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Invariance under finite Blaschke factors on $BMOA$

Ajay Kumar, Niteesh Sahni and Dinesh Singh

Abstract. This paper describes completely the invariant subspaces of the operator of multiplication by a finite Blaschke factor on the Banach space $BMOA$ of analytic functions with bounded mean oscillation on the unit circle in the complex plane. As a simple application, we describe by very elementary means, the invariant subspaces of the co-analytic Toeplitz operator $T_{\overline{B}}$ on H^1. In the simplest case when $B(z) = z$, the invariant subspaces of $T_{\overline{B}}$ on H^1 were described by fairly deep arguments until the appearance of an elementary proof by two of the authors (Sahni & Singh). In recent times, the common invariant subspaces of the operators of multiplication by B^2 and B^3, first in the case of z^2 and z^3, and then for an arbitrary finite Blaschke B, have proved to be critical in the context of Nevanlinna–Pick type interpolation on H^2. Thus, keeping in mind the importance of invariant subspaces, we also offer a characterization of the common invariant subspaces of these operators on $BMOA$. Our proofs are that much more technical. Again, as an application, we obtain the common invariant subspaces of $T_{\overline{B^2}}$ and $T_{\overline{B^3}}$ on the Hardy space H^1.

Contents

1. Introduction

From the functional analytic viewpoint, the space of analytic functions of bounded mean oscillation, $BMOA$, derives its importance due to the fact that it is the dual of the Hardy space H^1. Of course, this duality

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relation famously known as Fefferman's theorem (see [9]), goes well beyond the classical Hardy space H^1 of the unit disk.

In our context, as manifested in [5], [17] and [20], duality plays an important role in characterizing the invariant subspaces of the backward shift on $BMOA$. This paper extends such results to a far more general situation. In fact, using elementary and simple techniques we characterize the invariant subspaces of the operator of multiplication by a finite Blaschke factor B on $BMOA$ and then using duality arguments we obtain in a simple way, the invariant subspaces of the co-analytic Toeplitz operator $T_{\overline{B}}$ on H^1 (Note: Multiplication by finite Blaschke factor B is a bounded operator on $BMOA$, see [13]). These results should be seen to be in the line of investigation of invariant subspaces that are — apart from being of interest in their own right — also interesting because of their applications to areas such as Nevanlinna–Pick interpolation (see [1], [2], [4], [8], [10], [11] and [12]). For more information on these areas the reader can refer to [14], [15], [16], and [20].

We wish to state here a key difference between the proofs of the special cases of the invariant subspace theorems relating to the operator of multiplication by z as in [17], and our theorem over here for the operator of multiplication by a finite Blaschke factor B. This difference relates to overcoming the absence of a gcd for B-inner functions that we consider in our proof for the operator of multiplication by B unlike in the case of the operator of multiplication by z, where we rely on the fact that any collection of inner functions has a gcd.

We also state and prove a second invariant subspace theorem, again in the context of $BMOA$, in which we describe completely the common invariant subspaces of the operators T_{B^2} and T_{B^3} that is of multiplication by $B(z)^2$ and $B(z)^3$ on $BMOA$. This theorem is similar in flavor to our first invariant subspace theorem and is important in its own right because it is a generalization of the H^2 version, Theorem 1.3 in [6], which in turn has proved to be very important in the context of constrained Nevanlinna–Pick interpolation. Furthermore, as an application, we produce the common invariant subspace characterization of the co-analytic Toeplitz operators $T_{\overline{B^2}}$ and $T_{\overline{B^3}}$ on the Hardy space H^1.

2. Notation and terminology

Let $\mathbb{D}$ stand for the unit disk in the complex plane and $\mathbb{T}$ for its boundary, namely the unit circle. For $p \geq 1$, the symbol H^p stands for the classical Hardy space of analytic functions defined on the disk $\mathbb{D}$, which can also be viewed as the following closed subspace of the Lebesgue space L^p of the circle:

$$\left\{ f \in L^p : \int_{\mathbb{T}} f(z) z^n \, dm = 0, \quad n = 1, 2, \ldots \right\},$$

where dm is the normalized Lebesgue measure. A function $I \in H^p$ is called inner if $|I| = 1$ $a.e.$ and a function $f \in H^p$ is called outer if $clos_p\{z^n f\} = H^p$. Here $clos_p$ is the closure in the p-norm.

A function $f \in L^1$ is said to be of bounded mean oscillation and written as $f \in BMO$ if

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I \left| f - \frac{1}{|I|} \int_I f \; dm \right| \; dm < \infty.$$

Here the supremum is taken over all subarcs I of the unit circle, and $|I|$ is the Lebesgue measure of the subarc I. BMO is a Banach space under the norm

$$\|f\| = \|f\|_* + |f(0)|.$$

A function g in BMO is said to be of vanishing mean oscillation or $g \in VMO$ if the above integral tends to zero as $|I|$ tends to zero. The space $BMOA = BMO \cap H^1$ and the space $VMOA = VMO \cap H^1$. We refer [19] for more details.

