New York Journal of Mathematics

New York J. Math. **24** (2018) 1039–1055.

Ordered invariant ideals of Fourier-Stieltjes algebras

S. Kaliszewski, Magnus B. Landstad and John Quigg

Abstract. For a locally compact group G, every G-invariant subspace E of the Fourier-Stieltjes algebra $B(G)$ gives rise to the following two ideals of the group C^*-algebra $C^*(G)$: the intersection of the kernels of the representations with many coefficient functions in E, and the preannihilator of E. We investigate the question of whether these two ideals coincide. This leads us to define and study two properties of E — *ordered* and *weakly ordered* — that measure how many positive definite functions E contains.

Contents

1. Introduction

In an effort to extend the class of groups for which the Baum-Connes conjecture is valid, Baum, Guentner, and Willett introduced in [2] crossed-product functors, which transform actions of a locally compact group G on C^*-algebras into C^*-algebras that lie between the full and reduced crossed products. Our approach to this has been to form crossed-product functors by applying coaction functors to the full crossed products. This in particular requires us to study exotic group C^*-algebras between $C^*(G)$ and $C^*_r(G)$ to form coaction functors. In [4] Brown and Guentner introduced a certain method of generating exotic group C^*-algebras of a discrete group G, starting with a G-invariant ideal D of $\ell^\infty(G)$. Their method carries

Received February 6, 2018.

2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 46L25, 22D25.

Key words and phrases. locally compact group, coaction, Fourier-Stieltjes algebra, positive definite function.

ISSN 1076-9803/2018

over to locally compact G, letting D be a G-invariant ideal of either $L^\infty(G)$ or the algebra $C_b(G)$ of continuous bounded functions. (This L^∞-or-C_b ambiguity is useful, because there are examples in which the ideal most naturally resides in one or the other.) They used D to define a class of unitary representations of G, and then applied standard C^*-representation theory to get an associated quotient of $C^*(G)$, denoted by $C^*_D(G)$, which was their main object of study. Recently there has been a flurry of activity regarding constructions using these group C^*-algebras (a brief sampling: [3, 5, 6, 12, 11, 13, 14, 17, 18, 20, 19]).

In [10] our strategy was to study the exotic group C^*-algebra $C^*_D(G)$ in terms of the dual space $C^*_D(G)^*$ of bounded linear functionals, which can be identified with a weak*-closed subspace of the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, namely the annihilator in $B(G)$ of the kernel of the quotient map $C^*(G) \to C^*_D(G)$.

However, recently a fundamental question has arisen, because the same D can be used to arrive at a potentially different weak*-closed subspace of $B(G)$ by taking the weak*-closure of $D \cap B(G)$ in $B(G)$.

Question 1.1. Does the weak*-closure $\overline{D \cap B(G)}$ coincide with the dual space $C^*_D(G)^*$?

In [10, Lemma 3.5 (1)] we thought that we had proven that the answer to Question 1.1 is "yes". But although we did give a correct proof of the inclusion $C^*_D(G)^* \subset \overline{D \cap B(G)}$, it has recently been pointed out to us by Buss, Echterhoff, and Willett that our argument for the reverse inclusion is incorrect. At this point we do not know whether [10, Lemma 3.5 (1)] is true in general; see Section 3 for a discussion. In this paper we investigate this question, although we emphasize that we do not have a complete solution.

Our incorrect proof of [10, Lemma 3.5 (1)] seemed to depend upon a nonzero G-invariant ideal E of $B(G)$ having the property that $E = \operatorname{span}\{E \cap P(G)\}$, where $P(G)$ denotes the set of continuous positive definite functions on G. In this paper we initiate the study of this property, which we have been unable to find in the literature. We call a subspace E of $B(G)$ *ordered* if it coincides with the linear span of the intersection $E \cap P(G)$, and we prove that if a nonzero G-invariant ideal E of $B(G)$ is ordered, then Question 1.1 has a positive answer.

Happily, Buss, Echterhoff, and Willett show in [5, Corollary 2.14] that a positive answer to Question 1.1 in general is equivalent to E having a somewhat weaker property, namely that the span of the intersection $E \cap P(G)$ be weak*-dense in E; when this happens we call E *weakly ordered*. In Section 3 we explore these two (new?) properties: ordered and weakly ordered. Although not every G-invariant ideal E of $B(G)$ is ordered (see Example 4.2), as far as we know it is an open question whether such E is always weakly ordered. It seems that G-invariance of E, or the requirement that E be an ideal, may be important here: in Example 3.23 we exhibit a weak*-dense subspace E of $B(\mathbb{T})$ for which $E \cap P(\mathbb{T}) = \{0\}$, and thus fails emphatically to be weakly ordered. We do not know an example of a subspace E that is either G-invariant or an ideal and is not weakly ordered. In any event, we believe that this paper, and [5], demonstrate that the properties ordered and (perhaps more importantly) weakly ordered deserve further study.

We begin in Section 2 by setting up the ingredients for our study: starting with a G-invariant subspace E of $B(G)$, we introduce *E-representations*, which are representations of G with a lot of coefficient functions in E (see Definition 2.3).

We show in Section 3 that Question 1.1 is equivalent to the question of whether the preannihilator ${}^{\perp}E$ coincides with the intersection of the kernels (in $C^*(G)$) of the E-representations, and also equivalent to the property that E is weakly ordered. Our ignorance concerning these ideas is deep: We do not even know whether G has nonzero E-representations, even assuming that E is an ideal of $B(G)$, although we give an affirmative answer in Proposition 3.20 for discrete groups. In Proposition 3.22 we prove that when G is amenable (which is in some sense of no interest with regard to our questions), the existence of just one nonzero E-representation implies that E is weakly ordered.

