Dense images of the power maps in Lie groups and minimal parabolic subgroups

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ABSTRACT. In this note, we study the density of the images of the k-th power maps $P_k : G \to G$ given by $g \to g^k$, for a connected Lie group G. We characterize $P_k(G)$ being dense in G in terms of the minimal parabolic subgroups of G. For a simply connected simple Lie group G, we characterize all integers k, for which $P_k(G)$ has dense image in G. We show also that for a simply connected semisimple Lie group weak exponentiality is equivalent to the image of the squaring map being dense.

CONTENTS

1. Introduction

Let G be a connected Lie group. For any positive integer k, let P_k denote the k-th *power map*, defined by $P_k(g) = g^k$ for all $g \in G$. There is a considerable amount of literature on the surjectivity of the individual power maps (for e.g. [Ch], [DM], and see references cited there), which can be applied, in particular, to study exponentiality of Lie groups. Here we consider a weaker question than surjectivity of the power maps, namely that of the image being dense. In the case of the exponential map, a connected Lie group for which the image is dense is said to be *weakly exponential*; this property is well-studied (for e.g. [HM], [H], [N], etc.). However, the question of density of images of the individual power maps has not been studied in detail so far.

In [BhM], it was shown that for a connected Lie group G, the image of the exponential map is dense in G if and only if $P_k(G)$ is dense in G for all

$k \in \mathbb{N}$. Criteria were given for the image of the individual power maps to be dense, in terms of conditions on regular elements and Cartan subgroups of G. It was proved that $P_k(G)$ being dense depends only on the semisimple quotient of G. For simple Lie groups, the set of integers k, for which $P_k(G)$ is dense, was analyzed.

Let $\tilde{G}$ be a simply connected simple Lie group. When either $\tilde{G}$ is compact or $\mathrm{Ad}(\tilde{G})$ is split, or $Z(\tilde{G})$ does not contain an infinite cyclic subgroup, conditions for $P_k(\tilde{G})$ to be dense were described in [BhM]. In general, the question remains open.

In this context, we obtain here a characterization of image of the density of the power map in terms of the minimal parabolic subgroups, analogous to the result of Jaworski for the exponential map (see Theorem 1.1). Using this, we characterize the integers k, for which P_k has dense image, for simply connected simple Lie groups (see Theorem 1.2). From the results, we deduce equivalence between weak exponentiality of $\tilde{G}$ and density of $P_2(\tilde{G})$, for a simply connected semisimple Lie group $\tilde{G}$ (see Corollary 1.3).

Let G be a connected semisimple Lie group and let $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ be the adjoint representation of G. Consider an Iwasawa decomposition $G = KAN$, where K, A, and N are closed subgroups of G, $\mathrm{Ad}(K)$ is a maximal compact subgroup of $\mathrm{Ad}(G)$, $\mathrm{Ad}(A)$ is a maximal connected subgroup consisting of elements diagonalisable over $\mathbb{R}$, and N is a simply connected nilpotent Lie subgroup normalised by A. Note that K contains the center of G, and K is compact if and only if the center is finite.

We denote by M the subgroup $Z_K(A)$, the centralizer of A in K. Then M is a closed (not necessarily connected) subgroup of G, and it normalizes N. The subgroup $P := MAN$ is a *minimal parabolic subgroup* of G associated with the Iwasawa decomposition $G = KAN$. We recall that all minimal parabolic subgroups of G are conjugate to each other.

We have the following theorem on density of the images of the power maps.

Theorem 1.1. *Let G be a connected semisimple Lie group and P be a minimal parabolic subgroup of G. For $k \in \mathbb{N}$, $P_k(G)$ is dense if and only if $P_k(P)$ is dense.*

From Theorem 1.1 we deduce the following results, using the structure of the subgroup $\tilde{M}$ of the minimal parabolic subgroup $\tilde{P} = \tilde{M}AN$ (notations as before) of $\tilde{G}$.