Now we record some important facts about the Hardy–Hilbert space H^2 of the circle which shall be used frequently. It is well known that $\{1, z, z^2, \dots\}$ is an orthonormal basis for H^2. Here $z = e^{i\theta}$. Throughout the paper, $B(z)$ shall stand for a fixed Blaschke factor of order n of the form:

$$B(z) = \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \qquad (\alpha_i \in \mathbb{D}; \alpha_1 = 0).$$

The following orthonormal basis in terms of $B(z)$ for H^2 has been described in [21]:

$$\left\{ e_{jm} = \frac{\sqrt{1 - |\alpha_{j+1}|^2}}{1 - \overline{\alpha_{j+1}} z} B_j(z) B(z)^m : 0 \le j \le n-1, m = 0, 1, 2, \dots \right\}.$$

The symbol $B_j(z)$ stands for the product $\prod_{i=1}^{j} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}$. As a consequence, any $f \in H^2$ can be written as $f = e_{0,0} f_0 + \dots + e_{n-1,0} f_{n-1}$, where $f_0, \dots, f_{n-1}$ belong to $H^2(B(z))$—the closed span of $\{1, B(z), B(z)^2, \dots\}$ in H^2. A function $\varphi \in H^\infty$ is called B-inner if $\{\varphi B(z)^m : m = 0, 1, 2, \dots\}$ is an orthonormal set in H^2.

For a finite Blaschke product $B(z)$, the Toeplitz operator T_B is defined by $T_B f(z) = B(z) f(z)$, for each $f \in BMOA$. A closed subspace $\mathcal{M}$ of $BMOA$ is T_B invariant if $T_B \mathcal{M} \subset \mathcal{M}$. The co-analytic Toeplitz operator with symbol $T_{\overline{B}}$ is the adjoint operator of the operator T_B. A closed subspace $\mathcal{K}$ of H^1 is said to be invariant under $T_{\overline{B}}$ if $T_{\overline{B}} \mathcal{K} \subset \mathcal{K}$.

In general, $H^p(B(z))$ shall denote the closure (weak star closure when $p = \infty$) of $span\{1, B(z), B(z)^2, \dots\}$ in H^p. For any subset X of H^p, we shall denote its closure in H^p as $clos_p X$. $BMOA(B(z))$ is the weak-star closed span of $\{1, B(z), B(z)^2, \dots\}$ in $BMOA$. If X is a subset of $BMOA$ then the weak-star closure of X in $BMOA$ will be denoted by $clos^* X$.

3. Preliminary results

A corner stone in the theory of $BMOA$ functions is the Fefferman's theorem which identifies the space $BMOA$ with the dual space of H^1. This theorem turns out a powerful tool in the characterization of invariant subspaces of $BMOA$. The precise statement runs as follows:

Theorem 3.1 (Fefferman's Theorem, [9])**.** *$BMOA$ is the dual of H^1 and the action of any $BMOA$ function f treated as a functional on H^1 is given by*

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{T}}\overline{f(re^{i\theta})}p(re^{i\theta})d\theta,$$

where p is any polynomial in H^1.

The authors in [5] and [17] make a significant use of a factorization result (stated as Corollary 3.3 below) in the proofs of their invariant subspace characterization. The lemma below is a generalization of this fact and will be crucial for the proof of our results.

Lemma 3.2. *Let f be in $BMOA$ and $q_1,\dots,q_r$ be B-inner functions, $r\le n$, such that $q_iH^2(B(z))\perp q_jH^2(B(z))$, $i\neq j$. If there exist functions $g_1,\dots,g_r$ belonging to $H^2(B(z))$ such that $f=q_1g_1+\cdots+q_rg_r$, then each $g_i\in BMOA(B(z))$.*

Proof. Since f is in $BMOA$, it acts as a bounded linear functional on H^1. Consequently for any polynomial p in $H^1(B(z))$, we have

$$\left|\int_{\mathbb{T}}\overline{f}q_1p\ dm\right|\le C\|q_1p\|_1\le C_1\|p\|_1.$$

Moreover,

$$\begin{aligned}\left|\int_{\mathbb{T}}\overline{f}q_1p\ dm\right| &= \left|\int_{\mathbb{T}}\overline{(q_1g_1+\cdots+q_rg_r)}q_1p\ dm\ \right|\\ &= \left|\int_{\mathbb{T}}\overline{q_1g_1}q_1p\ dm+\cdots+\int_{\mathbb{T}}\overline{q_rg_r}q_1p\ dm\right|\\ &= \left|\int_{\mathbb{T}}\overline{q_1g_1}q_1p\ dm\right|\\ &= \left|\int_{\mathbb{T}}\overline{g_1}p\ dm\right|.\end{aligned}$$

Except the first integral, all other integrals vanish because for each $i, j = 1, 2, \cdots, r$, $q_i H^2(B(z)) \perp q_j H^2(B(z))$, when $i \neq j$. The last step is a consequence of the fact that q_1 is B-inner. Therefore,

$$\left| \int_{\mathbb{T}} \overline{g_1} p \; dm \right| \leq C_1 \|p\|_1.$$

So the bounded linear functional $F_g(p) = \int_{\mathbb{T}} \overline{g} p \; dm$ can be extended to H^1. This means

$$\left| \int_{\mathbb{T}} \overline{g_1} h \; dm \right| \leq C_1 \|h\|_1$$

for all analytic polynomials h in H^1, and hence $g_1 \in BMOA$. The function g_1 has only powers of $B(z)$ because it lies inside $H^2(B(z))$, so it belongs to $BMOA(B(z))$. Similarly, $g_2, \ldots, g_r \in BMOA(B(z))$. □