In Section 4 we examine the two properties ordered and weakly ordered for certain "classical" ideals of $B(G)$, particularly those arising from L^p-spaces. We prove in Proposition 4.1 that the ideals given by the intersection of $B(G)$ with either $C_c(G)$ or $C_0(G)$ are ordered. On the other hand, regarding the ideals $E^p = L^p(G) \cap B(G)$ for $1 \le p \le \infty$ the situation is not so clear. We do not know whether they are all weakly ordered. In Example 4.2 we show that for some groups the ideal E^1 of $B(G)$ is not ordered, while in Proposition 4.4 we record the trivial fact that E^∞ is ordered. In Proposition 4.5 we show that for $1 \le p \le 2$ the ideal E^p is weakly ordered, and in Corollary 4.6 we observe the consequence that E^1 is weakly ordered but not ordered. In Corollary 4.7 we show that G has at least one nonzero E^p-representation for every p. In Proposition 4.9 we show that E^2 is ordered when G is abelian, and we conjecture that this carries over to all unimodular groups.

In Section 5 we indicate how our investigation into the ordering properties studied in Section 3 can be applied to fix [10, Lemma 3.5 (1)]. The main take-away from all this is the following: *The statement of Lemma 3.5 (1) in* [10] *should have included the hypothesis that the linear span of $E \cap P(G)$ is weak*-dense in E, and any result that appeals to that lemma should account for this additional hypothesis.*

2. The setup

Since the G-invariant set D discussed in Section 1 is only used to restrict the coefficient functions of representations, its intersection with the

Fourier-Stieltjes algebra is all that matters. So, let E be a G-invariant (not necessarily weak*- or even norm-closed) vector subspace of $B(G)$.

Remark 2.1. If E arises as in the introduction, by intersecting $B(G)$ with a G-invariant ideal of $L^\infty(G)$ or $C_b(G)$, then E will also be an ideal of $B(G)$. Although this property is in fact important to us for our main study of coaction functors, for the time being we only require E to be a G-invariant subspace of $B(G)$.

Definition 2.2. Let U be a unitary representation of G on a Hilbert space H, and let $\xi, \eta \in H$. Define the *coefficient function* $U_{\xi,\eta}$ by

$$U_{\xi,\eta}(x) = \langle U_x \xi, \eta \rangle \quad \text{for } x \in G.$$

We write U_ξ for $U_{\xi,\xi}$.

We will find it convenient to adopt the convention that the zero representation of G (on the 0-dimensional Hilbert space) is unitary.

Definition 2.3 (see [4, Definition 2.1])**.** An *E-representation* of G is a triple (U, H, H_0), where U is a unitary representation of G on a Hilbert space H and H_0 is a dense subspace of H such that $U_{\xi,\eta} \in E$ for all $\xi, \eta \in H_0$.

With our convention that the zero representation is unitary, we see that it is trivially an E-representation.

The fussy notation (U, H, H_0) will help us keep track of things; [4] just refers to U itself as the E-representation (actually, a D-representation where D is a G-invariant ideal of $\ell^\infty(G)$ for a discrete group G), and sometimes we will also do this.

Of course, U integrates to a nondegenerate representation of $C^*(G)$ on H, which we also denote by U. When we refer to the kernel of U, we mean the ideal

$$\ker U := \{a \in C^*(G) : U(a) = 0\}$$

of $C^*(G)$. Part (1) of the following definition is taken from [4, Definition 2.2], and part (2) from [10, Definition 3.2].

Definition 2.4. Let E be a G-invariant subspace of $B(G)$.

(1) Define an ideal of $C^*(G)$ by

$$J_E = \bigcap \{\ker U : U \text{ is an } E\text{-representation of } C^*(G)\},$$

and then let

$$C^*_{E,BG}(G) = C^*(G)/J_E.$$

(2) On the other hand, by G-invariance the preannihilator ${}^\perp E$ is also an ideal of $C^*(G)$, so we can define another quotient C^*-algebra by

$$C^*_{E,KLQ}(G) = C^*(G)/{}^\perp E.$$

Remark 2.5. Since E is a G-invariant subspace of $B(G)$, the weak*-closure $\overline{E}$ is also G-invariant, and hence is a $C^*(G)$-subbimodule of $B(G)$, so by [16, Corollary 3.10.8] the preannihilator ${}^\perp E = {}^\perp \overline{E}$ is a closed ideal of $C^*(G)$. This is also recorded in [5, Lemma 2.10]. The notation $C^*_{E,BG}$ and $C^*_{E,KLQ}$ comes from [5].

The following is an alternative version of Question 1.1, as we will see from the results in Section 3:

Question 2.6. If E is a G-invariant subspace of $B(G)$, is $J_E = {}^\perp E$? Equivalently, is $C^*_{E,BG}(G) = C^*_{E,KLQ}(G)$?

We are most interested in this question in the special case that E is actually an ideal of $B(G)$.

The inclusion $J_E \supset {}^\perp E$ always holds, as (correctly) shown in the second half of the proof of [10, Lemma 3.5 (1)], and here is the argument: Let $a \in {}^\perp E$, and let (U, H, H_0) be an E-representation. Then for all $\xi, \eta \in H_0$ we have $U_{\xi,\eta} \in E$, so

$$0 = U_{\xi,\eta}(a) = \langle U(a)\xi, \eta\rangle,$$

and so $U(a) = 0$ by density. Thus $a \in J_E$. Interestingly, this inclusion will also fall out of our investigation below (see Corollary 3.15).

In [10] we gave an incorrect argument for the containment $J_E \subset {}^\perp E$.

