New York Journal of Mathematics

New York J. Math. **24** (2018) 429–442.

The topology of local commensurability graphs

Khalid Bou-Rabee and **Daniel Studenmund**

ABSTRACT. We initiate the study of the p-local commensurability graph of a group, where p is a prime. This graph has vertices consisting of all finite-index subgroups of a group, where an edge is drawn between A and B if $[A : A \cap B]$ and $[B : A \cap B]$ are both powers of p. We show that any component of the p-local commensurability graph of a group with all nilpotent finite quotients is complete. Further, this topological criterion characterizes such groups. In contrast to this result, we show that for any prime p the p-local commensurability graph of any large group (e.g. a nonabelian free group or a surface group of genus two or more or, more generally, any virtually special group) has geodesics of arbitrarily long length.

CONTENTS

Let G be a group. The *commensurability index* of two commensurable subgroups $A, B \leq G$ is $[A : A \cap B][B : A \cap B]$. For a prime number p, the *p-local commensurability graph* of G, denoted $\Gamma_p(G)$, is the graph with vertices consisting of finite-index subgroups of G where two subgroups $A, B \leq G$ are adjacent if and only if their commensurability index is a power of p. For a warm-up example, see Figure 1.

The goal of this paper is to draw algebraic information of G from the topology of $\Gamma_p(G)$.

Theorem 1. *Let G be a finitely generated group. The following are equivalent:*

(1) *For any prime p, every component of $\Gamma_p(G)$ is complete.*
(2) *All of the finite quotients of G are nilpotent.*

Received March 4, 2018.

2010 *Mathematics Subject Classification.* Primary: 20E26 and 20E15; Secondary: 20B99 and 20F18.

Key words and phrases. commensurability, nilpotent groups, free groups, very large groups.

K.B. supported in part by NSF grant DMS-1405609.

D.S. supported in part by NSF grants DMS-1246989 and DMS-1547292.

ISSN 1076-9803/2018

The proof of Theorem 1 is in §2. The structure theory of solvable groups plays an important role in our proofs. Theorem 1 applies, for example, to Grigorchuk's group [Gri83], which is a 2-group and therefore has only nilpotent finite quotients.

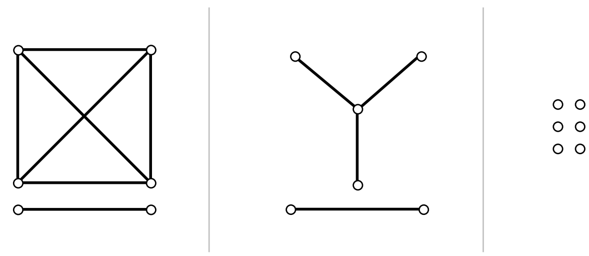

FIGURE 1. Let Sym_3 be the symmetric group on 3 elements (note Sym_3 is solvable and not nilpotent). The figure above displays $\Gamma_2(\mathrm{Sym}_3), \Gamma_3(\mathrm{Sym}_3)$, and $\Gamma_5(\mathrm{Sym}_3)$ in that order. All $\Gamma_p(\mathrm{Sym}_3)$ for primes $p > 3$ are discrete spaces.

In contrast to the above theorem, we show that components of the local commensurability graphs of free groups are far from complete:

Theorem 2. *Let F be a rank two free group. For any prime p and $N > 0$, there exist infinitely many geodesics γ, each in a different component of $\Gamma_p(F)$, such that the length of each γ is greater than N.*

We prove Theorem 2 in §3. A result of Robert Guralnick (which uses the classification of finite simple groups) concerning subgroups of prime power index in a nonabelian finite simple group is used in an essential way in our proof [Gur83]. Moreover, in our proof we get a clean description of an entire component of the p-local commensurability graph of many finite alternating groups. See Figure 2, for example.

Our next result demonstrates that arbitrarily long geodesics in the p-local commensurability graph of a free group cannot possibly all come from a single component. We prove this at the end of §1.

Proposition 3. *Let G be a finitely generated group. Let Ω be a connected component of $\Gamma_p(G)$. Then there exists $C > 0$ such that any two points in Ω are connected by a path of length less than C. That is, the diameter of Ω is finite. Moreover if any vertex of Ω is a normal subgroup of G then the diameter of Ω is bounded above by 3.*

As a consequence of Theorem 2 and Proposition 3, there exists components of the p-local commensurability graph of a nonabelian free group with no normal subgroups as vertices (see Corollary 22 at the end of §3).

Recall that a group is *large* if it contains a normal finite-index subgroup that admits a surjective homomorphism onto a non-cyclic free group. Such groups enjoy the conclusion of Theorem 2. See the end of §3 for the proof.

Corollary 4. *Let G be a large group. For any prime p and $N > 0$, there exists infinitely many geodesics γ, each in a different component of $\Gamma_p(G)$, such that the length of each γ is greater than N.*

Experiments that led us to the above theorems were done using GAP [GAP15] and Mathematica [W15].

The results of this paper are motivated by the authors' interest in studying the metric properties of the *commensurability graph* of a finitely generated group G, denoted $\Gamma(G)$ and defined to be the complete graph on the set of all finite-index subgroups of G with edges weighted by the commensurability indices. The weighted path metric gives $\Gamma(G)$ the structure of a metric space. This graph encodes, for instance, Alex Lubotzky and Dan Segal's *subgroup growth* [LS03] as the growth of balls in the graph-theoretic star of the vertex G in $\Gamma(G)$. The study of local commensurability graphs Γ_p is analogous to studying the local subgroup growth functions, where one only considers subgroups of index a power of a fixed prime p. In a forthcoming paper we will investigate the full geometry of commensurability graphs and explore connections between local and global aspects. Note, however, that in this paper we consider a local commensurability graph as a metric space with the standard path metric as an abstract graph; in particular, a 'geodesic' is a path which minimizes the edge lengths of paths between its vertices.

This paper sits in the broader program of studying infinite groups through their residual properties, which is an area of much activity (see, for instance, [KT16], [BRK12], [BRM11], [GK17], [BRHP15], [BRS16], [KM11], [Riv12], [Pat13], [LS03]). Specifically, a similar object is studied in [AAH+15]. There a graph is constructed with vertices consisting of subgroups of finite index, and an edge is drawn between two vertices if one is a prime-index subgroup (the prime is not fixed) of the other. They show that for every group G, their graph is bipartite with girth contained in the set $\{4, \infty\}$ and if G is a finite solvable group, then their graph is connected.