Corollary 3.3 ([5, Proposition 2.1.3])**.** *Let I be an inner function, and $g \in H^2$ such that $Ig \in BMOA$. Then $g \in BMOA$.*

Proof. Take $B(z) = z$ in Lemma 3.2. □

In proving Theorem 4.1, we need to show that $qBMOA(B(z)) \cap BMOA$ is weak-star closed in $BMOA$. We do this by showing that $qBMOA(B(z)) \cap BMOA$ is the annihilator of a subspace of H^1.

Lemma 3.4. *If q is a B-inner function, then $qBMOA(B(z)) \cap BMOA$ is the annihilator of the subspace, $clos_1[qH^2(B(z))]^{\perp}$ of H^1.*

Proof. Let f be an element of $qBMOA(B(z)) \cap BMOA$ and g be chosen from $[qH^2(B(z))]^{\perp}$. It is evident that $\int f\overline{g} \; dm = 0$. This means that f annihilates $[qH^2(B(z))]^{\perp}$ and hence it belongs to the annihilator of $clos_1[qH^2(B(z))]^{\perp}$.

On the other hand if $f \in Ann\left(clos_1[qH^2(B(z))]^{\perp}\right)$, then f will be in the dual space, i.e., in $BMOA$. Since $BMOA \subset H^2$, this f will also be in H^2. Further, f is orthogonal to $[qH^2(B(z))]^{\perp}$, thus $f \in qH^2(B(z))$.

So $f = qf_1$, for some $f_1 \in H^2(B(z))$. By Lemma 3.2, f_1 becomes a member of $BMOA(B(z))$ and hence

$$f \in qBMOA(B(z)) \cap BMOA.$$ □

Our next lemma plays an essential role in the proofs of Theorem 4.1 and Theorem 5.1. In this lemma, we show that $qH^{\infty}(B(z))$ is weak-star dense in $qBMOA(B(z)) \cap BMOA$.

Lemma 3.5. *If q is a B-inner function, then*

$$clos^*[qH^{\infty}(B(z))] = qBMOA(B(z)) \cap BMOA.$$

Proof. It is easy to see that

$$qH^{\infty}(B(z)) \subset qBMOA(B(z)) \cap BMOA.$$

Being the annihilator of the subspace $clos_1\left[[qH^2(B(z))]^{\perp}\right]$ of H^1 (see Lemma 3.4), the subspace

$$qBMOA(B(z)) \cap BMOA$$

is weak-star closed in $BMOA$. So it is obvious that

$$clos^*[qH^{\infty}(B(z))] \subseteq qBMOA(B(z)) \cap BMOA.$$

We prove the reverse inclusion. Chose an f in $qBMOA(B(z)) \cap BMOA$. Then $f = qg(B(z))$, for some g in $BMOA$. Since $g \in H^2$, there is a sequence of polynomials $\{g_n\}$ in H^2 such that

$$\|g_n - g\|_2 \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality assume that $g_n \to g$ a.e. (Actually a subsequence converges a.e. but we assume that we have replaced $\{g_n\}$ with that subsequence which we have relabelled as $\{g_n\}$ without loss of generality as the proof will show.)

Now, proceeding exactly as in the proof of Theorem 3.1 of [5], we construct a sequence of outer functions $\{O_n\}$ in H^{∞}. Let $\{O_n\}$ be the sequence of outer functions with

$$|O_n| = \begin{cases} \frac{1}{|g_n|}, & |g_n| > 1 \\ 1, & |g_n| \leq 1; \end{cases}$$

that is $\log|O_n| = -\log^+|g_n|$, and $O_n(0) > 0$. We note that $|O_n g_n| \leq 1$ and the sequence $\{O_n\}$ converges to 1 in the $\|\cdot\|_2$ norm. Taking composition of O_n and g_n with $B(z)$, we have

$$\|O_n(B(z)) - 1\|_2 \to 0 \qquad \text{and} \qquad \|g_n(B(z)) - g(B(z))\|_2 \to 0,$$

and $\|O_n(B(z))g_n(B(z))\|_{\infty} \leq 1$. There exist subsequences of $\{O_n(B(z))\}$ and $\{g_n(B(z))\}$ which converge almost everywhere to 1 and $g(B(z))$. For the same reason as mentioned above, we relabel these subsequences as $\{O_n(B(z))\}$ and $\{g_n(B(z))\}$. For the B-inner function q,

$$qO_n(B(z))g_n(B(z)) \to qg(B(z)) \qquad \text{a.e.}$$

and

$$\begin{aligned} \|qO_n(B(z))g_n(B(z))\|_{\text{BMOA}} &\leq \|qO_n(B(z))g_n(B(z))\|_{\infty} \\ &\leq \|q\|_{\infty}\|O_n(B(z))g_n(B(z))\|_{\infty} \\ &\leq \|q\|_{\infty}. \end{aligned}$$

This means

$$qO_n(B(z))g_n(B(z)) \to qg(B(z))$$

in the weak-star topology of $BMOA$. Since $qO_n(B(z))g_n(B(z))$ belongs to $qH^\infty(B(z))$, we conclude that $qg(B(z))$ belongs to the weak-star closure of $qH^\infty(B(z))$. □

Lastly we state two recent results that characterize subspaces of H^p invariant under the algebras $H^\infty(B(z))$ and $H_1^\infty(B(z))$. These shall be central to the proof of similar characterizations in the context of $BMOA$.

