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On the classification of certain inductive limits of real circle algebras

Andrew J. Dean, Dan Kucerovsky and Aydin Sarraf

Abstract. In this paper, a classification of simple unital real C^*-algebras that are inductive limits of certain real circle algebras such as $C(\mathbb{T}, M_{\frac{n}{2}}(\mathbb{H}))$ is given. The invariant consists of certain triples of real K-groups and the tracial state space of the complexification.

Contents

1. Introduction

For a *fixed* $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$, we say that a real C^*-algebra A is a *real* $A\mathbb{T}_J$*-algebra* if it is isomorphic to an inductive limit of a sequence

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots \longrightarrow A$$

where $A_i = \bigoplus_{k=1}^{m_i} A_k^j$, $j \in J$, and each A_k^j is of one of the following forms:

$$A_k^1 = C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{C})$$
$$A_k^3 = C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{\frac{n_k}{2}}(\mathbb{H})$$
$$A_k^4 = C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{R})$$
$$A_k^5 = C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{\frac{n_k}{2}}(\mathbb{H})$$

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where $C(\mathbb{T}, \eta_0) = \{f \in C(\mathbb{T}, \mathbb{C}) \mid f(\overline{z}) = \overline{f(z)}\}$.

The invariant for the classification of simple unital real $A\mathbb{T}_J$-algebras where $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$ consists of

$$\begin{array}{ccc} (K_0(A), [1_A]) & \xrightarrow{q_{\mathbb{C}}} & (K_0(A \otimes_{\mathbb{R}} \mathbb{C}), [1_{A\otimes_{\mathbb{R}}\mathbb{C}}]) \\ & & \downarrow q_{\mathbb{H}} \\ & & (K_0(A \otimes_{\mathbb{R}} \mathbb{H}) / \operatorname{Tor}(K_0(A \otimes_{\mathbb{R}} \mathbb{H})), [1_{A\otimes_{\mathbb{R}}\mathbb{H}}]) \end{array}$$

$$T(A \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{r} S(K_0(A \otimes_{\mathbb{R}} \mathbb{C}))$$

$$K_1(A)/\operatorname{Tor}(K_1(A)) \xrightarrow{\tilde{c}} K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\tilde{r}} K_1(A)/\operatorname{Tor}(K_1(A))$$

where $q_{\mathbb{C}}$, $q_{\mathbb{H}}$ are the canonical embedding maps and $\tilde{c}$, $\tilde{r}$ are defined as follows:

The complexification map $\mathfrak{c} : A \longrightarrow A \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{c}(a) = a \otimes 1$, and the realification map $\mathfrak{r} : A \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow M_2(A)$, $\mathfrak{r}(a+bi) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ induce the maps $\mathfrak{c}_* : K_1(A) \longrightarrow K_1(A \otimes_{\mathbb{R}} \mathbb{C})$, $\mathfrak{c}_*([a]) = [\mathfrak{c}(a)]$ and $\mathfrak{r}_* : K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(M_2(A)) \simeq K_1(A)$, $\mathfrak{r}_*([a]) = [\mathfrak{r}(a)]$. Since $K_1(A \otimes_{\mathbb{R}} \mathbb{C})$ is a finitely generated torsion-free abelian group, $\operatorname{Tor}(K_1(A))$ is a normal subgroup of $\operatorname{Ker}(\mathfrak{c}_*)$. We define $\tilde{c}$ as the composition of the following maps:

$$\begin{array}{ccc} K_1(A)/\operatorname{Tor}(K_1(A)) & & \\ \downarrow & & \\ K_1(A)/\operatorname{Tor}(K_1(A))/\operatorname{Ker}(\mathfrak{c}_*)/\operatorname{Tor}(K_1(A)) & & \\ \simeq \downarrow & & \\ K_1(A)/\operatorname{Ker}(\mathfrak{c}_*) & & \\ \simeq \downarrow & & \\ \operatorname{Im}(\mathfrak{c}_*) & \longrightarrow & K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

where the first map is the quotient map and the second map is inclusion. We define $\tilde{r}$ by $\tilde{r} := \pi \circ \mathfrak{r}_*$ where $\pi : K_1(A) \longrightarrow K_1(A)/\operatorname{Tor}(K_1(A))$ is the quotient map.

It is worth mentioning that the classification of real $A\mathbb{T}$-algebras (cf. Definition 2.1) is fundamentally different from the complex case in many ways. Period eight for real K-theory and the appearance of torsion are among the K-theoretical problems. Regarding regularity properties, there is a building block of stable rank greater than one and consequently a real circle algebra (cf. Definition 2.1) is not necessarily of stable rank one. Complex vector bundles over the circle are determined by their rank and their Chern class while real vector bundles over the circle are determined by their rank and Stiefel–Whitney class. The existence of a nontrivial line bundle (Möbius strip) over the circle is another difficulty. Disconnectedness of the orthogonal group in comparison with the unitary group is another obstruction. Furthermore, two of the eight basic building blocks of a real A$\mathbb{T}$-algebra have isomorphic K-groups (cf. Theorem 3.2).

2. Building blocks of real circle algebras

Definition 2.1. A complex C^*-algebra is called a complex circle algebra if it is isomorphic to a C^*-algebra of the form $C(\mathbb{T},\mathbb{C})\otimes F$ for some complex finite-dimensional C^*-algebra F. A real C^*-algebra A is called a real circle algebra if $A\otimes_{\mathbb{R}}\mathbb{C}$ is isomorphic to a complex circle algebra. An inductive limit of real circle algebras is called a real A$\mathbb{T}$-algebra.

Definition 2.2. Let A be a complex C^*-algebra. A $*$-antiautomorphism ϕ of A is a $*$-preserving $\mathbb{C}$-linear antimultiplicative bijective map from A to A. The map ϕ is called involutive if $\phi\circ\phi=id$. Moreover,

$$A_\phi=\{a\in A\mid \phi(a)=a^*\}$$

is a real C^*-algebra for which $A_\phi\cap iA_\phi=\{0\}$ and $A=A_\phi+iA_\phi$.

Theorem 2.3. *Let A be a prime complex C^*-algebra and ϕ be an involutive $*$-antiautomorphism of A. Then, A is simple if and only if A_ϕ is simple.*

Proof. Assume A_ϕ is not simple. Then, there exists a nontrivial ideal I in A_ϕ, and hence $I+iI$ is a nontrivial ideal of A. Conversely, assume that A is not simple and I is its nontrivial ideal. Then, $\phi(I)$ is also an ideal in A. Since A is prime, $J=I\cap\phi(I)$ is a nontrivial ideal of A and $\phi(J)=J$. Thus, $J_\phi=J\cap A_\phi$ is a nonzero proper ideal of A_ϕ. Therefore, A_ϕ is not simple. □

Theorem 2.4. *Let A be a complex unital C^*-algebra, $\mathfrak{L}(A)$ be the distributive complete lattice of closed ideals of A, ϕ be an involutive $*$-antiautomorphism of A,* $\operatorname{Max}(A)$ *be the set of maximal ideals of A and* $\operatorname{Prim}(A)$ *be the lattice of primitive ideals of A. Then:*

(i) *$\phi:\mathfrak{L}(A)\longrightarrow\mathfrak{L}(A)$ is an involutive lattice isomorphism.*
(ii) *ϕ induces an involutive homeomorphism of* $\operatorname{Max}(A)$*.*
(iii) *If A is separable then ϕ induces an involutive homeomorphism of* $\operatorname{Prim}(A)$*.*

Proof. (i) Obviously, ϕ takes a closed ideal to a closed ideal, preserves the ordering given by inclusion and the intersection operation. It also preserves the join, linearity of ϕ implies $\phi(I\vee J)=\phi(I+J)=\phi(I)+\phi(J)=\phi(I)\vee\phi(J)$. Therefore, ϕ is an involutive lattice isomorphism.

(ii) Let I be a maximal ideal in A. Assume that there exists a maximal ideal M such that $\phi(I)\subsetneqq M$ then $I\subsetneqq\phi(M)$ which is a contradiction. The map defined by $\widetilde{\phi}(I):=\phi(I)$ is an involutive homeomorphism of $\operatorname{Max}(A)$

because $F \subseteqq \mathrm{Max}(A)$ is closed if there exists a $M \subseteqq A$ such that $F = \mathrm{hull}(M) = \{P \in \mathrm{Max}(A) \mid M \subseteqq P\}$ and $\phi(F) = \phi^{-1}(F) = \mathrm{hull}(\phi(M))$ is a closed set.

(iii) Let $\mathbb{O}(\mathrm{Prim}(A))$ denote the lattice of open subsets of $\mathrm{Prim}(A)$. Define the lattice isomorphism map $h : \mathfrak{L}(A) \longrightarrow \mathbb{O}(\mathrm{Prim}(A))$ by

$$h(I) = U_I = \{J \in \mathrm{Prim}(A) \mid I \nsubseteq J\}.$$

Then $\widetilde{\phi} : \mathbb{O}(\mathrm{Prim}(A)) \longrightarrow \mathbb{O}(\mathrm{Prim}(A))$ defined by $\widetilde{\phi} := h \circ \phi \circ h^{-1}$ is an involutive lattice isomorphism. In particular, $\widetilde{\phi}$ preserves $U_A = \mathrm{Prim}(A)$. By [16, Corollary A.12], if A is separable then $\mathrm{Prim}(A)$ is point-complete in the sense that every closed prime (cf. [16, Definition A.1.ii]) subset is the closure of a singleton, and therefore ϕ induces an involutive homeomorphism of $\mathrm{Prim}(A)$. □

The above theorem insures the existence of the involutive homeomorphism $\tilde{\phi}$ referred to in the following theorem:

Theorem 2.5. *Let A be a unital separable complex C^*-algebra and let ϕ be an involutive $*$-antiautomorphism of A. Then, $Z(A_\phi)$ is isomorphic to the following real C^*-algebra*

$$C(X, \tilde{\phi}) = \{f \in C(X, \mathbb{C}) \mid f(\tilde{\phi}(x)) = \overline{f(x)}\}$$

where $X = \mathrm{Prim}(A)$ and $\tilde{\phi} : X \longrightarrow X$ is the involutive homeomorphism induced by ϕ.

Proof. Since $A = A_\phi + iA_\phi$, we conclude $Z(A_\phi) = (Z(A))_\phi$. By the Dauns–Hofmann Theorem, $Z(A) \simeq C(\mathrm{Prim}(A), \mathbb{C})$ and if we denote the isomorphism map by $\psi : Z(A) \longrightarrow C(\mathrm{Prim}(A), \mathbb{C})$ then $\acute{\phi} := \psi \circ \phi \circ \psi^{-1}$ is the involutive $*$-automorphism of $C(\mathrm{Prim}(A), \mathbb{C})$. By Theorem 2.4, $\acute{\phi}$ induces an involutive homeomorphism of $\mathrm{Prim}(A)$. Moreover, any maximal ideal of $C(\mathrm{Prim}(A), \mathbb{C})$ is of the form

$$I_{\tilde{\phi}(p)} = \{f \in C(\mathrm{Prim}(A), \mathbb{C}) \mid f(\tilde{\phi}(p)) = 0\}$$

for some $p \in \mathrm{Prim}(A)$ and $\acute{\phi}(I_{\tilde{\phi}(p)}) = I_{\tilde{\phi}(\tilde{\phi}(p))} = I_p$. Let e be the unit of $C(\mathrm{Prim}(A), \mathbb{C})$. Since for any $f \in C(\mathrm{Prim}(A), \mathbb{C})$ the function

$$g = f - f(\tilde{\phi}(p))e$$

vanishes at $\tilde{\phi}(p)$, $g \in I_{\tilde{\phi}(p)}$ and consequently $\acute{\phi}(g) \in \acute{\phi}(I_{\tilde{\phi}(p)}) = I_p$. Hence, $\acute{\phi}(f)(p) = f(\tilde{\phi}(p))$ and

$$\begin{aligned} Z(A_\phi) = (Z(A))_\phi &\simeq (C(\mathrm{Prim}(A), \mathbb{C}))_{\acute{\phi}} \\ &= \{f \in C(\mathrm{Prim}(A), \mathbb{C}) \mid f(\tilde{\phi}(p)) = \acute{\phi}(f)(p) = \overline{f(p)}\}. \end{aligned}$$ □

Theorem 2.6. *Let $A = C_0(X, M_n(\mathbb{C}))$ be a complex C^*-algebra where X is a locally compact Hausdorff space with Lebesgue covering dimension zero or one and let ϕ be an involutive $*$-antiautomorphism of A, then*

$$\phi(f)(x) = u_t(x)f_t(\psi(x))u_t(x)^*, f_t \in A, x \in X$$

where $f_t(x) = (f(x))^t$, t denotes the transpose, u is a unitary in $M(A)$ and ψ is an involutive homeomorphism of X. Moreover, $d(\phi(f)) = d(f \circ \psi)$ for any f in A_+ where d is a lower semicontinuous dimension function.

Proof. Define the map $T : A \longrightarrow A$ by $T(f) = f_t$ such that $f_t(x) = (f(x))^t$. Since T is an involutive $*$-antiautomorphism of A, $T \circ \phi$ is a $*$-automorphism of A. By a result of [6], the cohomology dimension of X with respect to the group $\mathbb{Z}$ is less than or equal to the covering dimension of X. Thus, $\check{H}^m(X; \mathbb{Z}) = 0$ for $m \geq 2$ and the result follows by [31, Corollary 5]. By the bijection between lower semicontinuous dimension functions and quasitraces [3, Theorem II.2.2], using the fact that quasitraces on exact C^*-algebras are traces, and the unitary invariance of traces we conclude that

$$\begin{aligned} d(\phi(f)) = d_\tau(\phi(f)) = \lim_{n\to\infty} \tau\left(\phi(f)^{\frac{1}{n}}\right) = \lim_{n\to\infty} \tau\left((f\circ\psi)^{\frac{1}{n}}\right) &= d_\tau(f \circ \psi) \\ &= d(f \circ \psi). \ \square \end{aligned}$$

Remark 2.7. In the case of the circle as a compact Hausdorff CW-complex, the Čech cohomology is naturally isomorphic to singular cohomology and it is well-known that $H^m(\mathbb{T}; \mathbb{Z}) = 0$ for $m \geq 2$.

Theorem 2.8. *Let F be a finite dimensional complex C^*-algebra and ϕ be an involutive $*$-antiautomorphism of $A = C(\mathbb{T}, F)$, then A_ϕ is of the following form:*

$$A_\phi \simeq \bigoplus_k A_k^j$$

where $j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, and

$$\begin{aligned}
A_k^1 &= C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{C}) \\
A_k^2 &= C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{R}) \\
A_k^3 &= C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{\frac{n_k}{2}}(\mathbb{H}) \\
A_k^4 &= C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{R}) \\
A_k^5 &= C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{\frac{n_k}{2}}(\mathbb{H}) \\
A_k^6 &= C(\mathbb{T}, \eta_1) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{R}) \\
A_k^7 &= \{f \in C([0,1], M_{n_k}(\mathbb{C})) \mid f(0) \in M_{n_k}(\mathbb{R}), f(1) \in M_{\frac{n_k}{2}}(\mathbb{H})\} \\
A_k^8 &= \left\{ f \in C([0,1], M_{n_k}(\mathbb{R})) \;\middle|\; f(1) = \begin{pmatrix} -1 & 0 \\ 0 & I_{n_k-1} \end{pmatrix} f(0) \begin{pmatrix} -1 & 0 \\ 0 & I_{n_k-1} \end{pmatrix} \right\}
\end{aligned}$$

where $\eta_1(z) = -z$.

Proof. It is well-known that F is isomorphic to $\bigoplus_l p_l F$ where p_l are central minimal projections of F. Therefore,

$$A = (C(\mathbb{T},\mathbb{C}) \otimes F) \simeq \left(C(\mathbb{T},\mathbb{C}) \otimes \left(\bigoplus_l p_l F \right) \right) \simeq \bigoplus_l (C(\mathbb{T},\mathbb{C}) \otimes p_l F)$$
$$\simeq \bigoplus_l e_l A$$

where $e_l = 1 \otimes p_l$ is a central minimal projection of A (since $\mathbb{T}$ is a connected compact Hausdorff space, the unit of $C(\mathbb{T},\mathbb{C})$ is the only nonzero minimal projection). Since $A \simeq \bigoplus_l e_l A \simeq \bigoplus_k (e_k + \phi(e_k))A$, we conclude

$$A_\phi \simeq \bigoplus_k ((e_k + \phi(e_k))A)_\phi$$

where ϕ on the components is defined by restriction. There are two cases to consider:

(1) If $\phi(e_k) \neq e_k$: In this case, we have
$$(e_k + \phi(e_k))A \simeq C(\mathbb{T}, M_{n_k}(\mathbb{C})) \oplus C(\mathbb{T}, M_{n_k}(\mathbb{C})).$$
Since ϕ interchanges the summands, the associated real C^*-algebra $\{(e_k a, \phi(e_k a)^*) : a \in A\}$ is isomorphic to $C(\mathbb{T}, M_{n_k}(\mathbb{C}))$. On the other hand,
$$C(\mathbb{T}, M_{n_k}(\mathbb{C})) \simeq C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{C}) \simeq C(\mathbb{T},\mathbb{C}) \otimes_{\mathbb{R}} M_{n_k}(\mathbb{R}).$$

(2) If $\phi(e_k) = e_k$: In this case, [29, Section 2] gives the other seven forms. □

Definition 2.9. For a *fixed* $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$, a real C^*-algebra A is called a *real* $A\mathbb{T}_J$*-algebra* if it is isomorphic to an inductive limit of a sequence
$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots \longrightarrow A$$
where $A_i = \bigoplus_{k=1}^{m_i} A_k^j$, $j \in J$, and the algebras A_k^j are defined in the statement of Theorem 2.8. The real C^*-algebra A is called *real* $A\mathbb{T}_1$*-algebra*, *real* $A\mathbb{T}_2$*-algebra* or *real* $A\mathbb{T}$*-algebra* if $J = \{1,2,3,4,5\}$, $J = \{1,2,3,4,5,6\}$ or $J = \{1,2,3,4,5,6,7,8\}$ respectively.

3. The existence theorem

Proposition 3.1. *For any exact real C^*-algebra A, we have*
$$K_n(C(\mathbb{T},\eta_0) \otimes_{\mathbb{R}} A) \simeq K_n(A) \oplus K_{n-1}(A),$$
$$K_n(C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} A) \simeq K_n(A) \oplus K_{n+1}(A).$$

Proof. By [27, Theorem 1.5.4], $K_n(C(\mathbb{T},\eta_0) \otimes_{\mathbb{R}} A) \simeq K_n(A) \oplus K_{n-1}(A)$.

To prove $K_n(C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} A) \simeq K_n(A) \oplus K_{n+1}(A)$, define the following sequence:
$$0 \longrightarrow C_0(\mathbb{R},\mathbb{R}) \xrightarrow{i} C(\mathbb{T},\mathbb{R}) \xrightarrow{ev} \mathbb{R} \longrightarrow 0.$$

It is known that

$$S\mathbb{R} := C_0(\mathbb{R},\mathbb{R}) \simeq C_0((0,1),\mathbb{R}) \simeq \{C([0,1],\mathbb{R}) \mid f(0) = f(1) = 0\}$$
$$\simeq \{C(\mathbb{T},\mathbb{R}) \mid f(1) = 0\}.$$

Let $h : C_0(\mathbb{R},\mathbb{R}) \longrightarrow \{C(\mathbb{T},\mathbb{R}) \mid f(1) = 0\}$ denote the isomorphism map. Define $i : C_0(\mathbb{R},\mathbb{R}) \longrightarrow C(\mathbb{T},\mathbb{R})$ by $i(f) := h(f)$, $ev : C(\mathbb{T},\mathbb{R}) \longrightarrow \mathbb{R}$ by $ev(f) := f(1)$ and $j : \mathbb{R} \longrightarrow C(\mathbb{T},\mathbb{R})$ by $j(\lambda) := \lambda e$ where e is the unit of $C(\mathbb{T},\mathbb{R})$. Since the map j satisfies $ev \circ j = id$, this is a split exact sequence. Therefore, it induces the following split exact sequences:

$$0 \longrightarrow C_0(\mathbb{R},\mathbb{R}) \otimes_{\mathbb{R}} A \longrightarrow C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} A \longrightarrow \mathbb{R} \otimes_{\mathbb{R}} A \longrightarrow 0$$
$$0 \longrightarrow K_n(C_0(\mathbb{R},\mathbb{R}) \otimes_{\mathbb{R}} A) \longrightarrow K_n(C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} A) \longrightarrow K_n(\mathbb{R} \otimes_{\mathbb{R}} A) \longrightarrow 0$$

Since $K_n(C_0(\mathbb{R},\mathbb{R}) \otimes_{\mathbb{R}} A) \simeq K_{n+1}(A)$, we conclude that

$$K_n(C(\mathbb{T},\mathbb{R}) \otimes_{\mathbb{R}} A) \simeq K_n(A) \oplus K_{n+1}(A). \qquad \square$$

Theorem 3.2. *Let F be a finite-dimensional complex C^*-algebra and ϕ be an involutory $*$-antiautomorphism of $A = C(\mathbb{T}, F)$, then the following table gives the K-groups of the building blocks of A_ϕ (cf. Theorem 2.8):*

n	0	1	2	3	4	5	6	7
$K_n(A^1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$K_n(A^2)$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$
$K_n(A^3)$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$K_n(A^4)$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$K_n(A^5)$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$
$K_n(A^6)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$K_n(A^7)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$K_n(A^8)$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$

Proof. The results for A^1 to A^5 follow from Proposition 3.1 and [17, Theorem III.5.19], and the results for A^6 to A^8 follow from [29, Section 2]. $\square$

Theorem 3.3. *Let X be a compact Hausdorff space and let τ be a topological involution of X. Denote the set of fixed points of τ by E. Then*

$$\operatorname{tsr}(C(X,\tau) \otimes_{\mathbb{R}} M_n(\mathbb{R})) = \left\lceil \frac{\max\{\lfloor \frac{\dim(X)}{2} \rfloor, \dim(E)\}}{n} \right\rceil + 1.$$

Proof. The result follows from [23, Theorem 5.9] and the proof of [26, Theorem 6.1]. $\square$

Corollary 3.4. *Let A^i denote the building block of a real $A\mathbb{T}_2$-algebra. Then,* $\operatorname{tsr}(A^2) = 2$ *and* $\operatorname{tsr}(A^i) = 1$ *for* $i \in \{1, 3, 4, 5, 6\}$.