Theorem 1.2. *Let $\tilde{G}$ be a simply connected simple Lie group. Then the following statements hold:*

(i) *If $\tilde{G} = \widetilde{\mathrm{SL}}(2, \mathbb{R})$, $\widetilde{\mathrm{SO}}^*(2n)$ (n even), $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$, $\widetilde{\mathrm{SU}}(p, p)$, $\widetilde{\mathrm{Spin}}^*(2, q)$ ($q \neq 2$), or $\tilde{\mathrm{E}}_{7(-25)}$, then $P_k(\tilde{G})$ is not dense in $\tilde{G}$ for any $k \in \mathbb{N}$.*

(ii) *If $\tilde{G} = \tilde{\mathrm{SL}}(n, \mathbb{R})$ (n > 2), $\tilde{\mathrm{Spin}}^*(p, q)$ (p $\neq$ 1, 2 and q $\neq$ 2, and p $\leq$ q), $\tilde{\mathrm{E}}_{6(6)}$, $\tilde{\mathrm{E}}_{6(2)}$, $\tilde{\mathrm{E}}_{7(7)}$, $\tilde{\mathrm{E}}_{7(-5)}$, $\tilde{\mathrm{E}}_{8(8)}$, $\tilde{\mathrm{E}}_{8(-24)}$, $\tilde{\mathrm{F}}_{4(4)}$, or $\tilde{\mathrm{G}}_{2(2)}$, then $P_k(\tilde{G})$ is dense in $\tilde{G}$ if and only if k is an odd integer.*

(iii) *If $\tilde{G} = \tilde{\mathrm{SO}}^*(2n)$ (n odd), $\mathrm{Sp}(p, q)$ (p $\leq$ q), $\mathrm{SU}^*(2n)$, $\tilde{\mathrm{SU}}(p, q)$ (1 $\leq$ p < q), $\tilde{\mathrm{Spin}}^*(1, q)$ (q > 3), $\tilde{\mathrm{E}}_{6(-14)}$, $\tilde{\mathrm{E}}_{6(-26)}$, or $\tilde{\mathrm{F}}_{4(-20)}$, then $P_k(\tilde{G})$ is dense in $\tilde{G}$ for all $k \in \mathbb{N}$.*

Let $\mathfrak{g}$ denote the Lie algebra of G and let $\exp : \mathfrak{g} \to \mathrm{G}$ be the associated exponential map. We recall that a connected Lie group G is said to be *weakly exponential* if $\exp(\mathfrak{g})$ is dense in G. It was proved by W. Jaworski that G is weakly exponential if and only if all its minimal parabolic subgroups are connected ([Ja, Theorem 12]). The result can be deduced from Theorem 1.1 (see Corollary 4.2).

For a simply connected semisimple Lie group, we further show the following.

Corollary 1.3. *Let $\tilde{G}$ be a simply connected semisimple Lie group. Then $\tilde{G}$ is weakly exponential if and only if $P_2(\tilde{G})$ is dense in $\tilde{G}$.*

The paper is organized as follows. In §2, we recall some definitions, prove some preliminary results about regular elements and minimal parabolic subgroups, and deduce Theorem 1.1. In §3, we prove Theorem 1.2. Corollaries 4.2 and 1.3 are proved in §4.

2. Characterization of density of the image of P_k

We begin by recalling some definitions, and noting some preliminary results.

Definition 1 ([H]). An element g in a Lie group G is said to be regular if the nilspace $N(\mathrm{Ad}_g - I)$ has minimal possible dimension.

The set of regular elements in G is denoted by $\mathrm{Reg}(G)$. The set $\mathrm{Reg}(G)$ is an open dense subset of G.

Let G be a group of $\mathbb{R}$-points of a complex semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. Let $g \in G = \mathbf{G}(\mathbb{R})$. Then $g = g_s g_u$, where g_s, and g_u are the Jordan semisimple and unipotent components of g, and the nilspace of $(\mathrm{Ad}_g - I)$ is equal to $\mathrm{Ker}(\mathrm{Ad}_{g_s} - I)$. The dimension of $\mathrm{Ker}(\mathrm{Ad}_{g_s} - I)$ is equal to the dimension of the centralizer $Z_G(g_s)$. Hence g is regular in G if and only if $Z_G(g_s)$ is of minimal possible dimension. Thus for the case of algebraic groups, Definition 1 coincides with Borel's definition (see[Bo, §12.2]) of a regular element. Also, we note that every regular element in G is necessarily semisimple.