Theorem 3.6 ([18, Theorem 4]). *Let $\mathcal{M}$ be a closed subspace of H^p, $0 < p \leq \infty$, such that $\mathcal{M}$ is invariant under $H^\infty(B)$. Then there exist B-inner functions $q_1, \ldots, q_r$, $r \leq n$, such that*

$$\mathcal{M} = \sum_{i=1}^{r} \oplus q_i H^p\left(B(z)\right).$$

Theorem 3.7 ([18, Theorem 3]). *Let $\mathcal{M}$ be a closed subspace of H^p, $0 < p \leq \infty$, such that $\mathcal{M}$ is invariant under $H_1^\infty(B)$ but not invariant $H^\infty(B(z))$. Then there exist B-inner functions $q_1, \ldots, q_r$, $r \leq n$, such that*

$$\mathcal{M} = \left(\sum_{j=1}^{k} \oplus \langle \varphi_j \rangle\right) \oplus \left(\sum_{i=1}^{r} \oplus B(z)^2 q_i H^p\left(B(z)\right)\right),$$

where $k \leq 2r-1, r \leq n, j = 1, 2, \ldots, k$ and

$$\varphi_j = (\alpha_{1j} + \alpha_{2j}B)q_1 + (\alpha_{3j} + \alpha_{4j}B)q_2 + \cdots + (\alpha_{2r-1,j} + \alpha_{2r,j}B)q_r.$$

4. T_B-invariant subspaces

The first one of the two invariant subspace results proved in this paper is as follows:

Theorem 4.1. *Let $B(z)$ be a finite Blaschke product of order n and $\mathcal{M}$ be a weak-star closed subspace of $BMOA$ which is invariant under T_B. Then there exist B-inner functions $q_1, \ldots, q_r$ with $r \leq n$, such that*

$$\mathcal{M} = \left(\sum_{i=1}^{r} \oplus q_i BMOA(B(z))\right) \cap BMOA.$$

A brief remark on the proof. For a weak-star closed subspace $\mathcal{M}$ of $BMOA$, Sahni and Singh in [17] first show that $\mathcal{M} \cap H^\infty$ is nontrivial and then establish that for every f in $\mathcal{M}$, there exists an outer function g such that $gf = \phi k$, for some k in H^∞. This function ϕ turns out to be the gcd of inner parts of all functions in $\mathcal{M}$ and the form of $\mathcal{M}$ is gcd ϕ times some subspace $\mathcal{N}$ of $BMOA$; i.e., $\mathcal{M} = \phi\mathcal{N}$. In the case of B-invariant subspaces, the structure of $\mathcal{M} \cap H^\infty$ is not so simple and no such divisor ϕ exists. In order to overcome this difficulty, we shall use Lemma 3.2, which is a generalization of Proposition 2.1.3 in [5] by Brown and Sadek, and Lemma 3.4 as well as Lemma 3.5 to establish that $qH^\infty(B(z))$ is weak-star dense in $qBMOA(B(z)) \cap BMOA$. A final argument will then describe the T_B -invariant subspaces of $BMOA$.

Proof. We shall first establish that $\mathcal{M}$ contains plenty of bounded analytic functions. Note that any $f(z) \in \mathcal{M}$ can be written as

$$f(z) = e_{00} f_0(B(z)) + \cdots + e_{n-1,0} f_{n-1}(B(z)),$$

for some $f_0(z), \ldots, f_{n-1}(z) \in H^2$.

For each $k = 0, \ldots, n-1$, define

$$g_k(z) = \exp(-|f_k(z)| - i|f_k(z)|^{\sim}),$$

where $|f_k(z)|^{\sim}$ stands for the harmonic conjugate, which exists for L^2 functions. Observe that $|g_k(z)| \leq 1$ and consequently $g_k(z) \in H^\infty$.

Let $h(z) = g_0(B(z)) \cdots g_{n-1}(B(z))$. Now

$$\begin{aligned}
|h(z)f(z)| &\leq \sum_{j=0}^{n-1} |e_{j0}||h(z)f_j(B(z))| \\
&\leq \sum_{j=0}^{n-1} |e_{j0}||g_j(B(z))f_j(B(z))| \\
&= \sum_{j=0}^{n-1} |e_{j0}| \; |f_j(B(z)) \exp(-|f_j(B(z))|)| \\
&\leq \sum_{j=0}^{n-1} |e_{j0}|
\end{aligned}$$

shows that $h(z)f(z) \in H^\infty$. We now claim that $h(z)f(z)$ also belongs to $\mathcal{M}$ and this in turn establishes that $\mathcal{M} \cap H^\infty \neq [0]$.