Proof. For $\tau = \eta_i$ where $i \in \{0, 1\}$, we have $\dim(E_{\eta_i}) = 0$ and clearly $\dim(E_{id}) = \dim(\mathbb{T}) = 1$. The result follows from the vector space isomorphism $M_{\frac{n}{2}}(\mathbb{H}) \simeq M_n(\mathbb{R})$. $\square$

Proposition 3.5. *The real C^*-algebra $C(\mathbb{T}, \eta_0)$ is singly generated by the function $g_0(z) = z$. The real C^*-algebras $C(\mathbb{T}, \mathbb{R})$ and $C(\mathbb{T}, \eta_1)$ are generated by two functions $g_1(z) = \mathrm{Re}(z)$, $g_2(z) = \mathrm{Im}(z)$ and $g_3(z) = i\,\mathrm{Re}(z)$, $g_4(z) = i\,\mathrm{Im}(z)$ respectively.*

Proof. The bivariate polynomial ring $\mathbb{R}[z, \overline{z}]$ is dense in $C(\mathbb{T}, \eta_0)$ by the real version of the Stone-Weierstrass theorem because it separates the points of $\mathbb{T}$. Similarly, $\mathbb{R}[i(\frac{z+\overline{z}}{2}), \frac{z-\overline{z}}{2}]$ is dense in $C(\mathbb{T}, \eta_1)$ and $\mathbb{R}[\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}]$ is dense in $C(\mathbb{T}, \mathbb{R})$. □

Theorem 3.6. *Let A^j denote the basic building block of a real $A\mathbb{T}_2$-algebra where $j \in \{1, \ldots, 6\}$ and $\mathbb{T}_+ := \{e^{i\theta} | 0 \le \theta \le \pi\}$ be the upper half-circle. Then, the following hold:*

(i) $$\mathrm{Aff}(T(A^j)) \simeq \mathrm{Aff}(M_1(\mathbb{T})) \simeq C(\partial_e M_1(\mathbb{T}), \mathbb{R}) \simeq C(\mathbb{T}, \mathbb{R})$$

for $j \in \{1, 2, 3, 6\}$.

(ii) $$\mathrm{Aff}(T(A^j)) \simeq \mathrm{Aff}(M_1(\mathbb{T}_+)) \simeq C(\partial_e M_1(\mathbb{T}_+), \mathbb{R}) \simeq C(\mathbb{T}_+, \mathbb{R})$$

for $j \in \{4, 5\}$.

(iii) $$\mathrm{Aff}(T(A^j \otimes_{\mathbb{R}} \mathbb{C})) \simeq \mathrm{Aff}(M_1(\mathbb{T})) \simeq C(\partial_e M_1(\mathbb{T}), \mathbb{R}) \simeq C(\mathbb{T}, \mathbb{R})$$

for $j \in \{2, \ldots, 6\}$.

(iv) $$\begin{aligned}\mathrm{Aff}(T(A^1 \otimes_{\mathbb{R}} \mathbb{C})) &\simeq \mathrm{Aff}(M_1(\mathbb{T})) \oplus \mathrm{Aff}(M_1(\mathbb{T})) \\ &\simeq C(\partial_e M_1(\mathbb{T}), \mathbb{R}) \oplus C(\partial_e M_1(\mathbb{T}), \mathbb{R}) \\ &\simeq C(\mathbb{T}, \mathbb{R}) \oplus C(\mathbb{T}, \mathbb{R}).\end{aligned}$$

Proof. The proof follows from the above theorem and the identifications

$$C(\mathbb{T}, \eta_0) \simeq \{f \in C(\mathbb{T}_+, \mathbb{C}) \mid f(\pm 1) \in \mathbb{R}\},$$
$$C(\mathbb{T}, \eta_1) \simeq \{f \in C(\mathbb{T}_+, \mathbb{C}) \mid f(-1) = \overline{f(1)}\},$$

together with the fact that states and traces are defined to be zero on the skew-adjoint elements of a real C^*-algebra (cf. [14]). □

Theorem 3.7. *Let $A = C(\mathbb{T}, \mathbb{R})$, let $\theta_1, \theta_2 \in \{id, \eta_0\}$ be homeomorphisms of $\mathbb{T}$, let $\hat{\phi}_1, \hat{\phi}_2$ be the associated involutions of A, i.e., $\hat{\phi}_i(f) = f \circ \theta_i$, and let $M : A \longrightarrow A$ be a Markov operator with $M\hat{\phi}_1 = \hat{\phi}_2 M$. Given $\epsilon > 0$ and a finite subset F of $C(\mathbb{T}, \mathbb{R})$, there exist $N > 0$ and continuous functions $\mu_1, \ldots, \mu_{2N}$ from $\mathbb{T}$ to $\mathbb{T}$ with $\mu_i \theta_2 = \theta_1 \mu_{2N+1-i}$ for each i such that*

$$\left\| M(f) - \frac{1}{2N} \sum_{i=1}^{2N} f \circ \mu_i \right\| < \epsilon$$

for all $f \in F$.

Proof. We just point out the important modifications to the proof of [21, Theorem 2.1]. The proof is divided into four cases:

(1) If $\theta_1 = \theta_2 = id$ then we can define $\mu_{2N+1-i} = \mu_i$ for $1 \leq i \leq N$ and the result follows from [21, Theorem 2.1].

(2) If $\theta_1 = id$ and $\theta_2 = \eta_0$ then $M(f)(z) = M(f)(\bar{z})$. Let $\mu_i : \mathbb{T}_+ \longrightarrow \mathbb{T}$ be the continuous map of [21, Theorem 2.1], we can extend μ_i by $(\mu_i)_{|\mathbb{T}_-}(z) = \mu_i(\bar{z})$ and we define $\mu_{2N+1-i} = \mu_i \circ \eta_0$ for $1 \leq i \leq N$.

(3) If $\theta_1 = \eta_0$ and $\theta_2 = id$ then $M(f) = M(f \circ \eta_0)$ which implies that M is a map from $C(\mathbb{T}_+, \mathbb{R})$ to $C(\mathbb{T}, \mathbb{R})$ and [21, Theorem 2.1] is not applicable to $C(\mathbb{T}_+, \mathbb{R})$. However, since

$$M(f) = M(f \circ \eta_0) = M\left(\frac{1}{2}f + \frac{1}{2}f \circ \eta_0\right),$$

we can apply [21, Theorem 2.1] to the elements $\frac{1}{2}f + \frac{1}{2}f \circ \eta_0$ of $C(\mathbb{T}, \mathbb{R})$ by considering the finite set $\{f, f \circ \eta_0 : f \in F\}$ in [21, Theorem 2.1]. Therefore, $M(f)$ can be approximated by

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{2}f + \frac{1}{2}f \circ \eta_0\right) \circ \mu_i$$

where $\mu_i : \mathbb{T} \longrightarrow \mathbb{T}_+$ and we define $\mu_{2N+1-i} = \eta_0 \circ \mu_i$ for $1 \leq i \leq N$.

(4) If $\theta_1 = \eta_0$ and $\theta_2 = \eta_0$ then we can proceed as follows:

For any $\epsilon > 0$, there is a $\delta_1 > 0$ such that for $x_1, x_2 \in X = \mathbb{T}_+$, $d(x_1, x_2) < \delta_1$ implies that $|f(x_1) - f(x_2)| < \frac{\epsilon}{3}$ for all $f \in F$. Choose a finite subset $\{x_1, \ldots, x_m\} \subset X$ which is δ_1-dense in X and $x_i \notin \{-1, 1\}$ for all $1 \leq i \leq m$. Choose a partition of X, denoting it by $\{X_1, X_2, \ldots, X_m\}$, such that X_1 contains 1, X_m contains -1 and with each X_i being a Borel set, satisfying:

(a) $x_i \in X_i$ for $i = 1, \ldots, m$;
(b) $X = \cup_{i=1}^m X_i$, $X_i \cap X_j = \varnothing$ for $i \neq j$;
(c) $d(x, x_i) < \delta_1$ if $x \in X_i$.

We extend this partition to $\mathbb{T}$ by $\tilde{X}_i = X_i$ for $2 \leq i \leq m-1$, $\tilde{X}_i = \eta_0(X_{2m-i})$ for $m+1 \leq i \leq 2m-2$, $\tilde{X}_m = X_m \cup \eta_0(X_m)$ and $\tilde{X}_1 = X_1 \cup \eta_0(X_1)$.

Therefore,

(a) $x_i \in \tilde{X}_i$ for $i = 2, \ldots, m-1$; $\eta_0(x_{2m-i}) \in \tilde{X}_i$ for $i = m+1, \ldots, 2m-2$; $x_1, \eta_0(x_1) \in \tilde{X}_1$ and $x_m, \eta_0(x_m) \in \tilde{X}_m$;
(b) $\mathbb{T} = \cup_{i=1}^{2m-2} \tilde{X}_i$, $\tilde{X}_i \cap \tilde{X}_j = \varnothing$ for $i \neq j$;
(c) $d(x, \tilde{x}_i) < \delta_1$ if $x \in \tilde{X}_i$ where $\tilde{x}_i = x_i$ for $i = 2, \ldots, m-1$, $\tilde{x}_i = \eta_0(x_{2m-i})$ for $i = m+1, \ldots, 2m-2$, $d(x, y) < 2\delta_1$ if $x \in \tilde{X}_1$, $y \in \{x_1, \eta_0(x_1)\}$ and $d(x, y) < 2\delta_1$ if $x \in \tilde{X}_m$ and $y \in \{x_m, \eta_0(x_m)\}$.

We proceed as on page 62 of [21] by picking the point $x_0 = 1$ and an integer $N > 0$ satisfying $\frac{1}{4N} < \delta^2$. Since $\mathbb{T}_+$ is path connected, there are maps $\beta_j : [0,1] \longrightarrow \mathbb{T}_+$ where $j = 1, \ldots, m$ such that $\beta_j(0) = x_0$ and $\beta_j(1) = x_j$. For $j = m+1, \ldots, 2m$, we define $\beta_j(t) = \eta_0(\beta_{2m-j+1}(t))$. The last paragraph on page 62 of [21] needs to be changed as well. We cover $Y = \mathbb{T}_+$ with $\{V_j\}_{j=1}^R$ such that 1 only belongs to V_1 and -1 only belongs to V_R and $y_j \in V_j$ such that

$$\left| M(f)(y) - \sum_{i=1}^{m} \lambda_{iy_i} f(x_i) \right| < \frac{\epsilon}{3}$$

for all $y \in V_j$ and $f \in F$.

Let $\{h_1, \ldots, h_R\}$ be a partition of unity subordinate to the cover $\{V_j\}_{j=1}^R$ such that $h_1(1) = h_R(-1) = 1$.

We extend this cover to $\mathbb{T}$ by defining $\tilde{V}_j = V_j$ for $2 \le j \le R-1$, $\tilde{V}_j = \eta_0(V_{2R-j})$ for $R+1 \le j \le 2R-2$, $\tilde{V}_R = V_R \cup \eta_0(V_R)$ and $\tilde{V}_1 = V_1 \cup \eta_0(V_1)$. We define $h_j = h_{2R-j} \circ \eta_0$ for $R+1 \le j \le 2R-2$, $h_1 = h_1 \circ \eta_0$ and $h_R = h_R \circ \eta_0$. On page 63 of [21], we can choose λ_i such that $\lambda_i(\eta_0(y)) = \lambda_{2m-i+1}(y)$ for $i = 1, \ldots, 2m$ and consequently $1 - G_{2m-j+1}(\eta_0(y)) = G_{j-1}(y)$ for $j = 1, \ldots, 2m$. Therefore, $G_{2m-j}(\eta_0(y)) < 1 - t < G_{2m-j+1}(\eta_0(y))$ if and only if $G_{j-1}(y) < t < G_j(y)$. Hence, α_j which is defined on page 64 of [21] satisfies $\alpha_j(y,t) = \alpha_{2m-j+1}(\eta_0(y), 1-t)$. We use the Greek letter μ for the map h which is defined on page 64 of [21]. It follows that

$$\begin{aligned}
\mu_i(\eta_0(y)) &= \beta_{2m-j+1}\left(\alpha_{2m-j+1}\left(\eta_0(y), \frac{2i-1}{4N}\right)\right) \\
&= \beta_{2m-j+1}\left(\alpha_j\left(y, 1 - \frac{2i-1}{4N}\right)\right) \\
&= \eta_0\left(\beta_j\left(\alpha_j\left(y, 1 - \frac{2i-1}{4N} = \frac{2(2N+1-i)-1}{4N}\right)\right)\right) \\
&= \eta_0(\mu_{2N+1-i})(y)
\end{aligned}$$

We can complete the proof as on pages 64–66 of [21]. □

Lemma 3.8. *Let $\mu_1, \mu_2 : \mathbb{T} \longrightarrow \mathbb{T}$ be continuous and let $\theta_1, \theta_2 \in \{id, \eta_0, \eta_1\}$ such that $\mu_1\theta_2 = \theta_1\mu_2$. Then, there exists a $*$-homomorphism*

$$\psi : C(\mathbb{T}, \mathbb{C}) \longrightarrow C(\mathbb{T}, \mathbb{C}) \otimes M_2(\mathbb{C})$$

such that $\psi \circ \phi_1 = T \circ \phi_2 \circ \psi$ where $\phi_i(f) = f \circ \theta_i$ and $T(f) = f^t$ where t denotes the transpose.

Proof. As in [30, Lemma 4.2], we can define $\psi(f) = W \operatorname{diag}(f \circ \mu_1, f \circ \mu_2) W^*$ where $W = 1 \otimes \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & -i \\ 1 & 1 \end{smallmatrix}\right)$ is a unitary element of $C(\mathbb{T}, \mathbb{C}) \otimes M_2(\mathbb{C})$. □

Theorem 3.9. *If $A = C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_n(\mathbb{R})$ and $p \in A$ is a projection of rank k then $pAp \otimes_{\mathbb{R}} M_2(\mathbb{R}) \simeq C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{2k}(\mathbb{R})$.*

Proof. By classification of vector bundles, $\mathrm{Vect}^1_{\mathbb{R}}(\mathbb{T}) \simeq H^1(\mathbb{T}; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Therefore, there are two real line bundles over the circle up to isomorphism, i.e., the trivial line bundle and the Möbius strip. Since $\mathrm{Vect}^2_{\mathbb{R}}(\mathbb{T}) \simeq \pi_0(SO(2, \mathbb{R})) = 0$, the Whitney sum of two Möbius line bundle is a trivial bundle of rank 2. On the other hand, there is a one-to-one correspondence between the isomorphism classes of real vector bundles over the space X and the Murray–von Neumann equivalence classes of projections in $C(X, K(H))$ where H is a real Hilbert space. Thus, it follows that the direct sum of two Möbius projections is Murray–von Neumann equivalent to a trivial projection. If p is a trivial projection then $pAp \simeq C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_k(\mathbb{R})$ and consequently $pAp \otimes_{\mathbb{R}} M_2(\mathbb{R}) \simeq C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{2k}(\mathbb{R})$. If p is the Möbius projection, then

$$\begin{aligned} pAp \otimes_{\mathbb{R}} M_2(\mathbb{R}) &\simeq (p \otimes I_2)(A \otimes_{\mathbb{R}} M_2(\mathbb{R}))(p \otimes I_2) \\ &\simeq I_{2k}(C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{2n}(\mathbb{R}))I_{2k} \\ &\simeq C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_{2k}(\mathbb{R}) \end{aligned}$$

where we used the fact that for a (complex or real) C^*-algebra A, if $p \sim q$ then $pAp \simeq qAq$. □

Remark 3.10. Let A be a real C^*-algebra. The order structure of

$$K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \oplus K_1(A \otimes_{\mathbb{R}} \mathbb{C})$$

is determined by the order structure in $K_0(A \otimes_{\mathbb{R}} \mathbb{C})$ together with the ideal structure of $K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \oplus K_1(A \otimes_{\mathbb{R}} \mathbb{C})$ and this is determined by the map $\alpha(I_0) = I_1$ associating to each ideal I_0 of $K_0(A \otimes_{\mathbb{R}} \mathbb{C})$ the unique subgroup I_1 of $K_1(A \otimes_{\mathbb{R}} \mathbb{C})$ such that $I = I_0 \oplus I_1$ is an ideal of $K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \oplus K_1(A \otimes_{\mathbb{R}} \mathbb{C})$ (cf. [11, 4.27]).

Theorem 3.11. *For a fixed $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$, let $A = \oplus_{i=1}^r A_i$ and $B = \oplus_{j=1}^s B_j$ where A_i and B_j are the building blocks of a real $A\mathbb{T}_J$-algebra. Let $T(A \otimes_{\mathbb{R}} \mathbb{C})$ and $T(B \otimes_{\mathbb{R}} \mathbb{C})$ be the tracial state spaces with involutions ϕ_A^*, ϕ_B^* defined by $\phi_A^*(\tau) = \tau \circ \phi_A$ and $\phi_B^*(\tau) = \tau \circ \phi_B$ where ϕ_A and ϕ_B are the involutive $*$-antiautomorphisms of $A \otimes_{\mathbb{R}} \mathbb{C}$ and $B \otimes_{\mathbb{R}} \mathbb{C}$. Let $\epsilon > 0$, let F be a finite subset of $\mathrm{Aff}(T(A \otimes_{\mathbb{R}} \mathbb{C}))$, and let*

$$M : \mathrm{Aff}(T(A \otimes_{\mathbb{R}} \mathbb{C})) \longrightarrow \mathrm{Aff}(T(B \otimes_{\mathbb{R}} \mathbb{C}))$$

be a Markov operator with $M\acute{\phi}_A = \acute{\phi}_B M$ where $\acute{\phi}_A$ and $\acute{\phi}_B$ are defined by $\acute{\phi}_A(g) = g \circ \phi_A^*$ and $\acute{\phi}_B(g) = g \circ \phi_B^*$. Let

$$\rho_A : K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \mathrm{Aff}(T(A \otimes_{\mathbb{R}} \mathbb{C})),$$
$$\rho_B : K_0(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \mathrm{Aff}(T(B \otimes_{\mathbb{R}} \mathbb{C})),$$

be the canonical maps defined by $\rho_A([p]) = r_A([p])$ and $\rho_B([p]) = r_B([p])$, where

$$r_A : T(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow S(K_0(A \otimes_{\mathbb{R}} \mathbb{C})),$$
$$r_B : T(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow S(K_0(B \otimes_{\mathbb{R}} \mathbb{C})),$$

are defined by $r_A([p])(\tau) = \tau([p])$ and $r_B([p])(\tau) = \tau([p])$. Suppose given order unit preserving positive group homomorphisms

$$h_0 : K_0(A) \longrightarrow K_0(B),$$

$$h_0^{\mathbb{C}}: K_0(A\otimes_{\mathbb{R}}\mathbb{C}) \longrightarrow K_0(B\otimes_{\mathbb{R}}\mathbb{C}),$$
$$h_0^{\mathbb{H}}: K_0(A\otimes_{\mathbb{R}}\mathbb{H})/\operatorname{Tor}(K_0(A\otimes_{\mathbb{R}}\mathbb{H})) \longrightarrow K_0(B\otimes_{\mathbb{R}}\mathbb{H})/\operatorname{Tor}(K_0(B\otimes_{\mathbb{R}}\mathbb{H}))$$

as well as a group homomorphism

$$h_1: K_1(A)/\operatorname{Tor}(K_1(A)) \longrightarrow K_1(B)/\operatorname{Tor}(K_1(B))$$

and a group homomorphism

$$h_1^{\mathbb{C}}: K_1(A\otimes_{\mathbb{R}}\mathbb{C}) \longrightarrow K_1(B\otimes_{\mathbb{R}}\mathbb{C})$$

that is compatible with $h_0^{\mathbb{C}}$ *in the sense of preserving the subgroups associated with the ideals of* K_0 *of complexification (see Remark* 3.10*), and suppose that the following diagrams commute:*

$$\begin{array}{ccccc} (K_0(A),[1_A]) & \xrightarrow{q_{\mathbb{C}}} & (K_0(A\otimes_{\mathbb{R}}\mathbb{C}),[1_{A\otimes_{\mathbb{R}}\mathbb{C}}]) & \xrightarrow{q_{\mathbb{H}}} & (K_0(A\otimes_{\mathbb{R}}\mathbb{H})/\operatorname{Tor}(K_0(A\otimes_{\mathbb{R}}\mathbb{H})),[1_{A\otimes_{\mathbb{R}}\mathbb{H}}]) \\ \downarrow h_0 & & \downarrow h_0^{\mathbb{C}} & & \downarrow h_0^{\mathbb{H}} \\ (K_0(B),[1_B]) & \xrightarrow{q_{\mathbb{C}}} & (K_0(B\otimes_{\mathbb{R}}\mathbb{C}),[1_{B\otimes_{\mathbb{R}}\mathbb{C}}]) & \xrightarrow{q_{\mathbb{H}}} & (K_0(B\otimes_{\mathbb{R}}\mathbb{H})/\operatorname{Tor}(K_0(B\otimes_{\mathbb{R}}\mathbb{H})),[1_{B\otimes_{\mathbb{R}}\mathbb{H}}]) \end{array}$$

$$\begin{array}{ccc} K_0(A\otimes_{\mathbb{R}}\mathbb{C}) & \xrightarrow{\rho_A} & \operatorname{Aff}(T(A\otimes_{\mathbb{R}}\mathbb{C})) \\ \downarrow h_0^{\mathbb{C}} & & \downarrow M \\ K_0(B\otimes_{\mathbb{R}}\mathbb{C}) & \xrightarrow{\rho_B} & \operatorname{Aff}(T(B\otimes_{\mathbb{R}}\mathbb{C})) \end{array}$$

$$\begin{array}{ccccc} K_1(A)/\operatorname{Tor}(K_1(A)) & \xrightarrow{\tilde{c}_A} & K_1(A\otimes_{\mathbb{R}}\mathbb{C}) & \xrightarrow{\tilde{r}_A} & K_1(A)/\operatorname{Tor}(K_1(A)) \\ \downarrow h_1 & & \downarrow h_1^{\mathbb{C}} & & \downarrow h_1 \\ K_1(B)/\operatorname{Tor}(K_1(B)) & \xrightarrow{\tilde{c}_B} & K_1(B\otimes_{\mathbb{R}}\mathbb{C}) & \xrightarrow{\tilde{r}_B} & K_1(B)/\operatorname{Tor}(K_1(B)) \end{array}$$

where $q_{\mathbb{C}}$*,* $q_{\mathbb{H}}$ *are the canonical induced maps, i.e.,* $q_{\mathbb{C}}([a]) = [a\otimes 1]$ *and* $q_{\mathbb{H}}([a\otimes(n+mi)]) = [a\otimes(n+mi+0j+0k)]$.