Remark 2.7. Just for fun, here is an alternative argument for the inclusion proved above: First, a set-theoretic technicality: there is a *set* $\mathcal{R}$ of representations of G such that

$$J_E = \bigcap\{\ker U : U \in \mathcal{R}\}.$$

(The issue here is that in Definition 2.4 the intersection is indexed by a proper class, i.e., not a set. But we are intersecting a *set* of ideals.) [7, Proposition 3.4.2 (i)] says that every state of $C^*(G)$ that vanishes on J_E is a weak*-limit of states of the form

$$\sum_{U\in\mathcal{F}} U_{\xi_{U,H,H_0}},$$

where $\mathcal{F} \subset \mathcal{R}$ is finite and $\xi_{U,H,H_0} \in H$ for all $(U, H, H_0) \in \mathcal{F}$. Now, by density each such state can be approximated in the weak*-topology by states of the same form but with $\xi_{U,H,H_0} \in H_0$ for each $(U, H, H_0) \in \mathcal{F}$. Thus every state in $J_E^\perp$ is in $\overline{E}$. Since $J_E^\perp$ is the dual space of the quotient C^*-algebra $C^*(G)/J_E$, every element of $J_E^\perp$ is a linear combination of states in $J_E^\perp$, and hence $J_E^\perp \subset \overline{E}$ by the preceding. Therefore $J_E \supset {}^\perp E$.

3. Ordered and weakly ordered subspaces

We now proceed to investigate Question 2.6. Throughout, E will denote a G-invariant subspace of $B(G)$. We want to find a reasonably general sufficient condition for $J_E = {}^\perp E$. In this section we illustrate one approach, mainly using the Hahn-Banach theorem. First we introduce some auxiliary notation. Recall from Definition 2.3 that our notation for an E-representation is a triple (U, H, H_0), where we keep track of the Hilbert space H and the dense subspace H_0.

Notation 3.1. We write

$$E_r := \{U_{\xi,\eta} : (U, H, H_0) \text{ is an } E\text{-representation}, \xi, \eta \in H_0\}.$$

Note: as a consequence of our convention that the zero representation is (unitary and hence is) an E-representation, we see that $0 \in E_r$.

Remark 3.2. Think of the elements of E_r as "representable" (which is our motivation for the notation). In [19, Section 4], Wiersma defines something similar but not quite the same — he would write $A_E(G)$ for the set of all coefficient functions $U_{\xi,\eta}$, where now ξ and η are allowed to be any vectors from the Hilbert space H of the E-representation U, not just from the dense subspace H_0.

Remark 3.3. Somehow irritating, we do not know whether G has any nonzero E-representations, equivalently whether $E_r \neq \{0\}$ (see Question 3.19 below).

Lemma 3.4. *For any G-invariant subspace E of $B(G)$, E_r is a vector subspace of E.*

Proof. It is obvious that E_r is closed under scalar multiplication. Note that any direct sum of E-representations is an E-representation, by the same reasoning as [4, Remark 2.4]. Let $f, g \in E$, and choose E-representations (U, H, H_0) and (V, K, K_0) and vectors $\xi, \eta \in H_0$ and $\kappa, \zeta \in K_0$ such that $f = U_{\xi,\eta}$ and $g = V_{\kappa,\zeta}$. Then $(U \oplus V, H \oplus K, H_0 \oplus K_0)$ is an E-representation, where $H_0 \oplus K_0$ stands for the algebraic direct sum of the vector subspaces H_0 and K_0. Since

$$U_{\xi,\eta} + V_{\kappa,\zeta} = (U \oplus V)_{(\xi,\kappa),(\eta,\zeta)},$$

we are done. □

Definition 3.5. For a (not necessarily G-invariant) subspace E of $B(G)$, put $E_0 = \operatorname{span}\{E \cap P(G)\}$. We say that E is *ordered* if $E_0 = E$, and we say that E is *weakly ordered* if E_0 is weak* dense in E.

Although in the above definition we temporarily removed the assumption that E is G-invariant, we will tacitly impose this assumption unless otherwise specified.

We will see that not every subspace of $B(G)$ is ordered, and we begin with an obvious obstruction: First recall that the involution in the Fourier-Stieltjes algebra $B(G)$ is given by

$$\widetilde{f}(x) = \overline{f(x^{-1})}.$$

It follows from the properties of duals of C^*-algebras that every ordered subspace E of $B(G)$ is self-adjoint: if $f \in E$ then also $\widetilde{f} \in E$.

Remark 3.6. The above terminology "ordered" makes sense because it is precisely what it means for the self-adjoint part of E to be a partially ordered (real) subspace of the self-adjoint part of $B(G)$. It is slightly less obvious that the terminology "weakly ordered" is sensible, but it is somehow related to the property "ordered" and is obviously weaker (in fact strictly weaker, as we will show).

We will show in Example 4.2 that in fact not every G-invariant ideal of $B(G)$ is ordered.

Question 3.7. Is every G-invariant subspace of $B(G)$ weakly ordered? Example 3.23 below gives some negative evidence, although it does not furnish a counterexample.

Lemma 3.8. *E is weakly ordered if and only if $^{\perp}E = {}^{\perp}E_0$.*

Proof. Since $E_0 \subset E$, this follows from the Hahn-Banach Theorem. □

The following is equivalent to [5, Lemma 2.15] (see also [19, Proposition 4.3]), with a somewhat different proof.

Proposition 3.9 ([5])**.** *If E is closed in the norm of $B(G)$, then it is ordered.*

Proof. By [1, Theorem 3.17] E is the set of all coefficient functions of some representation U of G, and then by [1, Proposition 2.2] E is the predual of the von Neumann algebra generated by $U(G)$. It then follows (see, for example, [16, Proposition 3.6.2] or [7, Theorem 12.3.3]) that E is the linear span of positive linear functionals. □

Remark 3.10. Proposition 3.9 obviously applies in particular to situations where E is relatively closed in $B(G)$ as a subset of $C_b(G)$ with the sup norm, most importantly $E = C_0(G) \cap B(G)$, as observed in [5, Lemma 2.15].

Remark 3.11. Here is an alternative, somewhat more elementary, argument: Since E is a subspace, trivially $E_0 \subset E$. For the opposite containment, let $f \in E$. To show that $f \in E_0$, without loss of generality assume that $f \neq 0$. By [19, Proposition 4.1] (for example), there is a representation U and vectors ξ, η such that $f = U_{\xi,\eta}$ and $\|f\| = \|\xi\|\|\eta\|$ (where the norm of f is taken in $B(G)$).