For all $t \in (0,1)$ define $h_t(z) = h(tz)$. Following the proof of Lemma 3.3 in [13] (see also Proposition 2.1 in [5]), there exists a sequence of polynomial $P_{tn}(B(z))$ such that $P_{tn}(B(z))f(z)$ converges weak-star to $h_t(z)f(z)$. Further, it is established that $h_t(z)f(z)$ converges weak-star to $h(z)f(z)$ as $t \to 1$. Therefore $P_{tn}(B(z))f(z)$ converges weak-star to $h(z)f(z)$. Since $\mathcal{M}$ is invariant under T_B, we observe that $P_n(B(z))f(z) \in \mathcal{M}$ and hence $h(z)f(z) \in \mathcal{M}$.

Since $\mathcal{M} \cap H^\infty$ is a weak star closed subspace of H^∞ which is invariant under multiplication by $B(z)$, by Theorem 3.6, there exist B-inner functions $q_1, \ldots, q_r$ with $r \leq n$ such that

$$\mathcal{M} \cap H^\infty = \sum_{i=1}^{r} \oplus q_i H^\infty(B(z)).$$

Now $q_i H^\infty(B(z)) \subset \mathcal{M}$ and by Lemma 3.5, $q_i BMOA(B(z)) \cap BMOA$ is the weak-star closure of $q_i H^\infty(B(z))$ in $BMOA$, for each $i = 1, 2, \ldots, r$. So we have

$$\left(\sum_{i=1}^{r} \oplus q_i BMOA(B(z)) \right) \cap BMOA \subset \mathcal{M}.$$

Our characterization will be complete if we show the containment from the other side.

Let f be an element of $\mathcal{M}$. Once again f can be written as

$$f(z) = e_{00}f_0(B(z)) + \cdots + e_{n-1,0}f_{n-1}(B(z)).$$

For each $j = 0, \ldots, n-1$, define

$$h_m^{(j)} = \exp\left(\frac{-|f_j(z)| - i|f_j(z)|^{\sim}}{m}\right).$$

Put $O_m(z) = h_m^{(0)}(z) \cdots h_m^{(n-1)}(z)$. Then $O_m(B(z))f \in \mathcal{M} \cap H^\infty$. Observe that $O_m(B(z)) \to 1$ *a.e.* which implies $|O_m(B(z))f - f| \to 0$ *a.e.*.

Since $|O_m(B(z))f - f|^2 \leq 4|f|^2$, we have by the dominated convergence theorem that $\int |O_m(B(z))f - f|^2 \to 0$; that is, $O_m(B(z))f \to f$ in H^2. This means that $f \in clos_2[\mathcal{M} \cap H^\infty]$; that is, $f \in q_1H^2(B(z)) \oplus \cdots \oplus q_rH^2(B(z))$. Therefore $f = q_1g_1 + \cdots + q_rg_r$ for some $g_1, \ldots, g_r \in H^2(B(z))$. By Lemma 3.2, the functions $g_1, \ldots, g_r$ all belong to $BMOA(B(z))$. Therefore, f belongs to $\left(\sum_{i=1}^{r} \oplus q_iBMOA(B(z))\right) \cap BMOA$. □

Corollary 4.2 ([5, Theorem 3.1], [17, Theorem 4.1] and [20, Theorem C]). *Let $\mathcal{M}$ be a weak-star closed subspace of $BMOA$ which is invariant under T_z. Then there exists an inner function q such that $\mathcal{M} = qBMOA \cap BMOA$.*

Proof. Taking $B(z) = z$ in Theorem 4.1, we get a z-inner function (which is nothing but an inner function) q such that $\mathcal{M} = qBMOA \cap BMOA$. □

As an application of the above theorem, we now derive the invariant subspaces of the co-analytic Toeplitz operator$T_{\overline{B}}$ on H^1.

Theorem 4.3. *Let $\mathcal{K}$ be a closed subspace of H^1 which is invariant under the co-analytic Toeplitz operator $T_{\overline{B}}$. Then there exist B-inner functions $q_1, \ldots, q_r$ with $r \leq n$ such that*

$$\mathcal{K} = clos_1\left(\bigcap_{i=1}^{r} \left[q_iH^2(B(z))\right]^{\perp}\right).$$

Proof. The annihilator of the subspace $\mathcal{K}$, denoted by $Ann(\mathcal{K})$, is a weak-star closed subspace of $BMOA$ and is also invariant under multiplication by $B(z)$. So by Theorem 4.1, there exist B-inner functions $q_1, \ldots, q_r$ (where $r \leq n$) such that

$$Ann(\mathcal{K}) = \left(\sum_{i=1}^{r} \oplus q_iBMOA(B(z))\right) \cap BMOA. \tag{4.1}$$

Since $q_iBMOA(B(z)) \subset q_iH^2(B(z))$ for each $i = 1, 2, \ldots, r$, we see that $Ann(\mathcal{K})$ annihilates

$$\left(\sum_{i=1}^{r} \oplus q_iH^2(B(z))\right)^{\perp},$$

and hence

$$\left(\sum_{i=1}^{r} \oplus q_i H^2(B(z))\right)^{\perp} \subset \mathcal{K}. \tag{4.2}$$

