New York Journal of Mathematics

New York J. Math. **22** (2016) 1249–1270.

Invariance under bounded analytic functions: generalizing shifts

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Abstract. In a recent paper, one of the authors — along with co-authors — extended the famous theorem of Beurling to the context of subspaces that are invariant under the class of subalgebras of H^∞ of the form IH^∞, where I is the inner function z^2. In recent times, several researchers have replaced z^2 by an arbitrary inner function I and this has proved important and fruitful in applications such as to interpolation problems of the Pick–Nevanlinna type. Keeping in mind the great deal of interest in such problems, in this paper, we provide analogues of the above mentioned IH^∞ related extension of Beurling's theorem in the setting of the Banach space $BMOA$, in the context of uniform algebras, on compact abelian groups with ordered duals and the Lebesgue space on the real line. We also provide a significant simplification of the proof of Beurling's theorem in the setting of uniform algebras and a new proof of Helson's generalization of Beurling's theorem in the context of compact abelian groups with ordered duals.

Contents

Received April 21, 2016.

2010 *Mathematics Subject Classification.* Primary 47B37; Secondary 47A25.

Key words and phrases. Invariant subspace, inner function, uniform algebra, compact abelian group, multiplier algebra of $BMOA$.

The research of the first author is supported by the Junior Research Fellowship of the Council of Scientific and Industrial Research, India (Grant no. 09/045(1232)/ 2012-EMR-I).

ISSN 1076-9803/2016

1. Introduction and statement of main theorem (Theorem C)

The results carried in this article stem from the famous and fundamental theorem of Beurling, [4], related to the characterization of the invariant subspaces of the operator of multiplication by the coordinate function z — also known as the shift operator — on the classical Hardy space H^2 of the open unit disk. This invariance is also equivalent to invariance under multiplication by each element of the Banach algebra H^∞ of bounded analytic functions on the disk, see [19, Lemma, p. 106]. The impetus for this article is the recent extension (on the open disk) of Beurling's theorem to the problem of characterizing invariant subspaces on H^2 where the invariance is under the context of multiplication by each element of the subalgebra IH^∞ of H^∞, where I is any inner function, i.e., I has absolute value 1 almost everywhere on the boundary $\mathbb{T}$ of the open unit disk. Such an extension has had important applications to interpolation problems and related issues for which we refer to [3], [6], [11], [14], [2], [20] and [21].

Our principal objective in this paper is to prove versions of the above mentioned extension of Beurling's theorem in the setting of the Banach space $BMOA$, the Hardy spaces on uniform algebras, on compact abelian groups and on the Lebesgue space of the real line. When dealing with uniform algebras, we first present a new, much simplified and elementary proof of Beurling's theorem on uniform algebras [12, p. 131]. We do this by eliminating, in the context of the Hardy spaces of uniform algebras, the use of a deep result of Kolmogoroff's on the weak 1-1 nature of the conjugation operator and also by eliminating the complicated technicalities of uniform integrability. Later on, in Section 5, we also present a new proof of the Helson–Lowdenslager version of Beurling's theorem on compact abelian groups (see [16], [17]).

With the purpose of making things clearer, we state below Beurling's theorem on the open unit disk and two other connected theorems that are relevant to the rest of this paper. All three theorems below are in the setting of the Hardy spaces of the open unit disk. At appropriate places we shall state the relevant versions of these theorems in the context of various Hardy spaces (mentioned above) and on $BMOA$. Our key objective is to show in the rest of the paper that the Theorem C below has valid versions in various Hardy spaces and on $BMOA$. It is this theorem that has proved to be important in interpolation problems of the open disk.

Let $\mathbb{D}$ denote the open unit disk and let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. We use $H^p(\mathbb{D})$, $1 \leq p < \infty$ to denote the classical Hardy space of analytic functions inside the unit disk $\mathbb{D}$ and $H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$. For $1 \leq p \leq \infty$, L^p denotes the Lebesgue space on the unit circle $\mathbb{T}$ and H^p stands for the closed subspace of L^p which consists of the functions in L^p whose Fourier coefficients for the negative indices are zero. Due to the fact that there is an isometric isomorphism between

$H^p(\mathbb{D})$ and H^p, on certain occasions we will identify $H^p(\mathbb{D})$ with H^p without comment (see [19]).

The shift operator S on the Hardy space H^2, as mentioned above, is defined as $(Sf)(z) = zf(z)$, for all $z \in \mathbb{T}$ and all f in H^2. The same definition extends to all Hardy spaces and S is an isometry on all of them. In fact, the operator S is well defined on the larger Lebesgue spaces L^p of which the Hardy spaces are closed subspaces and it is an isometry here as well. The space L^2 is a Hilbert space under the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f(z)\overline{g(z)}dm$$

where dm is the normalized Lebesgue measure. A proper nontrivial closed subspace $\mathcal{M}$ of a Banach space X is said to be invariant under a bounded linear transformation (operator) T acting on X if $T(\mathcal{M}) \subseteq \mathcal{M}$. Invariant subspaces and their characterization play an import role in operator theory and have numerous applications.

Note. All further necessary terminology and notation are given within the relevant sections that shall follow. Throughout the text, $clos_p$ stands for the closure in p-norm (weak-star when $p = \infty$) and [.] for the *span*.

Theorem A (Beurling's Theorem, [4])**.** *Every nontrivial shift invariant subspace of H^2 has the form ϕH^2, where ϕ is an inner function.*

Theorem B (Equivalent version of Beurling's Theorem, [19, Lemma, p. 106])**.** *A closed subspace of H^2 is shift-invariant iff it is invariant under multiplication by every bounded analytic function in H^∞.*

Theorem C (Extension of Beurling's Theorem, [28, Theorem 3.1])**.** *Let I be an inner function and let $\mathcal{M}$ be a subspace of L^p, $1 \leq p \leq \infty$ that is invariant under IH^∞. Either there exists a measurable set E such that $\mathcal{M} = \chi_E L^p$ or there exists a unimodular function ϕ such that $\phi IH^p \subseteq \mathcal{M} \subseteq \phi H^p$. In particular, if $p = 2$, then there exists a subspace $W \subseteq H^2 \ominus IH^2$ such that $\mathcal{M} = \phi(W \oplus IH^2)$.*

2. A brief preview

In Section 3, we present an analogue of Theorem C in the setting of the space $BMOA$. In Section 4, we present a simplification of the proof of Beurling's theorem and an analogue of Theorem C in the setting of uniform algebras. In Section 5, we produce a new and simple proof of the Helson–Lowdenslager analogue of Beurling's theorem and a version of Theorem C on compact abelian groups with ordered duals. Section 6 describes an avatar of Theorem C for the Lebesgue space of the real line.

3. Theorem C in the context of *BMOA*

Let $f \in H^1$, then we say that $f \in BMOA$ if

$$\|f\|_* = \sup_L \frac{1}{|L|} \int_L |f - f_L| d\theta < \infty$$

where L is a subarc of $\mathbb{T}$, $|L|$ is the normalized Lebesgue measure of L and $f_L = \frac{1}{|L|} \int_L f d\theta$. This $\|.\|_*$ is a pseudo norm. The space $BMOA$ is a Banach space under the norm $\|f\| = \|f\|_* + |f(0)|$ and $BMOA$ is the dual of H^1. The duality is due to a famous theorem of C. Fefferman which we state below.

Fefferman's Theorem (Disk version, [13, p. 261])**.** *Each $f \in BMOA$ is a linear functional on H^1 and its action is given by*

$$f(g) = \lim_{r \to 1-} \int_{\mathbb{T}} f(re^{i\theta})\overline{g(re^{i\theta})} d\theta, \quad \textit{for all } g \in H^1.$$

This duality induces the weak-star topology on $BMOA$. The weak star closed subspaces of $BMOA$ invariant under the operator of multiplication by the coordinate function z are well known, see [8], [30] and [31]. It is also easy to see that the appropriate version of Theorem B is valid in this context, i.e., the shift invariant subspaces are identical to those that are invariant under multiplication by each element of the algebra of multipliers of $BMOA$ (see Lemma 3.4) which we call the multiplier algebra of $BMOA$ and which we denote by $\mathfrak{M}_{\text{BMOA}}$. The point to be noted in the context of the space $BMOA$ is that in our version of Theorem C for this section, we replace H^∞ (of the original Theorem C) by $\mathfrak{M}_{\text{BMOA}}$. This is as it should be for H^∞ is the multiplier algebra of H^2 and $\mathfrak{M}_{\text{BMOA}}$ is the multiplier algebra of $BMOA$. Additionally we replace the arbitrary inner function I of Theorem C by any arbitrary finite Blaschke factor $B(z)$ since these are the only inner functions that reside inside of $\mathfrak{M}_{\text{BMOA}}$. The collection $\mathfrak{M}_{\text{BMOA}}$ is well known through the work of Stegenga [33]. This enables us to present here the appropriate version of Theorem C in the setting of $BMOA$. It will be relevant to point out some important and interesting references connected with the context of $BMOA$ and this section such as [1], [9], [22], [23], [24], [25], [26] and [27].