Then, there exists a $T\in\mathbb{N}$ *such that for each set* $\{r_1,\dots,r_R\}$ *of integers with* $2r_j \geq T$ *for each* j*, there is a unital* $*$*-homomorphism*

$$\lambda: A \longrightarrow B\otimes_{\mathbb{R}} H$$

where $H = M_{2r_1}(\mathbb{R})\oplus M_{2r_2}(\mathbb{R})\cdots\oplus M_{2r_R}(\mathbb{R})$*, such that* $\lambda_* = d_*\circ h_0$ *on* $K_0(A)$*,* $\lambda_*^{\mathbb{C}} = d_*\circ h_0^{\mathbb{C}}$ *on* $K_0(A\otimes_{\mathbb{R}}\mathbb{C})$*,* $\lambda_*^{\mathbb{H}} = d_*\circ h_0^{\mathbb{H}}$ *on* $K_0(A\otimes_{\mathbb{R}}\mathbb{H})$*,* $\lambda_* = d_*\circ h_1$ *on* $K_1(A)/\operatorname{Tor}(K_1(A))$*,* $\lambda_*^{\mathbb{C}} = d_*\circ h_1^{\mathbb{C}}$ *on* $K_1(A\otimes_{\mathbb{R}}\mathbb{C})$ *and*

$$\|\hat{\lambda}^{\mathbb{C}}(f) - \hat{d}^{\mathbb{C}}\circ M(f)\| < \epsilon$$

for all $f\in F$ *where for* $\tau\in T(B\otimes_{\mathbb{R}} H\otimes_{\mathbb{R}}\mathbb{C})$*,* $\hat{\lambda}^{\mathbb{C}}(f)(\tau) = f(\tau\circ\lambda_i^{\mathbb{C}})$*, and* d_* *arises from the diagonal embedding* $d: B\longrightarrow B\otimes_{\mathbb{R}} H$ *defined by* $d(b) = b\otimes 1_H$.

Proof. Let $\pi_j : B \longrightarrow B_j$ be the projection map and $id_i : A_i \longrightarrow A$ be the i^{th} coordinate embedding. If 1_i is the unit of A_i then $\pi_{j_*} \circ h_0 \circ id_{i_*}([1_i]) = [p_i]$ where p_i is a projection in $P_\infty(B_j)$. Since $\pi_{j_*} \circ h_0$ is unital we have

$$\begin{aligned}[1_{B_j}] = \pi_{j_*} \circ h_0([1_A]) = \pi_{j_*} \circ h_0([\oplus_{i=1}^r 1_i]) &= \sum_{i=1}^r \pi_{j_*} \circ h_0 \circ id_{i_*}[1_i] \\ &= \sum_{i=1}^r [p_i] = [\oplus_{i=1}^r p_i].\end{aligned}$$

Thus, $1_{B_j} \sim \oplus_{i=1}^r p_i$ and by [22, Lemma 3.4.2] there exist mutually orthogonal projections $\{q_i\}_{i=1}^r$ such that $\sum_{i=1}^r q_i = 1_{B_j}$ and $q_i \sim p_i$ for all $i \in \{1, \dots, r\}$. Hence, $\pi_{j_*} \circ h_0 \circ id_{i_*}[1_i] = [q_i]$. We can replace A by A_i and B by $q_i B_j q_i$ to reduce the problem to a single building block. Let

$$\alpha_0^{ij} : K_0(B_j) \longrightarrow K_0(q_i B_j q_i),$$
$$\alpha_0^{\mathbb{C}^{ij}} : K_0(B_j \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C}),$$

$$\begin{aligned}\alpha_0^{\mathbb{H}^{ij}} : K_0(B_j \otimes_{\mathbb{R}} \mathbb{H}) / \operatorname{Tor}(K_0(B_j \otimes_{\mathbb{R}} \mathbb{H})) \\ \longrightarrow K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{H}) / \operatorname{Tor}(K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{H})),\end{aligned}$$

be order unit preserving group homomorphisms,

$$\alpha_1^{ij} : K_1(B_j) / \operatorname{Tor}(K_1(B_j)) \longrightarrow K_1(q_i B_j q_i) / \operatorname{Tor}(K_1(q_i B_j q_i)),$$
$$\alpha_1^{\mathbb{C}^{ij}} : K_1(B_j \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C})$$

be group homomorphisms and let

$$\widehat{\alpha^{ij}} : \operatorname{Aff}(T(B_j \otimes_{\mathbb{R}} \mathbb{C})) \longrightarrow \operatorname{Aff}(T(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C}))$$

and $\gamma : K_0(B_j) \longrightarrow \mathbb{Z}$ be the canonical isomorphism maps. Then, we define the appropriate maps

$$\begin{aligned}
h_0^{ij} &:= \alpha_0^{ij} \circ \pi_{j_*} \circ h_0 \circ id_{i_*} : K_0(A_i) \longrightarrow K_0(q_i B_j q_i) \\
h_0^{\mathbb{C}^{ij}} &:= \alpha_0^{\mathbb{C}^{ij}} \circ \pi_{j_*} \circ h_0^{\mathbb{C}} \circ id_{i_*} : K_0(A_i \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C}) \\
h_0^{\mathbb{H}^{ij}} &:= \alpha_0^{\mathbb{H}^{ij}} \circ \pi_{j_*} \circ h_0^{\mathbb{H}} \circ id_{i_*} : K_0(A_i \otimes_{\mathbb{R}} \mathbb{H}) / \operatorname{Tor}(K_0(A_i \otimes_{\mathbb{R}} \mathbb{H})) \longrightarrow \\
&\qquad\qquad K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{H}) / \operatorname{Tor}(K_0(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{H})) \\
h_1^{ij} &:= \alpha_1^{ij} \circ \pi_{j_*} \circ h_1 \circ id_{i_*} : K_1(A_i) / \operatorname{Tor}(K_1(A_i)) \longrightarrow \\
&\qquad\qquad K_1(q_i B_j q_i) / \operatorname{Tor}(K_1(q_i B_j q_i)) \\
h_1^{\mathbb{C}^{ij}} &:= \alpha_1^{\mathbb{C}^{ij}} \circ \pi_{j_*} \circ h_1^{\mathbb{C}} \circ id_{i_*} : K_1(A_i \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C}) \\
M^{ij} &:= \frac{\gamma([1_j])}{\gamma([q_i])} \widehat{\alpha^{ij}} \circ \widehat{\pi_j} \circ M \circ \widehat{id_i} : \operatorname{Aff}(T(A_i \otimes_{\mathbb{R}} \mathbb{C})) \longrightarrow \\
&\qquad\qquad \operatorname{Aff}(T(q_i B_j q_i \otimes_{\mathbb{R}} \mathbb{C})).
\end{aligned}$$

Case 1. Assume that A_i and $q_iB_jq_i$ both are not of type 1, i.e., they are not of the form $C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}} M_n(\mathbb{C})$. Since the Markov map

$$M^{ij}:(\mathrm{Aff}(T(A_i\otimes_{\mathbb{R}}\mathbb{C})),\acute{\phi}^{i}_{A})\longrightarrow(\mathrm{Aff}(T(q_iB_jq_i\otimes_{\mathbb{R}}\mathbb{C})),\acute{\phi}^{j}_{B})$$

has the property $M^{ij}\acute{\phi}^{i}_{A}=\acute{\phi}^{j}_{B}M^{ij}$ where

$$\acute{\phi}^{i}_{A}=\widehat{\pi}_i\circ\acute{\phi}_A\circ\widehat{id}_i,$$
$$\acute{\phi}^{j}_{B}=\widehat{\pi}_j\circ\acute{\phi}_B\circ\widehat{id}_j\circ\widehat{\alpha^{ij}}^{-1},$$

if we denote the isomorphism maps (as order unit spaces) by

$$\psi_A:(\mathrm{Aff}(T(A_i\otimes_{\mathbb{R}}\mathbb{C})),\acute{\phi}^{i}_{A})\xrightarrow{\cong}(C(\mathbb{T},\mathbb{R}),\tilde{\phi}^{i}_{A}),$$
$$\psi_B:(\mathrm{Aff}(T(q_iB_jq_i\otimes_{\mathbb{R}}\mathbb{C})),\acute{\phi}^{j}_{B})\xrightarrow{\cong}(C(\mathbb{T},\mathbb{R}),\tilde{\phi}^{j}_{B}),$$

then we get the Markov map $\tilde{M}^{ij}:(C(\mathbb{T},\mathbb{R}),\tilde{\phi}^{i}_{A})\longrightarrow(C(\mathbb{T},\mathbb{R}),\tilde{\phi}^{j}_{B})$ defined by $\tilde{M}^{ij}:=\psi_B\circ M^{ij}\circ\psi_A^{-1}$ and we have $\tilde{M}^{ij}\tilde{\phi}^{i}_{A}=\tilde{\phi}^{j}_{B}\tilde{M}^{ij}$ where the involutions $\tilde{\phi}^{i}_{A}$ and $\tilde{\phi}^{j}_{B}$ are defined by $\tilde{\phi}^{i}_{A}=\psi_A\circ\acute{\phi}^{i}_{A}\circ\psi_A^{-1}$ and $\tilde{\phi}^{j}_{B}=\psi_B\circ\acute{\phi}^{j}_{B}\circ\psi_B^{-1}$. We define the relative finite set $\tilde{F}^{ij}:=\{f\circ\psi_A^{-1}\circ\widehat{id}_i\in(C(\mathbb{T},\mathbb{R}),\tilde{\phi}^{i}_{A})|f\in F\}$. The involutions $\tilde{\phi}^{i}_{A}$ and $\tilde{\phi}^{j}_{B}$ are of the form $\tilde{\phi}(f)=f\circ\theta$ where $\theta\in\{id,\eta_0\}$. Therefore, for δ by Theorem 3.7 there exist $N_{ij}>0$ and continuous functions $\tilde{\mu}_1,\ldots,\tilde{\mu}_{2N_{ij}}$ from $\mathbb{T}$ to $\mathbb{T}$ with $\tilde{\mu}_k\theta_2=\theta_1\tilde{\mu}_{2N+1-k}$ for each k such that

$$\left\|\tilde{M}^{ij}(f)-\frac{1}{2N_{ij}}\sum_{k=1}^{2N_{ij}}f\circ\tilde{\mu}_k\right\|<\delta$$

for all $f\in\tilde{F}^{ij}$. For $1\leq l\leq N_{ij}$, let

$$\psi^{ij}_{l}:(C(\mathbb{T},\mathbb{C}))_{\tilde{\phi}^{i}_{A}}\longrightarrow(C(\mathbb{T},\mathbb{C}))_{\tilde{\phi}^{j}_{B}}\otimes_{\mathbb{R}}M_2(\mathbb{R})$$

be the $*$-homomorphisms of Lemma 3.8. Let D_{A_i} be the triple

$$\begin{array}{ccc}(K_0(A_i),[1_{A_i}]) & \longrightarrow & (K_0(A_i\otimes_{\mathbb{R}}\mathbb{C}),[1_{A_i\otimes_{\mathbb{R}}\mathbb{C}}]) \\ & & \big\downarrow \\ & & (K_0(A_i\otimes_{\mathbb{R}}\mathbb{H})/\,\mathrm{Tor}(K_0(A_i\otimes_{\mathbb{R}}\mathbb{H})),[1_{A_i\otimes_{\mathbb{R}}\mathbb{H}}]).\end{array}$$

We define $D_{q_iB_jq_i}$ similarly. Here,

$$A_i=C(\mathbb{T},\eta_i)\otimes_{\mathbb{R}}M_{n_i}(\mathbb{F}_i),$$
$$q_iB_jq_i=C(\mathbb{T},\eta_j)\otimes_{\mathbb{R}}M_{n_j}(\mathbb{F}_j),$$

where $n_j=\mathrm{rank}(q_i)$, $\mathbb{F}_i,\mathbb{F}_j\in\{\mathbb{R},\mathbb{H}\}$, and $\eta_i,\eta_j\in\{\eta_0,id\}$.

Since $D_{A_i} \simeq D_{M_{n_i}(\mathbb{F}_i)}$ and $D_{q_iB_jq_i} \simeq D_{M_{n_j}(\mathbb{F}_j)}$, it follows from [28, Theorem 2.4] or [15, Theorem 14.1] that the homomorphism

$$\sigma : D_{M_{n_i}(\mathbb{F}_i)} \longrightarrow D_{M_{n_j}(\mathbb{F}_j)}$$

induces a standard $*$-homomorphism $\beta^{ij} : M_{n_i}(\mathbb{F}_i) \longrightarrow M_{n_j}(\mathbb{F}_j)$.

Therefore, we get a family of unital $*$-homomorphisms

$$\lambda_l^{ij} : (C(\mathbb{T},\mathbb{C}))_{\tilde{\phi}_A^i} \otimes_{\mathbb{R}} M_{n_i}(\mathbb{F}_i) \longrightarrow (C(\mathbb{T},\mathbb{C}))_{\tilde{\phi}_B^j} \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F}_j) \otimes_{\mathbb{R}} M_2(\mathbb{R})$$

where λ_l^{ij} is defined by $\lambda_l^{ij} := \psi_l^{ij} \otimes \beta^{ij}$ for $1 \leq l \leq N_{ij}$.

Let $\tilde{d}_*$ be the induced map from diagonal embedding in $M_2(\mathbb{R})$. Since $\operatorname{rank}(\psi_l^{ij}(p)) = 2\operatorname{rank}(p)$, it follows from [13, Theorem 8.3] that $(\psi_l^{ij} \otimes \beta^{ij})_* = \tilde{d}_* \circ h_0^{ij}$.

For $u \in U_\infty(A_i \otimes_{\mathbb{R}} \mathbb{C})$, we have

$$\lambda_{l*}^{ij}([u]) = (\psi_l^{ij} \otimes \beta^{ij})_*([u]) = (\psi_l^{ij} \otimes id)_*((id \otimes \beta^{ij})_*([u])).$$

Since $\operatorname{tsr}(A_i \otimes_{\mathbb{R}} \mathbb{C}) = 1$, it follows from [2, Theorem V.3.1.26] that

$$K_1(A_i \otimes_{\mathbb{R}} \mathbb{C}) \simeq U(A_i \otimes_{\mathbb{R}} \mathbb{C})/U_0(A_i \otimes_{\mathbb{R}} \mathbb{C}).$$

Since $U(M_n(\mathbb{C})) \simeq U_0(M_n(\mathbb{C}))$, we conclude that $(id \otimes \beta^{ij})_*([u]) = [u]$. Hence, $\lambda_{l*}^{ij}([u]) = (\psi_l^{ij} \otimes id)_*([u]) = [W \ \operatorname{diag}(u \circ \tilde{\mu}_1, u \circ \tilde{\mu}_2) \ W^*]$.

We first reduce the problem from A_i and $q_iB_jq_i$ to $\tilde{A}_i = Z(A_i)$ and $\tilde{B}_i = Z(q_iB_jq_i) \otimes_{\mathbb{R}} M_2(\mathbb{R})$. For each $1 \leq l \leq N_{ij}$, if $(\psi_l^{ij} \otimes id)_*$ doesn't have the correct K_1 behavior, we show that there exists a real $*$-homomorphisms ϕ_{ij} between basic building blocks giving rise to the following commutative diagram (i.e., ϕ_{ij} has the correct K_1 behavior):

$$\begin{array}{ccccc}
K_1(\tilde{A}_i)/\operatorname{Tor}(K_1(\tilde{A}_i)) & \xrightarrow{\tilde{c}_{\tilde{A}_i}} & K_1(\tilde{A}_i \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\tilde{r}_{\tilde{A}_i}} & K_1(\tilde{A}_i)/\operatorname{Tor}(K_1(\tilde{A}_i)) \\
\downarrow{\scriptstyle h_1^{ij}} & & \downarrow{\scriptstyle h_1^{\mathbb{C}ij}} & & \downarrow{\scriptstyle h_1^{ij}} \\
K_1(\tilde{B}_j)/\operatorname{Tor}(K_1(\tilde{B}_j)) & \xrightarrow{\tilde{c}_{\tilde{B}_j}} & K_1(q_iB_jq_i \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\tilde{r}_{\tilde{B}_j}} & K_1(\tilde{B}_j)/\operatorname{Tor}(K_1(\tilde{B}_j))
\end{array}$$

Since $K_1(\tilde{A}_i)/\operatorname{Tor}(K_1(\tilde{A}_i))$ and $K_1(\tilde{B}_j)/\operatorname{Tor}(K_1(\tilde{B}_j))$ are isomorphic to either $\mathbb{Z}$ or 0 and furthermore $K_1(\tilde{A}_i \otimes_{\mathbb{R}} \mathbb{C})$ and $K_1(\tilde{B}_j \otimes_{\mathbb{R}} \mathbb{C})$ are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$, any nonzero group homomorphism from $K_1(\tilde{A}_i)/\operatorname{Tor}(K_1(\tilde{A}_i))$ to $K_1(\tilde{A}_i \otimes_{\mathbb{R}} \mathbb{C})$ and from $K_1(\tilde{B}_j)/\operatorname{Tor}(K_1(\tilde{B}_j))$ to $K_1(\tilde{B}_j \otimes_{\mathbb{R}} \mathbb{C})$ is injective. Therefore, if a $*$-homomorphism from $\tilde{A}_i$ to $\tilde{B}_j$ gives rise to $h_1^{\mathbb{C}ij}$, it must give rise to h_1^{ij} as well so that the diagram commutes. We consider a case by case analysis. Note that $\tilde{r} \circ \tilde{c}$ is multiplication by 2.

In the following cases, the commutativity of the diagram gives a zero map from $K_1(\tilde{A}_i \otimes_{\mathbb{R}} \mathbb{C})$ to $K_1(\tilde{B}_j \otimes_{\mathbb{R}} \mathbb{C})$. Therefore, we can pick any real $*$-homomorphism from $\tilde{A}_i$ to $\tilde{B}_j$ (i.e., $\phi_{ij} = \psi_l^{ij} \otimes id$), since they all induce the zero map from $K_1(\tilde{A}_i \otimes_{\mathbb{R}} \mathbb{C})$ to $K_1(\tilde{B}_j \otimes_{\mathbb{R}} \mathbb{C})$; in the following diagrams, k in A^k denotes the type of $\tilde{A}_i$ or $\tilde{B}_j$:

$$\begin{array}{ccccc}
K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 & \longrightarrow & K_1(A^3 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 \\
\uparrow\downarrow & & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle 0} & & \uparrow\downarrow \\
K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z}
\end{array}$$

$$\begin{array}{ccccc}
K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 & \longrightarrow & K_1(A^3 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 \\
\uparrow\downarrow & & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle 0} & & \uparrow\downarrow \\
K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z}
\end{array}$$

For the following diagrams, the maps $\phi_{ij}^k : A^k \longrightarrow A^k$ where $k \in \{3, 4, 5\}$ defined by $\phi_{ij}^k(f) = \operatorname{diag}(f \circ \mu, f \circ \mu)$ where $\mu(z) = z^m$ do the job.

$$\begin{array}{ccccc}
K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 & \longrightarrow & K_1(A^3 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 \\
\downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\
K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0 & \longrightarrow & K_1(A^3 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^3)/\operatorname{Tor}(K_1(A^3)) \simeq 0
\end{array}$$

$$\begin{array}{ccccc}
K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\
K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^4)/\operatorname{Tor}(K_1(A^4)) \simeq \mathbb{Z}
\end{array}$$

$$\begin{array}{ccccc}
K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\
K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} & \longrightarrow & K_1(A^5)/\operatorname{Tor}(K_1(A^5)) \simeq \mathbb{Z}
\end{array}$$

In order to have the right effect on K_1, we can proceed as in the proof of [10, Theorem 3], i.e., if any of our maps doesn't give rise to $h_1^{\mathbb{C}^{ij}}$ and h_1^{ij}, we take out that map and replace it with one of the above constructed maps and these new maps will give rise to both h_1^{ij} and $h_1^{\mathbb{C}^{ij}}$. This replacements will not change the average of $*$-homomorphisms by more than $\frac{1}{N_{ij}}$. We can also make $\frac{1}{N_{ij}}$ smaller by repeating each map more than once, each one the same number of times so as not to change the average. Moreover,

$$\left\| \frac{1}{2N_{ij}} \sum_{l=0}^{2N_{ij}} \psi_l^{ij}(f) - \tilde{d} \circ \tilde{M}^{ij}(f) \right\| < \delta$$

for all $f \in \tilde{F}^{ij}$.

