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A remark on oscillatory integrals associated with fewnomials

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Abstract. We prove that the L^2 bound of an oscillatory integral associated with a polynomial phase depends only on the number of monomials that this polynomial consists of.

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1. Introduction

Let $d \in \mathbb{N}$. Consider the operator

$$H_Q f(x) := \int_{\mathbb{R}} f(x-t) e^{iQ(t)} \frac{dt}{t}, \tag{1}$$

with

$$Q(t) = a_1 t^{\alpha_1} + \cdots + a_d t^{\alpha_d}. \tag{2}$$

Here $a_i \in \mathbb{R}$ and α_i is a positive integer for each $1 \le i \le d$.

Theorem 1.1. *Given $d \in \mathbb{N}$, we have*

$$\|H_Q f\|_2 \le C_d \|f\|_2. \tag{3}$$

Here C_d is a constant that depends only on d, but not on any a_i or α_i.

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On $\mathbb{R}^2$, define the Hilbert transform along the polynomial curve

$$(t, Q(t))_{t\in\mathbb{R}}$$

by

$$\mathcal{H}_Q f(x,y) = \int_{\mathbb{R}} f(x-t, y-Q(t))\frac{dt}{t}. \tag{4}$$

As a corollary of Theorem 1.1, we have:

Corollary 1.2. *Given $d \in \mathbb{N}$, we have*

$$\|\mathcal{H}_Q f\|_2 \le C_d \|f\|_2. \tag{5}$$

Here C_d is a constant that depends only on d, but not on any a_i or α_i.

Corollary 1.2 follows from Theorem 1.1 via applying Plancherel's theorem to the second variable of $\mathcal{H}_Q f$. We leave out the details.

Denote by n the degree of the polynomial Q given by (2). Then it is well-known (see Stein and Wainger [SW70]) that the estimate (3) holds true if we replace C_d by C_n, a constant that is allowed to depend on the degree n. Moreover, Parissis [Par08] proved that

$$\sup_{P\in\mathcal{P}_n}\left| p.v. \int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right| \simeq \log n, \tag{6}$$

where $\mathcal{P}_n$ is the collection of all real polynomials of degree at most n. It would also be interesting to know whether the constant C_d in (3) can be made to $(\log d)^c$ for some $c > 0$.

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2. Reduction to monomials

We start the proof. In this section, we will split $\mathbb{R}$ into different intervals, and show that for all but finitely many of these intervals, there always exists a monomial which "dominates" our polynomial Q. In dimension one, this idea has been used extensively in the literature, for instance Folch-Gabayet and Wright [FW12]. Here we follow the formulation of Li and Xiao [LX16].

Notice that we can always let the function f absorb the linear term of Q. Hence we assume that $1 < \alpha_1 < \cdots < \alpha_d$. Denote by n the degree of the polynomial Q, that is $n = \alpha_d$. Let $\lambda = 2^{\frac{1}{n}}$. Define $b_j \in \mathbb{Z}$ such that

$$\lambda^{b_j} \le |a_j| < \lambda^{b_j+1}. \tag{7}$$

We define a few bad scales. For $1 \le j_1 < j_2 \le d$, define

$$\mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2) := \{l \in \mathbb{Z} : 2^{-\Gamma_0}|a_{j_2}\lambda^{\alpha_{j_2} l}| \le |a_{j_1}\lambda^{\alpha_{j_1} l}| \le 2^{\Gamma_0}|a_{j_2}\lambda^{\alpha_{j_2} l}|\}. \tag{8}$$

Here $\Gamma_0 := 2^{10d!}$. Notice that l satisfies

$$-2 - n\Gamma_0 + b_{j_2} - b_{j_1} \le (\alpha_{j_1} - \alpha_{j_2})l \le n\Gamma_0 + b_{j_2} - b_{j_1} + 2. \tag{9}$$

Hence $\mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)$ is a connected set whose cardinality is smaller than $4n\Gamma_0$. Define

$$\mathcal{J}^{(0)}_{\text{good}} := \left(\bigcup_{j_1 \neq j_2} \mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2) \right)^c. \tag{10}$$

Notice that $\mathcal{J}^{(0)}_{\text{good}}$ has at most d^2 connected components. Moreover, on each component, there is exactly one monomial which is "dominating".

Similarly, we define

$$\begin{aligned} &\mathcal{J}^{(1)}_{\text{bad}}(\Gamma_0, j_1, j_2) := \\ &\big\{ l \in \mathbb{Z} : 2^{-\Gamma_0} |\alpha_{j_2}(\alpha_{j_2} - 1) a_{j_2} \lambda^{\alpha_{j_2} l}| \le |\alpha_{j_1}(\alpha_{j_1} - 1) a_{j_1} \lambda^{\alpha_{j_1} l}| \\ &\quad \le 2^{\Gamma_0} |\alpha_{j_2}(\alpha_{j_2} - 1) a_{j_2} \lambda^{\alpha_{j_2} l}| \big\}. \end{aligned} \tag{11}$$

Moreover,

$$\mathcal{J}^{(1)}_{\text{bad}} := \bigcup_{j_1 \neq j_2} \mathcal{J}^{(1)}_{\text{bad}}(\Gamma_0, j_1, j_2) \quad \text{and} \quad \mathcal{J}_{\text{good}} := \mathcal{J}^{(0)}_{\text{good}} \setminus \mathcal{J}^{(1)}_{\text{bad}}. \tag{12}$$

Analogously, $\mathcal{J}_{\text{good}}$ has at most d^4 connected components.