We call a positive measure μ on the open unit disk, a Carleson measure if $\exists$ a positive constant N_μ such that $\mu(S_h) \leq N_\mu h$, for all $h \in (0, 1)$. Here

$$S_h = \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0|\}.$$

Remark 3.1. We will frequently be using the fact, given in Theorem 3.4, in [13, p. 233], that

$$f \in BMOA \iff d\mu_f = |f'(z)|^2(1 - |z|^2)dxdy$$

is a Carleson measure and the smallest constant $N_{\mu f}$ is such that $N_{\mu f}$ is equivalent to the pseudo norm $\|f\|_*^2$.

Since the paper [7] is not easily available we reproduce the details of the following two lemmas from it which are needed by us. While proving the forthcoming Lemma, we will use K to denote a constant which need not be the same at each occurrence. Let $f_t(z) = f(tz)$, for $t \in (0,1)$ and $z \in \mathbb{D}$, where f is any function analytic inside $\mathbb{D}$. We know that $H^\infty \subset BMOA \subset \mathcal{B}$, where $\mathcal{B}$ is the Bloch space. An analytic function f on $\mathbb{D}$ is said to be a Bloch function if $\sup_z |f'(z)|(1-|z|^2) < \infty$.

Lemma 3.2. *If $f \in \mathcal{B}$ and $g \in BMOA$, then for $z = re^{i\theta}$ in $\mathbb{D}$:*

(i) $|f(z) - f(tz)| \leq K.\log\left(\frac{1-rt}{1-r}\right)$, *where K is independent of t.*

(ii) $\int_0^1 \log^2\left(\frac{1-rt}{1-r}\right)\frac{1-r}{(1-rt)^2}dr < 1$, *for all t.*

(iii) $\int\int_{S_h} |g_t'(f-f_t)|^2(1-|z|^2)dxdy < Kh$*; where K is independent of t.*

(iv) $\|g_t\|_* \leq K$*, for some K independent of t.*

Proof. (i) Let $f \in \mathcal{B}$. Then

$$\sup_z |f'(z)|(1-|z|^2) < \infty \quad \text{and} \quad |f(z) - f(tz)| = \left|\int_{tz}^{z} f'(r)dr\right|.$$

This means

$$|f(z) - f(tz)| \leq \int_{tr}^{r} \frac{K}{1-x}dx = K.\log\left(\frac{1-rt}{1-r}\right).$$

(ii) Taking $\log\left(\frac{1-rt}{1-r}\right) = x$, in the integral, $\int_0^1 \log^2\left(\frac{1-rt}{1-r}\right)\frac{1-r}{(1-rt)^2}dr$, the integral becomes $\int_0^\infty \frac{x^2e^{-x}}{e^x-x}dx$. Since $x \leq e^x - 1 \leq e^x - t$, for $0 < t < 1$, the value of this integral will be less than $\int_0^\infty xe^{-x}dx = 1$.

(iii) From part (i), we see that $|f(z)-f(tz)|^2 \leq K.\log^2\left(\frac{1-rt}{1-r}\right)$. Thus the integral

$$\begin{aligned}
&\int\int_{S_h} |f(z) - f(tz)|^2|g_t'|^2(1-|z|^2)dxdy \\
&\leq K\int\int_{S_h} |g_t'|^2\log^2\left(\frac{1-rt}{1-r}\right)(1-r^2)dxdy \\
&\leq K\int\int_{S_h} \log^2\left(\frac{1-rt}{1-r}\right)\frac{(1-r)}{(1-rt)^2}dxdy \\
&\leq K\int_{\theta_0-h}^{\theta_0+h}\int_0^1 \log^2\left(\frac{1-rt}{1-r}\right)\frac{(1-r)}{(1-rt)^2}drd\theta \\
&\leq Kh.
\end{aligned}$$

(iv) By Fefferman's theorem, for each g in $BMOA$ there exists functions φ and ψ in L^∞ such that $g = \varphi + \tilde{\psi}$, where $\tilde{\psi}$ is the harmonic conjugate of

ψ. So we can write, $g_t = \varphi_t + \tilde{\psi}_t$. Thus

$$\|g_t\|_* \leq \|\varphi_t\|_* + \|\psi_t\|_* \leq \|\varphi\|_\infty + K.\|\psi\|_\infty. \qquad \square$$

Let $[f]$ denote the weak-star closure of $\{pf : p \text{ is a polynomial}\}$ in $BMOA$.

Lemma 3.3. *If $f \in BMOA$ and $g \in H^\infty$, then $fg \in BMOA$ implies $fg \in [f]$.*

Proof. Even though this proof is also available in [7, Lemma 2], we have chosen to reproduce the proof in details, since this technique is likely to prove useful in other situations. First we shall show that the integral

$$J = \int\int_{S_h} |(g_t f)'|^2 (1 - |z|^2) dxdy$$

is uniformly bounded for all $t \in (0, 1)$. Note that

$$J \leq K \left(\int\int_{S_h} |g_t f'|^2 (1 - |z|^2) dxdy + \int\int_{S_h} |g_t' f|^2 (1 - |z|^2) dxdy \right).$$

Take $J_1 = \int\int_{S_h} |g_t f'|^2 (1-|z|^2) dxdy$ and $J_2 = \int\int_{S_h} |g_t' f|^2 (1-|z|^2) dxdy$. We claim that both J_1 and J_2 are finite. By Remark 3.1, $f \in BMOA$ implies

$$\mu_f(S_h) = \int\int_{S_h} |f'(z)|^2 (1 - |z|^2) dxdy \leq Kh, \quad \forall\ h \in (0, 1).$$

As $g \in H^\infty$, so g_t also lies inside H^∞ and $\|g_t\|_\infty \leq \|g\|_\infty$, therefore

$$\begin{aligned} J_1 &= \int\int_{S_h} |g_t f'|^2 (1 - |z|^2) dxdy \\ &\leq \|g\|_\infty^2 \int\int_{S_h} |f'(z)|^2 (1 - |z|^2) dxdy \leq Kh. \end{aligned}$$

Now

$$\begin{aligned} |g_t' f|^2 &= |g_t' f - g_t' f_t - g_t f_t' + (g_t f_t)'|^2 \\ &\leq K \left(|g_t'(f - f_t)|^2 + |g_t f_t'|^2 + |(g_t f_t)'|^2 \right). \end{aligned}$$

Put

$$\begin{aligned} J_{2,1} &= \int\int_{S_h} |g_t'(f - f_t)|^2 (1 - |z|^2) dxdy, \\ J_{2,2} &= \int\int_{S_h} |g_t f_t'|^2 (1 - |z|^2) dxdy, \\ J_{2,3} &= \int\int_{S_h} |(g_t f_t)'|^2 (1 - |z|^2) dxdy. \end{aligned}$$

By part (iii) in Lemma 3.2, we have

$$J_{2,1} = \int\int_{S_h} |g_t'(f - f_t)|^2 (1 - |z|^2) dxdy < Kh.$$

By Remark 3.1 and part (iv) in Lemma 3.2, we have

$$\begin{aligned} J_{2,2} &= \int\int_{S_h} |g_t f_t'|^2 (1-|z|^2) dxdy, \\ &\leq \|g\|_\infty^2 \int\int_{S_h} |f_t'(z)|^2 (1-|z|^2) dxdy \\ &\leq \|g\|_\infty^2 \|f_t\|_*^2 . h \\ &\leq Kh. \end{aligned}$$

Since $g_t f_t$ is bounded for each $t \in (0,1)$, $g_t f_t \in BMOA$. Again using Remark 3.1 and part (iv) in Lemma 3.2, we have

$$\begin{aligned} J_{2,3} &= \int\int_{S_h} |(g_t f_t)'|^2 (1-|z|^2) dxdy \\ &\leq K. \int\int_{S_h} \left(|g_t f_t'|^2 + |g_t' f_t|^2\right)(1-|z|^2) dxdy \\ &= K\left(\int\int_{S_h} |g_t f_t'|^2 (1-|z|^2) dxdy + \int\int_{S_h} |g_t' f_t|^2 (1-|z|^2) dxdy\right) \\ &\leq K\left(\|g\|_\infty^2 \|f_t\|_*^2 . h + \|f_t\|_\infty^2 \|g_t\|_*^2 . h\right) \\ &\leq K\left(\|g\|_\infty^2 \|f_t\|_*^2 . h + \|f_t\|_*^2 \|g_t\|_*^2 . h\right) \\ &\leq Kh. \end{aligned}$$

By boundedness of all the above integrals, we have

$$\mu_{g_t f}(S_h) = J \leq K(J_1 + J_2) \leq (J_1 + J_{2,1} + J_{2,2} + J_{2,3}) \leq Kh.$$

Note that each of K's by virtue of Lemma 3.2 is independent of all $t \in (0,1)$, therefore $\mu_{g_t f}$ is uniformly bounded. Thus for each $t \in (0,1)$, $\mu_{g_t f}$ is a Carleson measure and by Remark 3.1, $g_t f \in BMOA$. Since $\{g_t f\}$ is uniformly bounded and converges point wise to gf as $t \to 1$, $g_t f$ converges weak-star to gf in $BMOA$.