We can construct $\lambda_l : A \longrightarrow B \otimes_{\mathbb{R}} M_2(\mathbb{R})$ as in [24, Lemma 4.2] and [24, Corollary 4.3] such that

$$\left\| \frac{1}{k} \sum_{l=0}^{k} \hat{\lambda}_l^{\mathbb{C}}(f) - M(f) \right\| < \varepsilon$$

for all $f \in F$ (refer to [24, Corollary 4.3] for the definition of $T \in \mathbb{N}$ and k).

Case 2. Assume that A_i and $q_iB_jq_i$ are both of type 1: In this case, we have:

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{(id,id)} & \mathbb{Z}^2 & \xrightarrow{id+id} & \mathbb{Z} \\
\downarrow {\scriptstyle h_0^{ij}} & & \downarrow {\scriptstyle h_0^{\mathbb{C}^{ij}}} & & \downarrow {\scriptstyle h_0^{\mathbb{H}^{ij}}} \\
\mathbb{Z} & \xrightarrow{(id,id)} & \mathbb{Z}^2 & \xrightarrow{id+id} & \mathbb{Z}
\end{array}$$

Assume $h_0^{\mathbb{C}^{ij}}(1,0) = (k,l)$ and $h_0^{\mathbb{C}^{ij}}(0,1) = (\acute{k}, \acute{l})$. By commutativity of the above diagram,

$$\begin{aligned}
k + l &= (id+id)(h_0^{\mathbb{C}^{ij}}(1,0)) = h_0^{\mathbb{H}^{ij}}((id+id)(0,1)) = h_0^{\mathbb{H}^{ij}}((id+id)(1,0)) \\
&= (id+id)(h_0^{\mathbb{C}^{ij}}(1,0)) = \acute{k} + \acute{l}
\end{aligned}$$

and $(k+\acute{k}, l+\acute{l}) = h_0^{\mathbb{C}^{ij}}(id,id)(1) = (id,id)(h_0^{ij}(1)) = (h_0^{ij}(1), h_0^{ij}(1))$ which implies $\acute{k} = l$ and $\acute{l} = k$. If we assume $\tilde{M}^{ij}(f,g) = (m_1(f,g), m_2(f,g))$ then the equation $(m_1(g,f), m_2(g,f)) = (m_2(f,g),\ m_1(f,g))$ follows from $\tilde{M}^{ij}\hat{\phi}_{A_i} = \hat{\phi}_{q_iB_jq_i}\tilde{M}^{ij}$. Thus, $\tilde{M}^{ij}(f,g) = (m(f,g), m(g,f))$. By commutativity of the following diagram,

$$\begin{array}{ccc}
\mathbb{Z}^2 & \xrightarrow{\rho_A} & C(\mathbb{T}, \mathbb{R}^2) \\
\downarrow {\scriptstyle h_0^{\mathbb{C}^{ij}}} & & \downarrow {\scriptstyle \tilde{M}^{ij}} \\
\mathbb{Z}^2 & \xrightarrow{\rho_B} & C(\mathbb{T}, \mathbb{R}^2)
\end{array}$$

it follows that $m(1,0) = \frac{k}{k+l}$ and $m(0,1) = \frac{l}{k+l}$. Therefore, the Markov maps

$$m_1, m_2 : C(\mathbb{T}, \mathbb{R}) \longrightarrow C(\mathbb{T}, \mathbb{R})$$

defined by $m_1(f) = \frac{k+l}{k} m(f,0)$ and $m_2(g) = \frac{k+l}{l} m(0,g)$ can be approximated by $\frac{1}{2N} \sum_{i=1}^{2N} f \circ \tilde{\mu}_i$ and $\frac{1}{2M} \sum_{i=1}^{2M} g \circ \tilde{\nu}_i$. If we let $R = lcm(2N, 2M)$, then $m(f,g)$ can be approximated by $\frac{1}{R(k+l)} \sum_{i=1}^{R} (lg \circ \tilde{\nu}_i + kf \circ \tilde{\mu}_i)$. If we define

$$\lambda_l^i : (C(\mathbb{T}, \mathbb{R}) \oplus C(\mathbb{T}, \mathbb{R})) \otimes_{\mathbb{R}} M_{n_i}(\mathbb{R}) \longrightarrow$$

$$(C(\mathbb{T},\mathbb{R}) \oplus C(\mathbb{T},\mathbb{R})) \otimes_{\mathbb{R}} M_{m_j = n_i(k+l)}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$$

by $\lambda_l^i(f,g) = (\mathrm{diag}(f \circ \tilde{\mu}_i \otimes I_k, g \circ \tilde{\nu}_i \otimes I_l), \mathrm{diag}(g \circ \tilde{\mu}_i \otimes I_k, f \circ \tilde{\nu}_i \otimes I_l)) \otimes I_2$ then

$$\left\| \tilde{d} \circ (\mathrm{diag}(m_1(f), m_2(g)), \mathrm{diag}(m_1(g), m_2(f))) - \frac{1}{R}\sum_{l=1}^{R} \lambda_l^i(f,g) \right\| < \delta.$$

Moreover, $\lambda_{l_*}^i([p \oplus q]) = \tilde{d}_* \circ h_0^{\mathbb{C}ij}([p] + [q])$ because $[p \circ \tilde{\mu}_i] = [p]$ and $[q \circ \tilde{\nu}_i] = [q]$. Similarly, the effect on h_0^{ij} and $h_0^{\mathbb{H}ij}$ is right. For the effect on K_1, we proceed as in Case 1 and [10, Theorem 3]. First, we reduce the problem to $Z(A_i)$ and $Z(q_i B_j q_i) \otimes_{\mathbb{R}} M_2(\mathbb{R})$.

For the following diagram, the commutativity of the diagram implies that the map has the form $\left(\begin{smallmatrix} m & n \\ n & m \end{smallmatrix}\right)$. The map $\phi : A^1 \longrightarrow A^1 \otimes_{\mathbb{R}} M_2(\mathbb{R})$ defined by $\phi(f) = diag(f \circ \mu, \overline{f \circ \nu})$ where $\mu(z) = z^m$ and $\nu(z) = z^n$ induces the map $\left(\begin{smallmatrix} m & n \\ n & m \end{smallmatrix}\right)$.

$$\begin{array}{ccccc}
K_1(A^1)/\operatorname{Tor}(K_1(A^1)) \simeq \mathbb{Z} & \xrightarrow{(id,id)} & K_1(A^1 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{id+id} & K_1(A^1)/\operatorname{Tor}(K_1(A^1)) \simeq \mathbb{Z} \\
\downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} m & n \\ n & m \end{smallmatrix}\right)} & & \downarrow \\
K_1(A^1)/\operatorname{Tor}(K_1(A^1)) \simeq \mathbb{Z} & \xrightarrow{(id,id)} & K_1(A^1 \otimes_{\mathbb{R}} \mathbb{C}) \simeq \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{id+id} & K_1(A^1)/\operatorname{Tor}(K_1(A^1)) \simeq \mathbb{Z}
\end{array}$$

If any of our maps doesn't give rise to $h_1^{\mathbb{C}ij}$ and h_1^{ij}, we can take out that map and replace it by the above constructed maps and proceed as in [10, Theorem 3]. □

4. The uniqueness theorem

Lemma 4.1. *Let A and B be direct sums of building blocks of a real $A\mathbb{T}_2$-algebra and let ϕ and ψ be $*$-homomorphisms from A to B giving rise to the same map from $K_0(A)$ to $K_0(B)$, then there exists a unitary $u \in B$ such that $\phi(a) = u\psi(a)u^*$ for each central minimal projection $a \in A$.*

Proof. Let $e \in A$ be a central minimal orthogonal projection. By equalities $[\phi(e)] = [\psi(e)]$, $[1 - \phi(e)] = [1 - \psi(e)]$, [1, Proposition 4.2.5] and [1, Proposition 4.6.5], there exists $u_e \in B$ such that $\phi(e) = u_e \psi(e) u_e^*$. If we let

$$u = \sum_{e \in A} \phi(e) u_e \psi(e)$$

then $\phi(a) = u\psi(a)u^*$ for each central minimal projection $a \in A$. □

Lemma 4.2. *If ϕ is an involutive $*$-antiautomorphism of $A = C(\mathbb{T}, M_n(\mathbb{C}))$ and $f \in U(A)$, then $w(\mathrm{Det}(\phi(f))) = w(\mathrm{Det}(f \circ \psi)) = \pm\, w(\mathrm{Det}(f))$ where w denotes the winding number map, and ψ is the associated involutive homeomorphism of $\mathbb{T}$.*

Proof. By Theorem 2.6, $\phi(f) = u_t(f_t \circ \psi)u_t^*$. Since

$$w(\mathrm{Det}(u_t^*)) = -w(\mathrm{Det}(u_t)),$$

we conclude $w(\mathrm{Det}(\phi(f))) = w(\mathrm{Det}(f \circ \psi))$. Since ψ is an involutive homeomorphism, it can just change the sign of winding number. □

Lemma 4.3. *Let A be a non-type-1 basic building block of a real $A\mathbb{T}_2$-algebra with a unital subalgebra C isomorphic to $M_n(\mathbb{R})$ or $M_{\frac{n}{2}}(\mathbb{H})$ for some n and whose commutant is the center. If ϕ and ψ are $*$-homomorphisms from A to a real algebra B which is a direct sum of building blocks with $\phi(1) = \psi(1) = e$, then there exists a unitary $\nu \in eBe$ with $\phi(c) = \nu\psi(c)\nu^*$ for each $c \in C$.*

Proof. By Lemma 4.1, it suffices to assume that eBe is a single building block which can be written as $Z \otimes_{\mathbb{R}} M_m(\mathbb{R})$ or $Z \otimes_{\mathbb{R}} M_{\frac{m}{2}}(\mathbb{H})$ where $Z \in \{C(\mathbb{T},\mathbb{R}), C(\mathbb{T},\mathbb{C}), C(\mathbb{T},\eta_0), C(\mathbb{T},\eta_1)\}$. Since

$$\phi_*, \psi_* : K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_0(eBe \otimes_{\mathbb{R}} \mathbb{C})$$

are positive order unit preserving group homomorphisms, we conclude that $n|m$, i.e., $m = nk$. Since $M_n(\mathbb{F})$ is simple, we conclude that $\phi(C) \simeq \psi(C) \simeq C$ (we denote the isomorphism map by $h : \psi(C) \longrightarrow \phi(C)$), and consequently there exists a subalgebra H of eBe isomorphic to $M_k(\mathbb{R})$ or $M_{\frac{k}{2}}(\mathbb{H})$ such that $eBe \simeq Z \otimes_{\mathbb{R}} H \otimes_{\mathbb{R}} \phi(C) \simeq Z \otimes_{\mathbb{R}} H \otimes_{\mathbb{R}} \psi(C)$. We define the map $\gamma \in \mathrm{Aut}(eBe)$ by $\gamma = id \otimes id \otimes h$. By [31, Corollary 5], $\gamma^{\mathbb{C}} \in \mathrm{Aut}(e(B \otimes_{\mathbb{R}} \mathbb{C})e)$ is inner, i.e., there exists a unitary $u \in e(B \otimes_{\mathbb{R}} \mathbb{C})e$ such that $\gamma^{\mathbb{C}} = \mathrm{Ad}(u)$. Let Φ be an involutive $*$-antiautomorphism of $e(B \otimes_{\mathbb{R}} \mathbb{C})e$ such that $(e(B \otimes_{\mathbb{R}} \mathbb{C})e)_\Phi \simeq eBe$. Then,

$$\gamma^{\mathbb{C}}(\Phi(a)) = \gamma^{\mathbb{C}}(a^*) = (\gamma^{\mathbb{C}}(a))^* = \Phi(\gamma^{\mathbb{C}}(a))$$

for each $a \in eBe$. Hence,

$$\gamma^{\mathbb{C}}(\Phi(a)) = u\Phi(a)u^* = \Phi(\gamma^{\mathbb{C}}(a)) = \Phi(uau^*) = \Phi(u^*)\Phi(a)\Phi(u)$$

for each $a \in eBe$ which implies $w = u^*\Phi(u^*) \in Z \otimes_{\mathbb{R}} \mathbb{C}$ and $\Phi(w) = w$. By Lemma 4.2, winding number of w is either zero or even. Moreover, if $e(B \otimes_{\mathbb{R}} \mathbb{C})e \simeq C(\mathbb{T}, M_n(\mathbb{C})) \oplus C(\mathbb{T}, M_n(\mathbb{C}))$ and ϕ switches the summands, then winding number of w will be even as well. In any case, the square root of a central unitary with winding number even or zero always exists; hence square root of w exists.

For $f \in Z \otimes_{\mathbb{R}} \mathbb{C}$, if $Z \otimes_{\mathbb{R}} \mathbb{C} \simeq C(\mathbb{T},\mathbb{C})$ then $\Phi(f) = f \circ \alpha$ where $\alpha \in \{\eta_0, \eta_1, id\}$ and if $Z \otimes_{\mathbb{R}} \mathbb{C} \simeq C(\mathbb{T},\mathbb{C}^2)$ then $\Phi(f,g) = (g,f)$ and in each case $\Phi(w^{1/2}) = w^{1/2}$. Thus,

$$\Phi(w^{1/2}u) = \Phi(u)w^{1/2} = u^*w^*w^{1/2} = u^*w^{1/2^*} = (w^{1/2}u)^*$$

and $\nu = w^{1/2}u \in eBe$ is the required unitary element. □

Lemma 4.4. *Let A be a basic building block of type 1 with the unital subalgebra C isomorphic to $M_n(\mathbb{C})$ and whose commutant is the center and let ϕ and ψ be real-linear $*$-homomorphisms from A to a real algebra B which is* *a direct sum of building blocks with $\phi(1) = \psi(1) = e$ giving rise to the same map from $K_0(A \otimes_{\mathbb{R}} \mathbb{C})$ to $K_0(B \otimes_{\mathbb{R}} \mathbb{C})$ then there exists a unitary $\nu \in eBe$ with $\phi(c) = \nu\psi(c)\nu^*$ for each $c \in C$.*

Proof. In [30, Lemma 2.3], it is enough to replace $[0,1]$ by $\mathbb{T}$. $\square$

Lemma 4.5. *Let*

$$C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_m(\mathbb{F}) \qquad \text{where} \quad \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\},$$
$$C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_m(\mathbb{F}) \qquad \text{where} \quad \mathbb{F} \in \{\mathbb{R}, \mathbb{H}\},$$

and $C(\mathbb{T}, \eta_1) \otimes_{\mathbb{R}} M_m(\mathbb{R})$ be the basic building blocks where $m \in \{n, \frac{n}{2}\}$ depending on the type of the block. Then we have the following identifications:

$$C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_m(\mathbb{F}) \simeq \{f \in C([0,1], \mathbb{R}) \otimes_{\mathbb{R}} M_m(\mathbb{F}) | f(0) = f(1)\}$$
$$C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_n(\mathbb{R}) \simeq \{f \in C([0,1], \mathbb{C}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) | f(0), f(1) \in M_n(\mathbb{R})\}$$
$$C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{\frac{n}{2}}(\mathbb{H}) \simeq \{f \in C([0,1], \mathbb{C}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) | f(0), f(1) \in M_{\frac{n}{2}}(\mathbb{H})\}$$
$$C(\mathbb{T}, \eta_1) \otimes_{\mathbb{R}} M_n(\mathbb{R}) \simeq \{f \in C([0,1], \mathbb{C}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) | f(1) = \overline{f(0)}\}.$$

Proof. The first isomorphism is given by the map $h(f) = g_f$ where $g_f(t) = f(e^{2\pi i t})$. As we mentioned before, $C(\mathbb{T}, \eta_0) \simeq \{f \in C(\mathbb{T}_+, \mathbb{C}) \mid f(\pm 1) \in \mathbb{R}\}$ and $C(\mathbb{T}, \eta_1) \simeq \{f \in C(\mathbb{T}_+, \mathbb{C}) \mid f(-1) = \overline{f(1)}\}$. The homeomorphism $\alpha : \mathbb{T}_+ \longrightarrow [0,1]$ defined by $\alpha(e^{i\pi t}) = t$ yields the other isomorphisms. $\square$

Remark 4.6. From now on, we may use the above isomorphisms without explicitly mentioning them.

Lemma 4.7. *Let A belong to $M_n(\mathbb{R})$:*

(i) *If A is skew-symmetric, then there are block diagonal matrices $D \in M_n(\mathbb{R})$, $\widetilde{D} \in M_n(\mathbb{C})$ and an orthogonal matrix $U \in M_n(\mathbb{R})$ such that $U^T AU = D$ and $W^* U^T AUW = \widetilde{D}$ where*

$$W = \operatorname{diag}(V_1, \ldots, V_m, 0_{n-2m}), \quad \widetilde{D} = \operatorname{diag}(\widetilde{D}_1, \ldots, \widetilde{D}_m, 0_{n-2m}),$$
$$D = \operatorname{diag}(D_1, \ldots, D_m, 0_{n-2m}),$$
$$D_j = \beta_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \widetilde{D}_j = \lambda_j \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, V_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

$\beta_j > 0$, $\mathrm{Sp}^{\mathbb{C}}(A) = \{\pm i\beta_j, 0\}$ and $\lambda_j \in \mathrm{Sp}^{\mathbb{C}}(A) - \{0\}$ for $1 \leq j \leq m$.

(ii) *If A is orthogonal, then there are block diagonal matrices*

$$K \in M_n(\mathbb{R}), \quad \widetilde{K} \in M_n(\mathbb{C})$$

and an orthogonal matrix $P \in M_n(\mathbb{R})$ such that

$$P^T AP = K \quad \text{and} \quad \tilde{W}^* P^T AP\tilde{W} = \widetilde{K}$$

where

$$\tilde{W} = \operatorname{diag}(V_1, \ldots, V_m, I_{n-2m}), \qquad \widetilde{K} = \operatorname{diag}(\widetilde{K}_1, \ldots, \widetilde{K}_m, J_{n-2m}),$$
$$K = \operatorname{diag}(K_1, \ldots, K_m, J_{n-2m}), \quad J_{n-2m} = \operatorname{diag}(\pm 1, \ldots, \pm 1),$$

$$K_j = \begin{pmatrix} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{pmatrix}, \quad \widetilde{K}_j = \begin{pmatrix} \overline{\mu}_j & 0 \\ 0 & \mu_j \end{pmatrix},$$

$0 < \theta_j < \pi$, $\mathrm{Sp}^{\mathbb{C}}(A) = \{e^{i\theta_j}, \pm 1\}$ *and* $\mu_j \in \mathrm{Sp}^{\mathbb{C}}(A) - \{\pm 1\}$ *for* $1 \leq j \leq m$.

Proof. The proof is well-known. Note that $V_j^* D_j V_j = \widetilde{D}_j$ and $V_j^* K_j V_j = \widetilde{K}_j$ for $1 \leq j \leq m$. □

Remark 4.8. If $f \in A$ is unitary, self-adjoint or skew-adjoint, where A is a basic building block, then its eigenfunctions are $\mathbb{T}$-valued, real-valued or purely imaginary-valued (other than zero) respectively, and its eigenprojections are orthogonal. Furthermore, assume that f has a spectral decomposition, i.e.,

$$f(z) = \sum_{i=1}^{n} \lambda_i(z) P_i(z)$$

where the eigenfunctions λ_i are distinct and P_i are the orthogonal eigenprojections with sum 1. Let ϕ be an involutive $*$-antiautomorphism of $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $(A \otimes_{\mathbb{R}} \mathbb{C})_\phi = A$. By orthogonality of eigenprojections, $fP_i = \lambda_i P_i$ for all $i = 1, .., n$. The involutive $*$-antiautomorphism

$$\phi : C(\mathbb{T}, M_n(\mathbb{C})) \longrightarrow C(\mathbb{T}, M_n(\mathbb{C})), \quad \phi(f) = (u^t)(f \circ \psi)^t (u^*)^t,$$

$u \in U(C(\mathbb{T}, M_n(\mathbb{C})))$ (cf. Theorem 2.6), is extendible to the involutive $*$-antiautomorphism

$$\tilde{\phi} : C([0,1], M_n(\mathbb{C})) \longrightarrow C([0,1], M_n(\mathbb{C}))$$

as follows (the map $\beta : [0,1] \longrightarrow \mathbb{T}$ is defined by $\beta(t) = e^{2\pi i t}$):

If $\psi : \mathbb{T} \longrightarrow \mathbb{T}, \psi = id$ then define

$$\tilde{\phi}(f) = (u \circ \beta)^t (f \circ \alpha)^t (u^* \circ \beta)^t$$

where $\alpha : [0,1] \longrightarrow [0,1], \alpha = id$.

If $\psi : \mathbb{T} \longrightarrow \mathbb{T}, \psi = \eta_0$ where $\eta_0(z) = \bar{z}$ then define

$$\tilde{\phi}(f) = (u \circ \beta)^t (f \circ \alpha)^t (u^* \circ \beta)^t$$

where $\alpha : [0,1] \longrightarrow [0,1], \alpha(t) = 1 - t$.

