right hand sides of inequalities $(1), (2), \ldots, (m)$ is 0.

Before we can do this, we will rewrite inequalities (1) and ($j$) to eliminate mention of $b_{m,s}$. We write $[j, u] = \{j, j+1, \ldots, u\}$, and if $u < j$ we let $[j, u] = \emptyset$.

**Definition 2.3.** Let $j < u$.

(a) We let $h_{u,j}(\emptyset) = t_{u,j}$ and for $\emptyset \neq D \subseteq [j+1, u-1]$ with $s = \min D$, let $h_{u,j}(D) = h_{u,s}(D \setminus \{s\})t_{s,j}$.

(b) $g_{u,j} = \sum_{D\subseteq[j+1,u-1]}(-1)^{|D|}h_{u,j}(D)$.

Thus, if $D = \{l_1, l_2, \ldots, l_s\} \subseteq [j+1, u-1]$ where $l_1 < l_2 < \cdots < l_s$, one has $h_{u,j} = t_{u,l_s}t_{l_s,l_{s-1}}st_{l_2,l_1}t_{l_1,j}$.

Also, for example, $g_{j+1,j} = h_{j+1,j}(\emptyset) = t_{j+1,j}$ and

$$\begin{aligned}
g_{j+3,j} &= h_{j+3,j}(\emptyset) - h_{j+3,j}(\{j+1\}) - h_{j+3,j}(\{j+2\})\\
&\quad + h_{j+3,j}(\{j+1,j+2\})\\
&= t_{j+3,j} - t_{j+3,j+1}t_{j+1,j} - t_{j+3,j+2}t_{j+2,j} + t_{j+3,j+2}t_{j+2,j+1}t_{j+1,j}.
\end{aligned}$$

Given $l+1 \leq j \leq u-1$, we have that

$$\begin{aligned}
\sum_{D\subseteq[j+1,u-1]}(-1)^{|D\cup\{j\}|}h_{u,l}(D\cup\{j\}) &= -\sum_{D\subseteq[j+1,u-1]}(-1)^{|D|}h_{u,j}(D)t_{j,l}\\
&= -t_{j,l}g_{u,j}.
\end{aligned}$$

Thus we have, for $l+2 \leq u \leq m$,

$$\begin{aligned}
g_{u,l} &= \sum_{D\subseteq[l+1,u-1]}(-1)^{|D|}h_{u,l}(D)\\
&= h_{u,l}(\emptyset) + \sum_{j=l+1}^{u-1}\sum_{D\subseteq[l+1,u-1],\min D=j}(-1)^{|D|}h_{u,l}(D)\\
&= t_{u,l} + \sum_{j=l+1}^{u-1}\sum_{D\subseteq[j+1,u-1]}(-1)^{|D\cup\{j\}|}h_{u,l}(D\cup\{j\})\\
&= t_{u,l} - \sum_{j=l+1}^{u-1} t_{j,l}g_{u,j}.
\end{aligned}$$

That is,

$$(\ddagger)\qquad \text{For } l+2 \leq u \leq m,\ g_{u,l} = t_{u,l} - \sum_{j=l+1}^{u-1} t_{j,l}g_{u,j}.$$

We now show by downward induction on $j$ that

$$(\dagger)\qquad \text{For } j \in \{1,2,\ldots,m-1\},\ b_{m,j} = a_{m,j} - \sum_{u=j+1}^{m} a_{m,u}g_{u,j}.$$