Now it remains to show that for each fixed $t \in (0,1)$, $g_t f \in [f]$. Observe that g_t is analytic on $\overline{\mathbb{D}}$, so there exists a sequence of polynomials P_n such that P_n converges to g_t and P_n' converges to g_t', uniformly on $\overline{\mathbb{D}}$. Write $(P_n f)' = P_n' f + P_n f'$ and

$$\mu_{P_n f}(S_h) = \int\int_{S_h} |(P_n f)'|^2 (1-|z|^2) dxdy.$$

As seen above for $J_{2,3}$, we have $\mu_{P_n f}(S_h) \leq Kh$. Here K is independent of n because both P_n and P_n' are uniformly bounded. This means $P_n f$ is uniformly bounded in $BMOA$ norm and hence by H^1-$BMOA$ duality, $\{P_n f\}$ is a uniformly bounded sequence of linear functional on H^1. Also, $P_n f$ converges pointwise to $g_t f$, so $P_n f$ converges weak-star to $g_t f$ in $BMOA$. For each n, $P_n f \in [f]$ and $g_t f$ is the weak-star limit of $P_n f$, so $g_t f \in [f]$ and hence $gf \in [f]$. $\square$

The following lemma shows that invariance under multiplication by z in $BMOA$ is equivalent to invariance under the multiplier algebra $\mathfrak{M}_{\mathrm{BMOA}}$.

Lemma 3.4. *If $\mathcal{M}$ is a weak-star closed subspace of $BMOA$, then $z\mathcal{M} \subset \mathcal{M}$ if and only if $\varphi\mathcal{M} \subset \mathcal{M}$, for each $\varphi \in \mathfrak{M}_{\mathrm{BMOA}}$.*

Proof. It is easy to see that $z\mathcal{M} \subset \mathcal{M}$, if $\varphi\mathcal{M} \subset \mathcal{M}$, for every $\varphi \in \mathfrak{M}_{\mathrm{BMOA}}$, since $z \in \mathfrak{M}_{\mathrm{BMOA}}$. To prove the converse, let us take an element φ inside the multiplier algebra $\mathfrak{M}_{\mathrm{BMOA}}$. By Theorem 1.2 in [33], we see that $\varphi \in H^\infty$. Let $f \in \mathcal{M}$. Now $\varphi f \in BMOA$, because $\varphi \in \mathfrak{M}_{\mathrm{BMOA}}$. So by Lemma 3.3, $\varphi f \in [f]$. But $[f] \subset \mathcal{M}$, because $z\mathcal{M} \subset \mathcal{M}$. So $\varphi f \in \mathcal{M}$. $\square$

Our proof of the main theorem of this section (Theorem 3.7) will make use of the following description of an orthonormal basis for H^2 in terms of a finite Blaschke factor $B(z)$ of order n:

Let $\alpha_1, \dots, \alpha_n \in \mathbb{D}$, and $B(z) = \prod_{i=1}^{n} \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$ be a Blaschke factor of order n. We assume that $\alpha_1 = 0$. Let $\hat{k}_i(z) = \frac{\sqrt{1-|\alpha_i|^2}}{1-\overline{\alpha_i}z}$, $B_0(z) = 1$ and $B_i(z) = \prod_{j=1}^{i} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}$, then $B_n(z) = B(z)$, $i = 1, 2, \dots, n$. Define $e_{j,m} = \hat{k}_{j+1}B_jB^m$; $0 \le j \le n-1, m = 0, 1, 2, \dots$.

Theorem 3.5. [32, Theorem 3.3]. *The set $\{e_{j,m}\}$ is an orthonormal basis for H^2.*

The space H^2 is decomposed in terms of its closed subspace $H^2(B)$, where $H^2(B)$ stands for the closed span of the set $\{1, B(z), B^2(z), \dots\}$ in H^2.

Theorem 3.6. [32, Corollary 3.4].

$$H^2 = e_{00}H^2(B) \oplus e_{10}H^2(B) \oplus \cdots \oplus e_{n-1,0}H^2(B).$$

Now we prove the main result of this section.

Theorem 3.7. *Let $B(z)$ be a finite Blaschke factor and $\mathcal{M}$ be a weak-star closed subspace of $BMOA$ which is invariant under $B(z)\mathfrak{M}_{\mathrm{BMOA}}$. Then, there exists a finite dimensional subspace W of $BMOA$ and an inner function φ such that*

$$\mathcal{M} = \varphi\,(W \oplus B(z)BMOA) \cap BMOA.$$

Proof. First, we shall show that $\mathcal{M}$ has nonempty intersection with H^∞. Using the fact that $\{e_{j,0}B(z)^m : m = 0, 1, 2, \dots\}$ is an orthonormal basis, in Theorem 3.6, any $f \in \mathcal{M}$ can be written as

$$f(z) = e_{00}f_0(B(z)) + \cdots + e_{n-1,0}f_{n-1}(B(z)), \tag{3.1}$$

for some $f_0(z), \dots, f_{n-1}(z)$ in H^2. For $k = 0, 1, \dots, n-1$, we define functions

$$g^{(k)}(z) = \exp\left(-|f_k(z)| - i|f_k(z)|^{\sim}\right),$$

where $\sim$ stands for the harmonic conjugate. Consider the function

$$g(z) = g^{(0)}(B(z)) \ldots g^{(n-1)}(B(z)).$$

It is easy to see that $g(z)f(z) \in H^\infty$. Define

$$g_t(z) = g(tz) = \prod_{k=0}^{n-1} g^{(k)}(t.B(z))$$

for $t \in (0,1)$.

For each such fixed t, $g^{(k)}(tz)$ is analytic on $\overline{\mathbb{D}}$, so there exists a sequence of polynomials $P_s^{(k)}(z)$ that converges uniformly to $g^{(k)}(tz)$ and hence there exists sequence $P_s^{(k)}(B(z))$ that converges uniformly to $g^{(k)}(tB(z))$ as $s \to \infty$. Taking $P_s(z) = \prod_{k=0}^{n-1} P_s^{(k)}(B(z))$ and $g_t(z) = g(tz) = \prod_{k=0}^{n-1} g^{(k)}(tB(z))$, we see that $P_s(z)$ converges to $g_t(z)$ uniformly and hence $P_s'(z)$ converges uniformly to $g_t'(z)$. As seen in Lemma 3.3, $P_s f$ is uniformly bounded sequence of linear functionals and converges pointwise to $g_t f$ as $s \to \infty$. Therefore, $P_s f$ converges weak-star to $g_t f$ in $BMOA$. For each natural number s, $P_s f \in \mathcal{M}$ because $\mathcal{M}$ is invariant under multiplication by $B(z)$. In addition, $\mathcal{M}$ is weak-star closed, so the weak-star limit $g_t f$ also belongs to $\mathcal{M}$. Again, as seen in Lemma 3.3, $g_t f$ converges weak-star to gf, so gf also belongs to $\mathcal{M}$. This establishes the claim that $\mathcal{M} \cap H^\infty$ is nonempty.

The space $\mathcal{M} \cap H^\infty$ is a weak-star closed subspace of H^∞ and is invariant under the algebra BH^∞, so by Theorem 3.1 in [28], there exists an inner function φ such that

$$\varphi B(z) H^\infty \subseteq \mathcal{M} \cap H^\infty \subseteq \varphi H^\infty. \tag{3.2}$$

It has been established in Theorem 4.1, in [30] that

$$\overline{IH^\infty} = IBMOA \cap BMOA$$

for any inner function I. Therefore,

$$\varphi B(z) BMOA \cap BMOA \subseteq \overline{\mathcal{M} \cap H^\infty} \subseteq \varphi BMOA \cap BMOA. \tag{3.3}$$

The bar in (3.3) denotes weak-star closure in $BMOA$.

We claim that $\overline{\mathcal{M} \cap H^\infty} = \mathcal{M}$. Consider the decomposition 3.1 for any $f \in \mathcal{M}$. For each $k = 0, 1, \ldots, n-1$, define a sequence ofH^∞ functions

$$g_m^{(k)}(z) = \exp\left(\frac{-|f_k(z)| - i|f_k(z)|^\sim}{m}\right).$$

Define

$$O_m(z) = \prod_{k=0}^{n-1} g_m^{(k)}(B(z)).$$

It can be seen that $O_m(z)f(z) \in H^\infty$, and $O_m(z) \to 1$ *a.e.*

As seen above, for each fixed m, $O_m(tz)f(z) \in \mathcal{M}$. Also, $O_m(tz)f(z)$ converges weak-star to $O_m(z)f(z)$ in $BMOA$, so $O_m(z)f(z) \in \mathcal{M}$ and hence in $\mathcal{M} \cap H^\infty$.

