New York Journal of Mathematics

New York J. Math. **24** (2018) 53–81.

Regularity for weak solutions to nondiagonal quasilinear degenerate parabolic systems with controllable growth conditions

Yan Dong and Dongyan Li

Abstract. The aim of this paper is to study regularity for weak solutions to the nondiagonal quasilinear degenerate parabolic systems related to Hörmander's vector fields, where the lower order items satisfy controllable growth conditions. Higher Morrey regularity is proved by establishing a reverse Hölder inequality for weak solutions, and then Hölder regularity is obtained by the isomorphic relationship.

Contents

1. Introduction

Regularity for weak solutions to divergence elliptic and parabolic systems in Euclidean spaces has been studied by many authors (see [1]–[3], [8], [15], [21], [23]–[24], [27] and the references therein). For diagonal elliptic systems, Giaquinta in [14] proved gradient estimates in Morrey spaces for weak solutions to linear elliptic systems with Hölder continuous coefficients. For nondiagonal elliptic systems, Wiegner in [25] considered Hölder estimates for weak solutions when the lower order items satisfy the natural growth

Received November 22, 2017.

2010 *Mathematics Subject Classification.* 35K40, 35D30.

Key words and phrases. Nondiagonal parabolic system; Hörmander's vector fields; controllable growth condition; regularity.

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11701162); Natural Science Foundation Research Project of Shaanxi Province (Grant No. 2016JQ1029); Natural Science Foundation of Education Department of Shaanxi Provincial Government(Grant No. 16JK1320); Research Fund for the Doctoral Program of Hubei University of Economics (Grant No. XJ16BS28).

ISSN 1076-9803/2018

conditions. Giaquinta and Struwe in [16] treated partial regularity for weak solutions to diagonal quasilinear parabolic systems with the natural growth conditions and got Hölder regularity.

Many scholars have studied degenerate elliptic and parabolic systems formed by Hörmander's (see [18]) vector fields. For linear diagonal elliptic systems, Di Fazio and Fanciullo in [5] obtained Morrey estimates for weak solutions, and then got Hölder estimates by Poincaré inequality and isomorphic relationship. For nonlinear nondiagonal elliptic systems, Dong and Niu in [10] considered nondiagonal quasilinear degenerate elliptic systems with the low order terms satisfy special growth conditions, they got higher Morrey estimates for weak solutions by the reverse Hölder inequality, and then obtained Hölder estimates by Morrey lemma. For diagonal parabolic systems the reader can refer to [6, 7]. And for some other studies, we quote [9, 11, 26] and references therein. Then the nondiagonal parabolic systems whether have a corresponding regular conclusion? This is the content of this paper.

In this paper, we consider quasilinear nondiagonal parabolic systems with the low order terms satisfying the controllable growth conditions. We generalized the results of [11, 16, 25]. As far as we know, when the low order terms satisfying the natural growth conditions, the weak solutions must be bounded, this condition can be abandoned when the lower order satisfying the controllable growth conditions. And due to the lack of a parabolic Poincaré inequality, it is more difficult to study. In order to solve these problems, we first introduce the average on the ball of weak solutions, and then get higher integrability for weak solutions and a parabolic Poincaré inequality. Concretely, we consider the following nondiagonal quasilinear degenerate parabolic system

$$u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta}(z,u)X_\beta u^j) = g_i(z,u,Xu) + X_\alpha^* f_i^\alpha(z,u), \tag{1.1}$$

where

$$X_\alpha = \sum_{k=1}^{n} b_{\alpha k}(x)\frac{\partial}{\partial x_k}$$

($b_{\alpha k}(x) \in C^\infty(\Omega)$) is a family of real smooth vector fields in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$ ($q \le n$) and satisfy Hörmander's condition (see Section 2) free up to the order s, $i, j = 1, 2, \dots, N$; $\alpha, \beta = 1, 2, \dots, q$;

$$X_\alpha^* = -X_\alpha + c_\alpha$$

($c_\alpha = -\sum_{k=1}^{n} \frac{\partial b_{\alpha k}}{\partial x_k} \in C^\infty(\Omega)$) is the transposed vector field of X_α. In this paper, we first establish higher integrability for weak solutions, and get a reverse Hölder inequality for weak solutions by the reverse Hölder inequality on the homogeneous space, and then obtain higher Morrey estimates, finally get Hölder estimates by isomorphic relationship.

Before stating our main results, we need several assumptions of (1.1).

(H1) Let coefficients

$$a_{ij}^{\alpha\beta}(z,u) = A^{\alpha\beta}(z)\delta_{ij} + B_{ij}^{\alpha\beta}(z,u),$$

where $A^{\alpha\beta}(z) \in VMO \cap L^\infty$, $A^{\alpha\beta}(z) = A^{\beta\alpha}(z)$ satisfy the ellipticity condition, $B_{ij}^{\alpha\beta}(z,u)$ are bounded and measurable, that is, there exist positive constants λ_0, μ_0 and $\delta, 0 < \lambda_0 \le \mu_0, 0 < \delta < 1$, such that for any $z \in Q_T$, $Q_T = \Omega \times (0,T)$, $\xi \in \mathbb{R}^{(q+1)N}$,

$$\lambda_0|\xi|^2 \le A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \mu_0|\xi|^2, \quad \lim_{R\to 0}\eta_R\left(A^{\alpha\beta}(z)\right) = 0,$$
$$\left|B_{ij}^{\alpha\beta}(z,u)\right| \le \delta\lambda_0.$$

(H2) Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$, $f_i^\alpha(z,u), g_i(z,u,Xu)$ satisfy

$$|f_i^\alpha(z,u)| \le \mu_1\left(|u|^{\frac{\gamma}{2}} + f^i(z)\right),$$
$$|g_i(z,u,Xu)| \le \mu_2\left(|Xu|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + g^i(z)\right),$$

where $f^i(z) \in L^\sigma(Q_T)\,(\sigma > Q+2)$, $g^i(z) \in L^\tau(Q_T)\,(\tau > Q+2)$, $\gamma = \frac{2(Q+2)}{Q}$. Let $g = \left(g^i\right)$, $f = \left(f^i\right)$, $\tilde{q} = \frac{2(Q+2)}{Q+4}$, the definitions of $VMO(Q_T)$, $\eta_R\left(A^{\alpha\beta}(z)\right)$, $W_2^{1,1}(Q_T, \mathbb{R}^N)$ and Q see Section 2.

If $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ and for any $\psi \in C_0^\infty(\Omega, \mathbb{R}^N)$,

$$\iint_{Q_T}\left[u_t^i\psi^i + a_{ij}^{\alpha\beta}X_\alpha\psi^i X_\beta u^j\right]dz = \iint_{Q_T}\left[g_i\psi^i + f_i^\alpha X_\alpha\psi^i\right]dz,$$

we say that u is a weak solution to (1.1).

Now the main results of the paper are the following.

Theorem 1.1. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ be a weak solution to* (1.1). *Suppose that assumptions* (H1)–(H2) *are satisfied. Then there exists a positive constant ε_0 such that for any $p \in [2, 2+\tilde{q}\varepsilon_0)$, we have*

$$u \in L_{\mathrm{loc}}^{\frac{p\gamma}{2}, Q+2-p+p\kappa}\left(Q_T, \mathbb{R}^N\right), \quad Xu \in L_{\mathrm{loc}}^{p, Q+2-p+p\kappa}\left(Q_T, \mathbb{R}^N\right),$$

where $\kappa = \min\left\{1 - \frac{\tilde{q}(Q+2)}{2\tau}, 1 - \frac{Q+2}{\sigma}\right\}$.

Theorem 1.2. *Under the assumptions in Theorem* 1.1*, we have*

$$u \in C_{\mathrm{loc}}^{\kappa}\left(Q_T, \mathbb{R}^N\right),$$

where $\kappa = \min\left\{1 - \frac{\tilde{q}(Q+2)}{2\tau}, 1 - \frac{Q+2}{\sigma}\right\}$.

This paper is organized as follows. In Section 2, we introduce Hörmander's vector fields and some related function spaces, and then recall several technical lemmas. Section 3 is devoted to establishing higher L^p regularity for gradient of weak solutions to (1.1). The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 4.

2. Preliminaries

Denote the commutator of vector fields $X_1, \dots, X_q$ by

$$X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \dots [X_{\beta_2}, X_{\beta_1}] \dots]], \quad \text{for } |\beta| = d.$$

We recall that d is the length of X_β.

Definition 2.1. If for every $x_0 \in \Omega \subset \mathbb{R}^n$, $\{X_\beta\,(x_0)\}_{|\beta|\leq s}$ spans $\mathbb{R}^n$, then we say that the system $\mathrm{X} = (X_1, \dots, X_q)$ satisfies Hörmander's condition of step s.