For $j = m-1$ this is immediate from the fact that $g_{m,m-1} = t_{m,m-1}$. So assume $j \in \{1,2,\ldots,m-2\}$ and $b_{m,l} = a_{m,l} - \sum_{u=l+1}^{m} a_{m,u}g_{u,l}$ for $l \in \{j+1, j+2, \ldots, m-1\}$. Then

$$\begin{aligned}
b_{m,j} &= a_{m,j} - \sum_{l=j+1}^{m} t_{l,j}b_{m,l}\\
&= a_{m,j} - t_{m,j}a_{m,m} - \sum_{l=j+1}^{m-1} t_{l,j}(a_{m,l} - \sum_{u=l+1}^{m} a_{m,u}g_{u,l})\\
&= a_{m,j} - \sum_{l=j+1}^{m} t_{l,j}a_{m,l} + \sum_{u=j+2}^{m} a_{m,u}\sum_{l=j+1}^{u-1} t_{l,j}g_{u,l}.
\end{aligned}$$

Now using (‡) with $j$ and $l$ interchanged, we have that

$$\begin{aligned}
b_{m,j} &= a_{m,j} - \sum_{l=j+1}^{m} t_{l,j}a_{m,l} + \sum_{u=j+2}^{m} a_{m,u}(t_{u,j} - g_{u,j})\\
&= a_{m,j} - t_{j+1,j}a_{m,j+1} - \sum_{u=j+2}^{m} a_{m,u}g_{u,j}\\
&= a_{m,j} - \sum_{u=j+1}^{m} a_{m,u}g_{u,j}.
\end{aligned}$$

Thus (†) is established.

Using (†) and (‡) we can rewrite (1) as

$$(1')\qquad t_{m,1}a_{m,m}k_1 > a_{m,1}k_1 - \sum_{u=2}^{m} a_{m,u}k_1g_{u,1} + a_{m,m}k_1t_{m,1}.$$

Now we deduce using (†) that, given $j \leq m-2$,

$$\begin{aligned}\textstyle\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j} &= \textstyle\sum_{s=j+1}^{m-1} t_{s,j}(a_{m,s} - \sum_{u=s+1}^{m} a_{m,u}g_{u,s}) \\ &= t_{j+1,j}a_{m,j+1} - a_{m,m}\textstyle\sum_{s=j+1}^{m-1} t_{s,j}g_{m,s} \\ &\quad + \textstyle\sum_{u=j+2}^{m-1} a_{m,u}(t_{u,j} - \sum_{s=j+1}^{u-1} t_{s,j}g_{u,s}).\end{aligned}$$

Then, using (‡) twice, we get that

$$\textstyle\sum_{s=j+1}^{m-1} b_{m,s}t_{s,j} = a_{m,j+1}t_{j+1,j} + \sum_{u=j+2}^{m-1} a_{m,u}g_{u,j} + a_{m,m}(-t_{m,j} + g_{m,j}).$$

Using this, we rewrite ($j$) as

$$(j') \qquad \begin{aligned} t_{m,j}a_{m,m}k_j > a_{m,j}k_j &- \textstyle\sum_{u=j+1}^{m} a_{m,u}k_jg_{u,j} + a_{m,m}t_{m,j}k_j \\ &- \textstyle\sum_{u=j+1}^{m}\sum_{l=1}^{j-1} a_{m,u}k_lg_{u,j}t_{j,l} + \sum_{l=1}^{j-1} a_{m,j}k_lt_{j,l}.\end{aligned}$$

Notice that each additive term in the right hand sides of inequalities $(1'), (2'), \ldots, (m-2'), (m-1), (m)$ includes some $a_{m,u}k_l$. We will show that each $a_{m,u}k_l$ which occurs as part of the sum of the right hand sides of inequalities $(1'), (2'), \ldots, (m-2'), (m-1), (m)$ has a sum of coefficients equal to 0, which will complete the proof.

Notice that among the sum of the right hand sides there are no occurrences of $a_{m,m}k_m$ or $a_{m,m}k_{m-1}$. With those exceptions for every $u$ and $l$ with $1 \leq l \leq u \leq m$, $a_{m,u}k_l$ occurs. One of the following cases must apply.