Acknowledgements. We are grateful to Ben McReynolds and Sean Cleary for useful and stimulating conversations. An anonymous referee provided the simple proof of Lemma 14 that appears here. Another anonymous referee provided several clarifications and corrections to the exposition, including the proof of Lemma 5.

1. Preliminaries and basic facts

In this section we record some basic facts that will be used throughout. We start with a couple of elementary results.

Lemma 5. *Let $N \leq G$ be a normal subgroup and $\pi : G \to G/N$ the quotient map. For subgroups $K \leq H \leq G$ we have*

$$[H : K] = [\pi(H) : \pi(K)][H \cap N : K \cap N].$$

Proof. Consider the action of $H \cap N$ on the coset space H/K by left translation. Let Z be the kernel of this action, and note that $Z \leq K \cap N$. This action has number of orbits equal to $[\pi(H):\pi(K)]$, so Burnside's lemma gives

$$|(H\cap N)/Z|\cdot[\pi(H):\pi(K)] = \sum_{g\in(H\cap N)/Z} |(H/K)^g|.$$

On the one hand, we clearly have

$$|(H\cap N)/Z| = [H\cap N : K\cap N]\cdot|(K\cap N)/Z|.$$

On the other hand, a coset $hK\in H/K$ is fixed precisely by the conjugate $h((K\cap N)/Z)h^{-1}$ under the action of $(H\cap N)/Z$, so we have

$$\begin{aligned}\sum_{g\in(H\cap N)/Z} |(H/K)^g| &= \sum_{hK\in H/K} |\{g\in(H\cap N)/Z \mid ghK = hK\}| \\ &= |H/K|\,|(K\cap N)/Z|.\end{aligned}$$

The desired result follows. □

Lemma 6. *Let N be a normal subgroup of G and p a prime. If A and N are both subgroups of index a power of p in G, then $[G:A\cap N]$ is also a power of p.*

Proof. Let $\pi: G\to G/N$ be the quotient map. Then $[A:A\cap N] = |\pi(A)|$. Because G/N is a p-group, it follows that $[A:A\cap N]$ is a power of p. Therefore $[G:A\cap N] = [G:A][A:A\cap N]$ is a power of p. □

Our next couple of lemmas give control of local commensurability graphs under some maps.

Lemma 7. *If G is a group, $\pi: G\to Q$ is a surjection, and γ a path in $\Gamma_p(G)$, then $\pi(\gamma)$ is a path in $\Gamma_q(Q)$ with length bounded above by the length of γ.*

Proof. If $K\le H\le G$ then $[\pi(H):\pi(K)]$ divides $[H:K]$ by Lemma 5. Therefore adjacent vertices in γ map to adjacent vertices in $\pi(\gamma)$, or are possibly identified in $\Gamma_p(Q)$. □

Lemma 8. *Suppose G is a group and p is prime.*

(1) *If N is a normal subgroup of G, then the quotient map $\pi: G\to G/N$ induces an isometric graph embedding $\Gamma_p(G/N)\to\Gamma_p(G)$ as an induced subgraph.*
(2) *If H is a finite-index subgroup of G, then the inclusion $i: H\to G$ induces a graph embedding $\Gamma_p(H)\to\Gamma_p(G)$ as an induced subgraph.*
(3) *If N is a finite-index normal subgroup of G, then the inclusion $i: N\to G$ induces an isometric graph embedding $\Gamma_p(N)\to\Gamma_p(G)$ as an induced subgraph.*

Proof. For 1, if $\pi: G\to G/N$ is a quotient map, then the assignment $K\mapsto\pi^{-1}(K)$ defines a graph embedding $\Gamma_p(G/N)\to\Gamma_p(G)$ whose image is an induced subgraph. This embedding is isometric by Lemma 7.

For 2, if $H\le G$ has finite index, then the assignment $K\mapsto i(K)$ defines a graph embedding $\Gamma_p(H)\to\Gamma_p(G)$ whose image is an induced subgraph.

For 3, let $N \lhd G$ be a finite-index subgroup, with assignment $\phi : K \mapsto i(K)$ defined over all subgroups K in N. Let $H_1, H_2 \in \phi(\Gamma_p(N))$ and let $H_1 = J_1, \dots, J_n = H_2$ be a path in $\Gamma_p(G)$ from H_1 to H_2. Then for each $i = 1, \dots, n-1$, we have that

$$[J_i : J_i \cap J_{i+1}][J_{i+1} : J_i \cap J_{i+1}]$$

is a power of p. By Lemma 7, $\pi(J_1), \dots, \pi(J_n)$ is a path in $\Gamma_p(G/N)$. Because $J_1 \leq N$, this is a path of p-subgroups of G/N. Therefore $[J_i : J_i \cap N]$ is a power of p for all $i = 1, \dots, n$. Thus, by Lemma 6 applied to $J_i \cap N$ and $J_{i+1} \cap J_i$, we have for $i = 1, \dots, n-1$,

$$[J_i : (J_i \cap N) \cap (J_i \cap J_{i+1})][J_{i+1} : (J_{i+1} \cap N) \cap (J_i \cap J_{i+1})],$$

is a power of p. Hence, for $i = 1, \dots, n-1$,

$$[J_i : N \cap J_i][N \cap J_i : N \cap J_i \cap J_{i+1}] = [J_i : N \cap J_i \cap J_{i+1}]$$

is a power of p giving that $[N \cap J_i : N \cap J_i \cap J_{i+1}]$ is a power of p, since above we showed that $[J_i : N \cap J_i]$ is a power of p. By a similar argument, we get that $[N \cap J_{i+1} : N \cap J_i \cap J_{i+1}]$ is a power of p, and thus $N \cap J_i$ and $N \cap J_{i+1}$ are adjacent in $\Gamma_p(G)$. It follows that the path $J_1, \dots, J_n$ can be replaced by the path (which possibly has repeated vertices) $J_1 = J_1 \cap N, J_2 \cap N, \dots, J_{n-1} \cap N, J_n \cap N = J_n$, which is entirely contained in $\Gamma_p(N)$. It follows that $\Gamma_p(N)$ is a geodesic metric space in the path metric induced from $\Gamma_p(G)$, as desired. □

Note that the hypothesis of normality in 3 cannot be removed. For example, suppose S and T are disjoint sets with $|S| = |T| = 5$ and consider the non-normal subgroup $\mathrm{Alt}_S \times \mathrm{Alt}_T \leq \mathrm{Alt}_{S \cup T}$. It can be shown using Lemma 18 below that Alt_S and Alt_T are in the same component of $\Gamma_5(\mathrm{Alt}_{S \cup T})$ but in different components of $\Gamma_5(\mathrm{Alt}_S \times \mathrm{Alt}_T)$.