We can rewrite f as follows (note that $\phi(f^*) = f$):

$$f = \frac{1}{2}(f + \phi(f^*)) = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i P_i + \phi\left(\sum_{i=1}^{n} \bar{\lambda}_i P_i\right)\right)$$
$$= \frac{1}{2}\sum_{i=1}^{n}(\lambda_i P_i + \tilde{\phi}(\bar{\lambda}_i)\tilde{\phi}(P_i)).$$

Let $g_i \in C(\mathrm{Sp}^{\mathbb{C}}(\lambda_i), \eta_0)$ be such that $\|g_i(\lambda_i) - \lambda_i\| < \frac{\epsilon}{n}$. Then, we can consider the function

$$\tilde{f} = \frac{1}{2}\sum_{i=1}^{n}(g_i(\lambda_i)P_i + g_i(\tilde{\phi}(\bar{\lambda}_i))\tilde{\phi}(P_i)).$$

Since $g_i(\tilde{\phi}(\bar{\lambda}_i)) = \overline{g_i(\tilde{\phi}(\lambda_i))}$, we conclude $\tilde{\phi}(\tilde{f}) = \tilde{f}^*$. Moreover, $\|\tilde{f} - f\| < \epsilon$.

Lemma 4.9. *Let A be a basic building block of a real $A\mathbb{T}_1$-algebra, $\epsilon > 0$ and $f \in A$ be a unitary (self-adjoint) such that only two of its eigenfunctions touch at only one point x_0, then there exists a unitary (self-adjoint) $g \in A$ such that g has distinct eigenfunctions and $\|g - f\| < \epsilon$.*

Proof. If $f(x_0) \in M_{\frac{n}{2}}(\mathbb{H})$, then we can decompose $f(x_0)$ as $f(x_0) = C + Dj$ where C and D are in $M_{\frac{n}{2}}(\mathbb{C})$ and we can embed $f(x_0)$ in $M_n(\mathbb{C})$ as a symplectic matrix by the injective $*$-homomorphism $h : M_{\frac{n}{2}}(\mathbb{H}) \longrightarrow M_n(\mathbb{C})$:

$$h(C + Dj) = \begin{pmatrix} C & D \\ -\overline{D} & \overline{C} \end{pmatrix}$$

If we define an antilinear unitary map $K : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by

$$K(x_1, x_2, \ldots, x_n) = (-\overline{x}_2, \overline{x}_1, \ldots, -\overline{x}_n, \overline{x}_{n-1})$$

and an involutive $*$-antiautomorphism $\phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ by

$$\phi(f(x_0)) = -K^* f(x_0)^* K^* = K f(x_0)^* K^*,$$

then $U = [V_1, \ldots, V_{\frac{n}{2}}, KV_1, \ldots, KV_{\frac{n}{2}}]$ belongs to $(M_n(\mathbb{C}))_\phi \simeq M_{\frac{n}{2}}(\mathbb{H})$ where V_j and KV_j, $1 \le j \le \frac{n}{2}$ are the eigenvectors of $h(f(x_0))$ and $Uh(f(x_0))U^* = D$ is the spectral decomposition of $h(f(x_0))$ in $M_n(\mathbb{C})$.

It follows that each real eigenvalue of $f(x_0)$ after embedding in $M_n(\mathbb{C})$ has even multiplicity and the complex eigenvalues of $f(x_0)$ appear as conjugate pairs [19]. If $f(x_0)$ is self-adjoint, then all eigenvalues are real and we have forced double degeneracy. Since the summation of $\frac{n}{2}$ geometric multiplicities should be equal to n, the eigenprojections of $f(x_0)$ are of rank two.

Assume λ_i, λ_j are two eigenfunctions of f that touch at the point x_0. If f is self-adjoint, we may choose real-valued functions $c_i, c_j \in C(\mathbb{T}, \mathbb{R})$ with norm less than one and supported in a neighborhood of x_0 such that g defined by $g = f + \frac{\epsilon}{4}(c_i P_i + c_j P_j + (c_i \circ \psi)\tilde{\phi}(P_i) + (c_j \circ \psi)\tilde{\phi}(P_j))$ meets our requirements (cf. Remark 4.8). If $f(x_0) \in M_{\frac{n}{2}}(\mathbb{H})$ and f is self-adjoint, then $g(x_0)$ has $\frac{n}{2}$ distinct eigenvalues, each of multiplicity two and $\frac{n}{2}$ rank two eigenprojections.

If f is unitary, we may choose real-valued functions $c_i, c_j \in C([0,1], \mathbb{R})$ with norm less than one and supported in a neighborhood of $arg(x_0)$ such that g defined by $g = f + \frac{\epsilon}{4}(e^{2\pi i c_i} P_i + e^{2\pi i c_j} P_j + e^{2\pi i c_i \circ \hat{\psi}} \tilde{\phi}(P_i) + e^{2\pi i c_j \circ \hat{\psi}} \tilde{\phi}(P_j))$, where $\hat{\psi}$ is the involutive homeomorphism of $[0,1]$ induced by ψ, meets our requirements. □

Lemma 4.10. *Let $\epsilon > 0$ and let B be a basic building block of a real $A\mathbb{T}_1$-algebra.*

(i) *If $f \in B$ is a self-adjoint element and B is of type 1, 2, 4, 6 (3) then there exists a self-adjoint element $g \in B$ such that $\|f - g\| < \epsilon$ and $g(z)$ has n ($\frac{n}{2}$) distinct eigenvalues for each $z \in \mathbb{T}$. If B is of type 5* *then g has $\frac{n}{2}$ distinct eigenvalues at the points* 0 *and* 1 *and it has n distinct eigenvalues everywhere else.*

(ii) *If $f \in B$ is a unitary element then there exists a unitary element $g \in B$ such that $\|f - g\| < \epsilon$ and $g(z)$ has n distinct eigenvalues for each $z \in \mathbb{T}$.*

Proof. (i) Let h be the piecewise analytic approximation of f (in the complex case, for unitary and self-adjoint elements, the proof of its existence is on page 186 of [9, Theorem 4.4] and on page 75 of [5, Theorem 4] respectively. In the real case, the essential difference is when h is unitary, in that case, on a suitable subinterval h is either of the form $h = e^k$ or of the form $h = e^k w$, depending on its winding number, where k is a skew-adjoint element and w is a constant unitary with winding number -1. By [18, Theorem II.6.1], the eigenfunctions and eigenprojections of h are piecewise analytic. It follows that unequal eigenfunctions of h coincide at finitely many points, because if they coincide at infinitely many points then by identity theorem they must be equal. By passing to subintervals, we may further assume that they coincide at one point. Moreover, we can reduce to the case that just two of the eigenfunctions coincide at the degenerate point. If at the remaining degenerate point the eigenfunctions touch but do not cross then we can remove this degeneracy by Lemma 4.9. If the eigenfunctions λ_j and λ_k cross at $t_0 \in [a, b] \subseteq [0, 1]$, i.e., $\lambda_j(a) < \lambda_k(a)$ and $\lambda_j(b) > \lambda_k(b)$, where the interval $[a, b]$ is picked such that $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ over $[a, b]$ is sufficiently close to $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ at t_0 and λ_j is sufficiently close to λ_k over $[a, b]$, then let $\{Q(t) : t \in [a, b]\}$ be a path of projections such that $Q \leq P_{\lambda_j} + P_{\lambda_k}$, $Q(a) = P_{\lambda_j}(a)$ and $Q(b) = P_{\lambda_k}(b)$. If we define $\tilde{h}$ by replacing $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ in h with $\min(\lambda_j, \lambda_k)Q + \max(\lambda_j, \lambda_k)(P_{\lambda_j} + P_{\lambda_k} - Q)$ over $[a, b]$ and setting $\tilde{h} = h$ everywhere else, then $\tilde{h}(a) = h(a)$, $\tilde{h}(b) = h(b)$, $\tilde{h}$ is sufficiently close to h over $[a, b]$ and its eigenfunctions touch but do not cross. By Lemma 4.9, we can construct the function g.

(ii) In this case, eigenfunctions are of the form $\exp(2\pi i F) : [0, 1] \longrightarrow \mathbb{T}$ where $F : [0, 1] \longrightarrow [0, 1]$ is a continuous function. If the eigenfunctions $\lambda_j = \exp(2\pi i F)$ and $\lambda_k = \exp(2\pi i G)$ cross at $t_0 \in [a, b] \subseteq [0, 1]$, i.e., $G(t) > F(t)$ for $t \in [a, t_0)$, $G(t) < F(t)$ for $t \in (t_0, b]$ and $G(t_0) = F(t_0)$, (where the interval $[a, b]$ is picked such that $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ over $[a, b]$ is sufficiently close to $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ at t_0 and λ_j is sufficiently close to λ_k over $[a, b]$) then let $\{Q(t) : t \in [a, b]\}$ be a path of projections such that $Q \leq P_{\lambda_j} + P_{\lambda_k}$, $Q(a) = P_{\lambda_j}(a)$ and $Q(b) = P_{\lambda_k}(b)$. If we construct $\tilde{h}$ by replacing $\lambda_j P_{\lambda_j} + \lambda_k P_{\lambda_k}$ in h with $\exp(2\pi i \min\{F, G\})Q + \exp(2\pi i \max\{F, G\})(P_{\lambda_j} + P_{\lambda_k} - Q)$ then $\tilde{h}(a) = h(a)$, $\tilde{h}(b) = h(b)$, $\tilde{h}$ is sufficiently close to h over $[a, b]$ and its eigenfunctions touch but do not cross. By Lemma 4.9, we can construct the function g. If eigenfunctions appear as conjugate pairs, then we replace

$$\lambda_j(P_{\lambda_j} - \tilde{\phi}(P_{\lambda_j})) + \lambda_k(P_{\lambda_k} - \tilde{\phi}(P_{\lambda_k}))$$

in h with

$$\begin{aligned}\exp(2\pi i\ \min\{F,G\})Q + \exp(2\pi i\ \max\{F,G\})(P_{\lambda_j} + P_{\lambda_k} - Q)\\ + \exp(2\pi i\ \min\{-F,-G\})\tilde{\phi}(Q)\\ + \exp(2\pi i\ \max\{-F,-G\})(\tilde{\phi}(P_{\lambda_j})\\ + \tilde{\phi}(P_{\lambda_k}) - \tilde{\phi}(Q)).\end{aligned}$$ □

Remark 4.11. Let B be a basic building block of a real $A\mathbb{T}_1$-algebra and $\phi : C(\mathbb{T},\mathbb{R}) \longrightarrow B$ be a unital $*$-homomorphism. There exists a unital $*$-homomorphism $\tilde{\phi} : C(\mathbb{T},\mathbb{R}) \longrightarrow B$ such that $\tilde{\phi}^{\mathbb{C}}(g_1 + ig_2)$ has distinct eigenfunctions and approximates $\phi^{\mathbb{C}}(g_1 + ig_2)$. Therefore, $\tilde{\phi}(g_1)$ approximates $\phi(g_1)$ and $\tilde{\phi}(g_2)$ approximates $\phi(g_2)$. Since $\tilde{\phi}(g_1)$ and $\tilde{\phi}(g_2)$ have the same set of eigenprojections, $\tilde{\phi}(g_1)$ commutes with $\tilde{\phi}(g_2)$. Hence, $\tilde{\phi}(g_1)$ and $\tilde{\phi}(g_2)$ are simultaneously diagonalizable.

Lemma 4.12. *Let $f \in B$ be unitary (self-adjoint) with distinct eigenfunctions where B is a basic building block of type 1, 3, 5 (or type 4 only if $f \in B$ is self-adjoint) of a real $A\mathbb{T}_J$-algebra and let*

$$f = \sum_{i=1}^{n} \lambda_i P_i$$

be its spectral decomposition. There exists a unitary $s \in B$ such that sfs^* is diagonal. Furthermore, if B is a building block of type 4 and $f \in B$ is a unitary, then there exists a unitary $s \in B$ such that sfs^* is block diagonal with two by two blocks.

Proof. Embed the building blocks of type 1 and 3 in $C([0,1], M_m(\mathbb{F}))$ where $\mathbb{F} \in \{\mathbb{C},\mathbb{H}\}, m \in \{n, \frac{n}{2}\}$ respectively and embed the building blocks of type 4 and 5 in $C([0,1],\eta_2) \otimes_{\mathbb{R}} M_m(\mathbb{F}))$ where $\mathbb{F} \in \{\mathbb{R},\mathbb{H}\}, m \in \{n, \frac{n}{2}\}, \eta_2(t) = 1-t$. The embedding map is $\iota(f) = f \circ \beta$ where $\beta(t) = e^{2\pi i t}$. According to [30, Lemma 2.5], for type 1 and 3 blocks there exists a unitary $u \in C([0,1], M_m(\mathbb{F}))$ such that ufu^* is diagonal. As in the proof of [30, Lemma 2.5], we can set $u = [e_1, \dots, e_n]$ where e_i are normalized eigenvector functions (i.e., for each $t \in [0,1], e_i(t)$ is an eigenvector). Since $f(0) = f(1)$, there exists a permutation $\sigma \in S_n$ such that $\lambda_{\sigma(i)}(0) = \lambda_i(1)$ and $P_{\sigma(i)}(0) = P_i(1)$. Since $P_{\sigma(i)}(0) = P_i(1)$, we conclude $e_{\sigma(i)}(0) = e_i(1)$. If f is self-adjoint, then $\sigma = id$ and hence u belongs to the real building block (type 1 or 3). If f is unitary then we proceed as follows:

Define $p \in C([0,1], M_m(\mathbb{F}))$ by $p(t) = P$ where P is an elementary permutation matrix (a column-switching transformation) where the permutation corresponds to $\sigma \in S_n$. It is known that P is a self-adjoint unitary and $\mathrm{Det}(P) = (-1)^d$ where d is the number of transpositions in the decomposition of σ. For a building block of type 1 (3), there exists a path of unitaries $z \in C([0,1],\mathbb{R}) \otimes_{\mathbb{R}} M_n(\mathbb{C})$ ($z \in C([0,1],\mathbb{R}) \otimes_{\mathbb{R}} M_{\frac{n}{2}}(\mathbb{H})$) that connects I to P. For example, we can connect I to iI through the path

$u_1(t) = I\cos(\frac{\pi t}{2}) + iI\sin(\frac{\pi t}{2})$ and we can connect iI to P through the path $u_2(t) = iI\cos(\frac{\pi t}{2}) + P\sin(\frac{\pi t}{2})$. Let's denote the composition of these two paths by u. If we set $s = uz$, then s belongs to the real building block (type 1 or 3).

For building blocks of type 4 and 5, we use the fact they are isomorphic to

$$\{f \in C([0,1],\mathbb{C}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) | f(0), f(1) \in M_n(\mathbb{R})\}$$

and

$$\{f \in C([0,1],\mathbb{C}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) | f(0), f(1) \in M_{\frac{n}{2}}(\mathbb{H})\},$$

respectively. If B is a building block of type 4 (5) and $f \in B$ is self-adjoint (unitary or self-adjoint), then the unitary $u = [e_1, \ldots, e_n]$ is not necessarily in the building block because $u(0), u(1)$ may not be in $M_n(\mathbb{R})$ ($M_{\frac{n}{2}}(\mathbb{H})$). However, since eigenvalues are distinct, there exist unitary diagonal matrices $\Lambda_1, \Lambda_2 \in M_n(\mathbb{C})$ such that $\Lambda_1 u(0), \Lambda_2 u(1) \in M_n(\mathbb{R})$ ($M_{\frac{n}{2}}(\mathbb{H})$). Let $\Lambda \in C([0,1], M_n(\mathbb{C}))$ be a path of unitary diagonal matrices that connects Λ_1 to Λ_2. Then $s = \Lambda^* u$ is a unitary in the building block 4 (5) and s diagonalizes f. If B is of type 4 and $f \in B$ is unitary, the same proof works with the difference that sfs^* is block diagonal instead of diagonal. □

Remark 4.13. If $A = C(\mathbb{T}, \eta_0)$, B is a basic building block of a real $A\mathbb{T}_2$-algebra, ϕ and ψ are unital $*$-homomorphisms from A to B such that they give rise to the same maps from $K_1(A \otimes_{\mathbb{R}} \mathbb{C})$ to $K_1(B \otimes_{\mathbb{R}} \mathbb{C})$. Then, $w(\mathrm{Det}(\phi(g_0))) = w(\mathrm{Det}(\psi(g_0)))$ because $\phi(g_0) = \phi^{\mathbb{C}}(g_0 \otimes 1)$ and $\psi(g_0) = \psi^{\mathbb{C}}(g_0 \otimes 1)$.

Lemma 4.14. *Let $A \in \{C(\mathbb{T},\mathbb{R}), C(\mathbb{T},\eta_0)\}$ and B be a basic building block of a real $A\mathbb{T}_J$-algebra and let ϕ, ψ be unital $*$-homomorphisms from A to B such that they give rise to the same maps from*

$$K_1(A)/\operatorname{Tor}(K_1(A)) \longrightarrow K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(A)/\operatorname{Tor}(K_1(A))$$

to

$$K_1(B)/\operatorname{Tor}(K_1(B)) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B)/\operatorname{Tor}(K_1(B)).$$

Let $\tilde{\phi}$ and $\tilde{\psi}$ be their multiplicity-free approximants on the set of canonical central generators $G = \{g_0, g_1, g_2\}$ such that

$$\tilde{\phi}(g) = \sum_{i=1}^{k} \theta_i p_i \quad \textit{and} \quad \tilde{\psi}(g) = \sum_{i=1}^{k} \beta_i q_i$$

are the corresponding spectral decompositions for $g \in G$. Then, there is a unitary $V \in B$ ($V \in B \otimes_{\mathbb{R}} \mathbb{C}$ if B is of type 4 and $\tilde{\phi}(g)$ is unitary) such that

$$\|V\tilde{\phi}(g)V^* - \tilde{\psi}(g)\| = \left\| \sum_{i=1}^{k} \theta_i q_i - \sum_{i=1}^{k} \beta_i q_i \right\|$$

Proof. (i) Let $A = C(\mathbb{T}, \eta_0)$:

If B is a building block of type 1, 3, or 5, then there exists a permutation $\sigma \in S_n$ for $\tilde{\phi}(g_0)$ such that $\theta_i(1) = \theta_{\sigma(i)}(0)$ where $i \in \{1, \ldots, n\}$. Since $\tilde{\phi}(g_0)$ is multiplicity-free, it follows from [32, Lemma 1.7] that σ is a cyclic permutation of some order m_1. Moreover, there exists an integer $x \in \mathbb{Z}$ such that $w(\mathrm{Det}(\tilde{\phi}(g_0))) = nx + m_1$ (cf. [24, Lemma 2.2]). Similarly, we have $w(\mathrm{Det}(\tilde{\psi}(g_0))) = ny + m_2$ where m_2 is the order of a cyclic permutation $\mu \in S_n$. As stated in Remark 4.13, $w(\mathrm{Det}(\tilde{\phi}(g_0))) = w(\mathrm{Det}(\tilde{\psi}(g_0)))$ and we conclude that $m_1 = m_2$ or equivalently $\sigma = \mu$. By Lemma 4.12, there is a unitary $V \in B$ with the required property. If B is of type 4, then there is a unitary $V \in B \otimes_{\mathbb{R}} \mathbb{C}$ with the required property.

(ii) Let $A = C(\mathbb{T}, \mathbb{R})$:

If B is a building block of type 1, 3, 4, or 5 and if $\sum_{i=1}^{n} \theta_i p_i$ and $\sum_{i=1}^{n} \beta_i q_i$ are the spectral decompositions of $\tilde{\phi}^{\mathbb{C}}(g_0)$ and $\tilde{\psi}^{\mathbb{C}}(g_0)$ respectively, then there exist cyclic permutations $\sigma, \mu \in S_n$ such that $\theta_i(1) = \theta_{\sigma(i)}(0)$ and $\beta_i(1) = \beta_{\mu(i)}(0)$ where $i \in \{1, \ldots, n\}$. Hence,

$$\tilde{\phi}(g_1) = \sum_{i=1}^{n} \mathrm{Re}(\theta_i) p_i, \qquad \tilde{\phi}(g_2) = \sum_{i=1}^{n} \mathrm{Im}(\theta_i) p_i,$$
$$\tilde{\psi}(g_1) = \sum_{i=1}^{n} \mathrm{Re}(\beta_i) q_i, \qquad \tilde{\psi}(g_2) = \sum_{i=1}^{n} \mathrm{Im}(\beta_i) q_i.$$

Therefore, $\sigma = \mu$. By Lemma 4.12, there is a unitary $V \in B$ with the required properties. ☐

Lemma 4.15. *Let $A \in \{C(\mathbb{T}, \mathbb{R}), C(\mathbb{T}, \eta_0)\}$ and B be a basic building block of a real $A\mathbb{T}_J$-algebra and let ϕ and ψ be unital $*$-homomorphisms from A to B such that they give rise to the same maps from*

$$K_1(A)/\operatorname{Tor}(K_1(A)) \longrightarrow K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(A)/\operatorname{Tor}(K_1(A))$$

to

$$K_1(B)/\operatorname{Tor}(K_1(B)) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B)/\operatorname{Tor}(K_1(B)).$$

Moreover, let $\tilde{\phi}$ and $\tilde{\psi}$ be their multiplicity-free approximants on the set of generators. Let g_0 be the canonical unitary generator of $C(\mathbb{T}, \mathbb{C})$ and let χ_j^r be the characteristic function of $I_j^r = \{e^{2\pi i t} | t \in [\frac{j-1}{r}, \frac{j}{r})\}$. If for every pair $m, n \in \mathbb{N}$ with $n > 12$ there is a finite subset $F \subset C(\mathbb{T} \cup \{0\}, [0,1])$ and $\delta > 0$ such that:

(i) $\tau(\chi_j^m(\tilde{\phi}^{\mathbb{C}}(g_0))) > \frac{1}{n}$ *for all $j = 1, \ldots, m$ and $\tau \in T(B \otimes_{\mathbb{R}} \mathbb{C})$,*

(ii) $\tau(\chi_j^{3n}(\tilde{\phi}^{\mathbb{C}}(g_0))) > 2\delta$ *for all $j = 1, \ldots, 3n$ and $\tau \in T(B \otimes_{\mathbb{R}} \mathbb{C})$,*

(iii) $\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z) = \lambda_1 z^r$ *and* $\mathrm{Det}(\tilde{\psi}^{\mathbb{C}}(g_0))(z) = \lambda_2 z^r$ *for some constants $\lambda_1, \lambda_2 \in \mathbb{T}$ and $r \in \mathbb{Z}$,*

(iv) $|\tau(\tilde{\phi}^{\mathbb{C}}(f(g_0))) - \tau(\tilde{\psi}^{\mathbb{C}}(f(g_0)))| \leq \delta, f \in F$ *and* $\tau \in T(B \otimes_{\mathbb{R}} \mathbb{C})$,

then there is a unitary $V \in B$ *such that*

$$\|V\tilde{\phi}(g)V^* - \tilde{\psi}(g)\| \leq \pi\left(\frac{28}{m} + \frac{6}{n}\right)$$

where $g \in \{g_0, g_1, g_2\}$ *is one of the canonical central generators of* A.