Following [26], we assume that Hörmander type vector fields $X_1, \dots, X_q$ are free up to the order s. For every multi-index $I = (i_1, i_2, \dots, i_k)$, the length of I is defined by $|I| = k$. If $i_k \leq q$, then we set

$$X_I = X_{i_i} X_{i_2} \dots X_{i_k}.$$

Definition 2.2 (Carnot–Carathéodory distance)**.** Let Ω be a bounded domain in $\mathbb{R}^n$. An absolutely continuous curve $\gamma : [0, T] \to \Omega$ is called a sub-unit curve with respect to $\mathrm{X} = (X_1, \cdots, X_q)$, if $\gamma'(t)$ exists for a.e. $t \in [0, T]$ and satisfies

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^{q} \langle X_j(\gamma(t)), \xi \rangle^2, \quad \text{for any } \xi \in \mathbb{R}^n.$$

We denote the length of this curve by $l_S\,(\gamma) = T$. Given any $x, y \in \Omega$, let $\Phi(x, y)$ be the collection of all sub-unit curves connecting x and y, define the Carnot–Carathéodory distance which induced by X by

$$d_{\mathrm{X}}(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

With this distance, we denote a metric ball of radius R centered at x_0 by

$$B_R(x_0) = B(x_0, R) = \{x \in \Omega : d(x_0, x) < R\}.$$

If one does not need to consider the center of the ball, then we also write B_R instead of $B(x_0, R)$.

It is well known that the doubling property (see [22]) for metric balls holds true, i.e., there exist positive constants c_D and R_D, such that for any $x_0 \in \Omega$, $0 < 2R < R_D$,

$$|B(x_0, 2R)| \leq c_D\,|B(x_0, R)|\,.$$

Furthermore, it follows that for any $R \leq R_D$ and $t \in (0, 1)$,

$$|B_{tR}| \geq c_D^{-1} t^Q\,|B_R|\,.$$

The number $Q = \log_2 c_D$ is called a locally homogeneous dimension relative to Ω. Clearly, $Q \geq n$.

As in [26], we assume that there exist two positive constants c_1 and c_2, such that

$$c_1 R^Q \leq |B_R| \leq c_2 R^Q.$$

Throughout this paper, we denote $z_0 = (x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$. A parabolic cylinder with vertex at z_0 is defined by

$$Q_R(z_0) = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right].$$

Let us denote $I_R(t_0) = \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right]$, and the parabolic boundary of Q_R by

$$\partial_p Q_R(z_0) = \left(\partial B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right]\right) \cup \left(B_R(x_0) \times \left\{t_0 - \frac{R^2}{2}\right\}\right).$$

We denote the Lebesgue measure of $B(x, R)$ in the n-dimensional space by $|B(x, R)|$, and the Lebesgue measure of $Q_R(z_0)$ in the $(n+1)$-dimensional space by $|Q_R(z_0)|$. To simplify the notations, in the sequel, $Q_R(z_0)$, $B_R(x_0)$, $I_R(t_0)$, $\sqrt{\sum_{i=1}^{q} |X_i u|^2}$ and (x, t) are written as Q_R, B_R, I_R, $|Xu|$ and z, respectively.

For any $(x, t), (y, s) \in Q_T$, we denote

$$d_p((x, t), (y, s)) = \sqrt{d_X(x, y)^2 + |t - s|}$$

as the parabolic distance in Q_T.

Definition 2.3 (Sobolev space)**.** Let m, k be 0 or 1, $1 \le p < +\infty$. The set

$$W_p^{m,k}\left(Q_T, \mathbb{R}^N\right) = \{u; X_\alpha u, \partial_t^r u \in L^p(Q_T), 0 \le |\alpha| \le m, 0 \le r \le k\}$$

is called a parabolic Sobolev space related to Hörmander's vector fields with the norm

$$\|u\|_{W_p^{m,k}} = \sum_{|\alpha| \le m} \|X_\alpha u\|_{L^p} + \sum_{r \le k} \|X_t^r u\|_{L^p}.$$

Definition 2.4 (Morrey space)**.** Let $1 \le p < +\infty$, $\lambda \ge 0$. We say that $f \in L^p(Q_T)$ belongs to the Morrey space $L^{p,\lambda}(Q_T, \mathbb{R}^N)$ if

$$\|f\|_{L^{p,\lambda}} = \sup_{z_0 \in Q_T, 0 \le \rho \le d_0} \left(\frac{1}{\rho^\lambda} \iint_{Q_T \cap Q_\rho(z_0)} |f|^p dz\right)^{\frac{1}{p}} < \infty,$$

where d_0 is the diameter of Q_T.

Definition 2.5 (Campanato space)**.** Let $1 \le p < +\infty$, $\lambda \ge 0$. A function $f \in L^p_{\mathrm{loc}}(Q_T)$ is said to belong to the Campanato space $\mathcal{L}^{p,\lambda}(Q_T, \mathbb{R}^N)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}} = [f]_{p,\lambda} + \|f\|_{L^p} < +\infty,$$

where $[f]_{p,\lambda} = \sup_{z_0 \in Q_T, 0 < \rho < d_0} \left(\frac{1}{\rho^\lambda} \iint_{Q_T \cap Q_\rho(z_0)} \left|f - f_{Q_T \cap Q_\rho(z_0)}\right|^p dz\right)^{\frac{1}{p}} < +\infty$, d_0 is the diameter of Q_T, and $f_{Q_T \cap Q_\rho(z_0)} = \frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} f(z) dz$.

Definition 2.6 (Hölder space)**.** For any $0 < k \leq 1$, the space $C^k(Q_T, \mathbb{R}^N)$ is the set of functions satisfying

$$[f]_{k,Q_T} \triangleq \sup_{z_1,z_2\in Q_T, z_1\neq z_2} \frac{|f(z_1)-f(z_2)|}{d_p(z_1,z_2)^k} < \infty.$$

We also define a norm by

$$|f|_{k,Q_T} \triangleq \sup_{Q_T} |f| + [f]_{k,Q_T}.$$

Definition 2.7 (BMO and VMO spaces)**.** For any $f \in L^1_{\text{loc}}(Q_T)$, we set

$$\eta_R(f) = \sup_{z_0\in Q, 0\leq\rho\leq R} \left(\frac{1}{|Q_T\cap Q_\rho(z_0)|} \iint_{Q_T\cap Q_\rho(z_0)} \left|f(z) - f_{Q_T\cap Q_\rho(z_0)}(z)\right| dz \right),$$

where $f_{Q_T\cap Q_\rho(z_0)} = \frac{1}{|Q_T\cap Q_\rho(z_0)|}\iint_{Q_T\cap Q_\rho(z_0)} f(z)dz$. If $\sup\limits_{R>0} \eta_R(f) < +\infty$, we say that f belongs to $BMO(Q_T)$ (Bounded Mean Oscillation). Moreover, if $\eta_R(f) \to 0$ as $R \to 0$, we say that f belongs to $VMO(Q_T)$ (Vanishing Mean Oscillation).

Lemma 2.8 (See [17])**.** *Let $H(\rho)$ be a nonnegative increasing function, and for any $0 < \rho < R \leq R_0 = dist(x_0, \partial\Omega)$,*

$$H(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\tilde{a}} + \varepsilon\right] H(R) + BR^{\tilde{b}},$$

where $A, \tilde{a}$ and $\tilde{b}$ are positive constants with $\tilde{a} > \tilde{b}$. Then there exist positive constants $\varepsilon_1 = \varepsilon_1(A, \tilde{a}, \tilde{b})$ and $c = c(A, \tilde{a}, \tilde{b})$, such that for any $\varepsilon < \varepsilon_1$, it follows

$$H(\rho) \leq c\left[\left(\frac{\rho}{R}\right)^{\tilde{b}} H(R) + B\rho^{\tilde{b}}\right].$$

Lemma 2.9 (Iterative lemma, see [4])**.** *Let $\varphi(t)$ be a nonnegative bounded function on $[T_0, T_1]$, where $T_1 > T_0 \geq 0$. Suppose that for any $s, t : T_0 \leq t < s \leq T_1$, φ satisfies*

$$\varphi(t) \leq \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B,$$

where θ, A, B and α are nonnegative constants, and $\theta < 1$. Then for any $T_0 \leq \rho < R \leq T_1$, one has

$$\varphi(\rho) \leq c\left[\frac{A}{(R-\rho)^\alpha} + B\right],$$

where c depends only on α and θ.

Lemma 2.10 (Sobolev–Poincaré inequality, see [19, 20])**.** *For any open domain Ω', $\bar{\Omega}' \subset\subset \Omega$, there exist positive constants R_0 and c, such that for any $0 < R \le R_0$, $B_R \subset \Omega$ and $u \in C^\infty(\overline{B_R})$, we have*

$$\left(\frac{1}{|B_R|}\int_{B_R}|u-u_R|^{p'}dx\right)^{\frac{1}{p'}} \le cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{1}{p}}, \tag{2.1}$$

where $1 < p < Q$, $1 \le p' \le \frac{pQ}{Q-p}$, $u_R = \frac{1}{|B_R|}\int_{B_R} u(x)dx$, R_0 and c depend on Ω' and Ω. In particular, if $u \in C_0^\infty(\overline{B_R})$, then

$$\left(\frac{1}{|B_R|}\int_{B_R}|u|^{p'}dx\right)^{\frac{1}{p'}} \le cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{1}{p}}. \tag{2.2}$$

Lemma 2.11 (Reverse Hölder inequality, see [13])**.** *Let $\hat{g}, \hat{f}$ be nonnegative on Q_T and satisfy*

$$\hat{g} \in L^{\hat{q}}(Q_T) \text{ and } \hat{f} \in L^{q'}(Q_T), \quad 1 < \hat{q} < q'.$$

Assume that there exist constants $\hat{b} > 1$ and $\hat{\theta}$ such that for any $Q_{2R} \subset\subset Q_T$ the following inequality holds

$$\frac{1}{|Q_R|}\iint_{Q_R}\hat{g}^{\hat{q}}dz \le \hat{b}\left[\left(\frac{1}{|Q_{4R/3}|}\iint_{Q_{4R/3}}\hat{g}dz\right)^{\hat{q}} + \frac{1}{|Q_{4R/3}|}\iint_{Q_{4R/3}}\hat{f}^{\hat{q}}dz\right] + \hat{\theta}\frac{1}{|Q_{4R/3}|}\iint_{Q_{4R/3}}\hat{g}^{\hat{q}}dz,$$

Then there exist positive constants ε_0 and $\theta_0 = \theta_0(\hat{q}, Q_T)$ such that if $\hat{\theta} < \theta_0$, then for any $\hat{p} \in [\hat{q}, \hat{q}+\varepsilon_0)$, $\hat{g} \in L^{\hat{p}}_{\text{loc}}(Q_T)$, and

$$\left(\frac{1}{|Q_R|}\iint_{Q_R}\hat{g}^{\hat{p}}dz\right)^{\frac{1}{\hat{p}}} \le c\left[\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}\hat{g}^{\hat{q}}dz\right)^{\frac{1}{\hat{q}}} + \left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}\hat{f}^{\hat{p}}dz\right)^{\frac{1}{\hat{p}}}\right],$$

where c and ε_0 depend on $\hat{b}, \hat{\theta}, \hat{q}$ and Q.