Let

$$H_\xi = \overline{\operatorname{span}}\{U_x\xi : x \in G\}$$
$$H_\eta = \overline{\operatorname{span}}\{U_x\eta : x \in G\},$$

and let P_ξ and P_η be the orthogonal projections onto these respective subspaces. Since P_η commutes with U,

$$f(x) = \langle U_x\xi, P_\eta\eta\rangle = \langle U_x P_\eta\xi, \eta\rangle.$$

Thus by construction we have

$$\|P_\eta\xi\|\|\eta\| \geq \|f\| = \|\xi\|\|\eta\|,$$

so $\|P_\eta \xi\| \geq \|\xi\|$, and hence, since P_η is a projection, we must have $P_\eta \xi = \xi$, giving $\xi \in H_\eta$. Similarly $\eta \in H_\xi$. Thus in fact $H_\xi = H_\eta$. Since E is G-invariant and norm-closed, the coefficient functions $U_{\xi',\eta'}$ are in E for all $\xi', \eta' \in H_\xi$. Thus

$$f = U_{\xi,\eta} = \frac{1}{4}\sum_{k=0}^{3} i^k U_{\xi+i^k\eta} \in E_0.$$

This argument should be compared to [19, proof of Proposition 4.3] and [5, proof of Lemma 2.15].

Proposition 3.12. $E_0 = E_r$.

Proof. Let $f \in E \cap P(G)$. Choose a cyclic representation U of $C^*(G)$ on a Hilbert space H, with cyclic vector ξ, such that $f = U_\xi$. Let

$$H_0 = \operatorname*{span}_{x\in G}\{U_x\xi\},$$

which is a dense subspace of H. For all $a \in C^*(G)$ and $x, y \in G$,

$$\langle U(a)U_x\xi, U_y\xi\rangle = x \cdot U_\xi \cdot y^{-1}(a),$$

and $x \cdot U_\xi \cdot y^{-1} \in E$, so (U, H, H_0) is an E-representation. Therefore $f \in E_r$. By Lemma 3.4 it follows that $E_0 \subset E_r$.

On the other hand, if (U, H, H_0) is an E-representation and $\xi, \eta \in H_0$, then

$$U_{\xi,\eta} = \frac{1}{4}\sum_{0}^{3} i^k U_{\xi+i^k\eta},$$

and $\xi + i^k\eta \in H_0$ for $k = 0, \ldots, 3$. Thus $E_r \subset E_0$. □

Remark 3.13. Recall from Remark 3.2 that in [19] Wiersma writes $A_E(G)$ for the set of all coefficient functions of E-representations. Using our notation, [19, Proposition 4.3] says that $A_E(G) = \overline{E_0}$. Thus by Proposition 3.12, $A_E(G) = \overline{E_r}$.

Corollary 3.14. $J_E = {}^\perp E_0$.

Proof. We have

$$\begin{aligned} J_E &= \bigcap\{\ker U : (U, H, H_0) \text{ is an } E\text{-representation}\} \\ &= \bigcap\{\ker U_{\xi,\eta} : (U, H, H_0) \text{ is an } E\text{-representation}, \xi, \eta \in H_0\} \\ &= {}^\perp E_r \\ &= {}^\perp E_0, \end{aligned}$$

where we used density of H_0 in the second step. □

Remark 3.15. We can use the above results to give an alternative proof that ${}^\perp E \subset J_E$: Since $E_0 \subset E$, we have ${}^\perp E \subset {}^\perp E_0$, so the inclusion follows from Corollary 3.14.

The following corollary is essentially the second half of [5, Proposition 2.13].

Corollary 3.16 ([5])**.** *If E is ordered then $J_E = {}^\perp E$.*

Proof. This follows immediately from Corollary 3.14 and the definition of ordered. □

We now recover [5, Corollary 2.14] (with a similar proof), which perfects Corollary 3.16:

Corollary 3.17 ([5])**.** *$J_E = {}^\perp E$ if and only if E is weakly ordered.*

Proof. By Corollary 3.14, $J_E = {}^\perp E$ if and only if ${}^\perp E = {}^\perp E_0$, which in turn is equivalent to $\overline{E} = \overline{E_0}$ (where the bars denote weak*-closures), and the result follows since $E_0 \subset E$. □

Remark 3.18. By [10, Lemma 3.14], if E is a nonzero G-invariant ideal of $B(G)$, then the norm closure of E contains $A(G)$, so the weak* closure contains $B_r(G)$.[1] Thus we always have ${}^\perp E \subset \ker\lambda$. Suppose that $E_0 \neq \{0\}$. Then E_0 is also a nonzero G-invariant ideal of $B(G)$, so by the same argument it follows that the (perhaps) smaller ideal E_0 is still weak* dense in $B_r(G)$, and so

$$ {}^\perp E \subset {}^\perp E_0 = J_E \subset {}^\perp B_r(G) = \ker\lambda. $$

Thus if $E_0 \neq \{0\}$ and ${}^\perp E = \ker\lambda$ then E is weakly ordered.

Question 3.19. If E is a nonzero G-invariant ideal of $B(G)$, does G have a nonzero E-representation? Equivalently, is the subspace E_r of E nontrivial? We find it hard to believe that this question is still open for general locally compact groups.

For G discrete, the answer is yes:

Proposition 3.20. *If G is discrete and E is a nonzero G-invariant ideal of $B(G)$, then λ is an E-representation.*

Proof. As (essentially) mentioned in [4, paragraph following Definition 2.6], if G is discrete then $E \supset c_c(G)$, and it follows that λ is an E-representation. □

Remark 3.21. Trivially, if E is weakly ordered, then $E_0 \neq \{0\}$, so G has a nonzero E-representation — and conversely if G is amenable (see Proposition 3.22 below). Although we here are only interested in nonamenable groups, some questions are also relevant for amenable groups, e.g., the classical ideals in Section 4. Also, if F is a nonzero G-invariant ideal of $B(G)$ containing E, and if G has a nonzero E-representation, then trivially it has a nonzero F-representation.