(1) $1 \leq l = u \leq m-1$;
(2) $1 \leq l = u-1 \leq m-2$;
(3) $l = 1$ and $u = m$;
(4) $2 \leq l \leq m-2$ and $u = m$;
(5) $l = 1$ and $u = m-1$;
(6) $2 \leq l \leq m-3$ and $u = m-1$;
(7) $l = 1$ and $3 \leq u \leq m-2$;
(8) $2 \leq l \leq u-2$ and $4 \leq u \leq m-2$.

In each of these cases except (1) and (2), the sum of the coefficients of $a_{m,u}k_l$ in the sum of the right hand sides of the inequalities

$$(1'), (2'), \ldots, (m-2'), (m-1), (m)$$

is $t_{u,l} - g_{u,l} - \sum_{j=l+1}^{u-1} t_{j,l}g_{u,j}$ which is equal to 0 by (‡).

It is interesting to note that the origins of the various parts of that sum depend on the case. For example, in case (6), the contribution to the sum of coefficients from inequality $(l')$ is $-g_{u,l}$, that from $(j')$ for $l < j \leq m-2$ is $-g_{u,j}t_{j,l}$, and the contribution from $(j')$ is $t_{u,l}$. On the other hand, in case (4), the contribution from inequality $(l')$ is $-g_{u,l} + t_{u,l}$, for $l < j \leq m-2$, the contribution from inequality $(j')$ is $-g_{u,j}t_{j,l}$, and the contribution from inequality $(m-1)$ is $-t_{u,m-1}t_{m-1,l} = -g_{u,m-1}t_{m-1,l}$.

In case (1), the contribution to the coefficient of $a_{m,u}t_l$ from inequality ($m$) is $-1$ and the contribution from inequality ($l'$) (or inequality ($m-1$) if $l = m-1$) is $+1$. In case (2), the contribution to the coefficient of $a_{m,u}t_l$ from inequality ($l'$) is $-g_{u,l}$ and the contribution from inequality ($l+1'$) (or inequality ($m-1$) if $l = m-2$) is $t_{u,l}$ so the coefficient of $a_{m,u}t_l$ is $-g_{u,u-1} + t_{u,u-1} = 0$. □

We are now ready to present the easy proof of our main theorem.

**Theorem 2.4.** *Let $u, v, n \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix of rank $n$ with integer entries. There is a $u \times n$ matrix $B$ with integer entries such that*

$$\{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^u.$$

*In particular, if $A$ is weakly image partition regular over $\mathbb{N}$, then $B$ is image partition regular over $\mathbb{N}$.*

**Proof.** By rearranging rows and columns, we may presume that the upper left $n \times n$ corner of $A$ has nonzero determinant. For $i \in \{1, 2, \ldots, u\}$, let $\vec{r}_i$ denote the $i^{\text{th}}$ row of $A$. For $i \in \{n+1, n+2, \ldots, u\}$ (if any), let $\langle \alpha_{i,j} \rangle_{j=1}^n$ be the sequence of rationals such that $\vec{r}_i = \sum_{j=1}^n \alpha_{i,j}\vec{r}_j$.

By repeatedly applying Lemma 2.1 (and switching columns if need be to make sure that each $c_{m,m} \neq 0$ for $m \in \{1, 2, \ldots, n\}$) we can obtain a $u \times v$ matrix $C$ with integer entries such that:

(1) $\{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} = \{C\vec{k} : \vec{k} \in \mathbb{Z}^v\}$.
(2) $C$ has the same linear dependencies among its rows as $A$ has.
(3) The transpose of $C$ is in row echelon form.

The matrix $C$ has the property that $c_{i,j} = 0$ whenever $i \in \{1, 2, \ldots, n\}$ and $j > i$ and $c_{i,j} = 0$ whenever $i$ and $j$ are greater than $n$. Further, since $\{C\vec{k} : \vec{k} \in \mathbb{Z}^v\}$ is unchanged if a column of $C$ is multiplied by $-1$, we may assume that for $m \in \{1, 2, \ldots, n\}$, $c_{m,m} > 0$.