Lemma 2.12 (See [12])**.** *The spaces*

$$\mathcal{L}^{2,Q+2+2\kappa}(Q_T, \mathbb{R}^N) \quad \text{and} \quad C^\kappa(\bar{Q}_T, \mathbb{R}^N)$$

($0 < \kappa < 1$) are topologically and algebraically isomorphic.

3. Higher integrability

We first introduce two cutoff functions $\xi(x)$ and $\eta(t)$(see [8]) such that for any $0 < \rho < R$, $B_R \subset \Omega$,

$$\xi(x) \in C_0^\infty(B_R), \quad 0 \le \xi \le 1, \quad |X\xi| \le \frac{C}{R-\rho} \quad \text{and} \quad \xi = 1 \text{ in } B_\rho;$$

$$\eta(t)=\begin{cases}\frac{2t-2\left(t_0-\frac{R^2}{2}\right)}{R^2-\rho^2}, t\in\left(t_0-\frac{R^2}{2}, t_0-\frac{\rho^2}{2}\right),\\ 1, \quad t\in\left[t_0-\frac{\rho^2}{2}, t_0+\frac{R^2}{2}\right].\end{cases}$$

Setting $\frac{1}{|B_R|}\int_{B_R}\xi^2dx=N_1$, we denote the average of $u(x,t)$ on B_R by

$$\bar{u}(t)=\left(\int_{B_R}\xi^2dx\right)^{-1}\int_{B_R}u\xi^2dx=\frac{1}{N_1|B_R|}\int_{B_R}u\xi^2dx.$$

Lemma 3.1. *Let $u\in W_2^{1,1}(Q_T,\mathbb{R}^N)$ be a weak solution to (1.1). Then $u\in L^{\gamma}_{\mathrm{loc}}(Q_T)$, and for any $Q_R\subset\subset Q_T$, we have*

$$\iint_{Q_R}|u|^{\gamma}dz\le c\sup_{I_R}\left(\int_{B_R}|u|^2dx\right)^{\frac{2}{Q}}\iint_{Q_R}|Xu|^2dz+c|Q_R|. \tag{3.1}$$

Proof. By Young's inequality and Hölder's inequality,

$$\begin{aligned}|\bar{u}(t)|&=\left|\frac{1}{N_1|B_R|}\int_{B_R}u\xi^2dx\right|\\&\le\frac{1}{N_1|B_R|}\left(\varepsilon\int_{B_R}|u|^2\xi^2dx+c_\varepsilon\int_{B_R}\xi^2dx\right)\\&\le\frac{\varepsilon}{N_1|B_R|}\int_{B_R}|u|^2\xi^2dx+c_\varepsilon,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\int_{B_R}|u|^2\xi^2dx&\le\left(\int_{B_R}|u|^{\gamma}dx\right)^{\frac{2}{\gamma}}\left(\int_{B_R}\xi^{Q+2}dx\right)^{\frac{\gamma-2}{\gamma}}\\&\le(N_1|B_R|)^{\frac{\gamma-2}{\gamma}}\left(\int_{B_R}|u|^{\gamma}dx\right)^{\frac{2}{\gamma}}.\end{aligned} \tag{3.3}$$

Then by (3.2) and (3.3), we get

$$\begin{aligned}&\iint_{Q_R}|\bar{u}(t)|^{\gamma}dz\le|B_R|\int_{I_R}|\bar{u}(t)|^{\gamma}dt\\&\le|B_R|\int_{I_R}\left(\frac{\varepsilon}{N_1|B_R|}\int_{B_R}|u|^2\xi^2dx+c_\varepsilon\right)^{\gamma}dt\\&\le\frac{\varepsilon}{{N_1}^{\gamma}|B_R|^{\gamma-1}}\int_{I_R}\left(\int_{B_R}|u|^2\xi^2dx\right)^{\gamma}dt+c_\varepsilon|Q_R|\\&\le\frac{\varepsilon}{{N_1}^{\gamma}|B_R|^{\gamma-1}}\sup_{I_R}\left(\int_{B_R}|u|^2dx\right)^{\frac{\gamma}{2}}\int_{I_R}\left(\int_{B_R}|u|^2\xi^2dx\right)^{\frac{\gamma}{2}}dt+c_\varepsilon|Q_R|\\&\le\frac{\varepsilon}{{N_1}^{\gamma}|B_R|^{\gamma-1}}\sup_{I_R}\left(\int_{B_R}|u|^2dx\right)^{\frac{\gamma}{2}}\\&\qquad\cdot\int_{I_R}\left((N_1|B_R|)^{\frac{\gamma-2}{\gamma}}\left(\int_{B_R}|u|^{\gamma}dx\right)^{\frac{2}{\gamma}}\right)^{\frac{\gamma}{2}}dt+c_\varepsilon|Q_R|\end{aligned} \tag{3.4}$$

$$\leq \frac{\varepsilon}{{N_1}^{\frac{\gamma+2}{2}} |B_R|^{\frac{\gamma}{2}}} \sup_{I_R} \left(\int_{B_R} |u|^2 dx \right)^{\frac{\gamma}{2}} \iint_{Q_R} |u|^\gamma dz + c_\varepsilon |Q_R| .$$

By (2.1) and Hölder's inequality, one has

$$\begin{aligned}
(3.5) \qquad \int_{B_R} |u-\bar{u}(t)|^{\frac{2(Q+1)}{Q-1}} dx &= \int_{B_R} \left(|u-\bar{u}(t)|^{\frac{2(Q+1)}{Q}} \right)^{\frac{Q}{Q-1}} dx \\
&\leq \left(\int_{B_R} \left| X \left(|u-\bar{u}(t)|^{\frac{2(Q+1)}{Q}} \right) \right| dx \right)^{\frac{Q}{Q-1}} \\
&\leq \left(\frac{2(Q+1)}{Q} \int_{B_R} \left| |u-\bar{u}(t)|^{\frac{Q+2}{Q}} Xu \right| dx \right)^{\frac{Q}{Q-1}} \\
&\leq c \left(\left(\int_{B_R} |u-\bar{u}(t)|^\gamma dx \right)^{\frac{1}{2}} \left(\int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}} \right)^{\frac{Q}{Q-1}} .
\end{aligned}$$

By Hölder's inequality and (3.5),

$$\begin{aligned}
&\iint_{Q_R} |u-\bar{u}(t)|^\gamma dz \\
&\leq \int_{I_R} \left(\int_{B_R} |u-\bar{u}(t)|^2 \right)^{\frac{1}{Q}} \left(\int_{B_R} |u-\bar{u}(t)|^{\frac{2(Q+1)}{Q-1}} \right)^{\frac{Q-1}{Q}} dt \\
&\leq c \int_{I_R} \left(\int_{B_R} |u-\bar{u}(t)|^2 \right)^{\frac{1}{Q}} \left(\int_{B_R} |u-\bar{u}(t)|^\gamma dx \right)^{\frac{1}{2}} \left(\int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \sup_{I_R} \left(\int_{B_R} |u|^2 dx \right)^{\frac{1}{Q}} \int_{I_R} \left(\int_{B_R} |u-\bar{u}(t)|^\gamma dx \right)^{\frac{1}{2}} \left(\int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq c \sup_{I_R} \left(\int_{B_R} |u|^2 dx \right)^{\frac{1}{Q}} \left(\iint_{Q_R} |u-\bar{u}(t)|^\gamma dz \right)^{\frac{1}{2}} \left(\iint_{Q_R} |Xu|^2 dz \right)^{\frac{1}{2}} .
\end{aligned}$$

So we have

$$(3.6) \qquad \iint_{Q_R} |u-\bar{u}(t)|^\gamma dz \leq c \sup_{I_R} \left(\int_{B_R} |u|^2 dx \right)^{\frac{2}{Q}} \iint_{Q_R} |Xu|^2 dz.$$

And by (3.4) and (3.6),

$$\begin{aligned}
&\iint_{Q_R} |u|^\gamma dz \\
&\leq c \iint_{Q_R} |u-\bar{u}(t)|^\gamma dz + c \iint_{Q_R} |\bar{u}(t)|^\gamma dz \\
&\leq c \sup_{I_R} \left(\int_{B_R} |u|^2 dx \right)^{\frac{2}{Q}} \iint_{Q_R} |Xu|^2 dz
\end{aligned}$$

$$+\frac{\varepsilon}{N_1^{\frac{\gamma+2}{2}}|B_R|^{\frac{\gamma}{2}}}\sup_{I_R}\left(\int_{B_R}|u|^2dx\right)^{\frac{\gamma}{2}}\iint_{Q_R}|u|^{\gamma}dz+c_{\varepsilon}\left|Q_R\right|.$$