The following results would be generally known to experts in the area. We include proofs for the convenience of the reader, for want of suitable references.

Lemma 2.1. *Let G be a connected semisimple Lie group and let P be a minimal parabolic subgroup of G. Then $\operatorname{Reg}(G) \cap P$ is dense in P.*

Proof. We first note that we may assume that G is linear: Let $G_1 = G/Z(G)$ and $\pi : G \to G_1$ be the natural covering map. Then G_1 is a linear group, with $P_1 := \pi(P)$ as a minimal parabolic subgroup and we have $\pi^{-1}(\operatorname{Reg}(G_1) \cap P_1) = \operatorname{Reg}(G) \cap P$, so it is enough to prove that $\operatorname{Reg}(G_1) \cap P_1$ is dense in P_1.

Let G be a linear semisimple Lie group and $\mathbf{G}$ be its Zariski closure. Let $P = MAN$ be a minimal parabolic subgroup of G. Let $T = HA$, where H is a maximal torus in M. Then T is an open subgroup of a maximal torus in G. Let $\mathbf{T}$ be the Zariski closure of T in $\mathbf{G}$. Then $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ and T is an open subgroup of $\mathbf{T}(\mathbb{R})$. Then $\operatorname{Reg}(G) = \operatorname{Reg}(\mathbf{G}(\mathbb{R})) \cap G$. Since $\mathbf{T}$ is a maximal torus in $\mathbf{G}$, it follows that $\mathbf{T}(\mathbb{R}) \cap \operatorname{Reg}(\mathbf{G}(\mathbb{R}))$ is dense in $\mathbf{T}(\mathbb{R})$. Since T is open in $\mathbf{T}(\mathbb{R})$, this further implies that $T \cap \operatorname{Reg}(G)$ is dense in T. The conjugates of T in P form a dense subset of P. As conjugates of regular elements are regular, this implies that $\operatorname{Rag}(G) \cap P$ is dense in P. $\square$

Lemma 2.2. *Let G be a connected semisimple Lie group and let P be a minimal parabolic subgroup of G. Let $g \in \operatorname{Reg}(G) \cap P$. Then there exists a unique Cartan subgroup C such that $g \in C \subset P$.*

Proof. Let $G_1 = G/Z(G)$ and $\pi : G \to G_1$ denote the natural projection map. Recall that $\pi^{-1}(\operatorname{Reg}(G_1)) = \operatorname{Reg}(G)$ ([Bou, Proposition 2, §2]) and C is a Cartan subgroup if and only if $C_1 = C/Z(G)$ is a Cartan subgroup. Hence we may assume G to be a linear group. Furthermore, we can assume that G is an algebraic group. Indeed, any connected linear semisimple Lie group is the connected component of the identity in the Hausdorff topology in an algebraic group, and the Cartan subgroup (resp. minimal parabolic subgroup) in G is the intersection with G of a Cartan subgroup (resp. minimal parabolic subgroup) in the algebraic group.

Let $g \in \operatorname{Reg}(G) \cap P$. It is known that any regular element of G is semisimple and $Z_G(g)^0$ is a Cartan subgroup of G (can be deduced from [Bo, Proposition 12.2], as centralizer commutes with base changes). It suffices to prove that $Z_G(g)^0$ is contained in P. By a conjugation we may assume that $P = MAN$, where $G = KAN$ is an Iwasawa decomposition, $M = Z_K(A)$, and that $g \in MA$. We note that for $g \in MA$, $Z_G(g)^0$ is contained in P. Indeed, g is regular, and hence 1 is not an eigenvalue of $\operatorname{Ad}(g)|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N. Since g is semisimple, by [Bo, Corollary 11.12], we have $g \in Z_{\mathbf{G}}(g)^0$ (where $\mathbf{G}$ is the Zariski closure of G), and hence $g \in Z_G(g)^0$ as required.