Our next lemma will lead to our first result concerning free groups.

Lemma 9. *Let A, B be vertices in $\Gamma_p(G)$. Suppose B shares an edge with A. If q^k divides $[G : A]$ for some prime $q \neq p$ then q^k divides $[G : B]$.*

Proof. In this case, we have

$$[G : A \cap B] = [G : A][A : A \cap B] = [G : B][B : A \cap B].$$

Hence, if q^k divides $[G : A]$, then q^k must divide $[G : B]$ because $[B : A \cap B]$ is a power of p. □

Proposition 10. *The p-commensurability graph of a free group has infinitely many components.*

Proof. Any free group has subgroups $\{N_1, N_2, N_3, \dots\}$ with distinct prime indices $\{q_1, q_2, q_3, \dots\}$. By the previous lemma, any vertex that is in the connected component of N_i has index divisible by q_i. Thus, no path exists between N_i and N_j for distinct i, j. □

We finish this section by proving a general result: for any group G, any component of $\Gamma_p(G)$ has finite diameter.

Proof of Proposition 3. Let G be any group and Ω a component of $\Gamma_p(G)$. Take any vertex A in Ω and let N be the normal core of A. Let $\pi : G \to G/N$ be the quotient map. Let $D = \{BN : B \in \Omega\}$. We claim that the diameter of $\Gamma_p(G)$ is less than $|D|+2$.

Let B be a subgroup in Ω. Let $V_1, \ldots, V_m$ be a path in $\Gamma_p(G)$ connecting A to B. Then by Lemma 5, $\pi(V_1), \ldots, \pi(V_m)$ is a path in $\Gamma_p(G/N)$ connecting $\pi(A)$ to $\pi(B)$. Hence

$$V_1N, \cdots, V_mN$$

is a path connecting A to BN, and so BN is an element of Ω. Further, if $[G : B] = np^r$ where $gcd(n, p^r) = 1$, then $[G : BN] = np^e$ by Lemma 9. Since $B \leq BN$ and $[G : BN][BN : B] = [G : B]$, we get

$$np^e[BN : B] = np^k$$

and therefore $[BN : B] = p^{k-e}$. Hence BN and B are adjacent in $\Gamma_p(G)$. It follows that there is an edge from any element in Ω to one in D, and so the diameter of Ω is bounded above by the diameter of the subgraph induced by D plus 2. This gives the desired bound $|D|+2$.

If Ω contains a normal subgroup as a vertex then we can pick $A = N$ in the above argument. Therefore D is the set of p-subgroups of G/N. Any two such subgroups are connected by an edge, so the diameter of Ω is bounded above by 3. □

2. Nilpotent groups: The Proof of Theorem 1

We will prove Theorem 1 in two steps, as Propositions 12 and 15 below. For a finite nilpotent group G let $S_p(G)$ denote the unique Sylow p-subgroup of G. Recall that G is the direct product of its Sylow subgroups.

Lemma 11. *Suppose $G = S_{p_1}(G) \times \cdots \times S_{p_k}(G)$ for primes $p_1, \cdots, p_k$. Let $\pi_i : G \to S_{p_i}(G)$ be the quotient map for each i. Then any subgroup $H \leq G$ has the form $H = \pi_1(H) \times \cdots \times \pi_k(H)$.*

Proof. Choose $\ell_1, \ldots, \ell_k$ so that $g^{p_i^{\ell_i}} = 1$ for all $g \in S_{p_i}(G)$. Choose N so that

$$Np_1^{\ell_1} \cdots p_{k-1}^{\ell_{k-1}} \equiv 1 \pmod{p_k^{\ell_k}}.$$

Take any $h \in H$ and write $h = (h_1, \ldots, h_k)$ for $h_i \in S_{p_i}(G)$ for all i. Then

$$h^{Np_1^{\ell_1} \cdots p_{k-1}^{\ell_{k-1}}} = (1, \ldots, 1, h_k).$$

Therefore $(1, \ldots, 1, h_k) \in H$, and so we may identify $\pi_k(H)$ with a subgroup of H. Applying this argument to each other factor, the result follows. □

Proposition 12. *If G is a finitely generated group such that every finite quotient of G is nilpotent, then every component of $\Gamma_p(G)$ is complete for all p.*

Proof. Suppose A and B are subgroups of G in the same component of $\Gamma_p(G)$ for some prime p and take any path $A = P_0, P_1, \ldots, P_n = B$ from A to B. Let N be a normal, finite-index subgroup of G contained in P_i for every i. Then G/N is a nilpotent group and $\pi(P_0), \pi(P_1), \ldots, \pi(P_n)$ is a path in $\Gamma_p(G/N)$, where $\pi : G \to G/N$ is the quotient map.

Let $\mathcal{P}$ be a finite set of primes so that $G/N = \prod_{q\in\mathcal{P}} S_q(G/N)$. By Lemma 11 we have decompositions $\pi(P_i) = \prod_{q\in\mathcal{P}} S_q(\pi(P_i))$ for each i. It is straightforward to see that

$$\pi(P_i) \cap \pi(P_{i+1}) = \prod_{q\in\mathcal{P}} S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))$$

for any i, and so for $j = i$ or $j = i+1$ we have

$$[\pi(P_j) : \pi(P_i) \cap \pi(P_{i+1})] = \prod_{q\in\mathcal{P}} [S_q(\pi(P_j)) : S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))].$$

Since $\pi(P_i)$ and $\pi(P_{i+1})$ are adjacent in the p-local commensurability graph of G/N, it follows that $S_q(\pi(P_i)) = S_q(\pi(P_{i+1}))$ for all i and all $q \neq p$. Therefore $S_q(\pi(A)) = S_q(\pi(B))$ for all $q \neq p$, and so $[\pi(A) : \pi(A)\cap\pi(B)][\pi(B) : \pi(A)\cap\pi(B)]$ is a power of p. Because $[K : L] = [\pi(K) : \pi(L)]$ for any subgroups $L \le K \le G$ containing N, this shows that A and B are adjacent in $\Gamma_p(G)$. □

Lemma 13. *If Q is a finite solvable group that is not nilpotent then there is some prime p so that a connected component of $\Gamma_p(Q)$ is not complete.*