Choosing ε small enough, then we get

$$\iint_{Q_R}|u|^{\gamma}dz\le c\sup_{I_R}\left(\int_{B_R}|u|^2dx\right)^{\frac{2}{Q}}\iint_{Q_R}|Xu|^2dz+c\left|Q_R\right|. \qquad \square$$

Lemma 3.2. *Let $u\in W_2^{1,1}(Q_T,\mathbb{R}^N)$ be a weak solution to* (1.1)*. Then for any $0<\rho<R$, $Q_R\subset\subset Q_T$, we have*

$$\begin{aligned}\sup_{I_\rho}\int_{B_\rho}|u-\bar{u}(t)|^2dx+\iint_{Q_\rho}|Xu|^2dz \\ \le\frac{c}{(R-\rho)^2}\iint_{Q_R}|u-\bar{u}(t)|^2dz+c\iint_{Q_R}\left(|u|^{\gamma}+|f|^2+|g|^{\tilde{q}}\right)dz.\end{aligned}\tag{3.7}$$

Proof. Let $B_\rho\subset B_R\subset\Omega$, multiplying both sides of (1.1) by the test function $(u-\bar{u}(t))\,\xi^2(x)\eta(t)$, and integrating on

$$Q'_R=B_R(x_0)\times\left(t_0-\frac{R^2}{2},s\right]$$

($s\le t_0+\frac{R^2}{2}$), we get

$$\begin{aligned}&\iint_{Q'_R}\left[u_t^i+X_\alpha^*\left(a_{ij}^{\alpha\beta}X_\beta u^j\right)\right]\left(u^i-\bar{u}(t)\right)\xi^2\eta dz \\ &=\iint_{Q'_R}\left[g_i+X_\alpha^*f_i^\alpha\right]\left(u^i-\bar{u}(t)\right)\xi^2\eta dz.\end{aligned}\tag{3.8}$$

By (H1), one has

$$\begin{aligned}&\iint_{Q'_R}\left[u_t^i+X_\alpha^*\left(a_{ij}^{\alpha\beta}X_\beta u^j\right)\right]\left(u^i-\bar{u}(t)\right)\xi^2\eta dz\\
&=\iint_{Q'_R}\left[u_t^i\left(u^i-\bar{u}(t)\right)\xi^2\eta+a_{ij}^{\alpha\beta}X_\beta u^jX_\alpha\left(\left(u^i-\bar{u}(t)\right)\xi^2\eta\right)\right]dz\\
&=\iint_{Q'_R}\Big[u_t^i\left(u^i-\bar{u}(t)\right)\xi^2\eta+a_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^iX_\beta u^j\\
&\qquad+2a_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j\Big]dz\\
&=\iint_{Q'_R}\Big[\left(\frac{1}{2}\left|u^i-\bar{u}(t)\right|^2\eta\right)_t\xi^2-\frac{1}{2}\left|u^i-\bar{u}(t)\right|^2\xi^2\eta_t\\
&\qquad+A^{\alpha\beta}\delta_{ij}\xi^2\eta X_\alpha u^iX_\beta u^j\Big]dz\\
&\quad+\iint_{Q'_R}B_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^iX_\beta u^j+2A^{\alpha\beta}\delta_{ij}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^jdz\end{aligned}$$

$$+\iint_{Q'_R} 2B_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j dz,$$

and

$$\begin{aligned}
&\iint_{Q'_R}\left[g_i+X_\alpha^* f_i^\alpha\right]\left(u^i-\bar{u}(t)\right)\xi^2\eta dz\\
&=\iint_{Q'_R}\left[g_i\left(u^i-\bar{u}(t)\right)\xi^2\eta+f_i^\alpha X_\alpha\left(\left(u^i-\bar{u}(t)\right)\xi^2\eta\right)\right]dz\\
&=\iint_{Q'_R}\left[g_i\left(u^i-\bar{u}(t)\right)\xi^2\eta+f_i^\alpha\xi^2\eta X_\alpha u^i+2\xi\eta\left(u^i-\bar{u}(t)\right)f_i^\alpha X_\alpha\xi\right]dz.
\end{aligned}$$

By the above, (3.8) can be written as

(3.9)

$$\begin{aligned}
&\iint_{Q'_R}\left[\left(\frac{1}{2}\left|u^i-\bar{u}(t)\right|^2\eta\right)_t\xi^2+A^{\alpha\beta}\delta_{ij}\xi^2\eta X_\alpha u^i X_\beta u^j\right]dz\\
&=\iint_{Q'_R}\left[\frac{1}{2}\left|u^i-\bar{u}(t)\right|^2\xi^2\eta_t-B_{ij}^{\alpha\beta}\xi^2\eta X_\alpha u^i X_\beta u^j\right]\\
&\quad-2\iint_{Q'_R}A^{\alpha\beta}\delta_{ij}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j dz\\
&\quad-2\iint_{Q'_R}B_{ij}^{\alpha\beta}\left(u^i-\bar{u}(t)\right)\xi\eta X_\alpha\xi X_\beta u^j dz\\
&\quad+\iint_{Q'_R}\left[g_i\left(u^i-\bar{u}(t)\right)\xi^2\eta+f_i^\alpha\xi^2\eta X_\alpha u^i+2\xi\eta\left(u^i-\bar{u}(t)\right)f_i^\alpha X_\alpha\xi\right]dz.
\end{aligned}$$

By (H2) and Young's inequality,

$$\begin{aligned}
(3.10)\quad &\iint_{Q'_R}g_i\left(u^i-\bar{u}(t)\right)\xi^2\eta dz\\
&\le\mu_2\iint_{Q'_R}\left(|Xu|^{2(1-\frac{1}{\gamma})}+|u|^{\gamma-1}+g^i(z)\right)\left(u^i-\bar{u}(t)\right)\xi^2\eta dz\\
&\le\varepsilon\iint_{Q'_R}|Xu|^2\xi^2\eta dz+c_\varepsilon\iint_{Q'_R}|u-\bar{u}(t)|^\gamma\xi^2\eta dz\\
&\quad+c_\varepsilon\iint_{Q'_R}|u|^\gamma\xi^2\eta dz+c_\varepsilon\iint_{Q'_R}|g|^{\tilde{q}}\xi^2\eta dz,
\end{aligned}$$

$$\begin{aligned}
(3.11)\quad &\iint_{Q'_R}\left[f_i^\alpha\xi^2\eta X_\alpha u^i+2\xi\eta\left(u^i-\bar{u}(t)\right)f_i^\alpha X_\alpha\xi\right]dz\\
&\le\mu_1\iint_{Q'_R}\left(|u|^{\frac{\gamma}{2}}+f^i(z)\right)\xi^2\eta X_\alpha u^i dz\\
&\quad+2\mu_1\iint_{Q'_R}\xi\eta\left(u-\bar{u}(t)\right)\left(|u|^{\frac{\gamma}{2}}+f^i(z)\right)X_\alpha\xi dz
\end{aligned}$$

$$\leq 2\varepsilon \iint_{Q'_R} |Xu|^2\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |u|^\gamma \xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |f|^2\xi^2\eta dz$$
$$+ 2\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2|X\xi|^2\eta dz.$$

Inserting (3.10) and (3.11) into (3.9), and by (H1) and Young's inequality, we get

$$\begin{aligned}
&\iint_{Q'_R} \left(\frac{1}{2}|u-\bar{u}(t)|^2\eta\right)_t \xi^2 dz + \lambda_0 \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\leq \iint_{Q'_R} \frac{1}{2}|u-\bar{u}(t)|^2\xi^2\eta_t dz + \delta\lambda_0 \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\quad + c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2|X\xi|^2\eta dz + 5\varepsilon \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\quad + c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^\gamma\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |u|^\gamma\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |g|^{\tilde{q}}\xi^2\eta dz \\
&\quad + c_\varepsilon \iint_{Q'_R} |f|^2\xi^2\eta dz + 2\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2|X\xi|^2\eta dz \\
&\leq \iint_{Q'_R} |u-\bar{u}(t)|^2 \left(\frac{1}{2}\xi^2\eta_t + c_\varepsilon|X\xi|^2\eta + 2\varepsilon|X\xi|^2\eta\right) dz \\
&\quad + (\delta\lambda_0 + 5\varepsilon) \iint_{Q'_R} |Xu|^2\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^\gamma\xi^2\eta dz \\
&\quad + c_\varepsilon \iint_{Q'_R} |u|^\gamma\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} \left(|f|^2 + |g|^{\tilde{q}}\right)\xi^2\eta dz.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{B_R} \frac{1}{2}|u-\bar{u}(t)|^2\xi^2\eta dx + (\lambda_0 - \delta\lambda_0 - 5\varepsilon) \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\leq \iint_{Q'_R} |u-\bar{u}(t)|^2 \left(\frac{1}{2}\xi^2\eta_t + c_\varepsilon|X\xi|^2\eta + 2\varepsilon|X\xi|^2\eta\right) dz \\
&\quad + c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^\gamma\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |u|^\gamma\xi^2\eta dz \\
&\quad + c_\varepsilon \iint_{Q'_R} \left(|f|^2 + |g|^{\tilde{q}}\right)\xi^2\eta dz.
\end{aligned}$$