The statement about the uniqueness follows from the fact that for a given regular element, there exists a unique Cartan subgroup containing it. $\square$

Proof of Theorem 1.1. Suppose that $P_k(G)$ is dense in G. By Lemma 2.1, it is enough to show that $\operatorname{Reg}(G) \cap P \subseteq P_k(P)$. Let $g \in \operatorname{Reg}(G) \cap P$. Then by Lemma 2.2, there exists a Cartan subgroup C containing g, and

contained in P. Therefore by applying [BhM, Theorem 1.1], we get that there exists h in C (and hence in P) such that $h^k = g$.

For the converse, let P be a minimal parabolic subgroup of G. We observe that $E = \cup_{g \in G} gPg^{-1}$ is dense in G. Since $\overline{P_k(G)}$ (the closure of $P_k(G)$ in G) is invariant under conjugation and contains P, we get that it contains E. Hence $\overline{P_k(G)} = G$. $\qquad\square$

For any connected Lie group G, let $R = \text{Rad(G)}$ (radical of G). A subgroup P of G is called a *minimal parabolic subgroup* of G, if the following hold:

(i) $P \supset R$.
(ii) P/R is a minimal parabolic subgroup of the connected semisimple group G/R.

Theorem 1.1 can be extended to all connected Lie groups. The following corollary is a straightforward application of [BhM, Proposition 3.3] and Theorem 1.1, and hence we omit the proof.

Corollary 2.3. *Let G be a connected Lie group and $k \in \mathbb{N}$. Then $P_k(G)$ is dense in G if and only if $P_k(P)$ is dense in P for every minimal parabolic subgroup P of G.*

3. Density of images for simple Lie groups

In this section, we determine conditions for density of the image of P_k for simply connected covering groups of simple Lie groups, and prove Theorem 1.2.

Suppose G is a linear Lie group and $G = KAN$ be an Iwasawa decomposition of G. Let $\pi : \tilde{G} \to G$ be the covering of G with $\tilde{K} = \pi^{-1}(K)$. Then $\tilde{G} = \tilde{K}AN$. If we set $\tilde{M} = \pi^{-1}(M)$, then $\tilde{M} = Z_{\tilde{K}}(A)$ and $\tilde{P} = \tilde{M}AN$ is the minimal parabolic subgroup of $\tilde{G}$.

Let us fix some notations. Let $\tilde{G}$ be a simply connected simple Lie group and $G = \text{Ad}(\tilde{G})$. Therefore $\tilde{K}$ is the pullback of the maximal compact subgroup of G.

Proposition 3.1. *Let the notations $\tilde{G}$, $\tilde{P}$, $\tilde{A}$ and N be as above. Suppose $\tilde{M}/\tilde{M}^*$ is a group of order 2^m for some $m > 0$. Let $k \in \mathbb{N}$. If $P_k : \tilde{M} \to \tilde{M}$ is surjective, then $P_k : \tilde{P} \to \tilde{P}$ is dense.*

Proof. We recall that N is a simply connected nilpotent Lie group. Let

$$N = N_0 \supset N_1 \supset \cdots \supset N_r = \{e\}$$

be the central series of N. Let $V_j := N_j/N_{j+1}$ for $j = 0, 1, \ldots, r-1$. We note that V_j is a real vector space for all j, as N is simply connected. Consider the representations $\psi_j : \tilde{M}A \to \text{GL(V}_j)$ for $j = 0, 1, \ldots, r - 1$, and let $K_j = \text{Ker}(\psi_j)$. Note that K_j is a closed subgroup of $\tilde{M}A$ for all j. Then one of the following two statements holds:

(i) $\dim(K_j) < \dim(\tilde{M}A)$ for all j.

(ii) For some j, $\dim(K_j) = \dim(\tilde{M}A)$.

Suppose (i) holds. Then $U := \tilde{M}A - \cup_j K_j$ is a dense open set in $\tilde{M}A$. It is easy to see that all elements in U act non trivially on all V_j's. Therefore by [DM, Theorem 1.1(i)], $gN \subset P_k(\tilde{P})$ for all $g \in U$. If we take $W = U \times N$, then W is a dense subset of $\tilde{P}$ such that $W \subset P_k(\tilde{P})$.