Let $C^*$ consist of the upper left $n \times n$ corner of $C$ and pick $B^*$ as guaranteed for $C^*$ by Theorem 2.2. Then $\{C^*\vec{k} : \vec{k} \in \mathbb{Z}^n\} \cap \mathbb{N}^n = \{B^*\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^n$.

If $u = n$, we may let $B = B^*$. So we assume that $u > n$. Note that for $i \in \{n+1, n+2, \ldots, u\}$ and $j \in \{1, 2, \ldots, n\}$,

$$c_{i,j} = \textstyle\sum_{m=1}^n \alpha_{i,m}c_{m,j} = \sum_{m=j}^n \alpha_{i,m}c_{m,j}.$$

For $i, j \in \{1, 2, \ldots, n\}$, let $b_{i,j} = b^*_{i,j}$. For $i \in \{n+1, n+2, \ldots, u\}$ and $j \in \{1, 2, \ldots, n\}$, let $b_{i,j} = \sum_{m=j}^n \alpha_{i,m}b_{m,j}$. We show first that each $b_{i,j}$ is an integer. This is immediate if $i, j \in \{1, 2, \ldots, n\}$. Recall from the proof of Theorem 2.2 that for $j < u < m \leq n$ there exists $g_{u,j} \in \mathbb{Z}$ such that for each $j < m \leq n$

$$b_{m,j} = c_{m,j} - \textstyle\sum_{u=j+1}^m c_{m,u}g_{u,j}. \tag{†}$$

Now let $i \in \{n+1, n+2, \dots, u\}$ and $j \in \{1, 2, \dots, n\}$. Then

$$\begin{aligned} b_{i,j} &= \textstyle\sum_{m=j}^{n} \alpha_{i,m} b_{m,j} \\ &= \alpha_{i,j} b_{j,j} + \textstyle\sum_{m=j+1}^{n} \alpha_{i,m}(c_{m,j} - \sum_{u=j+1}^{m} c_{m,u} g_{u,j}) \\ &= \textstyle\sum_{m=j}^{n} \alpha_{i,m} c_{m,j} - \sum_{u=j+1}^{n} g_{u,j} \sum_{m=u}^{n} \alpha_{i,m} c_{m,u} \\ &= a_{i,j} - \textstyle\sum_{u=j+1}^{n} g_{u,j} c_{i,u} \in \mathbb{Z}. \end{aligned}$$

To complete the proof, we show that

$$\{C\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^u.$$

To do this, we show that if $\vec{k} \in \mathbb{Z}^n$, $\vec{x} \in \mathbb{N}^n$, $C^*\vec{k} = B^*\vec{x}$, and $\vec{k}' \in \mathbb{Z}^v$ such that $k'_j = k_j$ for $j \in \{1, 2, \dots, n\}$, then $C\vec{k}' = B\vec{x}$. From this and the fact that $\{C^*\vec{k} : \vec{k} \in \mathbb{Z}^n\} \cap \mathbb{N}^n = \{B^*\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^n$ it follows immediately that $\{C\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^n = \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^n$.

So assume we have $\vec{k} \in \mathbb{Z}^n$, $\vec{x} \in \mathbb{N}^n$, and $\vec{k}' \in \mathbb{Z}^v$ such that $k'_j = k_j$ for $j \in \{1, 2, \dots, n\}$ and $C^*\vec{k} = B^*\vec{x}$. Recall that $c_{i,j} = 0$ whenever $i \in \{1, 2, \dots, u\}$ and $j > n$. To see that $C\vec{k}' = B\vec{x}$, let $i \in \{1, 2, \dots, v\}$. If $i \in \{1, 2, \dots, n\}$, then $\sum_{j=1}^{v} c_{i,j} k'_j = \sum_{j=1}^{n} c_{i,j} k_j = \sum_{j=1}^{n} c^*_{i,j} k_j = \sum_{j=1}^{n} b^*_{i,j} x_j = \sum_{j=1}^{n} b_{i,j} x_j$. So assume that $i > n$. Then