Now, $O_m(z)f(z) \to f(z)$ *a.e.* and

$$\|O_m(z)f(z)\|_{\text{BMOA}} \leq \|O_m(z)f(z)\|_\infty \leq K$$

for some constant K. By Dominated Convergence Theorem, for every $\epsilon > 0$,

$$\int_{\mathbb{T}} |O_m(z)f(z) - f(z)| < \epsilon \text{ for sufficiently large } m.$$

This means that for each polynomial $p \in H^1$ with upper bound M_p, we can find sufficiently large m, n such that

$$\int_{\mathbb{T}} |O_m(z)f(z) - O_n(z)f(z)| < \frac{\epsilon}{|M_p|}.$$

So

$$\begin{aligned}\int_{\mathbb{T}} |O_m(z)f(z)\overline{p(z)} - O_n(z)f(z)\overline{p(z)}| &= \int_{\mathbb{T}} |O_m(z)f(z) - O_n(z)f(z)||\overline{p(z)}| \\ &< \frac{\epsilon}{|M_p|}.|M_p| = \epsilon.\end{aligned}$$

Thus $(O_m f)\,(p)$ is a Cauchy sequence for each polynomial p in H^1. Moreover, $\|O_m(z)f(z)\|_{BMOA} \leq K$. By Ex. 13, in [5, p. 76], $\{O_m(z)f(z)\}$ converges weak-star to some $h(z)$ in $BMOA$.

We claim that $h(z)$ coincides with $f(z)$. Note that $(O_m f)(k)$ converges weak-star to $h(k)$, for each k in H^1. So $(O_m f)(k_{z_0})$ converges weak-star to $h(k_{z_0})$, where $k_{z_0} = \frac{1}{1-\bar{z_0}z}$ is the reproducing kernel in H^1, and z_0 is an arbitrary but fixed element of $\mathbb{D}$. Therefore, $O_m(z_0)f(z_0) = (O_m f)(k_{z_0})$ converges weak-star to $h(z_0) = h(k_{z_0})$. Since z_0 was arbitrarily chosen, so $O_m(z)f(z)$ converges weak-star to $h(z)$, for each $z \in \mathbb{D}$. But $O_m(z)f(z)$ converges to $f(z)$ a.e., so $h = f$ a.e.

This proves that $O_m f$ converges to f weak-star in $BMOA$, and hence $\overline{\mathcal{M} \cap H^\infty} = \mathcal{M}$. The inequality (3.3) now reads

$$\varphi B(z)BMOA \cap BMOA \subset \mathcal{M} \subset \varphi BMOA \cap BMOA. \tag{3.4}$$

Let $\overline{\overline{\mathcal{M}}}$ stand for the closure of $\mathcal{M}$ in H^2. Taking closure in H^2 throughout (3.4) we get

$$\varphi B(z)H^2 \subset \overline{\overline{\mathcal{M}}} \subset \varphi H^2. \tag{3.5}$$

From (3.5), we see that $\overline{\overline{\mathcal{M}}} \ominus \varphi B(z)H^2 \subset \varphi(H^2 \ominus B(z)H^2)$. So, there exists a subspace W_1 of $H^2 \ominus B(z)H^2$ such that $\overline{\overline{\mathcal{M}}} \ominus \varphi B(z)H^2 = \varphi W_1$. Moreover, $\dim W_1 \leq n$. Therefore

$$\overline{\overline{\mathcal{M}}} = \varphi W_1 \oplus \varphi B(z)H^2.$$

Since $\mathcal{M} \subset \overline{\overline{M}}$, we have the following form for $\mathcal{M}$:

$$\mathcal{M} = \varphi W \oplus \varphi B(z)\mathcal{N}; \tag{3.6}$$

where W is a subspace of W_1, and $\mathcal{N}$ is a subspace of H^2.

Now $H^2 \ominus B(z)H^2 = \{e_{0,0}, e_{1,0}, \ldots, e_{n-1,0}\} \subset H^\infty$ and consequently $W \subset H^\infty$. Thus in Equation (3.6) we have $W \subset BMOA$, and because $\varphi B(z)$ is inner, we also have $\mathcal{N} \subset BMOA$.

In light of (3.4) we see that $\varphi B(z)BMOA \cap BMOA \subset \varphi B(z)\mathcal{N}$. But $\mathcal{N} \subset BMOA$. So $\varphi B(z)\mathcal{N} = \varphi B(z)BMOA \cap BMOA$. This completes the proof of the theorem. □

If we take $B(z) = 1$, then invariance under $\mathfrak{M}_{\text{BMOA}}$ is equivalent to invariance under the operator S of multiplication by coordinate function z on $BMOA$, and the results in [8, Theorem 3.1], [30, Theorem 4.3] and [31, Theorem C] can be derived as corollaries of the above theorem.

Corollary 3.8. *Let $\mathcal{M}$ be a weak star closed subspace of $BMOA$ invariant under S. Then there exists a unique inner function φ such that*

$$\mathcal{M} = \varphi BMOA \cap BMOA.$$

Replacing $B(z)$ with z, we obtain common invariant subspaces of S^2 and S^3 and Theorem 3.1 in [30] is received as corollary of Theorem 3.7.

Corollary 3.9. *Let $\mathcal{M}$ be weak star closed subspace of $BMOA$ which is invariant under S^2 and S^3 but not invariant under S. Then there exists an inner function I and constants α, β such that*

$$\mathcal{M} = IBMOA_{\alpha,\beta} \cap BMOA.$$

Proof. This follows by taking $B(z) = z$ and W as subspace of *span*$\{1, z\}$. □

4. Theorem C in the setting of uniform algebras

Let X be a compact Hausdorff space and let A be a uniform algebra in $C(X)$, the algebra of complex valued continuous functions on X. Here, by a uniform algebra we mean a closed subalgebra of $C(X)$ which contains the constant functions and separates the points of X, i.e., for any $x, y \in X$, $x \neq y$, $\exists$ a function $f \in A$ such that $f(x) \neq f(y)$. For a multiplicative linear functional φ in the maximal ideal space of A, a *representing measure* m for φ is a positive measure on X such that $\varphi(f) = \int f dm$, for all $f \in A$. We shall denote the set of all *representing measures* for φ by M_φ. Let W be a convex subset of a vector space V, an element $x \in W$ is said to be a *core point* of W if whenever $y \in V$ such that $x + y \in W$, then for every sufficiently small $\epsilon > 0$, $x - \epsilon y \in W$. A *core measure* for φ is a measure which is a *core point* of M_φ.

For $1 \leq p < \infty$, $L^p(dm)$ is the space of functions whose p-th power in absolute value is integrable with respect to the representing measure m and $H^p(dm)$ is defined to be the closure of A in $L^p(dm)$. $L^\infty(dm)$ is the space of m-essentially bounded functions and $H^\infty(dm)$ is the weak-star closure of A in $L^\infty(dm)$. Let A_0 be the subalgebra of A such that $\int f dm = 0$, for all $f \in A$. $H_0^p(dm)$ is the closure of A_0 in $L^p(dm)$. The real annihilator of A in

L^p_R, $1 \le p \le \infty$, is the space N^p which consists of functions w in L^p_R such that $\int wf dm = 0$, for all $f \in A$. The conjugate function of a function f in $ReH^2(dm)$ is the function f^* in $ReH^2_0(dm)$ such that $f + if^* \in H^2(dm)$. The conjugation operator is the real linear operator which sends f to f^*.

We call a function I in $H^\infty(dm)$ *inner* if $|I| = 1$ m-almost everywhere. A subspace $\mathcal{M}$ of $L^p(dm)$ is said to be *invariant* under A if $A\mathcal{M} \subseteq \mathcal{M}$ or equivalently $A_0\mathcal{M} \subseteq \mathcal{M}$. We say $\mathcal{M}$ is *simply invariant* if $A_0\mathcal{M}$ is not dense in $\mathcal{M}$. We refer to [12] for more details.

Our purpose in the theorem given below is to demonstrate that the Theorem V.6.1 in [12, pg 131], which is the key result that essentially characterizes the invariant subspaces on uniform algebras, can actually be proved without the use of Kolmogoroff's theorem on the $\left(L^p, L^1\right)$ boundedness of the conjugation operator $(0 < p < 1)$ as defined above on uniform algebras and used in [12] for observing convergence in measure for the conjugate of a sequence of L^1 functions. We also eliminate the use of uniform integrability. The simplification is enabled since we make use of the geometry of the space L^2 and the annihilating space N^2 in L^2. In this setting, the conjugation operator is well defined and bounded without having to take recourse to the theorem of Kolmogoroff's. As our proof shows, we also do not use uniform integrability.