Proof. Note that $\tilde{\phi}^{\mathbb{C}}(g_0)$ and $\tilde{\psi}^{\mathbb{C}}(g_0)$ are well-defined because we can write $g_0 = g_1 \otimes 1 + g_2 \otimes i$, or $g_0 = g_0 \otimes 1$ depending on the type of A. By spectral mapping theorem,

$$\mathrm{Sp}^{\mathbb{C}}(\tilde{\phi}^{\mathbb{C}}(g_1)) = \mathrm{Re}(\mathrm{Sp}^{\mathbb{C}}(\tilde{\phi}^{\mathbb{C}}(g_0))), \;\; \mathrm{Sp}^{\mathbb{C}}(\tilde{\phi}^{\mathbb{C}}(g_2)) = \mathrm{Im}(\mathrm{Sp}^{\mathbb{C}}(\tilde{\phi}^{\mathbb{C}}(g_0))).$$

We use the notations $\lambda_i^1 = \mathrm{Re}(\lambda_i), \mu_i^1 = \mathrm{Re}(\mu_i), \lambda_i^2 = \mathrm{Im}(\lambda_i), \mu_i^2 = \mathrm{Im}(\mu_i)$, where λ_i and μ_i are the eigenfunctions of $\tilde{\phi}^{\mathbb{C}}(g_0)$ and $\tilde{\psi}^{\mathbb{C}}(g_0)$ respectively.

(i) If $A = C(\mathbb{T}, \eta_0)$ and B is a basic building block of type 1, 3, or 5 then by Lemma 4.14 there exists a unitary $V \in B$ such that

$$\|V\tilde{\phi}(g_0)V^* - \tilde{\psi}(g_0)\| = \left\|\sum_{i=1}^{k} \lambda_i q_i - \sum_{i=1}^{k} \mu_i q_i\right\|.$$

If B is a building block of type 4, then we proceed as in the last paragraph of [30, Proposition 2.6]. We block digonalize $\tilde{\phi}(g_0)$ and $\tilde{\psi}(g_0)$ by unitaries $U_{\tilde{\phi}} \in B$ and $U_{\tilde{\psi}} \in B$ taking into account the following:

$$\begin{aligned}\|U_{\tilde{\psi}}^* U_{\tilde{\phi}} \tilde{\phi}(g_0) U_{\tilde{\phi}}^* U_{\tilde{\psi}} - \tilde{\psi}(g_0)\| &= \|U_{\tilde{\phi}}\tilde{\phi}(g_0)U_{\tilde{\phi}}^* - U_{\tilde{\psi}}\tilde{\psi}(g_0)U_{\tilde{\psi}}^*\| \\ &= \|W(U_{\tilde{\phi}}\tilde{\phi}(g_0)U_{\tilde{\phi}}^* - U_{\tilde{\psi}}\tilde{\psi}(g_0)U_{\tilde{\psi}}^*)W^*\| \\ &= \|\,\mathrm{diag}(\lambda_1 - \mu_1, \ldots, \lambda_k - \mu_k)\|\end{aligned}$$

where W is the constant unitary of Lemma 4.7.

(ii) If $A = C(\mathbb{T}, \mathbb{R})$, then by Lemma 4.14 and Remark 4.11 for $g \in \{g_1, g_2\}$ there exists a unitary $V \in B$ such that

$$\|V\tilde{\phi}(g)V^* - \tilde{\psi}(g)\| = \left\|\sum_{i=1}^{k} \lambda_i^j q_i - \sum_{i=1}^{k} \mu_i^j q_i\right\|$$

where $j \in \{1, 2\}$.

Note that if $\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z) = \lambda_1 z^{r_1}$, $\mathrm{Det}(\tilde{\psi}^{\mathbb{C}}(g_0))(z) = \lambda_2 z^{r_2}$ and $r_1 \neq r_2$, then $w(\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z) \neq w(\mathrm{Det}(\tilde{\psi}^{\mathbb{C}}(g_0))(z)$ which is a contradiction. Thus, under our assumption, the continuous function α in condition (3) of [24, Lemma 2.3] is zero and $\mu = \lambda_1\lambda_2^{-1}$.

In all cases, conditions (i)–(iv) and [24, Lemma 2.3] implies that

$$\|\lambda_i - \mu_i\| \leq \pi\left(\frac{28}{m} + \frac{6}{n}\right)$$

for all $1 \leq i \leq k$. Hence,

$$\begin{aligned}\left\|\sum_{i=1}^{k} \lambda_i^j q_i - \sum_{i=1}^{k} \mu_i^j q_i\right\| &\leq \max\{\|\lambda_i^j - \mu_i^j\| : 1 \leq i \leq k\} \\ &\leq \max\{\|\lambda_i - \mu_i\| : 1 \leq i \leq k\} \leq \pi\left(\frac{28}{m} + \frac{6}{n}\right). \quad \square\end{aligned}$$

Theorem 4.16. *For a fixed $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$, let A and B be direct sums of basic building blocks of a real circle-quotient algebra associated to a real $A\mathbb{T}_J$-algebra (cf. Definition 5.2) and let ϕ and ψ be unital $*$-homomorphisms from A to B giving rise to the same map from the pair $K_0(A) \longrightarrow K_0(A \otimes_{\mathbb{R}} \mathbb{C})$ to the pair $K_0(B) \longrightarrow K_0(B \otimes_{\mathbb{R}} \mathbb{C})$ and from*

$$K_1(A)/\operatorname{Tor}(K_1(A)) \longrightarrow K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(A)/\operatorname{Tor}(K_1(A))$$

to

$$K_1(B)/\operatorname{Tor}(K_1(B)) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B)/\operatorname{Tor}(K_1(B)).$$

If the images of $\phi^{\mathbb{C}}$ and $\psi^{\mathbb{C}}$ on the basic building blocks having circle as their spectrum satisfy the conditions in the hypothesis of Lemma 4.15 and on the basic building blocks having interval as their spectrum satisfy the conditions in the hypothesis of [30, Proposition 2.6], then there exists a unitary $u \in B$ such that $\phi^{\mathbb{C}}$ and $(\operatorname{Ad}(u))\psi^{\mathbb{C}}$ agree to within $\pi(\frac{28}{m} + \frac{6}{n})$ on the canonical generators of $A \otimes_{\mathbb{R}} \mathbb{C}$.

Proof. This is the analogue of [10, Theorem 4] and its proof follows from Lemma 4.1, Lemma 4.3, Lemma 4.4, [30, Lemma 2.1], [30, Lemma 2.2], [30, Lemma 2.3], Lemma 4.15 and [30, Proposition 2.6]. $\square$

Remark 4.17. As it is pointed out on page 129 of [10], the determinant hypothesis in Lemma 4.15 can be weakened to the requirement that the images of canonical unitary generator under the two (complexified) maps have the same determinant.

5. The reduction theorem

Lemma 5.1. *Let $A = C(X, \tilde{\phi}) \otimes_{\mathbb{R}} M_n(\mathbb{F})$ and B be a real unital C^*-algebra where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, X is a compact Hausdorff space and $\tilde{\phi}$ is the involutive homeomorphism of X. If $\psi : A \longrightarrow B$ is a unital $*$-homomorphism, then $\psi(A) \simeq C(F \cup \tilde{\phi}(F), \tilde{\phi}) \otimes_{\mathbb{R}} M_n(\mathbb{F})$ where F is a closed subset of X.*

Proof. Clearly, $\psi(A) \simeq A/\operatorname{Ker}(\psi)$ and $\operatorname{Ker}(\psi)$ as a closed ideal of A is of the form $I \otimes_{\mathbb{R}} M_n(\mathbb{F})$ where

$$I = \{f \in C(X, \tilde{\phi}) \mid f_{|F} = 0\} = \{f \in C(X, \tilde{\phi}) \mid f_{|F \cup \tilde{\phi}(F)} = 0\}$$

for some closed subset F of X. The map

$$h : (C(X, \tilde{\phi})/I) \otimes_{\mathbb{R}} M_n(\mathbb{F}) \longrightarrow C(F \cup \tilde{\phi}(F), \tilde{\phi}) \otimes_{\mathbb{R}} M_n(\mathbb{F})$$

defined by $h([f]\otimes c) = f_{|F\cup\tilde{\phi}(F)}\otimes c$ is an isomorphism and

$$A/\operatorname{Ker}(\psi) \simeq (C(X,\tilde{\phi})/I)\otimes_{\mathbb{R}} M_n(\mathbb{F}). \qquad \square$$

In the above lemma, the space F is compact but it is not a CW-complex. Therefore, we need the following definition inspired by [24, Lemma 1.3] to reduce the problem to so-called good quotients:

Definition 5.2. A real C^*-algebra A is called a real circle-quotient algebra associated to a real $\mathrm{A}\mathbb{T}_2$-algebra if $A = \bigoplus_{j=1}^m A_j^i$, where $i \in \{1,\dots,6\} \cup \{9,\dots,16\}$ and A_j^i are of one of the following forms:

$$\begin{aligned}
A_j^1 &= C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}} M_{n_j}(\mathbb{C})\\
A_k^2 &= C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})\\
A_j^3 &= C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}} M_{\frac{n_j}{2}}(\mathbb{H})\\
A_j^4 &= C(\mathbb{T},\eta_0)\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})\\
A_j^5 &= C(\mathbb{T},\eta_0)\otimes_{\mathbb{R}} M_{\frac{n_j}{2}}(\mathbb{H})\\
A_j^6 &= C(\mathbb{T},\eta_1)\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})\\
A_j^9 &= M_{n_j}(\mathbb{C})\\
A_j^{10} &= M_{n_j}(\mathbb{R})\\
A_j^{11} &= M_{\frac{n_j}{2}}(\mathbb{H})\\
A_j^{12} &= C([0,1],\mathbb{R})\otimes_{\mathbb{R}} M_{n_j}(\mathbb{C})\\
A_j^{13} &= C([0,1],\mathbb{R})\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})\\
A_j^{14} &= C([0,1],\mathbb{R})\otimes_{\mathbb{R}} M_{\frac{n_j}{2}}(\mathbb{H})\\
A_j^{15} &= C([0,1],\eta_2)\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})\\
A_j^{16} &= C([0,1],\eta_2)\otimes_{\mathbb{R}} M_{\frac{n_j}{2}}(\mathbb{H})
\end{aligned}$$

where $\eta_2(t) = 1-t$ (cf. [30]). If $C(\mathbb{T},\eta_1)\otimes_{\mathbb{R}} M_{n_j}(\mathbb{R})$ is not in the list of building blocks then A is called a real circle-quotient algebra associated to a real $\mathrm{A}\mathbb{T}_1$-algebra. Moreover, A is called a real circle-quotient algebra associated to a real $\mathrm{A}\mathbb{T}_J$-algebra if $j \in J$ and

$$J \in \{\{1,9,12\},\{3,4,10,11,14,15\},\{3,5,11,14,16\}\}.$$

Remark 5.3. The functions $g_5 \in C([0,1],\mathbb{R})$ and $g_6 \in C([0,1],\eta_2)$ defined by $g_5(t) = t$ and $g_6(t) = i(\frac{1}{2}-t)$ are generators of $C([0,1],\mathbb{R})$ and $C([0,1],\eta_2)$ respectively (cf. [30]).

Lemma 5.4. *Let A be a real C^*-algebra and $\pi : A \longrightarrow B(H_{\mathbb{F}})$ be a finite-dimensional representation of A where $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$. Then, π is unitarily equivalent to a direct sum of irreducible representations.*

Proof. For complex C^*-algebras, this is proved on page 36 of [8, 2.3.5]. The proof carries to the real case as well. Note that a nondegenerate representation is a direct sum of cyclic representations. For $H_{\mathbb{F}} = H_{\mathbb{R}}$, combine [20, Proposition 5.8.8, (2)], [20, Proposition 5.2.7, (3)], and [20, Proposition 5.3.7, (3)]. $\square$

Lemma 5.5. *Let A be a real commutative C^*-algebras and let $B = A \otimes_{\mathbb{R}} M_n(\mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then, every irreducible representation of B is unitarily equivalent to $\pi_1 \otimes \pi_2$ where π_1 is an irreducible representation of A and π_2 is an irreducible representation of $M_n(\mathbb{F})$.*

Proof. For complex C^*-algebras, this is proved in [25, Lemma B.48] and its proof carries to any real GCR (postliminal) C^*-algebra including the real C^*-algebra B. $\square$

Proposition 5.6. *Let $\phi : C(X_i, \eta_i) \otimes_{\mathbb{R}} M_{n_i}(\mathbb{F}_i) \longrightarrow C(X_j, \eta_j) \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F}_j)$ be a $*$-homomorphism, $\mathbb{F}_i, \mathbb{F}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, X_i, X_j be compact Hausdorff spaces and η_i, η_j be the involutive homeomorphisms of X_i, X_j, then given*

$$f = g \otimes a \in C(X_i, \eta_i) \otimes_{\mathbb{R}} M_{n_i}(\mathbb{F}_i)$$

and $y \in X_j$, there exist $x_1, \ldots, x_k \in X_i$, a standard homomorphism (cf. [13, Definition 3.1]*) $\mu : M_{n_i}(\mathbb{F}_i) \longrightarrow M_m(\mathbb{F}_j)$ and a unitary $u \in \mathbb{F} \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F}_j)$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $n_j \geq mk$ such that*

$$\phi(f)(y) = \mathrm{Ad}(u)(\mathrm{diag}(g(x_1) \otimes \mu(a), \ldots, g(x_k) \otimes \mu(a), 0, .., 0)).$$

Proof. By Lemma 5.4, the representation

$$\pi := ev_y \circ \phi : C(X_i, \eta_i) \otimes_{\mathbb{R}} M_{n_i}(\mathbb{F}_i) \longrightarrow \mathbb{F} \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F}_j),$$

$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, is unitarily equivalent to $\oplus_{i=1}^k \pi_i$ where each π_i is irreducible (note that some of them may be zero). By Lemma 5.5, each π_i is unitarily equivalent to $\pi_i^1 \otimes \pi_i^2$ where π_i^1 is an irreducible representation of $C(X_i, \eta_i)$ and π_i^2 is an irreducible representation of $M_{n_i}(\mathbb{F}_i)$. An irreducible representation of $C(X_i, \eta_i)$ is a point-evaluation map. By [13, Lemma 3.5], any homomorphism (including irreducible representations) from $M_{n_i}(\mathbb{F}_i)$ into another real matrix algebra is unitarily equivalent to a standard homomorphism. In summary, there exist a unitary $u \in \mathbb{F} \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F}_j)$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, a standard homomorphism $\mu : M_{n_i}(\mathbb{F}_i) \longrightarrow M_m(\mathbb{F}_j)$ and points $x_1, \ldots, x_k \in X_i$ such that $\pi(f) = \phi(f)(y) = \mathrm{Ad}(u)(\mathrm{diag}(g(x_1) \otimes \mu(a), \ldots, g(x_k) \otimes \mu(a), 0, \ldots, 0))$ where $f = g \otimes a$. Note that there is no incompatibility or inconsistency issue because nonexistence of the representation π implies nonexistence of the homomorphism ϕ. $\square$

Definition 5.7. In Proposition 5.6, the set $\{x_1, \ldots, x_k\}$ is called the spectrum of $\phi(f)$ at y, and is denoted by $\mathrm{Spec}(\phi(f)(y))$. We define the spectrum of $\phi(f)$ by

$$\mathrm{Spec}(\phi(f)) := \cup_{y \in X_j} \mathrm{Spec}(\phi(f)(y)).$$

Moreover, if $A = \oplus_{i\in I} A_i$ where A_i and B are of the type defined in Proposition 5.6, $\phi : A \longrightarrow B$ is a $*$-homomorphism, and $y \in X_j$ then

$$\operatorname{Spec}(\phi(f)(y)) := \cup_{i\in I} \operatorname{Spec}(\phi^i(f)(y)).$$

Theorem 5.8. *For a fixed $J \in \{\{1\}, \{3, 4\}, \{3, 5\}\}$, let $A \simeq \varinjlim(A_i, \phi_{i,i+1})$ be a simple unital infinite-dimensional real $A\mathbb{T}_J$-algebra. Then,*

(i) *A is also the direct limit of a sequence of real circle-quotient algebras associated to the real $A\mathbb{T}_J$-algebra A with unital injective connecting maps.*

(ii) *The above inductive sequence with injective connecting maps can be perturbed so that its complexification satisfies the uniformly varying determinant property (preserving the injectivity and agreeing with the above sequence approximately at the level of traces):*

$$\operatorname{Det}(\phi^{\mathbb{C}}_{i,i+1}(g_0))(z) = \lambda z^k$$

for all i where $\lambda \in \mathbb{T}$, $k \in \mathbb{Z}$ are constants, and $g_0(z) = z$ is the generator of $C(\mathbb{T}, \mathbb{C})$ and $\operatorname{Det}(\phi^{\mathbb{C}}_{i,i+1}(g_5))(t) = c$ where $c \in \mathbb{R}$ is a constant and $g_5(t) = t$ is the generator of $C([0, 1], \mathbb{C})$. Moreover, A is also the direct limit of this new inductive system.

Proof. (i) We divide the proof into three steps:

Step 1. Let $I_1 \subset \operatorname{Ker}(\phi_{1,2})$ be an ideal of A_1 such that the spectrum of I_1 is a union of finitely many arc-segments and points. If we define

$$\psi_{1,2} : A_1 \xrightarrow{\pi_1} A_1/I_1 \xrightarrow{\beta_1} A_1/\operatorname{Ker}(\phi_{1,2}) \xrightarrow{\gamma_1} A_2$$

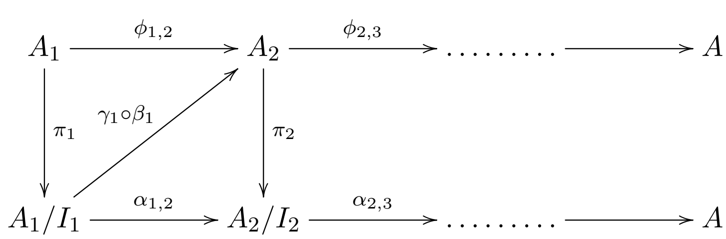
where π_1, β_1 are the unital surjective canonical homomorphisms and γ_1 is the canonical unital injective homomorphism, then $\psi_{1,2} = \phi_{1,2}$. Next, we define

$$\psi_{2,3} : A_2 \xrightarrow{\pi_2} A_2/I_2 \xrightarrow{\beta_2} A_2/\operatorname{Ker}(\phi_{2,3}) \xrightarrow{\gamma_2} A_3$$

and

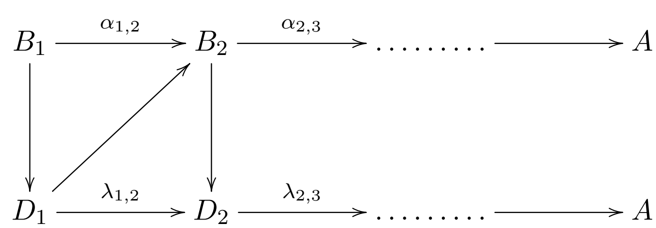
$$\alpha_{1,2} : A_1/I_1 \xrightarrow{\pi_2\circ\gamma_1\circ\beta_1} A_2/I_2.$$

Therefore, we have the following intertwining diagram:



Hence, $A = \varinjlim(B_i, \alpha_{i,i+1})$ where $B_i = A_i/I_i$ are real circle-quotient algebras.

Step 2. If there exits an B_i^l in $B_i = \oplus_{k=1}^m B_i^k$ such that spectrum of B_i^l is $\mathbb{T}$ and $\alpha_{i,\infty}^l$ is injective, then we leave it untouched. However, if there exists an B_i^j in $B_i = \oplus_{k=1}^m B_i^k$ such that spectrum of B_i^j is $\mathbb{T}$ and $\alpha_{i,i+1}^j$ is not injective, then we choose an ideal $I_j \subset \mathrm{Ker}(\alpha_{i,i+1}^j)$ such that B_i^j/I_j is a direct sum of blocks with spectrum the interval or point. Define D_i by replacing B_i^j in B_i with B_i^j/I_j and $\alpha_{i,i+1}^j$ in $\alpha_{i,i+1}$ with $\gamma_j \circ \beta_j$. Therefore, we have the following intertwining diagram:



Hence, $A = \varinjlim(D_i, \lambda_{i,i+1})$ where the maps $\lambda_{i,i+1}$ are injective on the summands with spectrum the circle.