[1] It might be worthwhile mentioning that there was a slight redundancy in our argument: it was not necessary to ensure that the ideal separates points in G.

Proposition 3.22. *Let G be amenable, and let E be a nonzero G-invariant ideal of $B(G)$. If G has a nonzero E-representation, then E is weakly ordered.*

Proof. By hypothesis, $E_0 \neq \{0\}$. As we point out in Remark 3.18, ${}^{\perp}E \subset \ker \lambda$. Since G is amenable, $\ker \lambda = \{0\}$. Consequently, ${}^{\perp}E = \ker \lambda$. Consulting Remark 3.18 again we conclude that E is weakly ordered. □

Example 3.23. Taking G to be the circle group $\mathbb{T}$, we will give an example of a weak*-dense subspace E of $B(G)$ for which $E_0 = \{0\}$, and so E fails to be weakly ordered in a very strong way. Note that our example is neither G-invariant nor an ideal of $B(G)$. Let

$$E = \operatorname{span}\{z^{n+1} - z^n : n \in \mathbb{N}\} \subset B(\mathbb{T}).$$

Then E is a subspace of $B(\mathbb{T})$, and we claim that $E \cap P(\mathbb{T}) = \{0\}$. By Bochner's theorem, it suffices to show that the Fourier transform

$$\widehat{E} = \{\widehat{f} : f \in E\} = \operatorname{span}\{\delta_{n+1} - \delta_n : n \in \mathbb{N}\}$$

contains no nonzero positive measure on $\mathbb{Z}$. Let

$$\mu = \sum_{n=-k}^{k} c_n(\delta_{n+1} - \delta_n),$$

and assume that μ is positive and nonzero. Clearly $k > 0$, and without loss of generality $c_k \neq 0$. For each $n \in \mathbb{Z}$ let $p_n = \chi_{\{n\}} \in c_0(\mathbb{Z})$. Then

$$0 \leq \langle p_{k+1}, f\rangle = c_k,$$

so $c_k > 0$. Next,

$$0 \leq \langle p_k, f\rangle = c_{k-1} - c_k,$$

so $c_{k-1} \geq c_k$. Continuing in this way, we find

$$c_{-k} \geq c_{-k+1} \geq \cdots \geq c_k > 0.$$

But

$$0 \leq \langle p_{-k}, f\rangle = -c_{-k},$$

giving a contradiction. Note that $\widehat{E}$ is weak*-dense in the space $\ell^1(\mathbb{Z})$ of complex measures on $\mathbb{Z}$, and so E is weak*-dense in $B(\mathbb{T})$. We thank J. Spielberg for fruitful discussion that led to this example.

Remark 3.24. Let E be a nonzero G-invariant ideal of $B(G)$. Suppose that there exists at least one nonzero E-representation U of G, equivalently $E_0 \neq \{0\}$. If V is any representation of G, then $U \otimes V$ is an E-representation since E is an ideal. In particular, $U \otimes \lambda$ is an E-representation. By Fell's trick, $U \otimes \lambda$ is unitarily equivalent to a multiple of λ. Thus this multiple of λ is an E-representation. We would like to conclude that λ is an E-representation, but this seems to require that the class of E-representations be closed under taking subrepresentations. It is not clear to us why this would be true. But if it were then we could conclude that E_0 contains every convolution $\xi * \eta$ for

$\xi, \eta \in L^2(G)$. Note that our assumptions imply that E_0 is also a nonzero G-invariant ideal of $B(G)$, so by Remark 3.18 we already knew — for different reasons — that the norm closure of E_0 contains $A(G)$.

4. The classical ideals

Inspired by the work of Brown and Guentner [4], the main examples of G-invariant subspaces of $B(G)$ that we want to include are actually ideals:

(1) $C_c(G) \cap B(G)$
(2) $C_0(G) \cap B(G)$
(3) $L^p(G) \cap B(G)$.

Proposition 4.1. *The ideals* (1) *and* (2) *are ordered.*

Proof. The first case is very well-known: by [8, Proposition 3.4],

$$\begin{aligned} C_c(G) \cap B(G) &= \operatorname{span}\{C_c(G) \cap P(G)\} \\ &= \operatorname{span}\{C_c(G) \cap B(G) \cap P(G)\}. \end{aligned}$$

The second case was mentioned in Remark 3.10. □

However, for ideals of type (3) things are murky. We do not even know whether they are all weakly ordered. Since this is an important source of ideals of $B(G)$, we examine this more closely. For $1 \le p \le \infty$ let

$$E^p = L^p(G) \cap B(G).$$

In particular, $E^\infty = B(G)$.

Note that if G is unimodular then every E^p is at least self-adjoint in $B(G)$. However, this does not hold generally, as the following example shows.

Example 4.2. Here we show that there are groups for which the ideal E^1 is not ordered, by showing that it is not self-adjoint in $B(G)$. It seems plausible, but not clear, that this carries over to arbitrary $p < \infty$ by embellishing the computations.

We want to find $f \in E^1$ such that $\widetilde{f} \notin E^1$. Let $g, h \in L^1(G) \cap L^2(G)$ be nonnegative, and put

$$f(x) = \langle \lambda_x g, h \rangle = \int f(x^{-1}y) g(y)\, dy.$$

Then $f \in B(G)$, and

$$\begin{aligned} \|f\|_1 &= \int\int g(x^{-1}y)h(y)\, dy\, dx \\ &= \int\int g(x^{-1})\, dx\, h(y)\, dy \quad \text{(after } x \mapsto yx) \\ &= \|g\Delta^{-1}\|_1 \|h\|_1, \end{aligned}$$

where here we write Δ^{-1} for the reciprocal $1/\Delta$.