Theorem 4.1. *Suppose the set of representing measures for φ is finite dimensional and m is a core measure for φ. Then there is a 1-1 correspondence between invariant subspaces $\mathcal{M}_p$ of $L^p(m)$ and closed (weak star closed if $q = \infty$) invariant subspaces $\mathcal{M}_q$ of $L^q(m)$ such that $\mathcal{M}_q = \mathcal{M}_p \cap L^q(dm)$ and $\mathcal{M}_p$ is the closure in $L^p(dm)$ of $\mathcal{M}_q$, $(0 < p < q \le \infty)$.*

Proof. It is enough to consider the case $q = \infty$ since the other values of q will have an identical proof. Let $\mathcal{M}_p$ be an invariant subspace of $L^p(dm)$. Put $M = \mathcal{M}_p \cap L^\infty(dm)$. By the Krein–Schmulian criterion, M is weak-star closed. We show $\overline{M}$ (in $L^p(dm)$) is equal to $\mathcal{M}_p$. By definition of N^2, $L^2 = H^2 \oplus \overline{H^2_0} \oplus N^2_c$ (see [12, p. 105]), therefore in case of real L^2, we have $L^2_R = ReH^2 \oplus N^2$. Let us choose any $f \in \mathcal{M}_p$ and let $P\left(|f|^{\frac{p}{2}}\right)$ be the projection in L^2_R of $|f|^{\frac{p}{2}}$ onto N^2. So $h = |f|^{\frac{p}{2}} - P\left(|f|^{\frac{p}{2}}\right) \in ReH^2$. Let

$$h_n = \exp\left(-\frac{(h + ih^*)}{n}\right),$$

where $*$ denotes the conjugation operator. Then $h_n \in H^\infty(dm)$ and $h_n f \in \mathcal{M}_p \cap L^\infty(dm)$. Further, $h_n f \to f$ in $L^p(dm)$ since $h_n \to 1$ boundedly and pointwise. This proves $\mathcal{M}_p \cap L^\infty(dm)$ is dense in $\mathcal{M}_p$.

Now suppose that M is a weak-star closed invariant subspace of $L^\infty(dm)$ and let $\mathcal{M}_p$ be the closure of M in $L^p(dm)$. We must show that

$$\mathcal{M}_p \cap L^\infty(dm) = M.$$

Clearly $\mathcal{M}_p \cap L^\infty(dm)$ is weak-star closed in $L^\infty(dm)$. Assume that

$$M \subsetneq \mathcal{M}_p \cap L^\infty(dm).$$

Then there exists $g \in {}^\perp M$ such that $g \notin {}^\perp\mathcal{M}_p \cap L^\infty(dm)$ $(g \in L^1(dm))$. We may assume without loss of generality that $g \in L^\infty(dm)$. This can be done by considering

$$g \exp\left(\frac{-\left(|g|^{\frac{1}{2}} + P\left(|g|^{\frac{1}{2}}\right)\right) - i\left(|g|^{\frac{1}{2}} + P\left(|g|^{\frac{1}{2}}\right)\right)^*}{n}\right).$$

Then for each $f \in \mathcal{M}_p$, there is a sequence $(f_n) \subset M$ such that

$$\int_X g f_n dm \to \int_X g f dm.$$

But $\int_X g f_n dm = 0$ so that $\int_X g f dm = 0$. This contradiction implies that $\mathcal{M}_p \cap L^\infty(dm) = M$. □

Theorem V.6.2 in [12, pg 132] is Beurling's theorem in the setting of uniform algebras and follows as a corollary to Theorem 4.1 (Theorem V.6.1 in [12]). We state it below and we will use it in the proof of our next theorem (Theorem 4.3).

Corollary 4.2 (Beurling's theorem in the setting of uniform algebras, [12, pg 132])**.** *Let m be a unique representing measure for φ. Let $\mathcal{M}_p$ be a simply invariant subspace of $L^p(dm)$. Then there is q in $\mathcal{M}_p$ such that $|q| = 1$ almost everywhere and $\mathcal{M}_p = qH^p(dm)$.*

Proof. The proof is identical to Theorem V.6.2 in [12]. □

Let I be an inner function in $H^\infty(dm)$, then $IH^\infty(dm)$ is a subalgebra of $H^\infty(dm)$. The following theorem is the version of Theorem C in the setting of uniform algebras, i.e., we characterize the subspaces of $L^p(dm)$, $1 \leq p \leq \infty$, which are invariant under $IH^\infty(dm)$.

Theorem 4.3. *Let I be an inner function and $\mathcal{M}$ be a subspace of $L^p(dm)$, $1 \leq p \leq \infty$, invariant under $IH^\infty(dm)$ such that $\int_X f dm \neq 0$, for some f in $\mathcal{M}$, then*

$$I.qH^p(dm) \subseteq \mathcal{M} \subseteq qH^p(dm)$$

where q is a m-measurable function such that $|q| = 1$ m-a.e. When $p = 2$,

$$\mathcal{M} = q\left(W \oplus IH^p(dm)\right)$$

for some subspace W of $H^2(dm)$.

Proof. Let us take $\mathcal{M}_1 = \mathrm{clos}_p[H^\infty(dm)\mathcal{M}]$, where the closure (weak-star when $p = \infty$) is taken in $L^p(dm)$, then

$$I\mathcal{M}_1 = I.\mathrm{clos}_p[H^\infty(dm)\mathcal{M}] = \mathrm{clos}_p[IH^\infty(dm)\mathcal{M}]. \tag{4.1}$$

Since $H^\infty(dm)$ is a Banach algebra under sup-norm, $A_0\mathcal{M}_1 \subseteq \mathcal{M}_1$. This implies $\mathcal{M}_1$ is an invariant subspace of $L^p(dm)$. Also $\mathcal{M} \subseteq \mathcal{M}_1$, so by hypothesis it follows $\int_X f dm \neq 0$, for some f in $\mathcal{M}_1$, and hence $A_0\mathcal{M}_1$ is not dense in $\mathcal{M}_1$. Therefore, $\mathcal{M}_1$ is simply invariant. Thus by Corollary 4.2 (Theorem V.6.2 in [12]), $\mathcal{M}_1 = qH^p(dm)$ for some $L^\infty(dm)$ function q with $|q| = 1$. So, (4.1) becomes

$$I.qH^p(dm) \subseteq \mathcal{M} \subseteq qH^p(dm).$$

When $p = 2$, then there exists a closed subspace $W_1 \subset \mathcal{M}$ such that

$$\mathcal{M} = W_1 \oplus I.qH^2(dm).$$

But $\mathcal{M} \subseteq qH^2(dm)$, so $W_1 = qW$, where W is a closed subspace of $H^2(dm)$, because q is unitary. Therefore

$$\mathcal{M} = q(W \oplus IH^2(dm)). \tag{4.2}$$

and the proof is complete. □

When $X = \mathbb{T}$ the unit circle, then the algebra A becomes the disk algebra and $L^p(dm) = L^p$, and we obtain the following part of Theorem 3.1, in [28], as a corollary.

Corollary 4.4. *Let I be an inner function and $\mathcal{M}$ be a subspace of L^p, $1 \leq p \leq \infty$, invariant under IH^∞ but not invariant under H^∞, then there exists a unimodular function q such that $I.qH^p \subseteq \mathcal{M} \subseteq qH^p$. When $p = 2$, there exists $W \subseteq H^2 \ominus IH^2$ and $\mathcal{M} = q(W \oplus IH^2)$.*

5. Theorem C for compact abelian groups

We use K to denote a compact abelian group dual to a discrete group Γ and σ to denote the Haar measure on K which is finite and normalized so that $\sigma(K) = 1$. For each λ in Γ, let χ_λ denote the character on K defined by $\chi_\lambda(x) = x(\lambda)$, for all x in K. $L^p(d\sigma)$, $1 \leq p < \infty$ denotes the space of functions whose p^{th}- power in absolute value is integrable on K with respect to the Haar measure σ. $L^\infty(d\sigma)$ is the space of essentially bounded functions w.r.t. the Haar measure σ. For $p = 2$, the space $L^2(d\sigma)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_K f(x)\overline{g(x)}d\sigma, \quad \forall\ f, g \in L^2(d\sigma)$$

and the set of characters $\{\chi_\lambda\}_{\lambda\in\Gamma}$ forms an orthonormal basis of $L^2(d\sigma)$. Every f in $L^1(d\sigma)$ has a Fourier series in terms of $\{\chi_\lambda\}_{\lambda\in\Gamma}$, i.e.,

$$f(x) \sim \sum_{\lambda\in\Gamma} a_\lambda(f)\chi_\lambda(x), \quad \text{where} \quad a_\lambda(f) = \int_K f(x)\overline{\chi_\lambda(x)}d\sigma.$$

Suppose Γ_+ is a semigroup such that Γ is the disjoint union $\Gamma_+ \cup \{0\} \cup \Gamma_-$, where 0 denote the identity element of Γ and $\Gamma_- = -\Gamma_+$. We say the elements of Γ_+ are positive and those of Γ_- are negative. The group Γ induces an order under these conditions. Details can be found in [29].