As $\mathcal{K}$ is a closed subspace of H^1, it is clear from (4.2) that

$$clos_1 \left(\bigcap_{i=1}^{r} \left[q_i H^2(B(z))\right]^{\perp}\right) \subset \mathcal{K}. \tag{4.3}$$

It remains to establish the inclusion from the other end. Let $f \in \mathcal{K}$. Then from (4.1), every element of $\left(\sum_{i=1}^{r} \oplus\, q_i BMOA(B(z))\right) \cap BMOA$ will annihilate f. It follows from Lemma 3.4 that the annihilator of the closed subspace

$$clos_1 \left(\left[\sum_{i=1}^{r} \oplus\, q_i H^2(B(z))\right]^{\perp}\right)$$

of H^1 is $\left(\sum_{i=1}^{r} \oplus q_i BMOA(B(z))\right) \cap BMOA$. Therefore

$$f \in clos_1 \left(\left[\sum_{i=1}^{r} \oplus\, q_i H^2(B(z))\right]^{\perp}\right),$$

which means that

$$f \in clos_1 \left(\bigcap_{i=1}^{r} \left[q_i H^2(B(z))\right]^{\perp}\right).$$

Hence

$$\mathcal{K} \subset clos_1 \left(\bigcap_{i=1}^{r} \left[q_i H^2(B(z))\right]^{\perp}\right). \qquad \square$$

The results proved in [17] and [20] on backward shift invariant subspace of H^1 follows as a corollary to the above theorem.

Corollary 4.4 ([17, Theorem 4.2] and [20, Theorem 3.1])**.** *Let $\mathcal{K}$ be a closed subspace of H^1 invariant under S^*. Then there exists a unique inner function I such that $\mathcal{K} = I\overline{H}_0^1 \cap H^1$. Here bar denotes complex conjugate.*

Proof. Taking $B(z) = z$ in Theorem 4.3 , there exists an inner function I such that $\mathcal{K} = clos_1 \left[IH^2\right]^{\perp}$. It is easy to see that the orthogonal complement of IH^2 in L^2 is the closed span of $\{I\bar{z}, I\bar{z}^2, \dots\}$ in L^2. This implies that $\left(IH^2\right)^{\perp} = I\overline{H_0^2} \cap H^2$. Taking closure in H^1 we get $\mathcal{K} = I\overline{H}_0^1 \cap H^1$. $\square$

5. Common invariant subspaces of T_{B^2} and T_{B^3}

As mentioned earlier, a very special case of Theorem 5.1, proved below, where $B(z) = z$ and the operators are acting on H^2 has led to the solution of a constrained Nevanlinna–Pick interpolation problem which in turn has proved to be a starting point of a fruitful area of research. We refer to [1], [2], [4], [8], [10], [11] and [12].

Theorem 5.1. *Let $B(z)$ be a finite Blaschke product of order n and $\mathcal{M}$ be a weak-star closed subspace of $BMOA$ which is invariant under T_{B^2} and T_{B^3} but not invariant under T_B. Then there exist B-inner functions $q_1, \dots, q_r$ with $r \leq n$, such that*

$$\mathcal{M} = \sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \left(\sum_{i=1}^{r} \oplus q_i B(z)^2 BMOA(B(z)) \right) \cap BMOA.$$

Here $\varphi_1, \dots, \varphi_k$, $1 \leq k \leq 2r-1$, are in H^∞ and each φ_j has the form

$$\varphi_j = (\alpha_{1j} + \alpha_{2j} B) q_1 + (\alpha_{3j} + \alpha_{4j} B) q_2 + \cdots + (\alpha_{2r-1,j} + \alpha_{2r,j} B) q_r.$$

Proof. Take the functions $g_k(z) = \exp(-|f_k(z)| - i|f_k(z)|^{\sim})$ described in the proof of Theorem 4.1, and define

$$h(z) = g_0(B^2(z)) \cdots g_{n-1}(B^2(z)).$$

It is easy to show that $h(z)f(z) \in H^\infty$. Proceeding as in the proof of Theorem 4.1 and using the invariance of $\mathcal{M}$ under T_B^2 we see that $h(z)f(z)$ belongs to $\mathcal{M}$. This shows that $\mathcal{M} \cap H^\infty$ is non trivial. Also $\mathcal{M} \cap H^\infty$ is a weak-star closed subspace of H^∞ which is invariant under T_B^2 and T_B^3.

The space $\mathcal{M} \cap H^\infty$ can not be invariant under T_B. For if $\mathcal{M} \cap H^\infty$ is T_B invariant, then by Theorem 3.6, there exist B-inner functions $q_1, q_2, \dots, q_r$ such that

$$\mathcal{M} \cap H^\infty = q_1 H^\infty(B(z)) \oplus q_2 H^\infty(B(z)) \oplus \cdots \oplus q_r H^\infty(B(z)). \tag{5.1}$$

Using lemma 3.5 and denseness of $\mathcal{M} \cap H^\infty$ in $\mathcal{M}$ we have

$$\mathcal{M} = \left(\sum_{i=1}^{r} \oplus q_i BMOA(B(z)) \right) \cap BMOA.$$

This is clearly not possible as $\mathcal{M}$ is not invariant under T_B.