$$\begin{aligned} \textstyle\sum_{j=1}^{v} c_{i,j} k'_j &= \textstyle\sum_{j=1}^{n} c_{i,j} k_j \\ &= \textstyle\sum_{j=1}^{n} k_j \sum_{m=j}^{n} \alpha_{i,m} c_{m,j} \\ &= \textstyle\sum_{m=1}^{n} \alpha_{i,m} \sum_{j=1}^{m} c_{m,j} k_j \\ &= \textstyle\sum_{m=1}^{n} \alpha_{i,m} \sum_{j=1}^{m} b_{m,j} x_j \\ &= \textstyle\sum_{j=1}^{n} x_j \sum_{m=j}^{n} \alpha_{i,m} b_{m,j} \\ &= \textstyle\sum_{j=1}^{n} x_j b_{i,j}. \end{aligned}$$

□

The "in particular" conclusion tells us that any configuration which can be shown to be partition regular in $\mathbb{N}$ using a (finite) weakly image partition regular matrix $A$ can in fact be shown to be partition regular in $\mathbb{N}$ using an image partition regular matrix which is no bigger than $A$.

We observe now that the converse to Theorem 2.4 fails badly. (In the statement of the theorem, $\{B\vec{x} : \vec{x} \in \mathbb{N}^2\} \subseteq \mathbb{N}^2$ so the final intersection is redundant. We keep it to preserve the form of the statement.)

**Theorem 2.5.** *Let* $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. *There do not exist* $v \in \mathbb{N}$ *and a* $2 \times v$ *matrix* $A$ *with integer entries such that*

$$\{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^2 = \{B\vec{x} : \vec{x} \in \mathbb{N}^2\} \cap \mathbb{N}^2.$$

**Proof.** Suppose that we have such $v$ and $A$. Trivially no row of $A$ has all zero entries. If the rank of $A$ is 1, then there is some $\alpha \in \mathbb{Q}$ such that for all $\vec{k} \in \mathbb{Z}^v$, if $A\vec{k} = \begin{pmatrix} a \\ b \end{pmatrix}$, then $b = \alpha a$, so we may assume that the rank of $A$ is

2. By rearranging columns, we may assume that the first two columns of $A$ are linearly independent. And by multiplying column 1 by $-1$ if necessary, we may assume that $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} > 0$.

Let $k_1 = a_{2,2} - a_{1,2}$, let $k_2 = a_{1,1} - a_{2,1}$, and for $j \in \{3, 4, \ldots, v\}$, if any, let $k_j = 0$. Then $A\vec{k} = \begin{pmatrix} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \\ a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{pmatrix}$. No element of $\{B\vec{x} : \vec{x} \in \mathbb{N}^2\}$ has both entries equal. $\square$

We remark that the proof of Theorem 2.4 can be modified to work if $A$ is allowed to be infinite with finitely many nonzero entries per row and $A$ is of infinite rank. However, then the result is weaker than what we already know to be true from Theorem 1.4. The conclusion of Theorem 1.4 with $u = v = \omega$ and $S = \mathbb{N}$ is then $\{A\vec{x} : \vec{x} \in \mathbb{Z}^\omega\} = \{C\vec{y} : \vec{y} \in \mathbb{N}^\omega\}$.

## 3. Preserving images over subrings

In this section we answer Question 1.5 for any $G$ which is a subring of $\mathbb{Q}$ with $1 \in G$, where $S = \{x \in G : x > 0\}$; in particular, for the case $S = \mathbb{N}$ and $G = \mathbb{Z}$. We will not use the assumption in the question that the matrix $A$ is weakly image partition regular over $S$.