For a subspace $\mathcal{M}$ of $L^p(d\sigma)$, we set

$$\mathcal{M}_- = \text{clos}_p \left[\bigcup_{\lambda > 0} \chi_\lambda . \mathcal{M} \right] \quad \text{and} \quad \mathcal{M}_\lambda = \chi_\lambda . \mathcal{M}.$$

We say a function in $L^2(d\sigma)$ is *analytic* if $a_\lambda(f) = 0$ for all $\lambda < 0$. $H^2(d\sigma)$ is the subspace of $L^2(d\sigma)$ consisting of all analytic functions in $L^2(d\sigma)$. For each $\lambda \in \Gamma$, χ_λ is an isometry on $H^2(d\sigma)$ and the adjoint operator of χ_λ is

$$\chi_\lambda^* f(x) = P\chi_{-\lambda} f(x)$$

where P is the orthogonal projection of $L^2(d\sigma)$ on $H^2(d\sigma)$.

A closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to be an *invariant subspace* under $\{\chi_\lambda\}_{\lambda \in \Gamma_1}$ if $\chi_\lambda \mathcal{M} \subset \mathcal{M}$ for all λ in Γ_1, where $\Gamma_1 \subseteq \Gamma$ such that $\Gamma_1 \cap \Gamma_1^{-1} = \{0\}$ and $\Gamma_1 \Gamma_1^{-1} = \Gamma$. $\mathcal{M}$ is said to be *doubly invariant* if $\chi_\lambda \mathcal{M} \subset \mathcal{M}$ and $\chi_\lambda^* \mathcal{M} \subset \mathcal{M}$ for all λ in Γ_1, where χ_λ^* denote the adjoint operator of χ_λ. We call a semigroup $\{\chi_\lambda\}_{\lambda \in \Gamma_1}$ of operators *unitary* if χ_λ is a unitary operator for each $\lambda \in \Gamma_1$ and *quasi-unitary* if the closure of

$$\left[\bigcup_{\lambda \notin \Gamma_1^{-1}} \chi_\lambda(\mathcal{H}) \right] = \mathcal{H}.$$

A semigroup $\{T_s\}_{s \in \Gamma_1}$ is called *totally nonunitary* if for any doubly invariant subspace $\mathcal{M}$ for which $\{T_s|\mathcal{M}\}_{s \in \Gamma_1}$ is quasi-unitary, we have $\mathcal{M} = \{0\}$.

First we present a new proof of the Helson–Lowdenslager generalization of Beurling's theorem in the setting of compact abelian groups. The statement of this theorem, in [16], runs as follows:

Theorem 5.1 ([16, Theorem 1])**.** *Let $\mathcal{M}$ be an invariant subspace larger than $\mathcal{M}_-$. Then $\mathcal{M} = q.H^2$, where q is measurable on K and $|q(x)| = 1$ almost everywhere.*

Our proof relies on the Suciu decomposition for a semigroup of isometries as stated below.

Theorem 5.2 (Suciu's Decomposition, [34, Theorem 2])**.** *Let $\{T_s\}_{s \in \Gamma_1}$ be a semigroup of isometries on a Hilbert space $\mathcal{H}$. The space $\mathcal{H}$ may be decomposed uniquely in the form*

$$\mathcal{H} = \mathcal{H}_q \oplus \mathcal{H}_t$$

in such a way that $\mathcal{H}_q$ and $\mathcal{H}_t$ are doubly invariant subspaces, $\{T_s|\mathcal{H}_q\}_{s \in \Gamma_1}$ is quasi-unitary and $\{T_s|\mathcal{H}_t\}_{s \in \Gamma_1}$ is totally nonunitary.

Theorem 5.3. *Let $\mathcal{M}$ be a closed subspace of $L^2(d\sigma)$ and $\mathcal{M}_- \subsetneq \mathcal{M}$. If $\mathcal{M}$ is invariant under the semigroup of characters $\{\chi_\lambda\}_{\lambda\geq 0}$ (i.e., $\lambda \in \Gamma_+ \cup \{0\}$), then*

$$\mathcal{M} = \varphi H^2(d\sigma)$$

where φ is a σ-measurable function and $|\varphi(x)| = 1$ almost everywhere.

Proof. $\mathcal{M}$ is a Hilbert space being a closed subspace of $L^2(d\sigma)$ and each χ_λ in the semigroup $\{\chi_\lambda\}_{\lambda\geq 0}$ is an isometry on $\mathcal{M}$. By Theorem 5.2, we can write

$$\mathcal{M} = \mathcal{L} \oplus \sum_{\lambda\geq 0} \chi_\lambda(\mathcal{N}) \tag{5.1}$$

where $\mathcal{N}$ is the orthogonal complement of

$$\mathrm{clos}_2\left[\bigcup_{\lambda>0} \chi_\lambda(\mathcal{M})\right]$$

in $L^2(d\sigma)$ and $\mathcal{L}$ is a quasi unitary subspace of $L^2(d\sigma)$. Clearly $\mathcal{N}$ is nonzero otherwise $\mathcal{M}_- = \mathcal{M}$.

Let φ be an element in $\mathcal{N}$. We claim that φ is nonzero almost everywhere. From Equation 5.1, we have

$$\langle \chi_\delta\varphi, \chi_\lambda\varphi\rangle = \int_K \chi_{\lambda-\delta}\varphi\bar{\varphi}d\sigma = 0 \quad \text{for all } \delta, \lambda \in \Gamma_+.$$

This means

$$\int_K \chi_\gamma|\varphi|^2 d\sigma = 0 \quad \text{for each nonzero } \gamma \in \Gamma.$$

and thus φ is constant almost everywhere. If we choose φ such that $\|\varphi\| = 1$, then $|\varphi| = 1$ a.e.

Next we assert that $\mathcal{N}$ is one dimensional. To see this assume the existence of a ψ in $\mathcal{N}$ which is orthogonal to φ. Then we have

$$\langle \chi_\delta\varphi, \chi_\lambda\psi\rangle = 0 \quad \text{for all } \delta, \lambda \geq 0,$$

which implies

$$\int_K \chi_{\delta-\lambda}\varphi\bar{\psi}d\sigma = 0$$

and thus

$$\int_K \chi_\gamma\varphi\bar{\psi}d\sigma = 0, \quad \gamma \in \Gamma.$$

Therefore, every Fourier coefficient of $\varphi\bar{\psi}$ is zero and hence $\varphi\bar{\psi} = 0$ a.e., which is possible only when ψ is zero almost everywhere, because φ is non-vanishing almost everywhere. So $\mathcal{N}$ is one-dimensional and Equation (5.1) can be written as

$$\mathcal{M} = \mathcal{L} \oplus \sum_{\lambda \geq 0} \chi_\lambda \varphi. \tag{5.2}$$

Since $\mathcal{L}$ is invariant under $\{\chi_\lambda\}_{\lambda\geq 0}$, for any f in $\mathcal{L}$, we have

$$\langle \chi_\delta f, \chi_\lambda \varphi \rangle = 0 \quad \text{for all } \delta, \lambda \geq 0.$$

A similar computation which we did above shows that $f = 0$ *a.e.*, which in turn implies $\mathcal{L}$ is zero and Equation (5.2) becomes

$$\mathcal{M} = \sum_{\lambda \geq 0} \chi_\lambda \varphi. \tag{5.3}$$

Now multiplication by φ is isometry on $L^2(d\sigma)$, so Equation 5.3 takes the form

$$\mathcal{M} = \varphi H^2(d\sigma)$$

which completes the proof. $\square$

Now we present an analogue of Theorem C in the setting of compact abelian groups with ordered duals.

Theorem 5.4. *Let $\mathcal{M}$ be a closed subspace of $L^p(d\sigma)$, $1 \leq p \leq \infty$ and $\tilde{\mathcal{M}} = \operatorname{clos}_p \left[\cup_{\lambda \geq 0} \chi_\lambda . \mathcal{M}\right]$. If $\tilde{\mathcal{M}}_- \subsetneq \tilde{\mathcal{M}}$ and for a fixed inner function I, $\chi_\lambda . I\mathcal{M} \subseteq \mathcal{M}$, for each $\lambda \geq 0$, then*

$$I\varphi H^p(d\sigma) \subseteq \mathcal{M} \subseteq \varphi H^p(d\sigma)$$

where φ is measurable on K and $|\varphi| = 1$ σ-almost everywhere. When $p = 2$, there exists a subspace $W \subseteq H^2(d\sigma)$ such that $\mathcal{M} = \varphi(W \oplus IH^2(d\sigma))$.