Choosing ε small enough such that $\lambda_0 - \delta\lambda_0 - 5\varepsilon > 0$, then by properties of ξ, η, we get

$$\begin{aligned}
&\sup_{I_\rho} \int_{B_\rho} |u-\bar{u}(t)|^2 dx + \iint_{Q_\rho} |Xu|^2 dz \\
&\leq c \iint_{Q'_R} |u-\bar{u}(t)|^2 \left(\frac{\xi^2}{R^2-\rho^2} + \frac{c\eta}{(R-\rho)^2} + \frac{c\eta}{(R-\rho)^2}\right) dz
\end{aligned}$$

$$
\begin{aligned}
&+ c \iint_{Q'_R} \left(|u - \bar{u}(t)|^\gamma + |u|^\gamma + |f|^2 + |g|^{\tilde{q}} \right) \xi^2 \eta dz \\
&\leq \frac{c}{(R-\rho)^2} \iint_{Q'_R} |u - \bar{u}(t)|^2 dz + c \iint_{Q'_R} \left(|u|^\gamma + |f|^2 + |g|^{\tilde{q}} \right) \xi^2 \eta dz. \quad \square
\end{aligned}
$$

Lemma 3.3. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ be a weak solution to* (1.1)*. Then for any $0 < \rho < R$, $Q_R \subset\subset Q_T$, we have*

$$
\begin{aligned}
&\iint_{Q_\rho} |u|^2 dz \\
&\leq \frac{cR^4}{(R-\rho)^2} \iint_{Q_R} |Xu|^2 dz + cR^2 \iint_{Q_R} \left(|u|^\gamma + |f|^2 + |g|^{\tilde{q}} \right) dz.
\end{aligned} \tag{3.12}
$$

Proof. Let $B_\rho \subset B_R \subset \Omega$, multiplying both sides of (1.1) by the test function $u\xi^2(x)\eta(t)$ and integrating on

$$
Q'_R = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, s \right]
$$

($s \leq t_0 + \frac{R^2}{2}$), one has

(3.13)

$$
\begin{aligned}
&\iint_{Q'_R} \left(\frac{1}{2} |u^i|^2 \eta \right)_t \xi^2 dz \\
&= \iint_{Q'_R} \frac{1}{2} |u^i|^2 \xi^2 \eta_t dz - \iint_{Q'_R} A^{\alpha\beta} \delta_{ij} \xi^2 \eta X_\alpha u^i X_\beta u^j dz \\
&\quad - \iint_{Q'_R} B_{ij}^{\alpha\beta} \xi^2 \eta X_\alpha u^i X_\beta u^j dz \\
&\quad - 2 \iint_{Q'_R} A^{\alpha\beta} \delta_{ij} u^i \xi \eta X_\alpha \xi X_\beta u^j dz - 2 \iint_{Q'_R} B_{ij}^{\alpha\beta} u^i \xi \eta X_\alpha \xi X_\beta u^j dz \\
&\quad + \iint_{Q'_R} \left[g_i u^i \xi^2 \eta + f_i^\alpha \xi^2 \eta X_\alpha u^i + 2\xi \eta u^i f_i^\alpha X_\alpha \xi \right] dz.
\end{aligned}
$$

By (H2) and Young's inequality,

$$
\begin{aligned}
&\iint_{Q'_R} g_i u^i \xi^2 \eta dz \\
&\leq c_\varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz + (2\varepsilon + \mu_2) \iint_{Q'_R} |u|^\gamma \xi^2 \eta dz + c_\varepsilon \iint_{Q'_R} |g|^{\tilde{q}} \xi^2 \eta dz,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{Q'_R} \left[f_i^\alpha \xi^2 \eta X_\alpha u^i + 2\xi \eta u^i f_i^\alpha X_\alpha \xi \right] dz \\
&\leq \mu_1 \iint_{Q'_R} \left(|u|^{\frac{\gamma}{2}} + f^i(z) \right) \xi^2 \eta X_\alpha u^i dz
\end{aligned}
$$

$$
\begin{aligned}
&+2\mu_1 \iint_{Q'_R} \xi\eta u^i \left(|u|^{\frac{\gamma}{2}} + f^i(z)\right) X_\alpha \xi dz \\
&\le 2\varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz + c_\varepsilon \iint_{Q'_R} |u|^\gamma \xi^2 \eta dz + c_\varepsilon \iint_{Q'_R} |f|^2 \xi^2 \eta dz \\
&+ 2\varepsilon \iint_{Q'_R} |u|^2 |X\xi|^2 \eta dz.
\end{aligned}
$$

Putting the above into (3.13) and by (H1), we get

$$
\begin{aligned}
&\int_{B_R} \frac{1}{2}|u|^2\xi^2\eta dx \\
&\le \iint_{Q'_R} \frac{1}{2}|u|^2\xi^2\eta_t dz + \mu_0 \iint_{Q'_R} |Xu|^2\xi^2\eta dz + \delta\lambda_0 \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\quad + 2\varepsilon \iint_{Q'_R} |u|^2|X\xi|^2\eta dz + 3c_\varepsilon \iint_{Q'_R} |Xu|^2\xi^2\eta dz \\
&\quad + (2\varepsilon + \mu_2 + c_\varepsilon) \iint_{Q'_R} |u|^\gamma \xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |g|^{\tilde{q}}\xi^2\eta dz \\
&\quad + 2\varepsilon \iint_{Q'_R} |Xu|^2\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} |f|^2\xi^2\eta dz + 2\varepsilon \iint_{Q'_R} |u|^2|X\xi|^2\eta dz \\
&\le \iint_{Q'_R} |u|^2 \left(\frac{1}{2}\xi^2\eta_t + 4\varepsilon|X\xi|^2\eta\right) dz \\
&\quad + \iint_{Q'_R} |Xu|^2 \left(\mu_0 + \delta\lambda_0 + 3c_\varepsilon + 2\varepsilon\right)\xi^2\eta dz \\
&\quad + \iint_{Q'_R} |u|^\gamma \left(2\varepsilon + \mu_2 + c_\varepsilon\right)\xi^2\eta dz + c_\varepsilon \iint_{Q'_R} \left(|f|^2 + |g|^{\tilde{q}}\right)\xi^2\eta dz.
\end{aligned}
$$

By properties of ξ, η,

$$
\begin{aligned}
&\iint_{Q_\rho} |u|^2 dz \le |I_\rho| \sup_{I_\rho} \int_{B_\rho} |u|^2 dx \\
&\le \rho^2 \iint_{Q_R} |u|^2 \left(\xi^2\eta_t + 8\varepsilon|X\xi|^2\eta\right) dz \\
&\quad + 2\rho^2 \iint_{Q_R} |Xu|^2 \left(\mu_0 + \delta\lambda_0 + 3c_\varepsilon + 2\varepsilon\right)\xi^2\eta dz \\
&\quad + 2\rho^2 \iint_{Q_R} |u|^\gamma \left(2\varepsilon + \mu_2 + c_\varepsilon\right)\xi^2\eta dz + 2c_\varepsilon\rho^2 \iint_{Q_R} \left(|f|^2 + |g|^{\tilde{q}}\right)\xi^2\eta dz \\
&\le \iint_{Q_R} |u|^2 \left(\frac{2\rho^2\xi^2}{R^2-\rho^2} + \frac{8c\varepsilon\rho^2\eta}{(R-\rho)^2}\right) dz + c_\varepsilon \frac{\rho^2(R-\rho)^2}{(R-\rho)^2} \iint_{Q_R} |Xu|^2\xi^2\eta dz \\
&\quad + 2\rho^2 \iint_{Q_R} |u|^\gamma \left(2\varepsilon + \mu_2 + c_\varepsilon\right)\xi^2\eta dz + 2c_\varepsilon\rho^2 \iint_{Q_R} \left(|f|^2 + |g|^{\tilde{q}}\right)\xi^2\eta dz
\end{aligned}
$$

$$\leq \theta \iint_{Q_R} |u|^2 dz + \frac{c_\varepsilon R^4}{(R-\rho)^2} \iint_{Q_R} |Xu|^2 dz$$
$$+ c_\varepsilon R^2 \iint_{Q_R} \left(|u|^\gamma + |f|^2 + |g|^{\tilde{q}}\right) dz,$$

where $\theta = \frac{2\rho^2\xi^2}{R^2-\rho^2} + \frac{8c\varepsilon\rho^2\eta}{(R-\rho)^2}$. By choosing ε small enough such that $\theta \in (0,1)$, then by Lemma 2.9 we obtain (3.12). □

Theorem 3.4. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ be a weak solution to (1.1). Then there exists a positive constant ε_0 such that for any $p \in [2, 2+\tilde{q}\varepsilon_0)$, we have $u \in L_{\mathrm{loc}}^{\frac{p\gamma}{2}}(Q_T)$, $Xu \in L_{\mathrm{loc}}^p(Q_T)$, and for any $Q_{2R} \subset\subset Q_T$,*

$$\frac{1}{|Q_R|} \iint_{Q_R} \left(|Xu|^2 + |u|^\gamma\right)^{\frac{p}{2}} dz$$
$$\leq c\Bigg[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(|Xu|^2 + |u|^\gamma\right) dz\right)^{\frac{p}{2}}$$
$$+ \frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(|f|^2 + |g|^{\tilde{q}} + 1\right)^{\frac{p}{2}} dz\Bigg].$$