Proof. Let Π be the set of prime divisors of the order of the finite solvable group Q. For any prime $q \in \Pi$ there is a Hall subgroup H_q so that $[Q : H_q] = q^k$ for some k and q does not divide the order of H_q. Because Q is not nilpotent, there is some prime p and a Hall subgroup H_p so that $g^{-1}H_p g \neq H_p$ for some $g \in Q$. Then both H_p and $g^{-1}H_p g$ are adjacent to Q in $\Gamma_p(Q)$, but there is no edge between H_p and $g^{-1}H_p g$ in $\Gamma_p(Q)$. □

Lemma 14. *If Q is a non-nilpotent finite group then Q contains a non-nilpotent solvable subgroup.*

Proof. Suppose every solvable subgroup of Q were nilpotent. Take any prime p and let S be a p-Sylow subgroup of Q. Let $T \le S$ be any nontrivial subgroup. Then $N_Q(T)/C_Q(T)$ is a p-group. If it were not, then there would be an element $x \in N_Q(T) - C_Q(T)$ and a prime number $q \neq p$ so that $x^q \in C_Q(T)$. Then the subgroup $H \le Q$ generated by x and S would have order with only two prime divisors, hence be solvable and therefore nilpotent. Since x is in a q-Sylow subgroup of H, this would mean $x \in C_Q(T)$.

By Frobenius' normal p-complement theorem, there is a normal subgroup $N \le Q$ of order prime to p so that $Q = SN$. Because this argument holds for any p, Q is solvable by Hall's theorem. This is a contradiction. □

Proposition 15. *Suppose G is a finitely generated group with a finite-index, normal subgroup N such that G/N is not nilpotent. Then there is some p so that a component of $\Gamma_p(G)$ is not complete.*

Proof. Take G and N as above, let $Q = G/N$ and let $\pi : G \to Q$ be the quotient map. If Q is solvable, then by Lemma 13 there is a prime p and subgroups $A, B \le Q$ in the same component of $\Gamma_p(Q)$ that are not adjacent. Then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are non-adjacent vertices in the same component of $\Gamma_p(G)$ by Lemma 8, so $\Gamma_p(G)$ is not complete.

Now consider the case that Q is not solvable. By Lemma 14 there is a non-nilpotent solvable subgroup $S \leq Q$. By Lemma 13 there is some prime p with a component Ω of $\Gamma_p(S)$ that is not complete. By Lemma 8 the component Ω fully embeds in a component of $\Gamma_p(G)$, which is therefore not complete. □

3. Free groups: The Proof of Theorem 2

Let F be the free group of rank two. Let p be a prime and $N \in \mathbb{N}$ be given. By Lemma 8, to prove Theorem 2 it suffices to find a finite quotient Q of F with subgroups $A, B \leq Q$ such that the length of any geodesic in $\Gamma_p(Q)$ connecting A to B is greater than N. Our candidate for Q is Alt_X, the alternating group on a set X of more than $p^k > N$ elements, and our candidates for A and B are conjugates of Alt_S for a subset $S \subseteq X$ with p^k elements.

We first need a couple technical group theoretic results. First, we give a description of a connected component in $\Gamma_p(\mathrm{Alt}_X)$. This requires a simple lemma.

Lemma 16. *If $|T_1 \cap T_2| \geq 2$ and $|T_1|, |T_2| \geq 4$, then $\langle Alt_{T_1}, Alt_{T_2} \rangle = Alt_{T_1 \cup T_2}$.*

Proof. We prove this by induction on $|T_1 \cup T_2|$. The case that $T_1 = T_2$ is clear, so suppose $T_1 \neq T_2$. The base case, when $|T_1| = |T_2| = 4$ and $|T_1 \cap T_2| \in \{2,3\}$, follows by computation (we did this in [GAP15]). For the inductive step, suppose without loss of generality that $x \in T_1 \setminus T_2$. By inductive hypothesis $\left\langle \mathrm{Alt}_{T_1 \setminus \{x\}}, \mathrm{Alt}_{T_2} \right\rangle = \mathrm{Alt}_{T_1 \cup T_2 \setminus \{x\}}$. Arguing similarly if $T_2 \setminus T_1$ is nonempty, we reduce to the case when $T_1 \cup T_2 \setminus T_1 \cap T_2$ consists of at most two points. To finish, we claim that any 3-cycle on points in $T_1 \cup T_2$ is in $\langle \mathrm{Alt}_{T_1}, \mathrm{Alt}_{T_2} \rangle$. Let v_1, v_2, v_3 be distinct points in $T_1 \cup T_2$. If $\{v_1, v_2, v_3\} \subseteq T_1$ or $\{v_1, v_2, v_3\} \subseteq T_2$, then we are done. Thus, by suitably relabeling, we may assume $v_1, v_2 \in T_1$ and $v_3 \in T_2$. Further, since $T_1 \cup T_2 \setminus T_1 \cap T_2$ consists of at most two points, then by relabeling again, we may assume $v_2 \in T_2$. Select $w_1, w_2 \in T_1 \cap T_2$ that are distinct from v_1, v_2, and v_3. Then, by the base case applied to $\mathrm{Alt}_{\{v_1,v_2,v_3,w_1\}} \leq \mathrm{Alt}_{T_1}$ and $\mathrm{Alt}_{\{v_1,v_2,v_3,w_2\}} \leq \mathrm{Alt}_{T_2}$, we obtain that $\mathrm{Alt}_{\{v_1,v_2,v_3,w_1,w_3\}}$ is contained in $\langle \mathrm{Alt}_{T_1}, \mathrm{Alt}_{T_2} \rangle$, and hence the desired 3-cycle is found. This completes the proof. □

For any subset $S \subseteq X$, we denote the symmetric group on S by Sym_S and the alternating group on S by Alt_S. For a subgroup $P \leq \mathrm{Sym}_S$ we define the *support* to be the complement of the fixed point set of the action of P on S.