**Definition 3.1.** Let $\mathbb{P}$ be the set of primes and let $F \subseteq \mathbb{P}$. Then

$$\mathbb{G}_F = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N} \text{ and all prime factors of } b \text{ are in } F\}.$$

Thus $\mathbb{G}_\emptyset = \mathbb{Z}$, $\mathbb{G}_{\{2\}} = \mathbb{D}$, the set of dyadic rationals, and $\mathbb{G}_\mathbb{P} = \mathbb{Q}$. It is easy to check that the sets of the form $\mathbb{G}_F$ are precisely the subrings of $\mathbb{Q}$ with 1. (Given a subring $R$ of $\mathbb{Q}$ with $1 \in R$, let $F = \{p \in \mathbb{P} : \frac{1}{p} \in R\}$. Given $\frac{a}{b} \in R$ with $(a, b) = 1$ pick $k$ and $l$ in $\mathbb{Z}$ such that $1 = ka + lb$. Then $\frac{1}{b} = k\frac{a}{b} + l \in R$. Consequently, if $p$ is a prime and $b = pc$, then $c\frac{1}{b} = \frac{1}{p} \in R$.)

**Theorem 3.2.** *Let $R$ be a subring of $\mathbb{Q}$ with $1 \in R$ and let*

$$S = \{x \in R : x > 0\}.$$

*Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with rational entries and rank $n$. There exists a $u \times n$ matrix $C$ such that*

$$\{A\vec{x} : \vec{x} \in R^v\} \cap S^u = \{C\vec{y} : \vec{y} \in S^n\} \cap S^u.$$

*If the entries of $A$ come from $R$, so do the entries of $B$.*

**Proof.** Pick $F \subseteq \mathbb{P}$ such that $R = \mathbb{G}_F$. Pick $m \in \mathbb{N}$ such that the entries of $mA$ are in $\mathbb{Z}$. If the entries of $A$ are in $\mathbb{G}_F$, choose such $m$ so that all of its prime factors are in $F$. By Theorem 2.4 pick a $u \times n$ matrix $B$ with integer entries such that $\{mA\vec{x} : \vec{x} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{y} : \vec{y} \in \mathbb{N}^n\} \cap \mathbb{N}^u$. Let $C = \frac{1}{m}B$ and note that, if all prime factors of $m$ are in $F$, then all entries of $B$ are in $\mathbb{G}_F$.

To see that $\{A\vec{x} : \vec{x} \in R^v\} \cap S^u \subseteq \{C\vec{y} : \vec{y} \in S^n\} \cap S^u$, let $\vec{x} \in R^v$ such that $A\vec{x} \in S^u$. Pick $b \in \mathbb{N}$ with all prime factors of $b$ in $F$ such that $b\vec{x} \in \mathbb{Z}^v$.

Then $mAb\vec{x} \in \mathbb{N}^u$ so pick $\vec{y} \in \mathbb{N}^n$ such that $mAb\vec{x} = B\vec{y}$. Then $A\vec{x} = \frac{1}{m}B\frac{1}{b}\vec{y}$ and $\frac{1}{b}\vec{y} \in S^n$.

To see that $\{C\vec{y} : \vec{y} \in S^n\} \cap S^u \subseteq \{A\vec{x} : \vec{x} \in R^v\} \cap S^u$, let $\vec{y} \in S^n$ such that $C\vec{y} \in S^u$. Pick $b \in \mathbb{N}$ such that all prime factors of $b$ are in $F$ such that $b\vec{y} \in \mathbb{N}^n$. Since entries of $\frac{1}{m}B\vec{y}$ are positive and entries of $Bb\vec{y}$ are in $\mathbb{Z}$ we have that $Bb\vec{y} \in \mathbb{N}^u$. Pick $\vec{x} \in \mathbb{Z}^v$ such that $mA\vec{x} = Bb\vec{y}$, Then $\frac{1}{b}\vec{x} \in R^v$ and $A\frac{1}{b}\vec{x} = \frac{1}{m}B\vec{y}$. □

## 4. Image partition regular is a stronger notion

In the introduction to [6], Rado used van der Waerden's Theorem [8] as motivation for the problem which he solved, namely proving that a finite matrix $A$ with rational entries is kernel partition regular over $\mathbb{N}$ if and only if it satisfies the *columns condition*. The details of the columns condition are not relevant for this paper. One may find them in [6] (provided one can read German) or in [2, pp. 73-74].