Proof. By (3.7) and (2.1),

$$\sup_{I_{4R/5}} \left(\int_{B_{4R/5}} |u - \bar{u}(t)|^2 dx\right)^{\frac{1}{2}} \tag{3.14}$$
$$\leq \left(\frac{c}{R^2} \iint_{Q_R} |u - \bar{u}(t)|^2 dz + c \iint_{Q_R} |u|^\gamma dz\right)^{\frac{1}{2}}$$
$$+ c\left(\iint_{Q_R} \left(|f|^2 + |g|^{\tilde{q}}\right) dz\right)^{\frac{1}{2}}$$
$$\leq c\left(\iint_{Q_R} \left(|Xu|^2 + |u|^\gamma\right) dz\right)^{\frac{1}{2}} + c\left(\iint_{Q_R} \left(|f|^2 + |g|^{\tilde{q}}\right) dz\right)^{\frac{1}{2}}.$$

By Hölder's inequality, (2.1) and (2.2),

$$\int_{I_{4R/5}} \left(\int_{B_{4R/5}} |u - \bar{u}(t)|^2 dx\right)^{\frac{1}{2}} dt \tag{3.15}$$
$$\leq \int_{I_R} \left(\int_{B_R} |u - \bar{u}(t)|^{\tilde{q}} dx\right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_R} |u - \bar{u}(t)|^\gamma dx\right)^{\frac{1}{2\gamma}} dt$$
$$\leq cR^{\frac{1}{\tilde{q}}} \int_{I_R} \left(\int_{B_R} |Xu|^{\tilde{q}} dx\right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_R} |Xu|^2 dx\right)^{\frac{1}{4}} dt$$

$$\le cR^{\frac{1}{\tilde q}}\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{1}{2\tilde q}}\left(\int_{I_R}\left(\int_{B_R}|Xu|^2dx\right)^{\frac12\frac{\tilde q}{2\tilde q-1}}dt\right)^{\frac{2\tilde q-1}{2\tilde q}}$$
$$\le cR^{\frac32}\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{1}{2\tilde q}}\left(\iint_{Q_R}|Xu|^2dz\right)^{\frac14}.$$

By (3.14) and (3.15),

(3.16)
$$\iint_{Q_{4R/5}}|u-\bar u(t)|^2dz=\int_{I_{4R/5}}\left(\int_{B_{4R/5}}|u-\bar u(t)|^2dx\right)dt$$
$$\le \sup_{I_{4R/5}}\left(\int_{B_{4R/5}}|u-\bar u(t)|^2dx\right)^{\frac12}\cdot\left(\int_{I_{4R/5}}\left(\int_{B_{4R/5}}|u-\bar u(t)|^2dx\right)^{\frac12}dt\right)$$
$$\le cR^{\frac32}\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{1}{2\tilde q}}\left(\iint_{Q_R}\left(|Xu|^2+|u|^\gamma\right)dz\right)^{\frac34}$$
$$+cR^{\frac32}\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{1}{2\tilde q}}\left(\iint_{Q_R}|Xu|^2dz\right)^{\frac14}$$
$$\cdot\left(\iint_{Q_R}\left(|f|^2+|g|^{\tilde q}\right)dz\right)^{\frac12}$$
$$\equiv A+B.$$

By Young's inequality,

$$A\le c_\varepsilon\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{2}{\tilde q}}+\varepsilon R^2\iint_{Q_R}\left(|Xu|^2+|u|^\gamma\right)dz,$$
$$B\le \varepsilon R\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{1}{\tilde q}}\left(\iint_{Q_R}\left(|Xu|^2+|u|^\gamma\right)dz\right)^{\frac12}$$
$$+c_\varepsilon R^2\iint_{Q_R}\left(|f|^2+|g|^{\tilde q}\right)dz$$
$$\le c_\varepsilon\left(\iint_{Q_R}|Xu|^{\tilde q}dz\right)^{\frac{2}{\tilde q}}+\varepsilon R^2\iint_{Q_R}\left(|Xu|^2+|u|^\gamma\right)dz$$
$$+c_\varepsilon R^2\iint_{Q_R}\left(|f|^2+|g|^{\tilde q}\right)dz.$$

Inserting the above two into (3.16), we get

(3.17) $$\iint_{Q_{4R/5}}|u-\bar u(t)|^2dz$$

$$
\begin{aligned}
&\leq c_\varepsilon\Big(\iint_{Q_R}|Xu|^{\tilde q}dz\Big)^{\frac{2}{\tilde q}}+\varepsilon R^2\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+c_\varepsilon R^2\iint_{Q_R}\Big(|f|^2+|g|^{\tilde q}\Big)dz.
\end{aligned}
$$

By (3.7) and (3.17),

$$
\begin{aligned}
&\frac{1}{|Q_{3R/4}|}\iint_{Q_{3R/4}}|Xu|^2dz\\
&\leq\frac{c}{R^2}\frac{1}{|Q_{3R/4}|}\iint_{Q_{4R/5}}|u-\bar u(t)|^2dz\\
&\quad+\frac{c}{|Q_{3R/4}|}\iint_{Q_{4R/5}}\Big(|u|^\gamma+|f|^2+|g|^{\tilde q}\Big)dz\\
&\leq\frac{c_\varepsilon|Q_R|^{\frac{2}{\tilde q}}}{|Q_{3R/4}|\,R^2}\Big(\frac{1}{|Q_R|}\iint_{Q_R}|Xu|^{\tilde q}dz\Big)^{\frac{2}{\tilde q}}\\
&\quad+\frac{\varepsilon}{|Q_{3R/4}|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+\frac{c_\varepsilon}{|Q_{3R/4}|}\iint_{Q_R}\Big(|u|^\gamma+|f|^2+|g|^{\tilde q}\Big)dz\\
&\leq c_\varepsilon\Big(\frac{1}{|Q_R|}\iint_{Q_R}|Xu|^{\tilde q}dz\Big)^{\frac{2}{\tilde q}}+\frac{\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+\frac{c_\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|u|^\gamma+|f|^2+|g|^{\tilde q}\Big)dz.
\end{aligned}
\tag{3.18}
$$

By (3.1) and (3.18),

$$
\begin{aligned}
&\frac{1}{|Q_{3R/4}|}\iint_{Q_{3R/4}}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\leq c_\varepsilon\Big(\frac{1}{|Q_R|}\iint_{Q_R}|Xu|^{\tilde q}dz\Big)^{\frac{2}{\tilde q}}+\frac{\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+\frac{c_\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|u|^\gamma+|f|^2+|g|^{\tilde q}\Big)dz\\
&\leq c_\varepsilon\Big(\frac{1}{|Q_R|}\iint_{Q_R}|Xu|^{\tilde q}dz\Big)^{\frac{2}{\tilde q}}+\frac{\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+\frac{c_\varepsilon}{|Q_R|}\sup_{I_R}\Big(\int_{B_R}|u|^2dz\Big)^{\frac{2}{Q}}\iint_{Q_R}|Xu|^2dz+c_\varepsilon\\
&\quad+\frac{c_\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|f|^2+|g|^{\tilde q}\Big)dz
\end{aligned}
$$

$$
\begin{aligned}
&\le c_\varepsilon\Big(\frac{1}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)^{\frac{\tilde q}{2}}dz\Big)^{\frac{2}{\tilde q}}\\
&\quad+\left[\varepsilon+c_\varepsilon\sup_{I_R}\left(\int_{B_R}|u|^2dz\right)^{\frac{2}{Q}}\right]\frac{1}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)dz\\
&\quad+\frac{c_\varepsilon}{|Q_R|}\iint_{Q_R}\Big(|f|^2+|g|^{\tilde q}+1\Big)dz.
\end{aligned}
$$

If $R\to 0$, $\sup\limits_{I_R}\left(\int_{B_R}|u|^2dz\right)^{\frac{2}{Q}}\to 0$, so when R small, we can choose ε small enough such that $\theta=\varepsilon+c_\varepsilon\sup\limits_{I_R}\left(\int_{B_R}|u|^2dz\right)^{\frac{2}{Q}}\in(0,1)$. Let

$$
\begin{aligned}
\hat g&=\Big(|Xu|^2+|u|^\gamma\Big)^{\frac{\tilde q}{2}},\\
\hat f&=\Big(|f|^2+|g|^{\tilde q}+1\Big)^{\frac{\tilde q}{2}},\\
\hat q&=\frac{2}{\tilde q}=\frac{Q+4}{Q+2}>1,
\end{aligned}
$$

then the above can be written as

$$
\begin{aligned}
&\frac{1}{|Q_{3R/4}|}\iint_{Q_{3R/4}}\hat g^{\hat q}dz\\
&\le c\left[\Big(\frac{1}{|Q_R|}\iint_{Q_R}\hat g dz\Big)^{\hat q}+\frac{1}{|Q_R|}\iint_{Q_R}\hat f^{\hat q}dz\right]+\frac{\theta}{|Q_R|}\iint_{Q_R}\hat g^{\hat q}dz.
\end{aligned}
$$