On the other hand, $\widetilde{f} \in B(G)$, and

$$\widetilde{f}(x) = f(x^{-1}) = \int g(xy)h(y)\,dy,$$

so

$$\begin{aligned}\|\widetilde{f}\|_1 &= \int\int g(xy)\,dx\,h(y)\,dy \\ &= \int\int g(x)\,dx\,\Delta(y)^{-1}h(y)\,dy \\ &= \|g\|_1\|\Delta^{-1}h\|_1.\end{aligned}$$

Now, we impose further assumptions on g, h:

$$\begin{aligned}&\|g\|_1 = \infty \\ &\|g\Delta^{-1}\|_1 < \infty \\ &0 < \|h\|_1, \|\Delta^{-1}h\|_1 < \infty.\end{aligned}$$

Then $f \in E^1$ and $\widetilde{f} \notin E^1$, so E^1 is not ordered.

We can easily choose a suitable h — for instance, let $h \in C_c(G)$ be nonnegative and not identically 0. It seems likely that we can also choose a suitable g in any nonunimodular group. For a specific example, let G be the $ax + b$ group $\mathbb{R}^+ \times \mathbb{R}$, with operation

$$(x, y)(u, v) = (xu, xv + y).$$

Recall that the Haar measure and modular function are given by

$$\begin{aligned}d(x, y) &= \frac{dx\,dy}{x^2} \\ \Delta(x, y) &= \frac{1}{x}.\end{aligned}$$

We look for g of the form

$$g(x, y) = \phi(x)\psi(y),$$

with $\phi, \psi \geq 0$. We need $g \in L^2$, which means

$$\infty > \int_G g(x, y)^2\,d(x, y) = \int_0^\infty \frac{\phi(x)^2}{x^2}\,dx \int_{\mathbb{R}} \psi(y)^2\,dy.$$

We also need $g\Delta^{-1}$ integrable but g nonintegrable, which means

$$\infty = \int_G g(x, y)\,d(x, y) = \int_0^\infty \frac{\phi(x)}{x^2}\,dx \int_{\mathbb{R}} \psi(y)\,dy$$

and

$$\infty > \int_G \frac{g(x, y)}{\Delta(x, y)}\,d(x, y) = \int_0^\infty \frac{\phi(x)}{x}\,dx \int_{\mathbb{R}} \psi(y)\,dy.$$

These conditions are all met with, e.g.,

$$\phi(x) = xe^{-x} \quad \text{and} \quad \psi(y) = e^{-y^2}.$$

Question 4.3. When is E^p

(1) ordered?
(2) weakly ordered?

Trivially:

Proposition 4.4. *E^∞ is ordered.*

Proposition 4.5. *If $1 \le p \le 2$ then E^p is weakly ordered.*

Proof. [10, Proposition 4.2] says that ${}^\perp E^p = \ker \lambda$, and the result follows. □

Corollary 4.6. *For some groups G, there are G-invariant ideals of $B(G)$ that are weakly ordered but not ordered.*

Proof. This follows immediately from Example 4.2 and Proposition 4.5. □

We can at least answer Question 3.19 affirmatively for E^p:

Corollary 4.7. *For every p, there is a nonzero E^p-representation of G.*

Proof. By Proposition 4.5, E^p is weakly ordered for all $p \le 2$, and so G has a nonzero E^p-representation, and hence has a nonzero E^p representation for all $p > 2$ as well, because if $p > q$ then $E^p \supset E^q$, since $B(G)$ consists of bounded functions. □

Remark 4.8. Proposition 4.5 is also implied by [19, Proposition 4.4 (i)], which says that $A_{L^p}(G) = A(G)$ for all $p \in [1, 2]$.

Proposition 4.9. *If G is abelian, then E^2 is ordered.*

Proof. The Fourier transform takes $L^2(\widehat{G}) \cap L^1(\widehat{G})$ bijectively onto $L^2(G) \cap A(G)$. Now, $L^2(\widehat{G}) \cap L^1(\widehat{G})$ is the linear span of the nonnegative functions it contains, so $L^2(G) \cap A(G)$ is the linear span of the positive definite functions it contains. The result now follows from Proposition 4.10 below. □

In the above proof we appealed to the following elementary fact, which is perhaps folklore:

Proposition 4.10. *For any locally compact group G, if $1 \le p \le 2$ then $L^p(G) \cap B(G) = L^p(G) \cap A(G)$.*

Proof. It suffices to show that if $f \in L^p(G) \cap B(G)$ then $f \in A(G)$. Since $B(G)$ consists of bounded functions, we have

$$L^p(G) \cap B(G) \subset L^2(G) \cap B(G)$$

so it suffices to prove the result for the special case $p = 2$. Choose a representation U of G and vectors ξ, η such that $f = U_{\xi,\eta}$. For any $g \in L^1(G) \cap L^2(G)$, define $\psi_g \in A(G)$ by

$$\psi_g(x) = \langle \lambda_x g, \overline{f} \rangle$$

$$\begin{aligned}
&= \int g(x^{-1}y)f(y)\,dy \\
&= \int g(y)\langle U_{xy}\xi, \eta\rangle\,dy \\
&= \int \langle U_x U_y g(y)\xi, \eta\rangle\,dy \\
&= \langle U_x U_g \xi, \eta\rangle.
\end{aligned}$$

It follows that for any $a \in C^*(G)$ we have

$$\psi_g(a) = \langle U_a U_g \xi, \eta\rangle.$$

Letting $U_g\xi \to \xi$ in the norm of the Hilbert space of U, we have $\psi_g \to f$ in the $B(G)$-norm, and therefore $f \in A(G)$. □

Remark 4.11. We conjecture that the conclusion of Proposition 4.9 holds for all unimodular groups.