It is interesting that Rado's Theorem does not allow one to prove the partition regularity of, say, $\mathcal{F} = \big\{\{a, a+d, a+2d, a+3d\} : a, d \in \mathbb{N}\big\}$ by any of the most natural methods. Given a finite coloring of $\mathbb{N}$ one is looking for monochromatic $\{x_1, x_2, x_3, x_4\}$ where $x_1 = a$, $x_2 = a + d$, $x_3 = a + 2d$, and $x_4 = a + 3d$. A natural way of capturing this information is via the equations

$$x_2 - x_1 = x_3 - x_2$$
$$x_3 - x_2 = x_4 - x_3$$

which correspond to the kernel partition regularity of the matrix

$$\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{pmatrix}.$$

This matrix does indeed satisfy the columns condition but (in terms of Rado's proof) only because

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is in the kernel.

Since he was not looking for a constant arithmetic progression, Rado augmented the equations by introducing $x_5 = d$ so that the equation $x_5 = x_2 - x_1$ could be added to the matrix. This yielded the strengthened version of van der Waerden's Theorem which got not only a monochromatic arithmetic progression but also such a progression with its increment.

One might think that a clever choice of equations would allow one to just prove van der Waerden's Theorem (without strengthenings of it) using kernel partition regular matrices. This is not so.

**Theorem 4.1.** *Let $u \geq 3$. There do not exist $m \in \mathbb{N}$ and a $m \times u$ matrix $A$ with rational entries such that $A$ is kernel partition regular over $\mathbb{N}$ and whenever $\vec{x} \in \mathbb{N}^u$ and $A\vec{x} = \vec{0}$, the entries of $\vec{x}$ can be arranged to form a nontrivial $u$-term arithmetic progression.*

**Proof.** [3, Theorem 2.6]. □

We show now that a similar situation holds with respect to weak image partition regularity.

**Theorem 4.2.** *Let $u \geq 3$. There do not exist $v \in \mathbb{N}$ and a $u \times v$ matrix $A$ with rational entries which is weakly image partition regular over $\mathbb{N}$ and whenever $\vec{y} \in \{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^u$, the entries of $\vec{y}$ can be arranged to form a nontrivial $u$-term arithmetic progression.*

**Proof.** Suppose we have such a matrix $A$ and let $l = \text{rank}(A)$. By rearranging rows and columns, we may presume that the upper left $l \times l$ corner $A^*$ of $A$ has nonzero determinant.

Assume first that $l = u$. Pick $\vec{x} \in \mathbb{Q}^l$ such that

$$A^*\vec{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Pick $d \in \mathbb{N}$ such that the entries of $d\vec{x}$ are integers and define $\vec{k} \in \mathbb{Z}^v$ by, for $i \in \{1, 2, \ldots, v\}$,

$$k_i = \begin{cases} dx_i & \text{if } i \leq l \\ 0 & \text{if } i > l. \end{cases}$$

Then

$$A^*\vec{k} = \begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix}.$$

Thus we may assume that $u > l$. Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_u$ denote the rows of $A$. For each $t \in \{l+1, l+2, \ldots, u\}$, let $\gamma_{t,1}, \gamma_{t,2}, \ldots, \gamma_{t,l}$ denote the elements of $\mathbb{Q}$ determined by $\vec{r}_t = \sum_{i=1}^{l} \gamma_{t,i}\vec{r}_i$. Let $D$ be the $(u-l) \times u$ matrix such that for $t \in \{1, 2, \ldots, u-l\}$ and $i \in \{1, 2, \ldots, u\}$,

$$d_{t,i} = \begin{cases} \gamma_{l+t,i} & \text{if } i \leq l \\ -1 & \text{if } i = l+t \\ 0 & \text{otherwise.} \end{cases}$$

Since $A$ is weakly image partition regular over $\mathbb{N}$, we have by [4, Lemma 2.3] that $D$ is kernel partition regular over $\mathbb{N}$.