Lemma 17. *Let p be a prime number and k an integer so that $p^k > 4$. Let X be a finite set, $S \subseteq X$, and $P \leq Sym_X$ a p-group with support disjoint from S. Let E be an index p^j subgroup of $Alt_S \times P$. If $|S| = p^k$ or $|S| = p^k - 1$, then we have the decomposition $E = Alt_T \times P'$ for some $P' \leq P$ and some $T \subseteq S$ with $|T| = p^k$ or $|T| = p^k - 1$.*

Proof. Let $\pi : \mathrm{Alt}_S \times P \to \mathrm{Alt}_S$ be the projection map. By Lemma 5 we have

$$[\mathrm{Alt}_S \times P : E] = [\mathrm{Alt}_S : \pi(E)][1 \times P : E \cap (1 \times P)].$$

The left hand side of this equation is a power of p, so $[\mathrm{Alt}_S : \pi(E)]$ is a power of p. Because $|S| = p^k$ or $|S| = p^k - 1$ by assumption, Theorem 1(a) in [Gur83] immediately implies that either $\pi(E) = \mathrm{Alt}_S$ or $|S| = p^k$ and $\pi(E) = \mathrm{Alt}_{S\setminus\{v\}}$ for some $v \in S$. Let T denote the set such that $\pi(E) = \mathrm{Alt}_T$. Let q be 3 if $p \neq 3$ and q be 2 if $p = 3$. For the case $p \neq 3$, recall that Alt_T is generated by 3-cycles by elementary properties of alternating groups. In the case $p = 3$, note that $p^k > 6$. Because Alt_6 is generated by an element of order 2 and one of order 4, Lemma 16 implies that Alt_T is generated by elements of order 2 or 4 in this case. Therefore in either case it follows that Alt_T is generated by elements $g_1, \ldots, g_k$ each with order dividing a power of q. Since π maps onto Alt_T, we have that for each $i = 1, \ldots, k$, there exists $v_i \in P$ such that $(g_i, v_i) \in E$. Since $v_i \in P$, we have that the order of v_i is coprime with g_i, hence as $q \neq p$, there exists ℓ such that

$$(g_i, v_i)^\ell = (g_i, 1).$$

It follows then that E contains all of $\mathrm{Alt}_T \times 1$, and hence $E = \mathrm{Alt}_T \times P'$ where $P' \leq P$, as desired. □

Let $\Omega_{S,X}$ be the component of $\Gamma_p(\mathrm{Alt}_X)$ containing Alt_S, and let $B_{S,X}$ denote the set of subgroups in $\Omega_{S,X}$ isomorphic to Alt_T for some $|T| \in \{p^k, p^k - 1\}$. For odd primes p, we get the following description:

Lemma 18. *Let $S \subseteq X$ be a set of cardinality p^k for some odd prime p such that $p^k > 4$. Vertices of the component $\Omega_{S,X}$ in $\Gamma_p(Alt_X)$ consist of two classes of subgroups:*

Type 1. *subgroups of the form $\langle Alt_T, P\rangle$, where $|T| = p^k$ and $P \leq Alt_X$, and*
Type 2. *subgroups of the form $\langle Alt_T, P\rangle$, where $|T| = p^k - 1$ and $P \leq Alt_X$.*

In either case, the subgroup is $Alt_T \times P$, where P is a p-group with support in T^c. Moreover, for all primes p, if V is a vertex of Type 1 or Type 2, the set T is uniquely determined by V.

Proof. We first show uniqueness of T. This implies that Type 1 and Type 2 are disjoint classes. Let V be a vertex with distinct decompositions $\mathrm{Alt}_{T_i} \times P_i$ with $|T_i| > 3$ and p-group P_i with support in T_i^c for $i = 1, 2$ such that $T_1 \neq T_2$. If $T_1 \cap T_2$ is empty, then

$$[V : \mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1][\mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1 : \mathrm{Alt}_{T_1} \times 1] = [V : \mathrm{Alt}_{T_1} \times 1] = |P_1|,$$

and thus $[\mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1 : \mathrm{Alt}_{T_1} \times 1] = |\mathrm{Alt}_{T_2}|$ must be a power of p. But this is impossible as $|\mathrm{Alt}_{T_2}|$ is either $(p^k)!/2$ or $(p^k - 1)!/2$ for $p^k > 4$. Thus, T_1 and T_2 overlap. If $T_1 \neq T_2$ then $\mathrm{Alt}_{T_1} \times 1$ cannot be normal in V because Alt_{T_2} acts transitively on T_2. But $\mathrm{Alt}_{T_1} \times 1$ is clearly normal in $\mathrm{Alt}_{T_1} \times P_1$, so this is a contradiction. Therefore $T_1 = T_2$.

Since elements in $B_{S,X}$ are of Type 1 or 2 and Alt_S itself lies in $B_{S,X}$, it suffices to show that any E that is adjacent to an element of Type 1 or 2 must itself be of Type 1 or 2.

Let E be adjacent to $V = \mathrm{Alt}_T \times P$ where P is a p-group with support in T^c and $|T| = p^k$ or $|T| = p^k - 1$. Then $E \cap V$ is a subgroup of $\mathrm{Alt}_T \times P$ of index a power of p. By Lemma 17, $E \cap V = \mathrm{Alt}_T \times P'$ or $E \cap V = \mathrm{Alt}_{T \setminus \{v\}} \times P'$ where $P' \leq P$ and $v \in T$. We will therefore assume without loss of generality that E contains $\mathrm{Alt}_T \times 1 = \mathrm{Alt}_T$ as a subgroup of p power index.

Suppose that E does not leave T invariant. Let $T_1, T_2, \cdots, T_k$ be the orbit of E acting on T and note that E contains Alt_{T_i} for each i. Suppose $T_i \cap T_{i+1}$ has fewer than two elements for some i. The group Alt_{T_i} contains $\mathrm{Alt}_{T_i \setminus T_i \cap T_{i+1}}$, which includes a permutation of order 2 since $|T_i| > 4$. Hence E contains $\mathrm{Alt}_{T_i} \times \mathbb{Z}/2\mathbb{Z} \geq \mathrm{Alt}_{T_i}$. This is impossible, as Alt_{T_i} is of index p^k in E for an odd prime p. We therefore know that $T_i \cap T_{i+1}$ has more than two elements for every i. Then by applying Lemma 16 we conclude that E contains $\mathrm{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$. Since E contains Alt_T as a subgroup of prime power index and $T_1 \cup \cdots \cup T_k \neq T_1$, it follows that $|T_1 \cup \cdots \cup T_k| = p^k$ and in fact E contains $\mathrm{Alt}_{T_1 \cup \cdots \cup T_k}$ as a subgroup of index p^ℓ for some ℓ.