3. Bad scales

Due to the control on the cardinalities of various bad sets, the contributions from those $l \notin \mathcal{J}_{\text{good}}$ can be controlled by a multiple of the Hardy–Littlewood maximal function.

Let us be more precise. Suppose that we are working on the collection of bad scales $\mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)$ for some j_1 and j_2. Define

$$H_l f(x) = \int_{\mathbb{R}} f(x-t) e^{iQ(t)} \psi_l(t) \frac{dt}{t}. \tag{13}$$

Here ψ_0 is a nonnegative smooth bump function supported on

$$[-\lambda^2, -\lambda^{-1}] \cup [\lambda^{-1}, \lambda^2]$$

such that

$$\sum_{l \in \mathbb{Z}} \psi_l(t) = 1 \text{ for every } t \neq 0, \text{ with } \psi_l(t) := \psi_0\left(\frac{t}{\lambda^l}\right). \tag{14}$$

By the triangle inequality, we have

$$\left| \sum_{l \in \mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)} H_l f(x) \right| \le \sum_{l \in \mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)} \int_{\mathbb{R}} |f(x-t)| \psi_l(t) \frac{dt}{|t|}. \tag{15}$$

Recall that the cardinality of $\mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)$ is at most $4n\Gamma_0$. Now we partition the set $\mathcal{J}^{(0)}_{\text{bad}}(\Gamma_0, j_1, j_2)$ into subsets of consecutive elements, and such that each subset contains exactly n elements, with possibly one exception which can be handled in the same way. The scale that these n elements can see is about $\lambda^n = 2$, in the sense that for every $l_0 \in \mathbb{Z}$, $\text{supp}\Big(\sum_{l=l_0}^{l_0+n} \psi_l\Big)$ has Lebesgue measure about λ^n. Hence the contribution from each of these subsets can be controlled by $2Mf(x)$. Here M denotes the Hardy–Littlewood maximal operator. Hence the right hand side of (15) can be controlled by $8\Gamma_0 \cdot Mf(x)$. This takes care of the contribution from bad scales.

4. Good scales

Suppose we are working on one connected component of $\mathcal{J}_{\text{good}}$, and for each integer l in such a component, we assume that $a_{j_1} t^{\alpha_{j_1}}$ dominates $Q(t)$ in the sense of (8), that is,

$$|a_{j_1} \lambda^{\alpha_{j_1} l}| \geq 2^{\Gamma_0} |a_{j_1'} \lambda^{\alpha_{j_1'} l}| \quad \text{for every } j_1' \neq j_1, \tag{16}$$

and $a_{j_2}\alpha_{j_2}(\alpha_{j_2}-1)t^{\alpha_{j_2}-2}$ dominates $Q''(t)$ in the sense of (11), that is,

$$|a_{j_2}\alpha_{j_2}(\alpha_{j_2}-1)\lambda^{\alpha_{j_2} l}| \geq 2^{\Gamma_0} |a_{j_2'}\alpha_{j_2'}(\alpha_{j_2'}-1)\lambda^{\alpha_{j_2'} l}| \text{ for every } j_2' \neq j_2. \tag{17}$$

Let us call such a set $\mathcal{J}_{\text{good}}(j_1, j_2)$. Under this assumption, we have the estimates

$$|Q(t)| \leq 2|a_{j_1} t^{\alpha_{j_1}}| \text{ and } |Q''(t)| \geq |a_{j_1} t^{\alpha_{j_1}-2}|, \tag{18}$$

for every $t \in [\lambda^{l-2}, \lambda^{l+1}]$ with $l \in \mathcal{J}_{\text{good}}(j_1, j_2)$. Recall that $\lambda = 2^{\frac{1}{n}}$ is the smallest scale that we will work with. This scale is only visible when $a_n t^n$ dominates. When some other monomial dominates, at such a small scale, our polynomial will not have enough room to see the oscillation. This will be reflected when we come to the stage of applying van der Corput's lemma (see (27) below). Define

$$\lambda_{j_1} := 2^{\frac{1}{\alpha_{j_1}}}.$$

We choose this scale because the monomial $a_{j_1} t^{\alpha_{j_1}}$ dominates. Let

$$\Phi_{j_1, j_2}(t) = \sum_{l \in \mathcal{J}_{\text{good}}(j_1, j_2)} \psi_l(t). \tag{19}$$

