

# On generalized Jørgensen inequality in infinite dimension

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ABSTRACT. In [5], Li has obtained an analogue of the Jørgensen inequality in the infinite-dimensional Möbius group. We show that this inequality is strict.

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## 1. Introduction

The Möbius group  $M(n)$  acts by isometries on the  $n$ -dimensional real hyperbolic space. The Jørgensen inequality is a pioneer result in the theory of discrete subgroups of Möbius groups. The classical Jørgensen inequality gives a necessary criterion to detect the discreteness of a two-generator subgroup in  $M(2)$  and  $M(3)$ . There have been several generalizations of the Jørgensen inequality in higher dimensional Möbius groups, e.g. [3], [8], [9].

The Clifford algebraic formalism to Möbius group was initiated by Ahlfors in [1]. In this approach the  $2 \times 2$  matrices over finite dimensional Clifford algebras act by linear fractional transformations on the  $n$ -sphere. Waterman used the Clifford algebraic formalism of Möbius groups to obtain some Jørgensen type inequalities in [9]. Frunză initiated a framework for infinite dimensional Möbius groups in [2]. This framework is an extension of the Clifford algebraic viewpoint by Ahlfors. In [5, 6, 4], Li has used this viewpoint further to obtain discreteness criteria in infinite dimension.

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In [5], Li has obtained an analogue of Jørgensen inequality in the infinite-dimensional Möbius group. The aim of this note is to show that this inequality is strict. In Section 2, we briefly recall basic notions of the infinite dimensional theory and note down the Jørgensen type inequality of Li. In Section 3 we prove that Li's inequality is strict, see Theorem 3.1.

## 2. Preliminaries

**2.1. Infinite dimensional Clifford group.** The Clifford algebra  $\mathcal{C}$  is the associative algebra over  $\mathbb{R}$  generated by a countable family  $\{i_k\}_{k=0}^\infty$  subject to the relations:

$$i_h i_k = -i_k i_h, \quad h \neq k, \quad i_k^2 = -1,$$

and no others. Every element of  $\mathcal{C}$  can be expressed as  $a = \sum a_I I$ , where  $I = i_{k_1} i_{k_2} \dots i_{k_p}$ ,  $1 \leq k_1 < k_2 < \dots < k_p \leq n$ ,  $n$  is a fixed natural number depending upon  $a$ ,  $a_I \in \mathbb{R}$ , and  $\sum_I a_I^2 < \infty$ . If  $I = \emptyset$ , then  $a_I$  is the real part of  $a$  and the remaining part is the 'imaginary part' of  $a$ . In  $\mathcal{C}$  the Euclidean norm is given as usual by

$$|a| = \sqrt{|Re(a)|^2 + |Im(a)|^2}.$$

As in the finite-dimensional Clifford algebra,  $\mathcal{C}$  has three special involutions, defined by the following.

$*$ : In  $a \in \mathcal{C}$  as above, replace in each  $I = i_{v_1} i_{v_2} \dots i_{v_k}$  by  $i_{v_k} \dots i_{v_1}$ .  $a \mapsto a^*$  is an anti-automorphism.

$'$ : Replace  $i_k$  by  $-i_k$  in  $a$  to obtain  $a'$ .

The conjugate  $\bar{a}$  of  $a$  is now defined as:  $\bar{a} = (a^*)' = (a')^*$ .

Elements of the following type:

$$a = a_0 + a_1 i_1 + \dots + a_n i_n + \dots,$$

are called *vectors*. The set of vectors is denoted by  $\ell_2$ . Let  $\bar{\ell}_2 = \ell_2 \cup \{\infty\}$ . For any  $x \in \ell_2$ , we have  $x^* = x$  and  $\bar{x} = x'$ . Every non-zero vector is invertible and  $x^{-1} = \bar{x}/|x|^2$ . The set of products of finitely many non-zero vectors is a multiplicative group, called the Clifford group, and denoted by  $\Gamma$ .

A Clifford matrix  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  over  $\ell_2$  is defined as follows:

- (1)  $a, b, c, d \in \Gamma \cup \{0\}$ ;
- (2)  $\Delta(g) = ad^* - bc^* = 1$ ;
- (3)  $ab^*, d^*b, cd^*, c^*a \in \ell_2$ .

The set of all such matrices forms a group, denoted by  $SL(\Gamma)$ . For  $g$  as above,  $g^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ . Note that  $gg^{-1} = g^{-1}g = I$ .

The group  $PSL(\Gamma) = SL(\Gamma)/\{\pm I\}$  acts on  $\bar{\ell}_2$  by the following transformation:

$$g : x \mapsto (ax + b)(cx + d)^{-1}.$$

**2.2. Classification of elements in  $SL(\Gamma)$ .** Let  $f$  be in  $SL(\Gamma)$ . Then

- $f$  is *loxodromic* if it is conjugate in  $SL(\Gamma)$  to  $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$ , where  $r \in \mathbb{R} - \{0\}$ ,  $|r| \neq 1$ ,  $\lambda \in \Gamma$ . If  $\lambda = \pm 1$ , then  $f$  is called *hyperbolic*.
- $f$  is *parabolic* if it is conjugate in  $SL(\Gamma)$  to  $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$ , where  $a, b \in \Gamma$ ,  $|a| = 1$ ,  $b \neq 0$ , and  $ab = ba'$ .
- Otherwise  $f$  is *elliptic*.