By Lemma 2.11, we know that there exists a positive constant ε_0 such that for any $\hat p\in[\hat q,\hat q+\varepsilon_0)$, we get

$$
\begin{aligned}
&\Big(\frac{1}{|Q_R|}\iint_{Q_R}\Big(|Xu|^2+|u|^\gamma\Big)^{\frac{\hat p\tilde q}{2}}dz\Big)^{\frac{1}{\hat p}}\\
&\le c\Bigg[\Big(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}\Big(|Xu|^2+|u|^\gamma\Big)dz\Big)^{\frac{\tilde q}{2}}\\
&\qquad+\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}\Big(|f|^2+|g|^{\tilde q}+1\Big)^{\frac{\hat p\tilde q}{2}}dz\right)^{\frac{1}{\hat p}}\Bigg].
\end{aligned}
$$

Let $p=\hat p\tilde q\in[2,2+\tilde q\varepsilon_0)$, the proof is finished. □

4. Proofs of main theorems

Let v be a weak solution to the homogeneous system

$$\begin{cases} v_t^i + X_\alpha^* \left(\left(A^{\alpha\beta}(z) \right)_{S(Q_R)} \delta_{ij} X_\beta v^j \right) = 0 \\ v - u \in W_2^{1,1}(S(Q_R), \mathbb{R}^N) \end{cases} \tag{4.1}$$

and then $w = u - v \in W_{2,0}^{1,1}(S(Q_R), \mathbb{R}^N)$ satisfies:

$$\begin{aligned} w_t^i &+ X_\alpha^* \left(\left(A^{\alpha\beta}(z) \right)_{S(Q_R)} \delta_{ij} X_\beta w^j \right) \\ &= X_\alpha^* \left(\left(\left(A^{\alpha\beta}(z) \right)_{S(Q_R)} - A^{\alpha\beta}(z) \right) \delta_{ij} X_\beta u^j \right) \\ &\quad - X_\alpha^* \left(B_{ij}^{\alpha\beta} X_\beta u^j \right) + g_i(z, u, Xu) + X_\alpha^* f_i^\alpha(z), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} S^2 = SS' &= \left(A^{\alpha\beta}(z) \right)_{Q_R} \\ &= \frac{1}{|Q_R|} \iint_{Q_R} A^{\alpha\beta}(z) dz, \\ S(Q_R) &= \{ Sz : z \in Q_R \}. \end{aligned}$$

Similar to Lemma 4.1 in [7] we have the following lemma.

Lemma 4.1. *Let $v \in W_2^{1,1}(Q_T, \mathbb{R}^N)$ be a weak solution to (4.1). Then for any $0 < \rho < R$, $S(Q_R) \subset\subset Q_T$, we have*

$$\iint_{S(Q_\rho)} |SXv|^2 dz \le c \Big(\frac{\rho}{R} \Big)^{Q+2} \iint_{S(Q_R)} |SXv|^2 dz.$$

Proof of Theorem 1.1. Multiplying both sides of (4.2) by w^i and integrating on $S(Q_R)$,

$$\begin{aligned} &\iint_{S(Q_R)} \left(w_t^i w^i + \left(A^{\alpha\beta} \right)_{S(Q_R)} \delta_{ij} X_\beta w^j X_\alpha w^i \right) dz \\ &= \iint_{S(Q_R)} \left(\left(A^{\alpha\beta}(z) \right)_{S(Q_R)} - A^{\alpha\beta}(z) \right) \delta_{ij} X_\beta u^j X_\alpha w^i dz \\ &\quad - \iint_{S(Q_R)} B_{ij}^{\alpha\beta} X_\beta u^j X_\alpha w^i dz + \iint_{S(Q_R)} g_i w^i dz + \iint_{S(Q_R)} f_i^\alpha X_\alpha w^i dz. \end{aligned}$$

Since $\iint_{S(Q_R)} w_t^i w^i dz = \int_{S(B_R)} dx \int_{S(t_0 - R^2)}^{St_0} w^i dw^i = 0$, by (H1), we get

$$\begin{aligned} \lambda_0 &\iint_{S(Q_R)} |SXw|^2 dz \\ &\le \iint_{S(Q_R)} \left| A^{\alpha\beta}(z) - \left(A^{\alpha\beta}(z) \right)_{S(Q_R)} \right| |SXu| \, |SXw| dz \end{aligned} \tag{4.3}$$

$$+\,\delta\lambda_0 \iint_{S(Q_R)} |SXu|\,|SXw|dz$$
$$+\iint_{S(Q_R)} |Sg_i|\,|Sw|dz + \iint_{S(Q_R)} |Sf_i^\alpha|\,|SXw|dz.$$

By (H1), Young's inequality and Hölder's inequality, it has

(4.4)

$$\begin{aligned}
&\iint_{S(Q_R)} \left|A^{\alpha\beta}(z) - \left(A^{\alpha\beta}(z)\right)_{S(Q_R)}\right| |SXu|\,|SXw|dz \\
&\quad + \delta\lambda_0 \iint_{S(Q_R)} |SXu|\,|SXw|dz \\
&\le c_\varepsilon \iint_{S(Q_R)} \left|A^{\alpha\beta}(z) - \left(A^{\alpha\beta}(z)\right)_{S(Q_R)}\right|^2 |SXu|^2 dz \\
&\quad + \varepsilon \iint_{S(Q_R)} |SXw|^2 dz + \frac{\delta\lambda_0}{2} \iint_{S(Q_R)} |SXw|^2 dz \\
&\quad + \frac{\delta\lambda_0}{2} \iint_{S(Q_R)} |SXu|^2 dz \\
&\le c_\varepsilon \left(\iint_{S(Q_R)} \left|A^{\alpha\beta}(z) - \left(A^{\alpha\beta}(z)\right)_{S(Q_R)}\right|^{\frac{2p}{p-2}} dz \right)^{\frac{p-2}{p}} \\
&\qquad \cdot \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} + \left(\frac{\delta\lambda_0}{2} + \varepsilon\right) \iint_{S(Q_R)} |SXw|^2 dz \\
&\quad + \frac{\delta\lambda_0}{2} |S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} \\
&\le |S(Q_R)|^{\frac{p-2}{p}} \left(c_\varepsilon \Big(\eta_{S(Q_R)}\left(A^{\alpha\beta}\right)\Big)^{\frac{p-2}{p}} + \frac{\delta\lambda_0}{2} \right) \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} \\
&\quad + \left(\frac{\delta\lambda_0}{2} + \varepsilon\right) \iint_{S(Q_R)} |SXw|^2 dz.
\end{aligned}$$

By (2.2),

$$\begin{aligned}
&\left(\iint_{S(Q_R)} |Sw|^\gamma dz \right)^{\frac{1}{\gamma}} \le \left(\int_{I_R} cR^{\frac{4}{Q}} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{\gamma}{2}} dt \right)^{\frac{1}{\gamma}} \\
&\le cR^{\frac{4}{Q\gamma}} \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{2}{Q\gamma}} \left(\iint_{S(Q_R)} |SXw|^2 dz \right)^{\frac{1}{\gamma}}.
\end{aligned}$$

By (H2), Hölder's inequality, Young's inequality and the above inequality, we have

$$
\begin{aligned}
(4.5)\quad & \iint_{S(Q_R)} |Sg_i|\,|Sw| dz \\
&\leq \mu_2\,|S| \iint_{S(Q_R)} \left(|Xu|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + g^i(z)\right) |Sw| dz \\
&\leq \mu_2 |S|^{-\frac{2}{Q+2}} \iint_{S(Q_R)} |SXu|^{2(1-\frac{1}{\gamma})}\,|Sw| dz \\
&\quad + \mu_2 |S|^{-\frac{4}{Q}} \iint_{S(Q_R)} |Su|^{\gamma-1}\,|Sw| dz + \mu_2 \iint_{S(Q_R)} |Sg|\,|Sw| dz \\
&\leq \mu_2 \lambda_0^{-\frac{1}{Q+2}} \left(\iint_{S(Q_R)} |SXu|^2 dz\right)^{\frac{1}{\tilde q}} \left(\iint_{S(Q_R)} |Sw|^\gamma dz\right)^{\frac{1}{\gamma}} \\
&\quad + \mu_2 \lambda_0^{-\frac{2}{Q}} \left(\iint_{S(Q_R)} |Su|^\gamma dz\right)^{\frac{1}{\tilde q}} \left(\iint_{S(Q_R)} |Sw|^\gamma dz\right)^{\frac{1}{\gamma}} \\
&\quad + \mu_2 \left(\iint_{S(Q_R)} |Sg|^{\tilde q} dz\right)^{\frac{1}{\tilde q}} \left(\iint_{S(Q_R)} |Sw|^\gamma dz\right)^{\frac{1}{\gamma}} \\
&\leq c \lambda_0^{-\frac{1}{Q+2}} \left(\iint_{S(Q_R)} |SXu|^2 dz\right)^{\frac{1}{\tilde q}} R^{\frac{4}{Q\gamma}} \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx\right)^{\frac{2}{Q\gamma}} \\
&\qquad \cdot \left(\iint_{S(Q_R)} |SXw|^2 dz\right)^{\frac{1}{\gamma}} \\
&\quad + c \lambda_0^{-\frac{2}{Q}} \left(\iint_{S(Q_R)} |Su|^\gamma dz\right)^{\frac{1}{\tilde q}} R^{\frac{4}{Q\gamma}} \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx\right)^{\frac{2}{Q\gamma}} \\
&\qquad \cdot \left(\iint_{S(Q_R)} |SXw|^2 dz\right)^{\frac{1}{\gamma}} \\
&\quad + c \left(\iint_{S(Q_R)} |Sg|^{\tilde q} dz\right)^{\frac{1}{\tilde q}} R^{\frac{4}{Q\gamma}} \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx\right)^{\frac{2}{Q\gamma}} \\
&\qquad \cdot \left(\iint_{S(Q_R)} |SXw|^2 dz\right)^{\frac{1}{\gamma}} \\
&\leq c_\varepsilon R^{\frac{4}{Q+4}} \iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right) dz + c_\varepsilon R^{\frac{4}{Q+4}} \iint_{S(Q_R)} |Sg|^{\tilde q} dz
\end{aligned}
$$

$$+ 3\varepsilon \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{2}{Q}} \iint_{S(Q_R)} |SXw|^2 dz.$$