Notice that here we join all the small scales from $J_{\text{good}}(j_1, j_2)$ to form a larger scale. Next we will apply a new partition of unity to the function Φ_{j_1, j_2}. Define

$$H^{(j_1)}_{l'} f(x) = \int_{\mathbb{R}} f(x-t) e^{iQ(t)} \psi^{(j_1)}_{l'}(t) \Phi_{j_1, j_2}(t) \frac{dt}{t}. \tag{20}$$

Here $\psi_0^{(j_1)}$ is a nonnegative smooth bump function supported on

$$[-\lambda_{j_1}^2, -\lambda_{j_1}^{-1}] \cup [\lambda_{j_1}^{-1}, \lambda_{j_1}^2]$$

such that

$$\sum_{l' \in \mathbb{Z}} \psi_{l'}^{(j_1)}(t) = 1 \text{ for every } t \neq 0, \text{ with } \psi_{l'}^{(j_1)}(t) := \psi_0^{(j_1)}\left(\frac{t}{\lambda_{j_1}^{l'}}\right). \tag{21}$$

We define $B_{j_1} \in \mathbb{Z}$ such that

$$\lambda_{j_1}^{-B_{j_1}} \leq |a_{j_1}| < \lambda_{j_1}^{-B_{j_1}+1}, \tag{22}$$

denote $\gamma_{j_1} = B_{j_1}/\alpha_{j_1}$ and split the sum in l' into two cases.

$$\sum_{l' \in \mathbb{Z}} H_l^{(j_1)} f = \sum_{l' \leq \gamma_{j_1}} H_{l'}^{(j_1)} f + \sum_{l' > \gamma_{j_1}} H_{l'}^{(j_1)} f. \tag{23}$$

The first summand in (23) can be controlled by the maximal function and the maximal Hilbert transform. To be precise, we have a bound

$$\begin{aligned}
&\sum_{l' \leq \gamma_{j_1}} \left| \int_{\mathbb{R}} f(x-t) \psi_{l'}^{(j_1)}(t) \Phi_{j_1,j_2}(t) \frac{dt}{t} \right| \\
&\qquad + \sum_{l' \leq \gamma_{j_1}} \left| \int_{\mathbb{R}} f(x-t)(e^{iQ(t)}-1) \psi_{l'}^{(j_1)}(t) \Phi_{j_1,j_2}(t) \frac{dt}{t} \right| \\
&\lesssim H^* f(x) + Mf(x) \\
&\qquad + \sum_{l' \leq \gamma_{j_1}} \left| \int_{\mathbb{R}} f(x-t)(e^{iQ(t)}-1) \psi_{l'}^{(j_1)}(t) \Phi_{j_1,j_2}(t) \frac{dt}{t} \right|.
\end{aligned} \tag{24}$$

Here H^* stands for the maximally truncated Hilbert transform. The last summand in (24) can be further controlled by

$$\begin{aligned}
&\sum_{l' \leq \gamma_{j_1}} \int_{\mathbb{R}} |f(x-t)||a_{j_1} t^{\alpha_{j_1}}| \psi_{l'}^{(j_1)}(t) \frac{dt}{|t|} \\
&\leq \sum_{l \in \mathbb{N}} \int_{\lambda_{j_1}^{\gamma_{j_1}-l-2}}^{\lambda_{j_1}^{\gamma_{j_1}-l+1}} |f(x-t)||a_{j_1}||t|^{\alpha_{j_1}-1} dt \\
&\leq \sum_{l \in \mathbb{N}} \lambda_{j_1}^{(\gamma_{j_1}-l+1)(\alpha_{j_1}-1)} \int_{\lambda_{j_1}^{\gamma_{j_1}-l-2}}^{\lambda_{j_1}^{\gamma_{j_1}-l+1}} |f(x-t)||a_{j_1}| dt \leq 8Mf(x).
\end{aligned} \tag{25}$$

Hence it remains to handle the latter term from (23). We will prove that there exists $\delta > 0$ such that

$$\|H_{\gamma_{j_1}+l}^{(j_1)} f\|_2 \leq C_d 2^{-\delta l} \|f\|_2, \quad \text{for every } l \geq 0, \tag{26}$$

with a constant C_d depending only on d. This amounts to proving a decay for the multiplier

$$(27) \qquad \int_{\mathbb{R}} e^{iQ(t)+it\xi}\psi^{(j_1)}_{\gamma_{j_1}+l}(t)\frac{dt}{t} = \int_{\mathbb{R}} e^{iQ(\lambda_{j_1}^{\gamma_{j_1}+l}t)+i\lambda_{j_1}^{\gamma_{j_1}+l}t\xi}\psi^{(j_1)}_0(t)\frac{dt}{t}.$$

We calculate the second order derivative of the phase function:

$$(28) \qquad \lambda_{j_1}^{2\gamma_{j_1}+2l}|Q''(\lambda_{j_1}^{\gamma_{j_1}+l}t)| \geq \frac{1}{2}|a_{j_1}|\lambda_{j_1}^{B_{j_1}+\alpha_{j_1}l} \geq 2^{l-2}.$$

Hence the desired estimate follows from van der Corput's lemma, for which we refer to Proposition 2 in Page 332 [Stein93].

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