Therefore, by Theorem 3.7, there exist B-inner functions $q_1, \dots, q_r$ such that

$$\mathcal{M} \cap H^\infty = \sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \sum_{i=1}^{r} \oplus B(z)^2 q_i H^\infty(B(z)), \tag{5.2}$$

where the functions $\varphi_1, \dots, \varphi_k$, $k \leq 2r-1$, are in H^∞, and for each j,

$$\varphi_j = (\alpha_{1j} + \alpha_{2j} B) q_1 + (\alpha_{3j} + \alpha_{4j} B) q_2 + \cdots + (\alpha_{2r-1,j} + \alpha_{2r,j} B) q_r.$$

We finish off the argument by showing that $\mathcal{M} \cap H^\infty$ is weak-star dense in $\mathcal{M}$ and that its weak-star closure in $BMOA$ has the form described in (5.2).

Since the finite dimensional space $\sum_{j=1}^{k}\langle\varphi_j\rangle$ is weak-star closed and the weak-star closure of $q_iH^\infty(B(z))$ in $BMOA$ is $q_iBMOA(B(z))\cap BMOA$, we conclude that

$$clos^*[\mathcal{M}\cap H^\infty] = \sum_{j=1}^{k}\langle\varphi_j\rangle \oplus \left(\sum_{i=1}^{r}\oplus B^2q_iBMOA(B(z))\right)\cap BMOA.$$

It is trivial to see that $\mathcal{M}\cap H^\infty \subset \mathcal{M}$. Our proof will be complete once we establish the reverse containment. For that we again proceed in a manner similar to the proof of Theorem 4.1 by selecting an arbitrary $f\in\mathcal{M}$, and writing it as

$$f = e_{00}f_0(B(z)^2) + \cdots + e_{2n-1,0}f_{2n-1}(B(z)^2)$$

where $f_0(z),\ldots,f_{2n-1}(z)\in H^2(B(z)^2)$. Next, for each $j=0,\ldots,2n-1$, define a sequence of H^∞ functions

$$h_m^{(j)}(z) = \exp\left(\frac{-|f_j(z)|-i|f_j(z)|^\sim}{m}\right).$$

Put $O_m(z) = h_m^{(0)}(z)\cdots h_m^{(n-1)}(z)$, so that $O_m(B(z)^2)f(z)\in\mathcal{M}\cap H^\infty$, and $O_m(B(z)^2)\to 1$ *a.e.* as $m\to\infty$. An application of the dominated convergence theorem then yields $O_m(B(z)^2)f\to f$ in H^2 . This means that f belongs to $clos_2[\mathcal{M}\cap H^\infty]$. Thus $f=g+h$, for some $g\in\sum_{j=1}^{k}\langle\varphi_j\rangle$ and $h\in\sum_{i=1}^{r}\oplus B(z)^2q_iH^2(B(z))$. Further, h can be written as

$$h = B(z)^2(q_1h_1+q_2h_2+\cdots+q_rh_r),$$

where $h_1,h_2,\ldots,h_r\in H^2(B(z))$. By Corollary 3.3,

$$q_1h_1+q_2h_2+\cdots+q_rh_r\in BMOA.$$

Now apply Lemma 3.2 to conclude that $h_1,h_2,\ldots,h_r\in BMOA(B(z))$ and this completes the argument. □

In the context of H^p spaces, the common invariant subspaces of S^2 and S^3 were studied earlier in [6] and [14] and then generalized to a great deal in [15], [16], and [18]. The theorem which we proved above generalizes the main theorem in [17].

Corollary 5.2 ([17, Theorem 3.1])**.** *Let $\mathcal{M}$ be a weak-star closed subspace of $BMOA$ which is invariant under S^2 and S^3 but not invariant under S. Then there exists an inner function I, and constants α,β such that*

$$\mathcal{M} = I\cdot BMOA_{\alpha\beta}\cap BMOA.$$

Proof. Take $B(z) = z$ in Theorem 5.1, we have $\varphi = \langle \alpha + \beta z \rangle I$ and

$$\mathcal{M} = \langle \alpha + \beta z \rangle I \oplus z^2 I \cdot BMOA \cap BMOA = I \cdot BMOA_{\alpha\beta} \cap BMOA.$$

The symbol $BMOA_{\alpha\beta}$ is the weak-star closure in $BMOA$ of the space generated by $\{\alpha + \beta z, z^2 BMOA\}$. □

Next we present a backward shift version of Theorem 5.1.

Theorem 5.3. *Let $\mathcal{K}$ be a closed subspace of H^1 which is invariant under the co-analytic Toeplitz operators $T_{\overline{B^2}}$ and $T_{\overline{B^3}}$ but not invariant under $T_{\overline{B}}$. Then there exist B-inner functions $q_1, \ldots, q_r$ with $r \leq n$ and $k \leq 2r - 1$ such that*

$$\mathcal{K} = clos_1 \left[\left(\bigcap_{j=1}^{k} \langle \varphi_j \rangle^{\perp} \right) \bigcap \left(\bigcap_{i=1}^{r} \left(B^2 q_i H^2 (B(z))^{\perp} \right) \right) \right].$$

Here the functions φ_j are as in Theorem 5.1.