Proof. Since multiplication by I is an isometry on $L^p(d\sigma)$ and $\mathcal{M} \subseteq \tilde{\mathcal{M}}$, we have

$$I.\tilde{\mathcal{M}} = I.\operatorname{clos}_p \left[\bigcup_{\lambda \geq 0} \chi_\lambda . \mathcal{M}\right] = \operatorname{clos}_p \left[\bigcup_{\lambda \geq 0} \chi_\lambda . I\mathcal{M}\right] \subseteq \mathcal{M}$$

and thus we have

$$I.\tilde{\mathcal{M}} \subseteq \mathcal{M} \subseteq \tilde{\mathcal{M}}. \tag{5.4}$$

For $\delta > 0$ in Γ

$$\bigcup_{\delta > 0} \chi_\delta . \tilde{\mathcal{M}} = \bigcup_{\delta > 0} \chi_\delta . \operatorname{clos}_p \left[\bigcup_{\lambda \geq 0} \chi_\lambda . \mathcal{M}\right] = \operatorname{clos}_p \left[\bigcup_{\lambda > 0} \chi_\lambda . \mathcal{M}\right] \subseteq \tilde{\mathcal{M}}.$$

Now the subspace $\tilde{\mathcal{M}}$ is invariant and also $\tilde{\mathcal{M}}$ is larger than $\tilde{\mathcal{M}}_-$ by hypothesis. Therefore by Theorem $1'$, [16, p. 13], $\tilde{\mathcal{M}} = \varphi H^p(d\sigma)$, where φ is a σ-measurable function and $|\varphi| = 1$ σ-a.e. Thus Equation (5.4) becomes

$$I.\varphi H^p(d\sigma) \subseteq \mathcal{M} \subseteq \varphi H^p(d\sigma).$$

When $p = 2$, there exists a closed subspace V of $\mathcal{M}$ such that

$$\mathcal{M} = V \oplus I\varphi H^2(d\sigma).$$

But $V \subseteq \mathcal{M} \subseteq H^2(d\sigma)$, so $V = \varphi W$, where W is a closed subspace of H^2, because φ is unitary. Therefore

$$\mathcal{M} = \varphi(W \oplus IH^2(d\sigma)). \tag{5.5}$$

$\square$

When $I = \chi_{\lambda_0}$, for some λ_0 in Γ_+, we obtain the following as a corollary to Theorem 5.4.

Corollary 5.5. *Let $\mathcal{M}$ be a closed subspace of $L^p(d\sigma)$, $1 \leq p \leq \infty$ and $\tilde{\mathcal{M}} = \operatorname{clos}_p \left[\cup_{\lambda \geq 0} \chi_\lambda.\mathcal{M}\right]$. If $\tilde{\mathcal{M}}_- \subsetneq \tilde{\mathcal{M}}$ and for a fixed positive element λ_0 in Γ, $\chi_\lambda.\mathcal{M} \subseteq \mathcal{M}$, for each $\lambda \geq \lambda_0$, then*

$$\chi_{\lambda_0}.\varphi H^p(d\sigma) \subseteq \mathcal{M} \subseteq \varphi H^p(d\sigma)$$

where φ is measurable on K and $|\varphi| = 1$ σ-almost everywhere.

We observe that Theorem 1.3, in [10], becomes a special case of Corollary 5.5, when $p = 2$. If we take $\Gamma = \mathbb{Z}$ and $\lambda_0 = 2$, then $\chi_{\lambda_0} = z^2$ and $\chi_\lambda \mathcal{M} \subseteq \mathcal{M}, \forall\, \lambda \geq \lambda_0$ means invariance under H_1^∞. Here H_1^∞ is a subalgebra of H^∞ which is defined as

$$H_1^\infty = \{f \in H^\infty : f'(0) = 0\}.$$

Corollary 5.6 ([10, Theorem 1.3])**.** *Let $\mathcal{M}$ be a norm closed subspace of L^2 which is invariant for H_1^∞, but is not invariant for H^∞. Then there exist scalars α, β in $\mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha \neq 0$ and a unimodular function J, such that $\mathcal{M} = JH^2_{\alpha\beta}$.*

6. Theorem C for the Lebesgue space of the real line

Let $L^2(\mathbb{R})$ denote the space of square integrable functions on the real line $\mathbb{R}$. We consider $H^2(\mathbb{R})$ a closed subspace of $L^2(\mathbb{R})$ which consists of functions whose Fourier transform

$$F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

is zero almost everywhere for every $\lambda < 0$. A subspace $\mathcal{M}$ of $L^2(\mathbb{R})$ is said to be *invariant* if $e^{i\lambda x}\mathcal{M} \subseteq \mathcal{M}$, for all $\lambda > 0$ and *simply invariant* if $e^{i\lambda x}\mathcal{M} \subsetneq \mathcal{M}$, for $\lambda > 0$. If $e^{i\lambda x}\mathcal{M} = \mathcal{M}$ for all real λ, then we call $\mathcal{M}$ *doubly invariant.*

In this section, we give an extension along the lines of [10] and [28] of the Beurling–Lax theorem, [19, p. 114] for the Lebesgue space $L^2(\mathbb{R})$ of the real line.

Theorem 6.1. *Let $\mathcal{M}$ be a closed subspace of $L^2(\mathbb{R})$. If I is a measurable function with $I(x) = 1$ a.e. and $e^{i\lambda x} I\mathcal{M} \subseteq \mathcal{M}$, for all $\lambda \geq 0$, then either there exists a measurable subset E of $\mathbb{R}$ such that $\mathcal{M} = \chi_E L^2(\mathbb{R})$ or*

$$\mathcal{M} = q\left(W \oplus IH^2(\mathbb{R})\right)$$

where q is measurable function on the real line and $|q(x)| = 1$ almost everywhere.

Proof. Consider the subspace

$$\mathcal{N} = \operatorname{clos}_2 \left[\bigcup_{\lambda \geq 0} e^{i\lambda x}\mathcal{M}\right].$$

Our consideration of $\mathcal{N}$ implies that $\mathcal{M}$ is a subspace of $\mathcal{N}$ and $e^{i\lambda x}\mathcal{N} \subseteq \mathcal{N}$, for all $\lambda > 0$. Since multiplication by I is an isometry on $L^2(\mathbb{R})$ and $\mathcal{M}$ is a closed subspace of $L^2(\mathbb{R})$, $e^{i\lambda x} I\mathcal{M} \subseteq \mathcal{M}$, for $\lambda \geq 0$. So

$$I\mathcal{N} = I\operatorname{clos}_2 \left[\bigcup_{\lambda \geq 0} e^{i\lambda x}\mathcal{M}\right] = \operatorname{clos}_2 \left[\bigcup_{\lambda \geq 0} e^{i\lambda x} I\mathcal{M}\right] \subseteq \mathcal{M}.$$

Thus we obtain the inclusion

$$I\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{N}. \tag{6.1}$$

If $e^{i\lambda x}\mathcal{N} = \mathcal{N}$, for some λ and hence for all λ, then by [19, Theorem, p. 114], $\mathcal{N} = \chi_E L^2(\mathbb{R})$, for some fixed measurable subset E of the real line. Thus, $I\mathcal{N} = \mathcal{N}$ and by the inclusion in (6.1), we have $\mathcal{M} = \chi_E L^2(\mathbb{R})$.

On the other side, if $e^{i\lambda x}\mathcal{N} \subsetneq \mathcal{N}$, then again by [19, Theorem, p. 114], $\mathcal{N} = qH^2(\mathbb{R})$, where q is a measurable function on the real line and $|q(x)| = 1$ almost everywhere. So

$$I.qH^2(\mathbb{R}) \subseteq \mathcal{M} \subseteq qH^2(\mathbb{R}).$$

Now we see that

$$\begin{aligned} \mathcal{M} \ominus IqH^2(\mathbb{R}) &\subseteq qH^2(\mathbb{R}) \ominus I.qH^2(\mathbb{R}) \\ &= q\left(H^2(\mathbb{R}) \ominus IH^2(\mathbb{R})\right). \end{aligned}$$

So there exists a subspace $W \subseteq H^2(\mathbb{R}) \ominus IH^2(\mathbb{R})$ such that

$$qW = \mathcal{M} \ominus IqH^2(\mathbb{R})$$

or we can write

$$\mathcal{M} = q\left(W \oplus IH^2(\mathbb{R})\right). \qquad \square$$

Acknowledgments. The authors are grateful to the referee for making suggestions and critical remarks that have improved the presentation of the proof of Theorem 4.1 in the paper. The authors acknowledge the facilities and library support of the Mathematical Sciences Foundation, Delhi.

References

[1] Aleksandrov, A. B. Invariant subspaces of the shift operator. An axiomatic approach. Investigations on linear operators and the theory of functions, XI. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **113** (1981), 7–26, 264. MR0629832; *J. Sov. Math.* **22** (1983), 1695–1708. Zbl 0517.47019, doi: 10.1007/BF01882574.

[2] Ball, Joseph A.; Bolotnikov, Vladimir; ter Horst, Sanne. A constrained Nevanlinna–Pick interpolation problem for matrix-valued functions. *Indiana Univ. Math. J.* **59** (2010), no. 1, 15–52. MR2666471 (2011d:47037), Zbl 1196.47013, arXiv:0809.2345, doi: 10.1512/iumj.2010.59.3776.

[3] Ball, Joseph A.; Guerra Huamán, Moisés D. Convexity analysis and the matrix-valued Schur class over finitely connected planar domains. *J. Operator Theory* **70** (2013), no. 2, 531–571. MR3138369, Zbl 06254355, doi: 10.7900/jot.2011sep21.1940.