Remark 4.12. Here we show that Proposition 4.10 does not extend to $p > 2$: for $G = SL(2, \mathbb{R})$, by [19, Theorem 7.2]

$$\overline{L^p(G) \cap B(G)}^{\text{weak}^*} \neq \overline{L^2(G) \cap B(G)}^{\text{weak}^*},$$

whereas

$$\overline{L^p(G) \cap A(G)}^{\text{weak}^*} = B_r(G) = \overline{L^2(G) \cap B(G)}^{\text{weak}^*}.$$

Remark 4.13. In view of the discussion in this section, it might be worthwhile to consider three possible (re-)definitions of the G-invariant ideal E^p of $B(G)$:

(1) $L^p(G) \cap B(G)$;
(2) $\{f \in B(G) : f, \widetilde{f} \in L^p(G)\}$;
(3) $\operatorname{span}\{L^p(G) \cap P(G)\}$.

(1) is of course how we defined the notation E^p in this paper, and (2) is the self-adjoint part of (1). Since (3) is always self-adjoint, we obviously have (1) ⊃ (2) ⊃ (3).

(3) is the convention used in [6], with good reason.

By our definition, (1) = (3) only when (1) is ordered, which we have seen does not always occur; for example, it happens for $p = 2$ and G unimodular (in which case in fact (1) = (2) = (3)), but (1) is not ordered for some (all?) nonunimodular G, by Example 4.2.

If G is unimodular then (1) = (2).

Remark 4.14. Here is a frustrating illustration of our ignorance: First recall that [19, Theorems 7.2 and 7.3] show that for $G = SL(2, \mathbb{R})$ the large ideals of $B(G)$ consist precisely of $\overline{E^p}^{\text{weak}^*}$ for $1 \le p \le \infty$, and moreover for $2 \le p \le \infty$ these ideals are all distinct, the extremes being $B_r(G)$ for $p = 2$ and $B(G)$ for $p = \infty$. Now let $E = E^p$ for some $p \in (2, \infty)$. Then $\overline{E}_0^{\text{weak}^*}$ is a large ideal of $B(G)$, and so by Wiersma's results it coincides with $\overline{E^{p'}}^{\text{weak}*}$ for a unique $p' \in [2, p]$. But since we don't know whether E^p is weakly ordered, we can't determine whether $p' = p$.

5. Final comments

In this section we show how to apply the preceding discussion to fix [10, Lemma 3.5 (1)] and the references to it that have appeared already in the literature.

Most importantly, in all cases where [10, Lemma 3.5 (1)] is used, the hypothesis that E_0 be weak*-dense in E should be mentioned and verified. We emphasize that for large ideals E of $B(G)$ (or even just nonzero G-invariant norm-closed ideals), there is no problem: $C^*_{E,BG}(G) = C^*_{E,KLQ}(G)$. However, if E is just a nonzero G-invariant ideal, then it's probably best to use the Brown-Guentner convention for $C^*_E(G)$, namely take

$$C^*_E(G) = C^*_{E,BG}(G) = C^*(G)/J_E$$

rather than $C^*_{E,KLQ}(G) = C^*(G)/{}^\perp E$. As we have seen, if we replace the given E by $E_0 := \operatorname{span}(E \cap P(G))$ then the two approaches give the same group C^*-algebra. Note that this is the approach of [5, Example 2.16] for E^p.

We give a few examples of how results that mention [10, Lemma 3.5 (1)] should be adjusted. In that lemma itself, also item (2) depends upon the new hypothesis, since part of [10, Lemma 3.5 (2)] is equivalent to the equality $C^*(G)/{}^\perp E = C^*(G)/J_E$.

[10, Corollary 3.6 (1)] says that a representation U of G is an E-representation if and only if $\ker U \supset {}^\perp E$. Since this depends upon ${}^\perp E = J_E$, the new hypothesis $\overline{E_0} = \overline{E}$ should be added here.

Similarly, the new hypothesis should be added to [10, Observation 3.8 and Remark 3.18].

[10, Section 4] is explicitly about the classical ideals mentioned in Section 4, and in particular the problem arises in discussions of $C^*_{L^p(G)}(G)$. However, it follows from Proposition 4.5 that [10, Proposition 4.2] is correct as stated.

Remark 5.1. This might be a convenient place to correct another (relatively harmless) misstatement in [10, Remark 4.3], where it is asserted that, for discrete G, the weak*-closure of $C_0(G) \cap B(G)$ being strictly larger than $B_r(G)$ occurs precisely when G is a-T-menable but nonamenable, and that for perhaps the earliest result along these lines one can see [15]. This is garbled in a couple of ways. First of all, Menchoff's 1916 paper gives examples of singular measures whose Fourier coefficients tend to zero, thus showing that the intersection $E := C_0(G) \cap B(G)$ can properly contain the Fourier algebra $A(G)$, even for $G = \mathbb{T}$. This certainly does not, however, illustrate the phenomenon of the weak*-closure $\overline{E}$ being strictly larger than $B_r(G)$.

The second blunder here is the use of the word "precisely"; a-T-menability is equivalent to $J_E = 0$, and hence to $J_E^\perp = B(G)$. This property certainly implies $\overline{E} = B(G)$, which is strictly larger than $B_r(G)$ if G is nonamenable — so, when G is a-T-menable but nonamenable we have $\overline{E} \supsetneq B_r(G)$ for $E = C_0(G) \cap B(G)$. On the other hand, it is not clear to us that $\overline{E} \supsetneq B_r(G)$ implies a-T-menability.

While we are at it, we can mention one more minor slip in [10]: in the bibliographic entry for [15] the French word "développement" is misspelled.