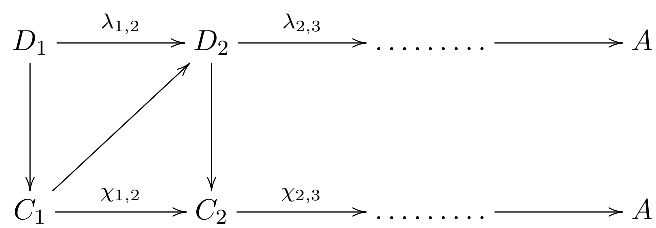
Step 3. In this step, all the partial maps on the summands with spectrum the circle or point are injective where the former is the consequence of Step 2 and the latter is the consequence of the simplicity of matrix algebras. Suppose that D_1^j is an interval building block summand of D_1 and that $\lambda_{1,\infty}^j := \lambda_{1,\infty}|_{D_1^j}$ is not injective. Let U_n denote the spectrum of $\mathrm{Ker}(\lambda_{1,n}^j)$ and U the spectrum of $\mathrm{Ker}(\lambda_{1,\infty}^j)$ identified in the canonical way as open subsets of the spectrum of D_1^j. Then, we have $U_n \subseteq U_{n+1}$ for all n, and $U = \cup_{n=1}^\infty U_n$. Let K denote the spectrum of $\lambda_{1,\infty}^j$ identified as a closed subset of the spectrum in the canonical way, i.e., K is the compliment of U. Choose a summable sequence $\{\delta_n\}$ of positive real numbers. Choose a finite set $J_1, \dots, J_l$ of pairwise disjoint closed subintervals of the spectrum of D_1^j with the following properties:

(1) The endpoints of the J_i's are in U.
(2) The compliment of $J_1 \cup J_2 \cup \cdots \cup J_l$ is contained in U. Denote this compliment by V.
(3) The set V is invariant under the involution.
(4) The set K which by (2) is contained in the union of the J's, is δ_1-dense in this union.

It follows that the closure of V is contained in U. Since the closure of V is a compact set, it follows that for some m, we have this closure being contained in U_m, and consequently V is contained in U_m. Let I_V denote the involution invariant ideal of D_1^j corresponding to the open set V. We then have that D_1^j/I_V is a finite direct sum of interval algebras, having spectra the J_i's. Furthermore, the map $\lambda_{1,m}^j$ factors through this quotient in the canonical way: $D_1^j \to D_1^j/I_V \to D_m$. Now, we do this for each interval summand of D_1 (possibly having to increase m). We get a new circle quotient algebra C_1 and maps $\pi_1 : D_1 \to C_1$ and $\psi_1 : C_1 \to D_m$ with the following properties:

(1) C_1 has the same circle summands as D_1, and the map π_1 is just the identity on all of the circle summands.
(2) $\psi_1 \circ \pi_1 = \lambda_{1,m}$.
(3) For each interval summand, C_1^s, the spectrum of $\lambda_{m,\infty} \circ \psi_{1,m}^s$ is δ_1-dense in the spectrum of C_1^s, when these latter spectra are identified with the J's.

Now, we relabel D_m with D_2, and proceed to find C_2, π_2, and ψ_2 in the same way with δ_2. Therefore, we have the following intertwining diagram:



where the D_n all have the same maps into the limit as they did before. Thus, all of the partial maps involving circle type summands are injective. Furthermore, passing to the subsequence of the C's, we have that the spectrum of the image in the limit is δ_n-dense in each interval type summand. In the new inductive system, suppose C_1^s is an interval type summand of C_1, and let g_s be the central generator of C_1^s. Then, $\chi_{1,2}(g_s)$ is a self-adjoint (skew-adjoint) element of C_2 whose spectrum is contained in the appropriate J, and, since it gets mapped to the image of the same old $\lambda_{2,\infty}^s$ in the new system, its spectrum is δ_1-dense in this J. Thus it may be perturbed to a new generator, commuting with the matrix units of the image of $\chi_{1,2}(C_1^s)$ having the whole of J as spectrum, and being still within some fixed multiple of δ away from $\chi_{1,2}(g_s)$. This defines a new partial map from C_1^s into C_2, which we may assume takes the unit of C_1^s to the same projection of C_2 as the old one, is injective, and agrees with the old map to within some fixed multiple of δ_1 on the generators. Define a map $\mu_{1,2} : C_1 \to C_2$ to be these new maps on the interval type summands of C_1, and equal to $\chi_{1,2}$ on the circle type summands. Then $\mu_{1,2}$ is injective, and agrees approximately on generators with $\chi_{1,2}$.

Now, we have a new inductive system $\{C_i, \mu_{i,i+1}\}$ with injective connecting maps and the following approximately intertwining diagram:

$$\begin{array}{ccccccc}
C_1 & \xrightarrow{\mu_{1,2}} & C_2 & \xrightarrow{\mu_{2,3}} & \cdots\cdots\cdots & \longrightarrow & A \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & & & \\
C_1 & \xrightarrow{\chi_{1,2}} & C_2 & \xrightarrow{\chi_{2,3}} & \cdots\cdots\cdots & \longrightarrow & A
\end{array}$$

(ii) Note that injectivity of ϕ implies injectivity of the complexification map $\phi^{\mathbb{C}} := \phi \otimes id$. We can perturb the injective maps such that their image on the set of canonical central generators has distinct eigenfuncions and this perturbation has no adverse effect on K_0-groups, K_1-groups of complexification, traces and injectivity of maps. It suffices to consider the center of a single building block in the source algebra which we denote it by A and a single building block in the target algebra which we denote it by B.

(1) If $A = C(\mathbb{T}, \mathbb{R})$ and $B = C(\mathbb{T}, \mathbb{R}) \otimes_{\mathbb{R}} M_n(\mathbb{C})$, then

$$\phi^{\mathbb{C}}(g_0) = \phi(g_1) \otimes 1 + \phi(g_2) \otimes i$$

and by spectral decomposition:

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i(t) P_i(t) = \phi^{\mathbb{C}}(g_0)(t) &= \phi\left(\frac{g_0 + g_0^*}{2}\right)(t) + i\phi\left(\frac{g_0 - g_0^*}{2i}\right)(t) \\
&= \phi(g_1)(t) + i\phi(g_2)(t) \\
&= \sum_{i=1}^{n} \mathrm{Re}(\lambda_i)(t) P_i(t) + i \sum_{i=1}^{n} \mathrm{Im}(\lambda_i)(t) P_i(t).
\end{aligned}$$

Let $\alpha = \frac{\mathrm{Det}(\phi^{\mathbb{C}}(g_0))(1) g_0^k}{\mathrm{Det}(\phi^{\mathbb{C}}(g_0))} = e^{2i\pi F}$ where $F \in C([0,1], \mathbb{R})$ and $k = w(\mathrm{Det}(\phi^{\mathbb{C}}(g_0)))$ so that winding number of α becomes zero. Let ψ be the involutive $*$-antiautomorphism of $B \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$(B \otimes_{\mathbb{R}} \mathbb{C})_{\psi} = B$$

and $\tilde{\psi}$ be its extension to its ambient interval algebra. Pick an eigenfunction $\lambda_j(t) = e^{2i\pi G_j(t)} \in \mathrm{Sp}(\phi^{\mathbb{C}}(g_0)(t))$ where $G_j \in C([0,1], \mathbb{R})$ and perturb it as follows:

$$\begin{aligned}
\tilde{\phi}(g_1)(t) = \frac{1}{2} &\left[\sum_{i \neq j}^{n} \mathrm{Re}(\lambda_i)(t) P_i(t) + \tilde{\psi}(\mathrm{Re}(\lambda_j))(t) \tilde{\psi}(P_i)(t)\right] \\
&+ \frac{1}{2}[\mathrm{Re}(\alpha\lambda_j)(t) P_j(t) + \tilde{\psi}(\mathrm{Re}(\alpha\lambda_j))(t) \tilde{\psi}(P_j)(t)] \\
\tilde{\phi}(g_2)(t) = \frac{1}{2} &\left[\sum_{i \neq j}^{n} \mathrm{Im}(\lambda_i)(t) P_i(t) + \tilde{\psi}(\mathrm{Im}(\lambda_i))(t) \tilde{\psi}(P_i)(t)\right]
\end{aligned}$$

$$+\frac{1}{2}[\mathrm{Im}(\alpha\lambda_j)(t)P_j(t)+\tilde{\psi}(\mathrm{Im}(\alpha\lambda_j))(t)\tilde{\psi}(P_j)(t)].$$

Hence,

$$\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z)=\left[\prod_{i=1,i\neq j}^{n}\lambda_i(z)\right]\alpha(z)\lambda_j(z)=\mathrm{Det}(\phi^{\mathbb{C}}(g_0))(1)z^k.$$

(2) If $A=C(\mathbb{T},\mathbb{R})$ and $B=C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}}M_{\frac{n}{2}}(\mathbb{H})$, then we have double degeneracy. Let ψ be the involutive $*$-antiautomorphism of $B\otimes_{\mathbb{R}}\mathbb{C}$ such that $(B\otimes_{\mathbb{R}}\mathbb{C})_\psi=B$. Since winding number of α (α is defined in case (1)) is zero, the fractional powers of α exist (here, we set $\tilde{\alpha}^4=\alpha$). On the other hand, one can check that $\tilde{\psi}$ permutes the eigenprojections (cf. Remark 4.8), i.e., $\tilde{\psi}(P_j)=P_{\sigma(j)}$ for some permutation σ where P_j is an eigenprojection in the spectral decomposition. We can consider the following perturbations:

$$\begin{aligned}\tilde{\phi}(g_1)(t)=\frac{1}{2}&\left[\sum_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n}\mathrm{Re}(\lambda_i)(t)(P_i(t)+\tilde{\psi}(P_i)(t))\right]\\&+\frac{1}{2}\mathrm{Re}(\tilde{\alpha}\lambda_j)(t)(P_j(t)+\tilde{\psi}(P_j)(t))\\&+\frac{1}{2}\mathrm{Re}(\tilde{\alpha}\lambda_{\sigma(j)})(t)(P_{\sigma(j)}(t)+\tilde{\psi}(P_{\sigma(j)})(t))\\&+\frac{1}{2}\mathrm{Re}(\tilde{\alpha}\lambda_l)(t)(P_l(t)+\tilde{\psi}(P_l)(t))\\&+\frac{1}{2}\mathrm{Re}(\tilde{\alpha}\lambda_{\sigma(l)})(t)(P_{\sigma(l)}(t)+\tilde{\psi}(P_{\sigma(l)})(t))\end{aligned}$$

where $\mathrm{Re}(\lambda_j)=\mathrm{Re}(\lambda_l)$ due to the double degeneracy.

$$\begin{aligned}\tilde{\phi}(g_2)(t)=\frac{1}{2}&\left[\sum_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n}\mathrm{Im}(\lambda_i)(t)(P_i(t)+\tilde{\psi}(P_i)(t))\right]\\&+\frac{1}{2}\mathrm{Im}(\tilde{\alpha}\lambda_j)(t)(P_j(t)+\tilde{\psi}(P_j)(t))\\&+\frac{1}{2}\mathrm{Im}(\tilde{\alpha}\lambda_{\sigma(j)})(t)(P_{\sigma(j)}(t)+\tilde{\psi}(P_{\sigma(j)})(t))\\&+\frac{1}{2}\mathrm{Im}(\tilde{\alpha}\lambda_l)(t)(P_l(t)+\tilde{\psi}(P_l)(t))\\&+\frac{1}{2}\mathrm{Im}(\tilde{\alpha}\lambda_{\sigma(l)})(t)(P_{\sigma(l)}(t)+\tilde{\psi}(P_{\sigma(l)})(t))\end{aligned}$$

where $\mathrm{Im}(\lambda_j)=\mathrm{Im}(\lambda_l)$ due to the double degeneracy. Hence,

$$\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z)=\left[\prod_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n}\lambda_i(z)\right](\tilde{\alpha}(z))^4\lambda_j(z)\lambda_{\sigma(j)}(z)\lambda_l(z)\lambda_{\sigma(l)}(z)$$

$$= \operatorname{Det}(\phi^{\mathbb{C}}(g_0))(1)z^k.$$

(3) If $A = C(\mathbb{T}, \mathbb{R})$ and $B = C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_n(\mathbb{R})$, and ψ is the involutive $*$-antiautomorphism of $B \otimes_{\mathbb{R}} \mathbb{C}$ such that $(B \otimes_{\mathbb{R}} \mathbb{C})_\psi = B$, then $\tilde{\psi}(P_j) = P_{\sigma(j)}$ for some permutation σ where P_j is an eigenprojection in the spectral decomposition. According to the diagram of K_1-triples, the map $\phi^{\mathbb{C}} : K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C})$ is zero. Therefore, $w(\operatorname{Det}(\phi^{\mathbb{C}}(g_0))) = 0$ implying that $\alpha = \frac{\operatorname{Det}(\phi^{\mathbb{C}}(g_0))(1)}{\operatorname{Det}(\phi^{\mathbb{C}}(g_0))}$ and $\alpha(\bar{z}) = \alpha(z)$. Since winding number of α is zero, the fractional powers of α exist (here, we set $\tilde{\alpha}^2 = \alpha$). We can consider the following perturbations (cf. Remark 4.8):

$$\begin{aligned}
\tilde{\phi}(g_1)(t) = \frac{1}{2}&\left[\sum_{i\notin\{j,\sigma(j)\}}^{n} \operatorname{Re}(\lambda_i)(t)P_i(t) + \tilde{\psi}(\operatorname{Re}(\lambda_i))(t)\tilde{\psi}(P_i)(t)\right] \\
&+ \frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_j)(t)P_j(t) + \frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_j))(t)\tilde{\psi}(P_j)(t) \\
&+ \frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(j)})(t)P_{\sigma(j)}(t) + \frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(j)}))(t)\tilde{\psi}(P_{\sigma(j)})(t) \\
\tilde{\phi}(g_2)(t) = \frac{1}{2}&\left[\sum_{i\notin\{j,\sigma(j)\}}^{n} \operatorname{Im}(\lambda_i)(t)P_i(t) + \tilde{\psi}(\operatorname{Im}(\lambda_i))(t)\tilde{\psi}(P_i)(t)\right] \\
&+ \frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_j)(t)P_j(t) + \frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_j))(t)\tilde{\psi}(P_j)(t) \\
&+ \frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(j)})(t)P_{\sigma(j)}(t) + \frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(j)}))(t)\tilde{\psi}(P_{\sigma(j)})(t).
\end{aligned}$$

Hence,

$$\operatorname{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z) = \left[\prod_{i\notin\{j,\sigma(j)\}}^{n} \lambda_i(z)\right](\tilde{\alpha}(z))^2\lambda_j(z)\lambda_{\sigma(j)}(z) = \operatorname{Det}(\phi^{\mathbb{C}}(g_0))(1).$$

(4) If $A = C(\mathbb{T}, \mathbb{R})$ and $B = C(\mathbb{T}, \eta_0) \otimes_{\mathbb{R}} M_{\frac{n}{2}}(\mathbb{H})$, and ψ is the involutive $*$-antiautomorphism of $B \otimes_{\mathbb{R}} \mathbb{C}$ such that $(B \otimes_{\mathbb{R}} \mathbb{C})_\psi = B$, then $\tilde{\psi}(P_j) = P_{\sigma(j)}$ for some permutation σ where P_j is an eigenprojection in the spectral decomposition. According to the diagram of K_1-triples, the map $\phi^{\mathbb{C}} : K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C})$ is zero. Therefore, $w(\operatorname{Det}(\phi^{\mathbb{C}}(g_0))) = 0$ implying that $\alpha = \frac{\operatorname{Det}(\phi^{\mathbb{C}}(g_0))(1)}{\operatorname{Det}(\phi^{\mathbb{C}}(g_0))}$ and $\alpha(\bar{z}) = \alpha(z)$. Since winding number of α is zero, the fractional powers of α exist (here, we set $\tilde{\alpha}^4 = \alpha$). We can consider the following perturbations (cf. Remark 4.8):

$$\tilde{\phi}(g_1)(t) = \frac{1}{2}\left[\sum_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n} \operatorname{Re}(\lambda_i)(t)P_i(t) + \tilde{\psi}(\operatorname{Re}(\lambda_i))(t)\tilde{\psi}(P_i)(t)\right]$$

$$
\begin{aligned}
&+\frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_j)(t)P_j(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_j))(t)\tilde{\psi}(P_j)(t)\\
&+\frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(j)})(t)P_{\sigma(j)}(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(j)}))(t)\tilde{\psi}(P_{\sigma(j)})(t)\\
&+\frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_l)(t)P_l(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_l))(t)\tilde{\psi}(P_l)(t)\\
&+\frac{1}{2}\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(l)})(t)P_{\sigma(l)}(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Re}(\tilde{\alpha}\lambda_{\sigma(l)}))(t)\tilde{\psi}(P_{\sigma(l)})(t)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\phi}(g_2)(t)=\frac{1}{2}&\left[\sum_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n}\operatorname{Im}(\lambda_i)(t)P_i(t)+\tilde{\psi}(\operatorname{Im}(\lambda_i))(t)\tilde{\psi}(P_i)(t)\right]\\
&+\frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_j)(t)P_j(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_j))(t)\tilde{\psi}(P_j)(t)\\
&+\frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(j)})(t)P_{\sigma(j)}(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(j)}))(t)\tilde{\psi}(P_{\sigma(j)})(t)\\
&+\frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_l)(t)P_l(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_l))(t)\tilde{\psi}(P_l)(t)\\
&+\frac{1}{2}\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(l)})(t)P_{\sigma(l)}(t)+\frac{1}{2}\tilde{\psi}(\operatorname{Im}(\tilde{\alpha}\lambda_{\sigma(l)}))(t)\tilde{\psi}(P_{\sigma(l)})(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\operatorname{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z)&=\left[\prod_{i\notin\{j,l,\sigma(j),\sigma(l)\}}^{n}\lambda_i(z)\right](\tilde{\alpha}(z))^4\lambda_j(z)\lambda_{\sigma(j)}(z)\lambda_l(z)\lambda_{\sigma(l)}(z)\\
&=\operatorname{Det}(\phi^{\mathbb{C}}(g_0))(1).
\end{aligned}
$$

(5) If $A=C(\mathbb{T},\eta_0)$ and $B=C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}}M_n(\mathbb{C})$, then consider the following perturbation:

$$
\tilde{\phi}(g_0)(z)=\left[\sum_{i=1,i\neq j}^{n}\lambda_i(z)P_i(z)\right]+\alpha(z)\lambda_j(z)P_j(z).
$$

(6) If $A=C(\mathbb{T},\eta_0)$ and $B=C(\mathbb{T},\mathbb{R})\otimes_{\mathbb{R}}M_{\frac{n}{2}}(\mathbb{H})$, then eigenfunctions appear as conjugate pairs and the determinant is automatically constant.

(7) If $A=C(\mathbb{T},\eta_0)$ and $B=C(\mathbb{T},\eta_0)\otimes_{\mathbb{R}}M_n(\mathbb{R})$, and ψ is the involutive $*$-antiautomorphism of $B\otimes_{\mathbb{R}}\mathbb{C}$ such that $(B\otimes_{\mathbb{R}}\mathbb{C})_\psi=B$, then $\tilde{\psi}(P_j)=P_{\sigma(j)}$ for some permutation σ where P_j is an eigenprojection in the spectral decomposition. Since winding number of α (α is defined in case (1)) is zero, the fractional powers of α exist (here, we set $\tilde{\alpha}(z)^4=\alpha(z)\bar{\alpha}(\bar{z})$). We can consider the following perturbation

(cf. Remark 4.8):

$$\begin{aligned}\tilde{\phi}(g_0)(z) = \frac{1}{2}\left[\sum_{i\notin\{j,\sigma(j)\}}^{n} \lambda_i(t)P_i(z) + \tilde{\psi}(\overline{\lambda_i})(t)\tilde{\psi}(P_i)(t)\right]\\ + \frac{1}{2}(\tilde{\alpha}\lambda_j)(t)P_j(t) + \frac{1}{2}\tilde{\psi}(\overline{\tilde{\alpha}\lambda_j})(t)\tilde{\psi}(P_j)(t)\\ + \frac{1}{2}(\tilde{\alpha}\lambda_{\sigma(j)})(t)P_{\sigma(j)}(t) + \frac{1}{2}\tilde{\psi}(\overline{\tilde{\alpha}\lambda_{\sigma(j)}})(t)\tilde{\psi}(P_{\sigma(j)})(t).\end{aligned}$$

Hence,

$$\mathrm{Det}(\tilde{\phi}^{\mathbb{C}}(g_0))(z) = \left[\prod_{i\notin\{j,\sigma(j)\}}^{n} \lambda_i(z)\right](\tilde{\alpha}(z))^2\lambda_j(z)\lambda_{\sigma(j)}(z) = z^k.$$

(8) If $A = C(\mathbb{T},\eta_0)$ and $B = C(\mathbb{T},\eta_0)\otimes_{\mathbb{R}} M_{\frac{n}{2}}(\mathbb{H})$, then we proceed similar to case (7) taking into account the double degeneracy.

If A is the center of a basic building block of a real AI-algebra and B is a basic building block of a real AI-algebra (the map from A to B must be allowable, see the definition of circle-quotient algebra associated to a real A$\mathbb{T}_J$-algebra, Definition 5.2), then we can proceed similar to the above (a case by case argument). If $\sum_{j=1}^{n} \lambda_j P_j$ is the spectral decomposition of $\phi^{\mathbb{C}}(g_5)$ then

$$\phi(g_6) = \sum_{j=1}^{n}\left(\frac{i}{2} - i\lambda_j\right)P_j.$$

The other cases, i.e., when A is a basic building block of a real AF-algebra or A is the center of a basic building block of a real AI-algebra (real A$\mathbb{T}_J$-algebra) and B is a basic building block of a real A$\mathbb{T}_J$-algebra (real AI-algebra, real AF-algebra), can be handled similarly (note that the map from A to B must be allowable, see the definition of circle-quotient algebra associated to a real A$\mathbb{T}_J$-algebra, Definition 5.2).