We may therefore assume, after replacing T with $T_1 \cup \cdots \cup T_k$ if necessary, that E leaves T invariant. Then $E \leq \mathrm{Sym}_T \times Q$ where Q is a group with support disjoint from T. Let $\pi : \mathrm{Sym}_T \times Q \to \mathrm{Sym}_T$ be the projection onto the first coordinate. By Lemma 5, $[\pi(E) : \mathrm{Alt}_T]$ divides $[E : \mathrm{Alt}_T]$ and hence is a power of p. It follows that $\pi(E) = \mathrm{Alt}_T$, as Alt_T is a maximal subgroup of Sym_T of index two. Further, since Alt_T is normal, we apply Lemma 5 to the map $\psi : \mathrm{Alt}_T \times Q \to Q$ to see that $|\psi(E)|$ is a power of p. Applying Lemma 17 we obtain the desired conclusion. □

The prime $p = 2$ requires relaxing the conclusion of Lemma 18, since any symmetric group on three or more elements contains an alternating group of index 2.

Lemma 19. *Let $S \subseteq X$ be a set of cardinality 2^k such that $k > 2$. Vertices of the component $\Omega_{S,X}$ in $\Gamma_2(Alt_X)$ consist of at least one of two types:*

> **Type 1'.** *subgroups V such that $Alt_T \times 1 \leq V \leq Sym_T \times P$, where $|T| = 2^k$ and $P \leq Alt_X$, and*
>
> **Type 2'.** *subgroups V such that $Alt_T \times 1 \leq V \leq Sym_T \times P$, where $|T| = 2^k - 1$ and $P \leq Alt_X$.*

In either case, P is a 2-group with support in T^c.

Proof. Since elements in $B_{S,X}$ are of Type 1' or 2', it suffices to show that any E that is adjacent to an element of one of the types must itself be of one of the types.

Let E be adjacent to some V with $\mathrm{Alt}_T \times 1 \leq V \leq \mathrm{Sym}_T \times P$ where P is a 2-group. Because V has index a power of 2 in $\mathrm{Sym}_T \times P$, we know that $E \cap V$ also has index a power of 2 in $\mathrm{Sym}_T \times P$. Since $\mathrm{Alt}_T \times P$ is a normal subgroup of $\mathrm{Sym}_T \times P$, we have by Lemma 6 that $(\mathrm{Alt}_T \times P) \cap E \cap V$ has index a power of 2 in $\mathrm{Sym}_T \times P$, and hence in $\mathrm{Alt}_T \times P$. By Lemma 17, $E \cap V \cap (\mathrm{Alt}_T \times P) = \mathrm{Alt}_T \times P'$ or $E \cap V \cap (\mathrm{Alt}_T \times P) = \mathrm{Alt}_{T \setminus \{v\}} \times P'$ where $P' \leq P$ and $v \in T$. We conclude that $\mathrm{Alt}_T \times 1$ or $\mathrm{Alt}_{T \setminus \{v\}} \times 1$ has index a power of 2 in $E \cap V \cap (\mathrm{Alt}_T \times P)$, and hence has index a power of 2 in E. We will therefore assume without loss of generality that E contains $\mathrm{Alt}_T \times 1 = \mathrm{Alt}_T$ as a subgroup with index a power of 2, where $|T| = 2^k$ or $|T| = 2^k - 1$.

Suppose that E does not leave T invariant. Let $T_1, T_2, \cdots, T_k$ be the orbit of E acting on T and note that E contains Alt_{T_i} for each i. Suppose $T_i \cap T_{i+1}$ has fewer

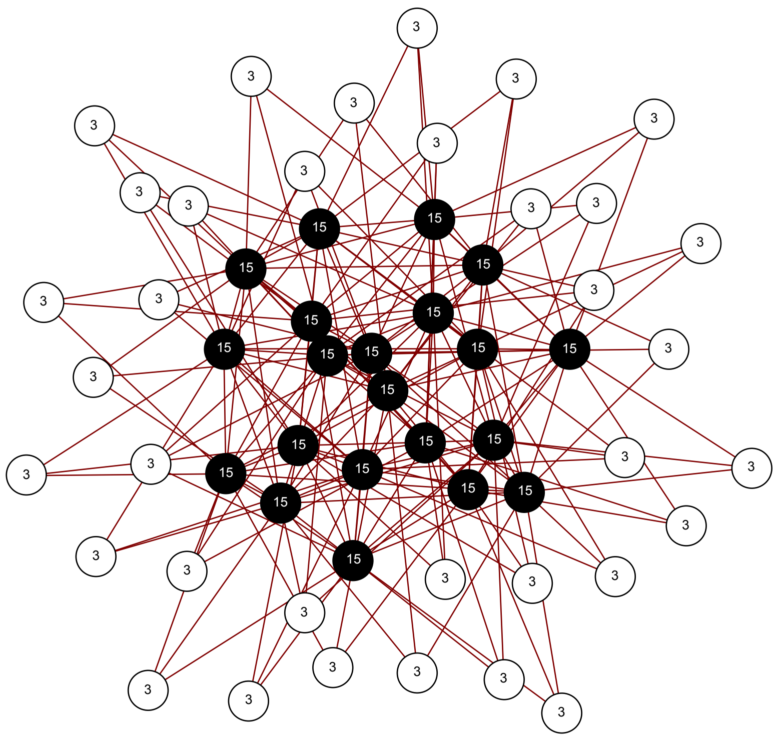

FIGURE 2. $\Omega_{S,X}$ with $|S| = 5$ and $|X| = 7$. The coloring gives the types and the numbers give the valence of each vertex. This figure was generated using GAP [GAP15] and Mathematica [W15]

than two elements for some i. The group Alt_{T_i} contains $\mathrm{Alt}_{T_i \setminus T_i \cap T_{i+1}}$, which includes a permutation of order 3 because $|T_i| > 3$. Hence E contains $\mathrm{Alt}_{T_i} \times \mathbb{Z}/3\mathbb{Z} \geq \mathrm{Alt}_{T_i}$. This is impossible, as Alt_{T_i} is of 2 power index in E. We therefore know that $T_i \cap T_{i+1}$ has more than two elements for every i. Then by applying Lemma 16 we conclude that E contains $\mathrm{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$. Since E contains Alt_T as a subgroup of prime power index and $T_1 \cup \cdots \cup T_k \neq T_1$, it follows that $|T_1 \cup \cdots \cup T_k| = 2^k$ and in fact E contains $\mathrm{Alt}_{T_1 \cup \cdots \cup T_k}$ as a subgroup of index 2^ℓ for some ℓ.