**Definition 2.1.** For  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , the trace of  $g$  is defined by

$$tr(g) = a + d^*.$$

A non-trivial element  $g \in SL(\Gamma)$  as above is called *vectorial* if  $b^* = b$ ,  $c^* = c$ , and  $tr(g) \in \mathbb{R}$ .

The real part of trace is a conjugacy invariant in  $SL(\Gamma)$ .

**Lemma 2.2.** [7, 5] If an element  $g$  in  $SL(\Gamma)$  is hyperbolic, then  $tr(g) \in \mathbb{R}$  and  $tr^2(g) > 4$ .

**Definition 2.3.** A subgroup  $G$  of  $SL(\Gamma)$  is called *elementary* if it has a finite orbit in  $\ell_2$ . Otherwise,  $G$  is called *non-elementary*. A subgroup  $G$  of  $SL(\Gamma)$  is *discrete* if a sequence  $f_i \rightarrow g$  in  $G$  implies that  $f_i = g$  for all sufficiently large  $i$ . Otherwise  $G$  is not discrete.

**2.3. Li-Jørgensen inequality.** The following is the generalized Jørgensen inequality in infinite dimension that was given by Li in [5].

**Theorem 2.4.** [5, Theorem 3.1] Let  $f, g \in SL(\Gamma)$  be such that  $f$  is hyperbolic, and  $[f, g] = fgf^{-1}g^{-1}$  is vectorial. Suppose that the two-generator group  $\langle f, g \rangle$  is discrete and non-elementary. Then

$$(2.1) \quad |tr^2(f) - 4| + |tr([f, g]) - 2| \geq 1.$$

### 3. Li-Jørgensen inequality is strict

**Theorem 3.1.** Let  $f, g \in SL(\Gamma)$  be such that  $f$  is hyperbolic and  $[f, g] = fgf^{-1}g^{-1}$  is vectorial. Suppose that the two-generator group  $\langle f, g \rangle$  is discrete and non-elementary. Then

$$(3.1) \quad |tr^2(f) - 4| + |tr([f, g]) - 2| > 1,$$

where the above inequality is strict.

**Proof.** It follows from Theorem 2.4 that

$$|tr^2(f) - 4| + |tr([f, g]) - 2| \geq 1.$$

If possible suppose that

$$(3.2) \quad |tr^2(f) - 4| + |tr([f, g]) - 2| = 1.$$

Up to conjugacy, we assume  $f = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ ,  $r > 1$ . Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Let  $J(f, g)$  denote the left hand side of (3.2).

By computation it is easy to see that

$$\text{tr}([f, g]) - 2 = -(r - r^{-1})^2 bc^*,$$

and

$$\text{tr}^2(f) - 4 = (r - r^{-1})^2.$$

So

$$J(f, g) = (r - r^{-1})^2(1 + |bc^*|) = 1.$$

Since  $[f, g]$  is vectorial, it follows from above that  $bc^*$  is a real number.

Let

$$g_0 = g, \quad g_{m+1} = g_m f g_m^{-1}, \quad g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}.$$

Also let  $K = (r - r^{-1})^2$  and  $w_m = b_m c_m^*$ .

Then by the equality in (3.2) we have  $K(1 + |w_0|) = 1$ . This implies  $K < 1$ .

Now note that

$$(3.3) \quad b_{m+1} c_{m+1}^* = -K(1 + b_m c_m^*) \cdot b_m c_m^*.$$

By induction,  $w_m = b_m c_m^*$  is a sequence of real numbers. Also

$$|w_{m+1}| \leq K|w_m|(1 + |w_m|).$$

If possible suppose  $K(1 + |w_m|) < 1$  for some  $m$ . Then using arguments similar to the proof of [5, Theorem 3.1], it can be shown that

$$|b_{m+1} c_{m+1}^*| \leq |b_m c_m^*|$$

and  $b_m c_m^* \rightarrow 0$  as  $m \rightarrow \infty$ , that would give a contradiction to the assumption that  $\langle f, g \rangle$  is non-elementary. So, we must have  $K(1 + |w_m|) \geq 1$  for all  $m$ .

Thus

$$1 \leq K(1 + |w_m|) \leq K(1 + |w_{m-1}|).$$

It is given that  $K(1 + |w_0|) = 1$ . By induction, it follows that for all  $m$ ,

$$(3.4) \quad J(f, g_m) = K(1 + |w_m|) = 1.$$

Note from (3.3) and (3.4) that

$$\begin{aligned} 1 - K &= K|w_{m+1}| \leq K \cdot K|w_m| \cdot |1 + w_m| \\ &\leq (1 - K)K|1 + w_m| \leq (1 - K)K(1 + |w_m|) \\ &\leq (1 - K), \end{aligned}$$

which implies

$$(3.5) \quad K|1 + w_m| = 1.$$

Observe that

$$\begin{aligned} |\text{tr}([f, g]) - 2 + 4 - \text{tr}^2(f)| &= K|1 + bc^*| = 1 \\ &= |\text{tr}([f, g]) - 2| + |4 - \text{tr}^2(f)|. \end{aligned}$$

Since  $4 - \operatorname{tr}^2(f) < 0$ , this implies  $w_0 > 0$ . Hence by induction from (3.4) and (3.5),  $w_m > 0$  for all  $m$ . Thus, we have from (3.4),  $K = 1/(1 + w_m)$ . In particular,  $w_m = w_{m+1}$ . Now, from (3.3), we have  $K(1 + w_m) = -1$ , i.e.  $K = -1/(1 + w_m)$ . This is a contradiction. Hence the inequality must be strict.  $\square$

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