References

[1] Arsac, Gilbert. Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire. *Publ. Dép. Math.* (Lyon) **13** (1976), no. 2, 1–101. MR444833, Zbl 0365.43005. 1045

[2] Baum, Paul; Guentner, Erik; Willett, Rufus. Expanders, exact crossed products and the Baum–Connes conjecture. *Ann. K-Theory* **1** (2016), no. 2, 155–208. MR3514939, Zbl 1331.46064, arXiv:1311.2343, doi: 10.2140/akt.2016.1.155. 1039

[3] Brannan, Michael; Ruan, Zhong-Jin. L_p-representations of discrete quantum groups.. *J. Reine Angew. Math.* **732** (2017), 165–210. MR3717091, Zbl 1378.46054, arXiv:1404.4133, doi: 10.1515/crelle-2014-0140. 1040

[4] Brown, Nathanial P.; Guentner, Erik P. New C^*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.* **45** (2013), no. 6, 1181–1193. MR3138486, Zbl 1337.46039, arXiv:1205.4649, doi: 10.1112/blms/bdt044. 1039, 1042, 1044, 1047, 1049

[5] Buss, Alcides; Echterhoff, Siegfried; Willett, Rufus. Exotic crossed products. *Operator Algebras and Applications* (Proceedings of the Abel Symposium 2015), 67–114, Contemporary Mathematics, 671. *American Mathematical Society, Providence, RI*, 2016. xii+267 pp. ISBN: 978-1-4704-1948-6. MR3546675, Zbl 1375.46048, arXiv:1510.02556, doi: 10.1007/978-3-319-39286-8_3. 1040, 1041, 1043, 1045, 1046, 1047, 1053

[6] Buss, Alcides; Echterhoff, Siegfried; Willett, Rufus. Exotic crossed products and the Baum–Connes conjecture. *J. Reine Angew. Math.* **740** (2018), 111–159. MR3824785, Zbl 06902328, arXiv:1409.4332, doi: 10.1515/crelle-2015-0061. 1040, 1052

[7] Dixmier, Jacques. C^*-algebras. Translated from the French by Francis Jellett. North-Holland Mathematical Library, 15. *North-Holland Publishing Co., Amsterdam-New York-Oxford*, 1977. xiii+492 pp. ISBN: 0-7204-0762-1. MR0458185, Zbl 0372.46058. 1043, 1045

[8] Eymard, Pierre. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–236. MR228628, Zbl 0169.46403, doi: 10.24033/bsmf.1607. 1049

[9] Hewitt, Edwin; Ross, Kenneth A. Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups. 2nd printing. Grundlehren der Mathematischen Wissenschaften, 152. *Springer-Verlag, Berlin*, 1994. ix+771 pp. MR0262773, Zbl 0830.43001.

[10] Kaliszewski, S.; Landstad, Magnus B.; Quigg, John. Exotic group C^*-algebras in noncommutative duality. *New York J. Math.* **19** (2013), 689–711. MR3141810, Zbl 1294.46047, arXiv:1211.4982. 1040, 1041, 1042, 1043, 1047, 1051, 1053, 1054

[11] Kaliszewski, S.; Landstad, Magnus B.; Quigg, John. Coaction functors. *Pacific J. Math.* **284** (2016), no. 1, 147–190. MR3530866, Zbl 1359.46059, arXiv:1505.03487, doi: 10.2140/pjm.2016.284.147. 1040

[12] Kaliszewski, S.; Landstad, Magnus B.; Quigg, John. Exact large ideals of $B(G)$ are downward directed. *Proc. Amer. Math. Soc.* **144** (2016), no. 10, 4401–4412. MR3531190, Zbl 1359.46058, arXiv:1508.00284, doi: 10.1090/proc/13100. 1040

[13] Kaliszewski, S.; Landstad, Magnus B.; Quigg, John. Exotic coactions. *Proc. Edinburgh Math. Soc.* (2) **59** (2016), no. 2, 411–434. MR3509238, Zbl 1370.46041, arXiv:1305.5489, doi: 10.1017/S0013091515000164. 1040

[14] Kyed, David; Sołtan, Piotr M. Property (T) and exotic quantum group norms. *J. Noncommut. Geom.* **6** (2012), no. 4, 773–800. MR2990124, Zbl 1277.46041, arXiv:1006.4044, doi: 10.4171/JNCG/105. 1040

[15] Menchoff, D. Sur l'unicité du développement trigonométrique. *C. R. Acad. Sci. Paris* **163** (1916), 433–436. JFM 46.0457.02. 1053, 1054

[16] Pedersen, Gert K. C^*-algebras and their automorphism groups. London Mathematical Society Monographs, 14. *Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York*, 1979. ix+416 pp. ISBN: 0-12-549450-5. MR0548006, Zbl 0416.46043, doi: 10.1017/S001309150001676X. 1043, 1045

[17] Ruan, Zhong-Jin; Wiersma, Matthew. On exotic group C^*-algebras. *J. Funct. Anal.* **271** (2016), no. 2, 437–453. MR3501853, Zbl 1342.22011, arXiv:1505.00805, doi: 10.1016/j.jfa.2016.03.002. 1040

[18] Wiersma, Matthew. C^*-norms for tensor products of discrete group C^*-algebras. *Bull. Lond. Math. Soc.* **47** (2015), no. 2, 219–224. MR3335116, Zbl 1328.46044, arXiv:1406.2654, doi: 10.1112/blms/bdu115. 1040

[19] Wiersma, Matthew. L^p-Fourier and Fourier–Stieltjes algebras for locally compact groups. *J. Funct. Anal.* **269** (2015), no. 12, 3928–3951. MR3418075, Zbl 1330.22014, arXiv:1409.2787, doi: 10.1016/j.jfa.2015.09.016. 1040, 1044, 1045, 1046, 1051, 1052

[20] Wiersma, Matthew. Constructions of exotic group C^*-algebras. *Illinois J. Math.* **60** (2016), no. 3–4, 655–667. MR3705441, Zbl 1373.22014, arXiv:1403.5223. 1040

(S. Kaliszewski) School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287, USA.
kaliszewski@asu.edu

(Magnus B. Landstad) Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway.
magnus.landstad@ntnu.no

(John Quigg) School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287, USA.
quigg@asu.edu

This paper is available via http://nyjm.albany.edu/j/2018/24-49.html.