We may therefore assume, after replacing T with $T_1 \cup \cdots \cup T_k$ if necessary, that E leaves T invariant. Then $\mathrm{Alt}_T \times 1 \leq E \leq \mathrm{Sym}_T \times Q$ where Q is a 2-group with support disjoint from T, as desired. □

Note that groups of Type 1 and Type 2 are of Type 1' and Type 2' respectively. The next result allows us to restrict attention to geodesics in $B_{S,X}$ when computing distances there.

Lemma 20. *Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime p and integer k. Then $B_{S,X}$ is a geodesic metric space in the path metric induced from $\Omega_{S,X}$.*

Proof. We first need a local fact. Let V_1, V_2 be two adjacent vertices in $\Omega_{S,X}$. If p is odd, then by Lemma 18 we have $V_i = \mathrm{Alt}_{T_i} \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of P_i is disjoint from T_i for $i = 1, 2$. If $p = 2$, then by Lemma 19, $\mathrm{Alt}_{T_i} \times 1 \le V_i \le \mathrm{Sym}_{T_i} \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of P_i is disjoint from T_i for $i = 1, 2$. We claim that in either case, Alt_{T_1} and Alt_{T_2} are connected by an edge in $\Gamma_p(\mathrm{Alt}_X)$.

Since V_1 and V_2 are adjacent, we have that

$$[V_1 : V_1 \cap V_2][V_2 : V_1 \cap V_2]$$

is a power of p. Thus, when p is odd, Lemma 17 applied twice along with the uniqueness in Lemma 18 gives that $V_1 \cap V_2$ is $\mathrm{Alt}_U \times P$ where $U \subseteq T_1 \cap T_2$ satisfies $|U| = p^k$ or $|U| = p^k - 1$ and $P \le P_1 \cap P_2$. Thus it is straightforward to see that Alt_{T_1} is adjacent to Alt_{T_2}.

When $p = 2$, set $H_i = \mathrm{Alt}_{T_i} \times 1$ and $\Lambda = V_1 \cap V_2$. Then H_i is normal in V_i, thus $H_i \cap \Lambda$ is normal in Λ. Since $[\mathrm{Sym}_{T_i} \times P_i : H_i]$ is a power of 2 and

$$[\mathrm{Sym}_{T_i} \times P_i : V_i][V_i : H_i] = [\mathrm{Sym}_{T_i} \times P_i : H_i],$$

we get $[V_i : H_i]$ is a power of 2. Since H_i is normal in V_i, Lemma 6 implies that $[V_i : H_i \cap \Lambda]$ is a power of 2. Further, as $[V_i : \Lambda]$ is a power of 2 and

$$[V_i : \Lambda][\Lambda : H_i \cap \Lambda] = [V_i : H_i \cap \Lambda]$$

we conclude that $[\Lambda : H_i \cap \Lambda]$ is a power of 2 for $i = 1, 2$. Thus, applying Lemma 6 to $H_1 \cap \Lambda \lhd \Lambda$ and $H_2 \cap \Lambda \lhd \Lambda$, we have that $H_1 \cap H_2 \cap \Lambda$ has index a power of 2 in Λ. As

$$[V_i : \Lambda][\Lambda : H_1 \cap H_2 \cap \Lambda] = [V_i : H_1 \cap H_2 \cap \Lambda],$$

it follows that $[V_i : H_1 \cap H_2 \cap \Lambda]$ is a power of 2. Because $[V_i : H_i]$ is also a power of 2 (shown above) and

$$[V_i : H_i][H_i : H_1 \cap H_2 \cap \Lambda] = [V_i : H_1 \cap H_2 \cap \Lambda]$$

we have $[H_i : H_1 \cap H_2 \cap \Lambda]$ is a power of 2 for each i. By applying Theorem 1(a) in [Gur83] and the uniqueness in Lemma 18, we have $H_1 \cap H_2 \cap \Lambda$ is Alt_S for some $S \subseteq T_1 \cap T_2$ with $|S| = p^k$ or $|S| = p^k - 1$. Thus, Alt_{T_1} is adjacent to Alt_{T_2}, as claimed.

Now let γ be a path in $\Omega_{S,X}$ that, except for its endpoints, is entirely in the complement of $B_{S,X}$. Enumerate the vertices of γ in the order they are traversed,

$$V_1, V_2, \dots, V_m, \text{ where } \mathrm{Alt}_{T_i} \times 1 \le V_i \le \mathrm{Sym}_{T_i} \times P_i \text{ for all } i = 1, \dots, m$$

Then by the previous claim, we may form a new path (after throwing out repeated vertices)

$$\mathrm{Alt}_{T_{i_1}}, \mathrm{Alt}_{T_{i_2}}, \dots, \mathrm{Alt}_{T_{i_n}}.$$

that is entirely contained in $B_{S,X}$ and has the same endpoints as γ. It follows that $B_{S,X}$ is geodesic in $\Omega_{S,X}$, as desired. □

Proposition 21. *Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime p and integer k. There exists $V, W \in B_{S,X}$ such that any path in $\Omega_{S,X}$ connecting V to W has length at least $p^k - \max\{0, 2p^k - |X|\}$.*

Proof. By Proposition 20, it suffices to show that there exists $V, W \in B_{S,X}$ such that any path in $B_{S,X}$ has length greater than $|X| - p^k$. Let $O_1, O_2 \subseteq X$ with $|O_1 \cap O_2| \leq \max\{0, 2p^k - |X|\}$ and either $|O_i| = p^k$ or $|O_i| = p^k - 1$ for each i. Let $E_1, E_2, \ldots, E_m$ be distinct vertices in a non-back-tracking path in $B_{S,X}$ connecting Alt_{O_1} to Alt_{O_2}. Let $T_1, T_2, \ldots, T_m$ be subsets of X such that $E_i = \mathrm{Alt}_{T_i}$ for $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we have one of three cases:

(1) E_i is Type 1 and E_{i+1} is Type 1: In this case, $|T_{i+1} \cap T_i| = |T_i| - 1 = p^k - 1$.
(2) E_i is Type 1 and E_{i+1} is Type 2: In this case, $T_{i+1} \subset T_i$ and $|T_{i+1}| = |T_i| - 1 = p^k - 1$.
(3) E_i is Type 2 and E_{i+1} is Type 1: In this case, $T_{i+1} \supset T_i$ and $|T_{i+1}| = |T_i| + 1 = p^k$.
(4) E_i is Type 2 and E_{i+1} is Type 2: This case never occurs, as $[\mathrm{Alt}_T : \mathrm{Alt}_U]$ is not a power of p for any proper subset $U \subset T$ with $|T| = p^k - 1$.