The above perturbation has the following properties:

(I) It has no effect on K_0-groups because it changes the eigenfunctions and not the eigenprojections.

(II) Since $w(\alpha) = w(\tilde{\alpha}) = 0$, it has no effect on the induced map from $K_1(A\otimes_{\mathbb{R}}\mathbb{C})$ to $K_1(B\otimes_{\mathbb{R}}\mathbb{C})$.

(III) The perturbed sequence agrees approximately at the level of traces of the complexification (cf. [10, Theorem 6]).

Moreover, A is also the direct limit of this new inductive system because the approximate intertwining argument used in [10, Theorem 6] is exactly applicable to the diagram of complexifications and the maps constructed there are all real C^*-algebra maps preserving the real structures (of course, we are dealing with different generators but the key point is the relationship of other generators (g_i, $i = 1, \dots, 4$) with the unitary generator g_0). The only significant change is the replacement of Theorem 4 in that proof with Theorem 4.16. $\square$

6. Approximate divisibility

Definition 6.1 ([4])**.** A C^*-algebra A is said to be approximately divisible if for any finite subset $F \subset A$, any $\epsilon > 0$ and any integer $N > 0$ there is a finite-dimensional sub-C^*-algebra $A_0 \subset A$ and a finite subset $F_0 \subset A$ such that F_0 commutes with $A_0, F \subset_\epsilon F_0$ (i.e., for any $f \in F$, $\mathrm{dist}(f, F_0) < \epsilon$) and each simple direct summand of A_0 is of order at least N.

Theorem 6.2. *Let A be a simple unital infinite-dimensional real $\mathrm{A}\mathbb{T}_J$-algebra. It follows that A is approximately divisible.*

Proof. By Theorem 5.8, $A \simeq \varinjlim(A_n, \phi_{n,n+1})$ where each $\phi_{n,n+1}$ is injective and unital, and each A_n is a real circle-quotient algebra (associated to the real $\mathrm{A}\mathbb{T}_J$-algebra A). We do not discuss the summands that involve basic building blocks of real interval algebras as the argument for these building blocks is not different from what is discussed in [30, Proposition 3.6], see also [10, Theorem 2]. We need to prove that for each $n \in \mathbb{N}$, each $N \in \mathbb{N}$, each finite set $F \subseteq A_n$ and each $\epsilon > 0$ there exists $m > n$ and a $*$-homomorphism $\psi : A_n \longrightarrow A_m$ such that $\|\phi_{n,m}(f) - \psi(f)\| < \epsilon$ for $f \in F$ where $\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n,n+1}$, and a unital finite-dimensional C^*-subalgebra H of $A_m \cap (\psi(A_n))'$ such that each summand of H has order at least N.

It suffices to consider A_n to be the center of a single basic building block (i.e., a basic building block of a real $\mathrm{A}\mathbb{T}_J$-algebra). Simplicity of the limit together with injectivity of the connecting maps gives the approximate density of the eigenvalues in the primitive ideal space of the source algebra. In the complex case, this is proved in [7, Proposition 2.1]. We sketch the proof here to affirm its validity for real C^*-algebras. Let $B_\delta(z) \subseteq \mathrm{Prim}(A_n)$ be open and nonempty. Take any $f \in A_n$ with $\emptyset \neq supp(f) \subseteq B_\delta(z)$. Assume that for any $m > n$ there exists $y_m \in \mathrm{Prim}(A_m)$ such that $\mathrm{Spec}(\phi_{n,m}(f)(y_m))$ is not δ-dense in $\mathrm{Prim}(A_n)$. In other words, $\mathrm{Spec}(\phi_{n,m}(f)(y_m)) \cap B_\delta(z) = \emptyset$ or equivalently $\phi_{n,m}(f)(y_m) = 0$. Hence, $I_{y_m} = \{\phi_{n,m}(f) \in A_m \mid \phi_{n,m}(f)(y_m) = 0\}$ is a nontrivial proper closed two-sided ideal of A_m (it is nontrivial because $f \neq 0$ implies $\phi_{n,m}(f) \neq 0$ by injectivity). Set $J = \overline{\bigcup_{i=1}^{\infty} I_{y_i}}$, then J is a nontrivial proper closed two-sided ideal of A because if J is not proper then $1_A \in J$ which implies $1_{A_k} \in I_{y_k}$ at some finite stage.

Once we established δ-density, we no longer need injectivity of connecting maps and we can define the map $\phi : A_n \longrightarrow A_m^j$ by $\phi := \pi_j \circ \phi_{n,m}$ where π_j is the projection map and $A_m^j = C(\mathbb{T}, \eta_j) \otimes_{\mathbb{R}} M_{n_j}(\mathbb{F})$, $\eta_j = id$, $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ or $\eta_j = \eta_0$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$, is a single basic building block of A_m.

If p is an eigenprojection for the map $\phi^j : A_n \to A_m^j$ of multiplicity k, then, as in the complex case, we may find a subalgebra of A_m^j isomorphic to $M_k(\mathbb{R})$ and commuting with the image $\phi^j(A_n)$ by taking as matrix units k trivial projections whose sum is p, and partial isometries implementing the equivalences.

We divide the proof into two cases (by K_1, we mean the K_1 of the complexification):

(i) $A_n = C(\mathbb{T}, \eta_0)$:

Case 1. We approximate $\phi(g_0)$ by $\tilde{\phi}(g_0)$ with distinct eigenfunctions such that this approximation does not change the K_1-class. If the K_1-class of the $\tilde{\phi}(g_0)$ or equivalently $w(\mathrm{Det}(\tilde{\phi}(g_0)))$ is a multiple of the rank of the unit of the target algebra, then the coalescing process produces eigenfunctions with large multiplicity, i.e., we can repeat each one at least N times (eigenfunctions are $2N\delta$-dense) and this perturbation will not change $w(\mathrm{Det}(\tilde{\phi}(g_0)))$ or equivalently the K_1-class.

If $\tilde{\phi}(g_0)$ does not belong to a type 4 building block, then we can consider its diagonalization $\tilde{\phi}(g_0) = u\,\mathrm{diag}(\lambda_1, \dots, \lambda_{n_j})u^*$ and define

$$\psi(g_0) = u\,\mathrm{diag}(\mu_1 \otimes I_{l_1}, \dots, \mu_k \otimes I_{l_k})u^*$$

where μ_i are the perturbed eigenfunctions and $l_i \geq N$, and $\psi(A_n)$ commutes with a finite-dimensional sub-C^*-algebra

$$H = \bigoplus_{i=1}^{k} M_{l_i}(\mathbb{R})$$

of A_m^j.

Since $\|\mu_i - \lambda_s\| \leq 2N\delta$ where $i = 1, \dots, k$, $s = 1, \dots, n_j$,

$$\begin{aligned} &\|\psi(g_0) - \tilde{\phi}(g_0)\| \\ &= \|u\,\mathrm{diag}(\mu_1 \otimes I_{l_1}, \dots, \mu_k \otimes I_{l_k})u^* - u\,\mathrm{diag}(\lambda_1, \dots, \lambda_{n_j})u^*\| \\ &\leq \max\{\|\mu_i - \lambda_s\|\} \leq 2N\delta \leq \frac{\epsilon}{2}. \end{aligned}$$

Hence,

$$\|\phi(g_0) - \psi(g_0)\| \leq \|\tilde{\phi}(g_0) - \phi(g_0)\| + \|\psi(g_0) - \tilde{\phi}(g_0)\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

If $\tilde{\phi}(g_0)$ belongs to a type 4 building block A_m, then we can proceed as in [30, Proposition 3.6], i.e., instead of diagonalizing $\tilde{\phi}(g_0)$ which may not be in A_m anymore, we perturb it such that the dimension of each of its eigenprojections P_i is at least N and we then cut it by the projections of the form $(P_i + P_j) \in A_m$ (if α is an involutive $*$-antiautomorphism of $A_m \otimes_{\mathbb{R}} \mathbb{C}$ such that $(A_m \otimes_{\mathbb{R}} \mathbb{C})_\alpha = A_m$ then $P_j = \tilde{\alpha}(P_i)$).

Case 2. If K_1-class of the image of the canonical central generator is not a multiple of n_j, then we can reduce this case to Case 1 through a procedure called "eigenvalue crossover". The eigenvalue crossover is explained on pages 101-107 of [10]. The difference with the complex case is that we can now have two types of eigenfunctions: those that satisfy $\tilde{\alpha}(\lambda_i) = \overline{\lambda_i}$ and those that do not, but they undergo a permutation such that $\tilde{\alpha}(\lambda_i) = \overline{\lambda_j}$ for some $j \neq i$. Nevertheless, this issue can be resolved as follows:

Let $\pi : A \to B$ be a map of the form

$$\pi(f) = \sum_i f \circ \lambda_i P_i + \sum_j (f \circ \gamma_j Q_j + f \circ (\Phi_B \circ \gamma_j \circ \Phi_A)\Phi_B(Q_j))$$

where the P_i have even multiplicity, and $\{P_i, Q_j, \phi_B(Q_j)\}$ is a pairwise orthogonal system. Form a (not necessarily real) homomorphism π_1 as follows. First split each P_i into two equivalent subprojections $P_i = p_i^1 \oplus p_i^2$, and take the first one from each pair. Then take one projection from each $Q_j, \phi_B(Q_j)$ pair. Take as π_1 the cut down of π by the sum of the selected projections. We then have $\pi = \pi_1 \oplus \pi_2$, were π_2 is completely determined by π_1. Now use Elliott's method on π_1 to get a new π_1' with all eigenvalues having a certain multiplicity, and use the correspondence to do exactly the matching perturbation to π_2, so that $\pi_1' \oplus \pi_2'$ is real. Notice that the eigenvalue density of π_2 is exactly the same as that of π_1.

We provide a sketch of the "eigenvalue crossover" process for the sake of completeness. Let $w(\mathrm{Det}(\tilde{\phi}(g_0))) = pn_j + q$ and $r = \lceil \frac{n_j}{8} \rceil$ where $p, q \in \mathbb{Z}$, $|q| \leq \frac{n_j}{2}$, and choose n_j large enough, i.e., $n_j \geq 16$. Divide $\mathrm{Sp}^{\mathbb{C}}(\tilde{\phi}(g_0))$ into three subsets where each of them is strictly $\frac{\delta}{2}$-dense in $\mathbb{T}$; namely

$$\begin{aligned} G_1 &= \{\lambda_1, \dots, \lambda_{|q|+r}\}, \\ G_2 &= \{\lambda_{|q|+r+1}, \dots, \lambda_{|q|+2r}\}, \\ G_3 &= \{\lambda_{|q|+2r+1}, \dots, \lambda_{n_j}\}. \end{aligned}$$

First, cross over each element of G_3 with $\lambda_{|q|+2r}, \lambda_{|q|+2r-1}, \dots, \lambda_{2r+1}$ of G_2 (and possibly G_1 depending on $|q|$). Next, cross over

$$\lambda_{|q|+2r}, \lambda_{|q|+2r-1}, \dots, \lambda_{2r+1}$$

with the remaining $2r$ elements of G_2 and G_1. We always cross over with the closest eigenvalue to G_3 (by relabeling as necessary) in the clockwise direction. In other words, we consider the sub-algebras $M_{|q|+r}(\mathbb{F})$, $M_r(\mathbb{F})$ and $M_{n_j-|q|-2r}(\mathbb{F})$ of $M_{n_j}(\mathbb{F})$ and inside of each sub-algebra the K_1-class of the image of the canonical generators (namely, $p + \mathrm{sign}(q)$, $p - \mathrm{sign}(q)$ and p respectively) is a multiple of the rank of the unit of that sub-algebra and consequently we are back to Case 1.

(ii) $A_n = C(\mathbb{T}, \mathbb{R})$: In this case, it is sufficient to repeat the above argument for $\phi^{\mathbb{C}}(g_0)$. We consider $\phi^{\mathbb{C}}(g_0) = \phi(g_1) \otimes 1 + \phi(g_2) \otimes i$. As above, we have two cases, for Case 1 we perturb both $\mathrm{Re}(\lambda_i)$ and $\mathrm{Im}(\lambda_i)$. For Case 2, we perturb the functions $F_i \in C([0,1], \mathbb{R})$ in $\lambda_i = e^{2\pi i F_i}$ such that we have the appropriate coalescing of λ_i with other eigenfunctions of $\tilde{\phi}^{\mathbb{C}}(g_0)$. □

7. The classification theorem

Theorem 7.1. *For a fixed $J \in \{\{1\}, \{3,4\}, \{3,5\}\}$, let $A \simeq \varinjlim(A_n, \phi_{n,n+1})$ and $B \simeq \varinjlim(B_n, \psi_{n,n+1})$ be simple unital real infinite-dimensional $A\mathbb{T}_J$-algebras. Suppose the following diagrams commute*

$$\begin{array}{ccccc}
(K_0(A), [1_A]) & \xrightarrow{q_{\mathbb{C}}} & (K_0(A \otimes_{\mathbb{R}} \mathbb{C}), [1_{A\otimes_{\mathbb{R}}\mathbb{C}}]) & \xrightarrow{q_{\mathbb{H}}} & (K_0(A \otimes_{\mathbb{R}} \mathbb{H})/\operatorname{Tor}(K_0(A \otimes_{\mathbb{R}} \mathbb{H})), [1_{A\otimes_{\mathbb{R}}\mathbb{H}}]) \\
\downarrow h_0 & & \downarrow h_0^{\mathbb{C}} & & \downarrow h_0^{\mathbb{H}} \\
(K_0(B), [1_B]) & \xrightarrow{q_{\mathbb{C}}} & (K_0(B \otimes_{\mathbb{R}} \mathbb{C}), [1_{B\otimes_{\mathbb{R}}\mathbb{C}}]) & \xrightarrow{q_{\mathbb{H}}} & (K_0(B \otimes_{\mathbb{R}} \mathbb{H})/\operatorname{Tor}(K_0(B \otimes_{\mathbb{R}} \mathbb{H})), [1_{B\otimes_{\mathbb{R}}\mathbb{H}}])
\end{array}$$

$$\begin{array}{ccccc}
K_1(A)/\operatorname{Tor}(K_1(A)) & \xrightarrow{\tilde{c}_A} & K_1(A \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\tilde{r}_A} & K_1(A)/\operatorname{Tor}(K_1(A)) \\
\downarrow h_1 & & \downarrow h_1^{\mathbb{C}} & & \downarrow h_1 \\
K_1(B)/\operatorname{Tor}(K_1(B)) & \xrightarrow{\tilde{c}_B} & K_1(B \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\tilde{r}_B} & K_1(B)/\operatorname{Tor}(K_1(B))
\end{array}$$

$$\begin{array}{ccc}
K_0(A \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\rho_A} & \operatorname{Aff}(T(A \otimes_{\mathbb{R}} \mathbb{C})) \\
\downarrow h_0^{\mathbb{C}} & & \downarrow M \\
K_0(B \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{\rho_B} & \operatorname{Aff}(T(B \otimes_{\mathbb{R}} \mathbb{C}))
\end{array}$$

where the maps $h_0, h_0^{\mathbb{C}}, h_0^{\mathbb{H}}$ are positive order unit preserving group isomorphisms,

$$h_1 : K_1(A)/\operatorname{Tor}(K_1(A)) \longrightarrow K_1(B)/\operatorname{Tor}(K_1(B)),$$
$$h_1^{\mathbb{C}} : K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow K_1(B \otimes_{\mathbb{R}} \mathbb{C}),$$

are group isomorphisms, $\phi_T : T(B \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow T(A \otimes_{\mathbb{R}} \mathbb{C})$ is a continuous affine isomorphism and $\phi_T \phi_B^* = \phi_A^* \phi_T$.

Then, there exists a $*$-isomorphism $\phi : A \longrightarrow B$ giving rise to the maps $h_0, h_0^{\mathbb{C}}, h_0^{\mathbb{H}}$, h_1, $h_1^{\mathbb{C}}$, and ϕ_T.

Proof. For each i, let D_A^i, D_B^i, E_A^i, E_B^i be the followings triples respectively:

$$\begin{array}{ccc}
(K_0(A_i), [1_{A_i}]) & \longrightarrow & (K_0(A_i \otimes_{\mathbb{R}} \mathbb{C}), [1_{A_i\otimes_{\mathbb{R}}\mathbb{C}}])(K_0(A_i \otimes_{\mathbb{R}} \mathbb{H})/\operatorname{Tor}(K_0 \\
 & & \downarrow \\
 & & (A_i \otimes_{\mathbb{R}} \mathbb{H})), [1_{A_i\otimes_{\mathbb{R}}\mathbb{H}}])
\end{array}$$

$$\begin{array}{ccc} (K_0(B_i),[1_A]) & \longrightarrow & (K_0(B_i \otimes_{\mathbb{R}} \mathbb{C}),[1_{B_i \otimes_{\mathbb{R}} \mathbb{C}}]) \\ & & \downarrow \\ & & (K_0(B_i \otimes_{\mathbb{R}} \mathbb{H})/\operatorname{Tor}(K_0(B_i \otimes_{\mathbb{R}} \mathbb{H})),[1_{B_i \otimes_{\mathbb{R}} \mathbb{H}}]) \end{array}$$

$$K_1(A)/\operatorname{Tor}(K_1(A)) \to K_1(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_1(A)/\operatorname{Tor}(K_1(A))$$

$$K_1(B)/\operatorname{Tor}(K_1(B)) \to K_1(B \otimes_{\mathbb{R}} \mathbb{C}) \to K_1(B)/\operatorname{Tor}(K_1(B))$$

The diagrams in the statement of the theorem induce the following three diagrams. The argument in sections 5.1.1-5.1.3 of [10] applies directly to the current situation (this step is the analogue of [30, Lemma 5.1]).

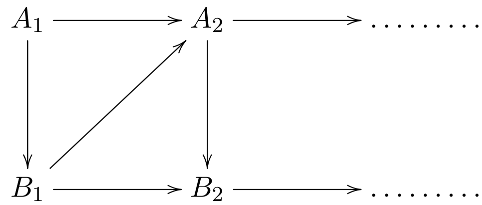
$$\begin{array}{ccccccc} D_A^1 & \longrightarrow & D_A^2 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & D_A \\ \downarrow & \nearrow & \downarrow & & & & \uparrow\downarrow \\ D_B^1 & \longrightarrow & D_B^2 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & D_B \end{array}$$

$$\begin{array}{ccccccc} E_A^1 & \longrightarrow & E_A^2 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & E_A \\ \downarrow & \nearrow & \downarrow & & & & \uparrow\downarrow \\ E_B^1 & \longrightarrow & E_B^2 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & E_B \end{array}$$

$$\begin{array}{ccccccc} \operatorname{Aff}(T(A_1 \otimes_{\mathbb{R}} \mathbb{C})) & \longrightarrow & \operatorname{Aff}(T(A_2 \otimes_{\mathbb{R}} \mathbb{C})) & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & \operatorname{Aff}(T(A \otimes_{\mathbb{R}} \mathbb{C})) \\ \downarrow & \nearrow & \downarrow & & & & \uparrow\downarrow \\ \operatorname{Aff}(T(B_1 \otimes_{\mathbb{R}} \mathbb{C})) & \longrightarrow & \operatorname{Aff}(T(B_2 \otimes_{\mathbb{R}} \mathbb{C})) & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & \operatorname{Aff}(T(B \otimes_{\mathbb{R}} \mathbb{C})) \end{array}$$

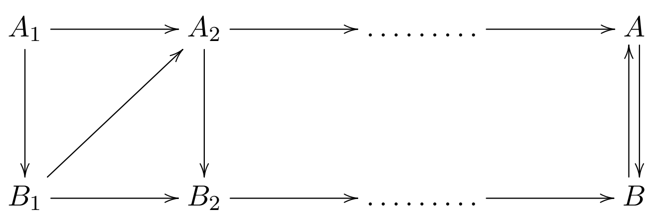
where the first two diagrams are commutative and the third diagram is approximately commutative and the three diagrams giving rise to the diagrams in the statement of the theorem. Furthermore, exactly as in [10], since the limit algebra is simple, we may assume, by passing to subsequences if necessary, that the condition in the existence theorem (Theorem 3.11) on the K_1 groups associated to ideals of K_0 is met.

In applying the existence theorem, we use approximate divisibility and by passing to subsequences and relabeling, we get the following diagram such that the induced diagrams at the level of K-groups commute and at the level of affine function spaces approximately commute. The argument in sections 5.1.4-5.1.7 of [10] applies directly to the current situation (this step is the analogue of [30, Lemma 5.2]).



By reduction theorem, we can write A and B as the inductive limit of direct sums of real circle-quotient algebras with injective connecting maps satisfying the uniformly varying determinant property at the level of complexification.

The above diagram is not approximately commutative, by the uniqueness theorem and passing to subsequences and relabeling we however get the following approximately commutative diagram which is commuting at the level of K-groups and approximately commuting at the level of affine function spaces and satisfying the necessary compatibility conditions. The argument in sections 5.2.1-5.2.4 of [10] applies directly to the current situation (this step is the analogue of [30, Theorem 5.3.]).



$\square$

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(Andrew J. Dean) Department of Mathematics, Lakehead University, Thunder Bay, Ontario, P7B 5E1, Canada
ajdean@lakeheadu.ca

(Dan Kucerovsky) Department of Mathematics, University of New Brunswick, Fredericton, NB, E3B 5A3, Canada
dkucerov@unb.ca

(Aydin Sarraf) Department of Mathematics, University of New Brunswick, Fredericton, NB, E3B 5A3, Canada
Aydin.Sarraf@unb.ca

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