[4] Beurling, Arne. On two problems concerning linear transformations in Hilbert space. *Acta Math.* **81** (1948), 17 pp. MR0027954 (10,381e), Zbl 0033.37701, doi: 10.1007/BF02395019.

[5] Bhatia, Rajendra. Notes on functional analysis. Texts and Readings in Mathematics, 50. *Hindustan Book Agency, New Delhi*, 2009. x+237 pp. ISBN: 978-81-85931-89-0. MR2477477 (2010a:46001), Zbl 1175.46001.

[6] Broschinski, Adam. Eigenvalues of Toeplitz operators on the annulus and Neil algebra. *Complex Anal. Oper. Theory* **8** (2014), no. 5, 1037–1059. MR3208801, Zbl 1331.47047, doi: 10.1007/s11785-013-0331-5.

[7] Brown, Leon; Sadek, Jawad. On cyclic vectors in $BMOA$ and $VMOA$. *Izv. Tekhn. Univ. Plovdiv. Fund. Nauk. Prilozhen.* **2** (1996), 15–19. MR1420110 (98k:30046), Zbl 0878.46016.

[8] Brown, Leon; Sadek, Jawad. Invariant subspaces in the space of analytic functions of bounded mean oscillation. *Houston J. Math.* **27** (2001), no. 4, 883–886. MR1874679 (2002j:30053), Zbl 1005.30041.

[9] Cima, Joseph A.; Ross, William T. The backward shift on the Hardy space. Mathematical Surveys and Monographs, 79. *American Mathematical Society, Providence, RI*, 2000. xii+199 pp. ISBN: 0-8218-2083-4. MR1761913 (2002f:47068), Zbl 0952.47029, doi: 10.1090/surv/079.

[10] Davidson, Kenneth R.; Paulsen, Vern I.; Raghupathi, Mrinal; Singh, Dinesh. A constrained Nevanlinna–Pick interpolation problem. *Indiana Univ. Math. J.* **58** (2009), no. 2, 709–732. MR2514385 (2010j:30077), Zbl 1167.47013, doi: 10.1512/iumj.2009.58.3486.

[11] Dritschel, Michael A.; Pickering, James. Test functions in constrained interpolation. *Trans. Amer. Math. Soc.* **364** (2012), no. 11, 5589–5604. MR2946923, Zbl 1294.47024, doi: 10.1090/S0002-9947-2012-05515-2.

[12] Gamelin, Theodore W. Uniform algebras. *Prentice-Hall, Inc., Englewood Cliffs, N. J.*, 1969. xiii+257 pp. MR0410387 (53 #14137), Zbl 0213.40401.

[13] Garnett, John B. Bounded analytic functions. Revised first edition. Graduate Texts in Mathematics, 236. *Springer, New York*, 2007. xiv+459 pp. ISBN: 978-0-387-33621-3; 0-387-33621-4. MR2261424 (2007e:30049), Zbl 1106.30001, doi: 10.1007/0-387-49763-3.

[14] Hamilton, Ryan; Raghupathi, Mrinal. The Toeplitz corona problem for algebras of multipliers on a Nevanlinna–Pick space. *Indiana Univ. Math. J.* **61** (2012), no. 4, 1393–1405. MR3085612, Zbl 1308.47017, arXiv:1104.3821, doi: 10.1512/iumj.2012.61.4685.

[15] Helson, Henry. Lectures on invariant subspaces. *Academic Press, New York - London*, 1964. xi+130 pp. MR0171178 (30 #1409), Zbl 0119.11303.

[16] Helson, H. Analyticity on compact abelian groups. *Algebras in analysis* (Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973), 1–62. *Academic Press, London*, 1975. MR0427959 (55 #989), Zbl 0321.43009.

[17] Helson, Henry; Lowdenslager, David. Invariant subspaces. *Proc. Internat. Sympos. Linear Spaces* (Jerusalem, 1960), 251–262. *Jerusalem Academic Press, Jerusalem; Pergamon, Oxford*, 1961. MR0157251 (28 #487), Zbl 0142.10305.

[18] Hoffman, Kenneth. Analytic functions and logmodular Banach algebras. *Acta Math.* **108** (1962), 271–317. MR0149330 (26 #6820), Zbl 0107.33102, doi: 10.1007/BF02545769.

[19] Hoffman, Kenneth. Banach spaces of analytic functions. *Prentice Hall, Inc., Englewood Cliffs, N.J.*, 1962. xiii+217 pp. MR0133008 (24 #A2844), Zbl 0117.34001.

[20] Jury, Michael T.; Knese, Greg; McCullough, Scott. Nevanlinna–Pick interpolation on distinguished varieties in the bidisk. *J. Funct. Anal.* **262** (2012), no. 9, 3812–3838. MR2899979, Zbl 1241.32002, arXiv:1009.4144, doi: 10.1016/j.jfa.2012.01.028.

[21] Knese, Greg. Polynomials defining distinguished varieties. *Trans. Amer. Math. Soc.* **362** (2010), no. 11, 5635–5655. MR2661491 (2011f:47022), Zbl 1205.14041, arXiv:0909.1818, doi: 10.1090/S0002-9947-2010-05275-4.

[22] Kumar, Ajay; Sahni, Niteesh; Singh Dinesh; Thukral, Virender. Factorising non-square matrix-valued analytic functions. In preparation.

[23] Lata, Sneh; Mittal, Meghna; Singh, Dinesh. A finite multiplicity Helson–Lowdenslager–de Branges theorem. *Studia Math.* **200** (2010), no. 3, 247–266. MR2733268 (2012i:47012), Zbl 1210.47021, arXiv:0910.5416, doi: 10.4064/sm200-3-3.

[24] Nikol'skiĭ, N. K. Treatise on the shift operator. Spectral function theory. Grundlehren der Mathematischen Wissenschaften, 273. *Springer-Verlag, Berlin*, 1986. xii+491 pp. ISBN: 3-540-15021-8. MR0827223 (87i:47042), Zbl 0587.47036, doi: 10.1007/978-3-642-70151-1.

[25] Nikol'skiĭ, Nikolai K. Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz. Mathematical Surveys and Monographs, 92. *American Mathematical Society, Providence, RI*, 2002. xiv+461 pp. ISBN: 0-8218-1083-9. MR1864396 (2003i:47001a), Zbl 1007.47001, doi: 10.1090/surv/092.
Nikol'skiĭ, Nikolai K. Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems. Mathematical Surveys and Monographs, 93. *American Mathematical Society, Providence, RI*, 2002. xiv+439 pp. ISBN: 0-8218-2876-2. MR1892647, Zbl 1007.47002, doi: 10.1090/surv/093.

[26] Paulsen, Vern I.; Singh, Dinesh. A Helson–Lowdwnslager–de Branges theorem in L^2. *Proc. Amer. Math. Soc.* **129** (2001), no. 4, 1097–1103. MR1814147 (2002b:46038), Zbl 1008.47009, doi: 10.1090/S0002-9939-00-05716-6.

[27] Paulsen, Vern I.; Singh, Dinesh. Modules over subalgebras of the disk algebra. *Indiana Univ. Math. J.* **55** (2006), no. 5, 1751–1766. MR2270936 (2007h:46064), Zbl 1120.46033, doi: 10.1512/iumj.2006.55.2773.

[28] Raghupathi, Mrinal. Nevanlinna–Pick interpolation for $\mathbb{C} + BH^\infty$. *Integral Equations Opererator Theory* **63** (2009), no. 1, 103–125. MR2480640 (2010f:47033), Zbl 1198.47026, arXiv:0803.1278, doi: 10.1007/s00020-008-1639-9.

[29] Rudin, Walter. Fourier analysis on groups. Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1990. x+285 pp. ISBN: 0-471-52364-X. MR1038803 (91b:43002), Zbl 0698.43001.

[30] Sahni, Niteesh; Singh, Dinesh. Multiplication by monomials on $BMOA$. *Houston J. Math.* **40** (2014), no. 3, 875–883. MR3275629, Zbl 1327.46027.

[31] Singh, Dinesh; Singh, U. N. Invariant subspaces in $VMOA$ and $BMOA$. *Michigan Math. J.* **41** (1994), no. 2, 211–218. MR1278428 (95b:46039), Zbl 0854.46024, doi: 10.1307/mmj/1029004989.

[32] Singh, Dinesh; Thukral, Virender. Multiplication by finite Blaschke factors on de Branges spaces. *J. Operator Theory* **37** (1997), no. 2, 223–245. MR1452275 (98h:46025), Zbl 0912.46027.

[33] Stegenga, David A. Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation. *Amer. J. Math.* **98** (1976), no. 3, 573–589. MR0420326 (54 #8340), Zbl 0335.47018, doi: 10.2307/2373807.

[34] Suciu, I. On the semi-groups of isometries. *Studia Math.* **30** (1968), 101-110. MR0229093 (37 #4671), Zbl 0184.15906.

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This paper is available via http://nyjm.albany.edu/j/2016/22-55.html.