Proof. Let $Ann(\mathcal{K})$ be the annihilator of $\mathcal{K}$ which is a weak-star closed subspace of $BMOA$ and is also invariant under T_B^2 and T_B^3. If possible assume that $Ann(\mathcal{K})$ is invariant under T_B, then this forces $\mathcal{K}$ to be invariant under $T_{\bar{B}}$ which is a contradiction.

Now in view of Theorem 5.1, there exist B-inner functions $q_1, \ldots, q_r$ ($r \leq n$) such that

$$Ann(\mathcal{K}) = \sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \left(\sum_{i=1}^{r} \oplus B^2 q_i BMOA(B(z)) \right) \cap BMOA. \tag{5.3}$$

Since $q_i BMOA(B(z))$ is contained in $q_i H^2(B(z))$, observe that $Ann(\mathcal{K})$ annihilates every element of the orthogonal complement

$$\left(\sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \sum_{i=1}^{r} \oplus B^2 q_i H^2(B(z)) \right)^{\perp}.$$

Therefore,

$$\left(\sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \sum_{i=1}^{r} \oplus B^2 q_i H^2(B(z)) \right)^{\perp} \subset \mathcal{K}.$$

and hence

$$clos_1 \left[\left(\bigcap_{j=1}^{k} \langle \varphi_j \rangle^{\perp} \right) \bigcap \left(\bigcap_{i=1}^{r} \left(B^2 q_i H^2 (B(z))^{\perp} \right) \right) \right] \subset \mathcal{K}. \tag{5.4}$$

To establish the reverse inclusion, let $f \in \mathcal{K}$. Then from (5.3), f will be annihilated by $\sum_{j=1}^{k} \langle \varphi_j \rangle \oplus \left(\sum_{i=1}^{r} \oplus B^2 q_i BMOA(B(z)) \right) \cap BMOA$.

It follows from Lemma 3.4 that the annihilator of the closed subspace

$$clos_1\left(\left[\sum_{j=1}^{k}\langle\varphi_j\rangle \oplus \sum_{i=1}^{r} \oplus B^2 q_i H^2(B(z))\right]^{\perp}\right)$$

of H^1 is $\sum_{j=1}^{k}\langle\varphi_j\rangle \oplus \left(\sum_{i=1}^{r} \oplus B^2 q_i BMOA(B(z))\right) \cap BMOA$. Therefore

$$f \in clos_1\left(\left[\sum_{j=1}^{k}\langle\varphi_j\rangle \oplus \sum_{i=1}^{r} \oplus B^2 q_i H^2(B(z))\right]^{\perp}\right)$$

and hence

$$\mathcal{K} \subset clos_1\left[\left(\bigcap_{j=1}^{k}\langle\varphi_j\rangle^{\perp}\right) \bigcap \left(\bigcap_{i=1}^{r}\left(B^2 q_i H^2(B(z))^{\perp}\right)\right)\right]. \qquad \square$$

In the spirit of Corollary 4.4, we now work out subspaces of H^1 which are invariant under the backward shift operators S^{*2} and S^{*3}.

Corollary 5.4. *Let $\mathcal{K}$ be a closed subspace of H^1 invariant under S^{*2} and S^{*3} but not under S^*. Then there exists a unique inner function I, and constants α, β such that $\mathcal{K} = \langle(\alpha + \beta z)I\rangle \oplus I\overline{H}_0^1 \cap H^1$. Here the symbol $\langle\ .\ \rangle$ denotes the linear span and bar represents the complex conjugate.*

Proof. Taking $B(z) = z$ in Theorem 5.3 we see that $\mathcal{K}$ is of the form:

$$\mathcal{K} = clos_1\left[\langle(\gamma + \delta z)I\rangle^{\perp} \cap (z^2 I H^2)^{\perp}\right]. \tag{5.5}$$

Here γ, δ are complex numbers and $\perp$ denotes orthogonal complement in H^2. It is easy to see that $\langle(\gamma + \delta z)I\rangle^{\perp} = \left(I\overline{H_0^2} \oplus z^2 I H^2 \oplus \langle(\alpha + \beta z)I\rangle\right) \cap H^2$, where α, β satisfy $\alpha\overline{\gamma} + \beta\overline{\delta} = 0$. Also, $(z^2 I H^2)^{\perp} = \left(I\overline{H_0^2} \oplus \langle I, Iz\rangle\right) \cap H^2$. Consequently, (5.5) simplifies to

$$\mathcal{K} = \langle(\alpha + \beta z)I\rangle \oplus I\overline{H_0^1} \cap H^1. \qquad \square$$

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(Ajay Kumar) Department of Mathematics, University of Delhi, Delhi (India) 110007
nbkdev@gmail.com

(Niteesh Sahni) Department of Mathematics, Shiv Nadar University, Dadri, Uttar Pradesh (India) 201314
niteeshsahni@gmail.com

(Dinesh Singh) Department of Mathematics, University of Delhi, Delhi (India) 110007
dineshsingh1@gmail.com

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