We claim that whenever $\vec{x} \in \mathbb{N}^u$ and $D\vec{x} = \vec{0}$, the entries of $\vec{x}$ can be arranged to form a nontrivial $u$-term arithmetic progression. This contradiction to Theorem 4.1 will complete the proof. So let $\vec{x} \in \mathbb{N}^u$ such that $D\vec{x} = \vec{0}$. Pick $\vec{w} \in \mathbb{Q}^l$ such that

$$A^*\vec{w} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{pmatrix}.$$

Pick $c \in \mathbb{N}$ such that the entries of $c\vec{w}$ are integers. Then

$$A^*c\vec{w} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_l \end{pmatrix}.$$

Define $\vec{k} \in \mathbb{Z}^u$ by, for $i \in \{1, 2, \ldots, u\}$,

$$k_i = \begin{cases} cw_i & \text{if } i \leq l \\ 0 & \text{if } i > l. \end{cases}$$

Let $t \in \{l+1, l+2, \ldots, u\}$. Then $0 = \sum_{i=1}^{u} d_{t-l,i}x_i = \sum_{i=1}^{l} \gamma_{t,i}x_i - x_t$ so $x_t = \sum_{i=1}^{l} \gamma_{t,i}x_i$. Thus

$$\begin{aligned} \textstyle\sum_{j=1}^{u} a_{t,j}k_j &= \textstyle\sum_{j=1}^{l} a_{t,j}cw_j \\ &= \textstyle\sum_{j=1}^{l} \sum_{i=1}^{l} \gamma_{t,i}a_{i,j}cw_j \\ &= \textstyle\sum_{i=1}^{l} \gamma_{t,i} \sum_{j=1}^{l} a_{i,j}cw_j \\ &= \textstyle\sum_{i=1}^{l} \gamma_{t,i}cx_i = cx_t. \end{aligned}$$

Let $\vec{y} = c\vec{x}$. Then $\vec{y} = A\vec{k}$ and $\vec{y} \in \mathbb{N}^u$ so the entries of $\vec{y}$ can be arranged to form a nontrivial arithmetic progression and therefore so can the entries of $\vec{x}$. □

As we have noted, Rado proved van der Waerden's Theorem using kernel partition regular matrices by extending the theorem to require that the increment be the same color. The same thing can be done using weakly image partition regular matrices. Consider for example the following matrix.

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\{C\vec{k} : \vec{k} \in \mathbb{Z}^2\} \cap \mathbb{N}^4 = \{C\vec{x} : \vec{x} \in \mathbb{N}^2\} \cap \mathbb{N}^4 = \left\{ \begin{pmatrix} d \\ a \\ a+d \\ a+2d \end{pmatrix} : a, d \in \mathbb{N} \right\}.$$

But the matrix $C$ is in fact image partition regular. If one wants to accomplish the same thing with a matrix which is weakly image partition regular over $\mathbb{N}$ but not image partition regular over $\mathbb{N}$, there is a simple switch. Let

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & -1 \\ 1 & -2 \end{pmatrix}.$$

Then

$$\{D\vec{k} : \vec{k} \in \mathbb{Z}^2\} \cap \mathbb{N}^4 = \{C\vec{x} : \vec{x} \in \mathbb{N}^2\} \cap \mathbb{N}^4 = \left\{ \begin{pmatrix} d \\ a \\ a+d \\ a+2d \end{pmatrix} : a, d \in \mathbb{N} \right\}.$$

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