Now suppose (ii) holds. Then K_j is the union of some connected components of $\tilde{M}A$. Since $\tilde{M}A/\tilde{M}^*A$ is a group of order 2^m ($m > 0$) and P_k is surjective, we obtain that k is odd. Also, it follows that $P_k(K_j) = K_j$ for an odd integer k. Let $\mathcal{F}$ be the set of $j \in \{0, 1, \ldots, r-1\}$ such that $\dim(K_j) = \dim(\tilde{M}A)$. Then for all $i \in \mathcal{F}$, $P_k(K_i) = K_i$ and hence for any element x in K_i, we have $xN \subset P_k(\tilde{P})$ by [DM, Theorem 1.1(i)]. Now $\cup_{i \notin \mathcal{F}} K_i$ is a proper closed analytic subset of $\tilde{M}A$ of smaller dimension. Hence as in case (i), there exists a dense open set W' in $\tilde{M}A$ such that $W_1 \subset P_k(\tilde{P})$, where $W_1 = W' \times N$. $\square$

Remark 3.2. Let G be a connected semisimple linear Lie group and $P = MAN$ be a minimal parabolic subgroup of G. Then $M = F \times M^*$, where F is a group whose elements are of order 2, and M^* is the connected component of the identity in M ([K, Theorem 7.53(c)]). Then for odd k, $P_k(P)$ is dense in P (by Proposition 3.1). For even k, $P_k(P) = P^*$ and hence $P_k : P \to P$ is not dense. The reader may compare this with [BhM, Remark 3.1]. From this, one can also reprove [BhM, Corollary 1.5].

Proposition 3.3. *Let $\tilde{G}$ be a simply connected Lie group, and let the notations be as in Proposition* 3.1. *Then the following statements hold:*

(i) *If $\tilde{M}/\tilde{M}^*$ has $\mathbb{Z}$ as a factor, then $P_k(\tilde{G})$ is not dense for any k.*

(ii) *If $\tilde{M}/\tilde{M}^*$ is a group of order 2^m for some $m > 0$, then $P_k(\tilde{G})$ is dense if and only if k is odd.*

(iii) *If $\tilde{M}$ is connected, then $P_k(\tilde{G})$ is dense for all k.*

Proof. (i) The condition in the hypothesis implies that, P_k is not surjective for the group $\tilde{M}/\tilde{M}^*$. Hence $P_k : \tilde{P} \to \tilde{P}$ is not dense. Therefore the result follows from Theorem 1.1.

(ii) It is immediate from Theorem 1.1 that P_k is not dense for any even integer k. For odd k, the statement follows from Proposition 3.1 and Theorem 1.1.

(iii) In this case, $\tilde{P}$ is connected and hence by [Ja, Theorem 12], $\tilde{G}$ is weakly exponential. This implies P_k is dense in $\tilde{G}$ for all k. $\square$

We denote by $E_{6,\mathbb{C}}$, $E_{7,\mathbb{C}}$, $E_{8,\mathbb{C}}$, $F_{4,\mathbb{C}}$ and $G_{2,\mathbb{C}}$ the complex simply connected simple exceptional Lie groups.

$E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$ denote connected noncompact real forms of $E_{6,\mathbb{C}}$.

$E_{7(7)}$, $E_{7(-5)}$ and $E_{7(-25)}$ denote connected noncompact real forms of $E_{7,\mathbb{C}}$.

$E_{8(8)}$ and $E_{8(-24)}$ denote connected noncompact real forms of $E_{8,\mathbb{C}}$.

$F_{4(4)}$ and $F_{4(-20)}$ denote connected noncompact real forms of $F_{4,\mathbb{C}}$.

$G_{2(2)}$ denotes connected noncompact real forms of $G_{2,\mathbb{C}}$.

Among these, $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $F_{4(4)}$ are split exceptional simple Lie groups.

Proof of Theorem 1.2. (i) In these cases, from [Jo, Proposition 17.1, 17.4, 17.6, 17.7, 17.9, 14.1 respectively], it follows that $\tilde{M}/\tilde{M}^*$ has $\mathbb{Z}$ as a factor group. Therefore the assertion follows from Proposition 3.3.