Thus, we see that for each i, we see that T_i and T_{i+1} differ by moving, adding, or removing at most one element. It follows that $m \geq p^k - |O_1 \cap O_2| \geq p^k - \max\{0, 2p^k - |X|\}$. □

Proof of Theorem 2. Let F be a rank two free group and p a prime. Given $N > 0$, choose k so that $p^k > N$ and $p^k > 4$. For any finite set X with $|X| > 2p^k$, let γ_X be a path of length p^k in $\Gamma_p(\mathrm{Alt}_X)$ guaranteed by Proposition 21. Then pulling back γ_X over any surjection $\pi : F \to \mathrm{Alt}_X$ produces a path of length p^k in $\Gamma_p(F)$ by Lemma 8. By Lemma 9, sets X_1 and X_2 with relatively prime cardinalities will produce geodesics in different components of $\Gamma_p(F)$. □

Proof of Corollary 4. Let G be a large group, p a prime, and $N > 0$. Since a finite-index subgroup of a nonabelian free group is nonabelian, there exists a normal finite-index subgroup $H \leq G$ that surjects onto F, the free group of rank 2. By Lemma 7 and Theorem 2, there exists vertices $V, W \in \Gamma_p(H)$ such that any path connecting them in $\Gamma_p(G)$ has length greater than N. The result now follows from Lemma 8, as $\Gamma_p(H)$ isometrically embeds into $\Gamma_p(G)$. □

Corollary 22. *Let G be a large group and p be a prime. There exists a connected component of $\Gamma_p(G)$ that does not contain any normal subgroup.*

Proof. By Proposition 3, any component of $\Gamma_p(G)$ containing a normal subgroup as a vertex has diameter at most 3. By Corollary 4, there are components of G with arbitrarily long geodesics. □

References

[AAH+15] Akbari, Saieed; Ashtab, A.; Heydari, F.; Rezaee, Mehdi; Sherafati, F. The prime index graph of a group. Preprint, 2015. arXiv:1508.01133. 431

[BRHP15] BOU-RABEE, KHALID; HAGEN, MARK F.; PATEL, PRIYAM. Residual finiteness growths of virtually special groups. *Math. Z.* **279** (2015), no. 1–2, 297–310. MR3299854, Zbl 1317.20036, arXiv:1402.6974, doi: 10.1007/s00209-014-1368-5. 431

[BRK12] BOU-RABEE, KHALID; KALETHA, TASHO. Quantifying residual finiteness of arithmetic groups. *Compos. Math.* **148** (2012), no. 3, 907–920. MR2925403, Zbl 1256.20030, arXiv:1008.3579, doi: 10.1112/S0010437X11007469. 431

[BRM11] BOU-RABEE, KHALID; MCREYNOLDS, D. B. Asymptotic growth and least common multiples in groups. *Bull. Lond. Math. Soc.* **43** (2011), no. 6, 1059–1068. MR2861528, Zbl 1253.20029, arXiv:0907.3681, doi: 10.1112/blms/bdr038. 431

[BRS16] BOU-RABEE, KHALID; SEWARD, BRANDON. Arbitrarily large residual finiteness growth. *J. Reine Angew. Math.* **710** (2016), 199–204. MR3437564, Zbl 1359.20022, arXiv:1304.1782, doi: 10.1515/crelle-2013-0127. 431

[GAP15] THE GAP GROUP. GAP – Groups, Algorithms, and Programming, Version 4.7.8. 2015. `https://www.gap-system.org`. 431, 436, 439

[Gri83] GRIGORCHUK, ROSTISLAV I. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR* **271** (1983), no. 1, 30–33. MR712546(85g:20042), Zbl 0547.20025. 430

[GK17] GUPTA, NEHA; KAPOVICH, ILYA. The primitivity index function for a free group, and untangling closed curves on hyperbolic surfaces. With an appendix by Khalid Bou-Rabee. To appear in *Mathematical Proceedings of the Cambridge Philosophical Society*. Preprint, 2017. arXiv:1411.5523v4. 431

[Gur83] GURALNICK, ROBERT M. Subgroups of prime power index in a simple group. *J. Algebra* **81** (1983), no. 2, 304–311. MR700286(84m:20007), Zbl 0515.20011, doi: 10.1016/0021-8693(83)90190-4. 430, 437, 440

[KM11] KASSABOV, MARTIN; MATUCCI, FRANCESCO. Bounding the residual finiteness of free groups. *Proc. Amer. Math. Soc.* **139** (2011), no. 7, 2281–2286. MR2784792(2012g:20085), Zbl 1230.20045, arXiv:0912.2368, doi: 10.1090/S0002-9939-2011-10967-5. 431

[KT16] KOZMA, GADY; THOM, ANDREAS. Divisibility and laws in finite simple groups. *Math. Ann.* **364** (2016), no. 1–2, 79–95. MR3451381, Zbl 1344.20024, arXiv:1403.2324, doi: 10.1007/s00208-015-1201-4. 431

[LS03] LUBOTZKY, ALEXANDER; SEGAL, DAN. Subgroup growth. Progress in Mathematics, 212. *Birkhäuser Verlag, Basel*, 2003. xxii+453 pp. ISBN: 3-7643-6989-2. MR1978431(2004k:20055), Zbl 1071.20033. 431

[Pat13] PATEL, PRIYAM. Quantifying algebraic properties of surface groups and 3-manifold groups. Ph.D. thesis - Rutgers, The State University of New Jersey - New Brunswick. 2013, 52 pp. ISBN: 978-1303-46244-3. MR3192999. 431

[Riv12] RIVIN, IGOR. Geodesics with one self-intersection, and other stories. *Adv. Math.* **231** (2012), no. 5, 2391–2412. MR2970452, Zbl 1257.57024, arXiv:/0901.2543, doi: 10.1016/j.aim.2012.07.018. 431

[W15] WOLFRAM RESEARCH, INC.. Mathematica. Champaign, IL, 2015. 431, 439

(Khalid Bou-Rabee) DEPARTMENT OF MATHEMATICS, CCNY CUNY
`khalid.math@gmail.com`

(Daniel Studenmund) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME
`dstudenm@nd.edu`

This paper is available via `http://nyjm.albany.edu/j/2018/24-24.html`.