(ii) When $G = E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ or $F_{4(4)}$, the assertion follows from [BhM, Case-5]. In the rest of the cases, $\tilde{M}/\tilde{M}^*$ is a group of order 2^m ($m = p$ for $\tilde{\text{Spin}}^*(p,q)$ ($p \neq 1$, 2 and $q \neq 2$, and $p \leq q$), $m = 3$ for $\tilde{E}_{6(2)}$, $\tilde{E}_{7(-5)}$, $\tilde{E}_{8(-24)}$, and $\tilde{G}_{2(2)}$) ([Jo, Proposition 17.1, 17.5, 13.1, 12.1, 10.4]). Hence by Proposition 3.1, the result follows.

(iii) In this case $\tilde{M}$ is known to be connected; see [Jo, Proposition 17.2, 17.3, 17.6, 17.8, §17 (9), 15.1, §16]. The theorem therefore follows from Proposition 3.3(iii). □

4. Application to weak exponentiality

In this section, we deduce Corollaries 4.2 and 1.3.

Let G be a group. An element g in G is said to be *divisible* if for all $k \in \mathbb{N}$, there exists $h_k \in G$ such that $h_k^k = g$. If all elements of G are divisible, then G is said to be a divisible group.

Proposition 4.1. *Let G be a connected semisimple Lie group and let P be a minimal parabolic subgroup of G. If $P_k(P)$ is dense for every positive integer k, then P is connected.*

Proof. Let $G_1 = G/Z(G)$ and $\pi : G \to G_1$ be the natural projection map. Let $P_1 = \pi(P)$. We note that $Z(G)$ is contained in P. Then we get the following short exact sequence.

$$1 \to Z(G) \to P \to P_1 \to 1.$$

Let P^* and P_1^* be the connected components of the identity in P and P_1 respectively. Then we have the following short exact sequence.

$$1 \to Z(G)/Z(G) \cap P^* \to P/P^* \to P_1/P_1^* \to 1.$$

As G_1 is a connected linear group, P_1/P_1^* is finite. Since P/P^* is divisible, so is P_1/P_1^*. We observe that any finite divisible group is trivial and hence $P_1 = P_1^*$. Thus P/P^* is a finitely generated abelian group (as $Z(G)$ is a finitely generated abelian group). Since any finitely generated divisible abelian group is trivial, we get that $P = P^*$. □

Theorem 1.1 can be used to deduce the following Corollary, which is well known ([Ja, Theorem 12]).

Corollary 4.2. *Let G be a connected semisimple Lie group. Then the following are equivalent:*

(i) *G is weakly exponential.*

(ii) *All minimal parabolic subgroups of G are weakly exponential.*

Proof. (i)$\Rightarrow$(ii) We note that G being weakly exponential implies $P_k(G)$ is dense in P for all k. Thus by Theorem 1.1, $P_k(P)$ is dense in P for all k, and hence by Proposition 4.1, we get that P is connected. Now by applying [BhM, Corollary 1.3], we get that (ii) holds.

(ii)$\Rightarrow$(i) Let P be a minimal parabolic subgroup. By hypothesis, P is connected and $P_k(P)$ is dense in P for all k. Therefore by Theorem 1.1, $P_k(G)$ is dense in G for all k. Then by [BhM, Corollary 1.3], it follows that G is weakly exponential. □

Proof of Corollary 1.3. Let $G_1, G_2, \ldots, G_r$ be the simple factors of $\tilde{G}$. Since $\tilde{G}$ is simply connected, G_i is simply connected for all i, and

$$\tilde{G} = G_1 \times G_2 \times \cdots \times G_r.$$

Now $P_k(\tilde{G})$ is dense in $\tilde{G}$ if and only if $P_k(G_i)$ is dense in G_i for all $i = 1, 2, \ldots, r$. By Theorem 1.2, it follows that G_i is weakly exponential if and only if $P_2(G_i)$ is dense in G_i. Since $\tilde{G}$ is weakly exponential if and only if each G_i is weakly exponential, the assertion as in the Corollary follows. □

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