By (H2) and Young's inequality,

$$\begin{aligned}
(4.6) \quad \iint_{S(Q_R)} |Sf_i^\alpha| \, |SXw| dz &\leq \mu_1 \iint_{S(Q_R)} |S| \left(|u|^{\frac{\gamma}{2}} + f^i(z) \right) |SXw| dz \\
&\leq \mu_1 |S|^{-\frac{2}{Q}} \iint_{S(Q_R)} |Su|^{\frac{\gamma}{2}} \, |SXw| dz + \mu_1 \iint_{S(Q_R)} |Sf| \, |SXw| dz \\
&\leq \mu_1 \lambda_0^{-\frac{1}{Q}} \iint_{S(Q_R)} |Su|^{\frac{\gamma}{2}} \, |SXw| dz + \mu_1 \iint_{S(Q_R)} |Sf| \, |SXw| dz \\
&\leq c_\varepsilon \iint_{S(Q_R)} |Su|^\gamma dz + 2\varepsilon \iint_{S(Q_R)} |SXw|^2 dz + c_\varepsilon \iint_{S(Q_R)} |Sf|^2 dz.
\end{aligned}$$

Inserting (4.4), (4.5), (4.6) into (4.3), and by (H1), we have

$$\begin{aligned}
&\lambda_0 \iint_{S(Q_R)} |SXw|^2 dz \\
&\leq |S(Q_R)|^{\frac{p-2}{p}} \left(c_\varepsilon \Big(\eta_{S(Q_R)} \left(A^{\alpha\beta} \right) \Big)^{\frac{p-2}{p}} + \frac{\delta\lambda_0}{2} \right) \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} \\
&\quad + \left(\frac{\delta\lambda_0}{2} + 3\varepsilon + 3\varepsilon \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{2}{Q}} \right) \iint_{S(Q_R)} |SXw|^2 dz \\
&\quad + c_\varepsilon R^{\frac{4}{Q+4}} \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz + c_\varepsilon R^{\frac{4}{Q+4}} \iint_{S(Q_R)} \Big(|SXu|^2 + |Su|^\gamma \Big) dz \\
&\quad + c_\varepsilon \iint_{S(Q_R)} |Su|^\gamma dz + c_\varepsilon \iint_{S(Q_R)} |Sf|^2 dz.
\end{aligned}$$

Set $\theta_1 = \dfrac{\frac{\delta\lambda_0}{2}}{\lambda_0 - \frac{\delta\lambda_0}{2} - 3\varepsilon - 3\varepsilon \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{2}{Q}}}$.

Taking ε small enough such that

$$\lambda_0 - \frac{\delta\lambda_0}{2} - 3\varepsilon - 3\varepsilon \sup_{I_R} \left(\int_{B_R} |SXw|^2 dx \right)^{\frac{2}{Q}} > 0$$

and $\theta_1 \in (0,1)$, it follows that

$$\begin{aligned}
(4.7) \quad &\iint_{S(Q_R)} |SXw|^2 dz \\
&\leq |S(Q_R)|^{\frac{p-2}{p}} \left(c \Big(\eta_{S(Q_R)} \left(A^{\alpha\beta} \right) \Big)^{\frac{p-2}{p}} + \theta_1 \right) \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} \\
&\quad + c R^{\frac{4}{Q+4}} \iint_{S(Q_R)} \Big(|SXu|^2 + |Su|^\gamma \Big) dz + c R^{\frac{4}{Q+4}} \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz
\end{aligned}$$

$$+ c \iint_{S(Q_R)} |Su|^\gamma dz + c \iint_{S(Q_R)} |Sf|^2 dz.$$

By (2.1) and Hölder's inequality,

$$
\begin{aligned}
(4.8)\qquad & \iint_{S(Q_R)} |Su|^\gamma dz \\
& \le c \iint_{S(Q_R)} \left|Su - \overline{Su}(t)\right|^\gamma dz + c\,|S(B_R)| \int_{S(I_R)} \left|\overline{Su}(t)\right|^\gamma dt \\
& \le c \int_{I_R} cR^{\frac{4}{Q}} \left(\int_{B_R} |SXu|^2 dx\right)^{\frac{\gamma}{2}} dt \\
& \quad + c\,|S(B_R)| \int_{S(I_R)} \left| \frac{1}{N_1\,|S(B_R)|} \int_{S(B_R)} Su\xi^2 dx \right|^\gamma dt \\
& \le cR^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx\right)^{\frac{2}{Q}} \iint_{S(Q_R)} |SXu|^2 dz \\
& \quad + c\,|S(B_R)| \frac{1}{(N_1\,|S(B_R)|)^{\frac{2}{p}}} \int_{S(I_R)} \left(\int_{S(B_R)} |Su|^{\frac{p\gamma}{2}} dx\right)^{\frac{2}{p}} dt \\
& \le c|S(Q_R)|^{\frac{p-2}{p}} R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx\right)^{\frac{2}{Q}} \\
& \qquad \cdot \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
& \quad + c\,|S(B_R)| \frac{|S(I_R)|^{\frac{p-2}{p}}}{(N_1\,|S(B_R)|)^{\frac{2}{p}}} \left(\iint_{S(Q_R)} |Su|^{\frac{p\gamma}{2}} dz\right)^{\frac{2}{p}} \\
& \le c|S(Q_R)|^{\frac{p-2}{p}} R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx\right)^{\frac{2}{Q}} \\
& \qquad \cdot \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
& \quad + c|S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} |Su|^{\frac{p\gamma}{2}} dz\right)^{\frac{2}{p}}.
\end{aligned}
$$

Inserting (4.8) into (4.7), and by Hölder's inequality, we have

$$(4.9)\qquad \iint_{S(Q_R)} |SXw|^2 dz$$

$$
\begin{aligned}
&\leq |S(Q_R)|^{\frac{p-2}{p}} \left(c\Big(\eta_{S(Q_R)}\left(A^{\alpha\beta}\right)\Big)^{\frac{p-2}{p}} + \theta_1 \right) \left(\iint_{S(Q_R)} |SXu|^p dz \right)^{\frac{2}{p}} \\
&\quad + cR^{\frac{4}{Q+4}} \iint_{S(Q_R)} \Big(|SXu|^2 + |Su|^\gamma \Big) dz + cR^{\frac{4}{Q+4}} \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx \right)^{\frac{2}{Q}} \\
&\qquad \cdot \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma \right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} |Su|^{\frac{p\gamma}{2}} dz \right)^{\frac{2}{p}} + c \iint_{S(Q_R)} |Sf|^2 dz \\
&\leq |S(Q_R)|^{\frac{p-2}{p}} \left(c\Big(\eta_{S(Q_R)}\left(A^{\alpha\beta}\right)\Big)^{\frac{p-2}{p}} + \theta_1 \right) \\
&\quad \cdot \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma \right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + cR^{\frac{4}{Q+4}} |S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma \right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx \right)^{\frac{2}{Q}} \\
&\quad \cdot \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma \right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} |Su|^{\frac{p\gamma}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz + c \iint_{S(Q_R)} |Sf|^2 dz \\
&\leq |S(Q_R)|^{\frac{p-2}{p}} \theta_2 \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma \right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} |Su|^{\frac{p\gamma}{2}} dz \right)^{\frac{2}{p}} + c \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz \\
&\quad + c \iint_{S(Q_R)} |Sf|^2 dz,
\end{aligned}
$$

where $\theta_2 = c\big(\eta_{S(Q_R)}\left(A^{\alpha\beta}\right)\big)^{\frac{p-2}{p}} + \theta_1 + cR^{\frac{4}{Q+4}} + cR^{\frac{4}{Q}} \sup_{I_R}\left(\int_{B_R}|SXu|^2dx\right)^{\frac{2}{Q}}$.
Since $0 < \theta_1 < 1$, so we can choose R small enough such that $\theta_2 \in (0,1)$.
By (4.9) and Lemma 4.1, we know that for any $2\rho < R$,

$$
\begin{aligned}
&\iint_{S(Q_{2\rho})}|SXu|^2dz \le 2\iint_{S(Q_{2\rho})}|SXv|^2dz + 2\iint_{S(Q_{2\rho})}|SXw|^2dz \\
&\le c\Big(\frac{\rho}{R}\Big)^{Q+2}\iint_{S(Q_R)}|SXv|^2dz + c\iint_{S(Q_{2\rho})}|SXw|^2dz \\
&\le c\Big(\frac{\rho}{R}\Big)^{Q+2}\iint_{S(Q_R)}|SXu|^2dz \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}}\theta_2\left(\iint_{S(Q_R)}\left(|SXu|^2+|Su|^\gamma\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}}\left(\iint_{S(Q_R)}|Su|^{\frac{p\gamma}{2}}dz\right)^{\frac{2}{p}} \\
&\quad + c\iint_{S(Q_R)}|Sg|^{\tilde{q}}dz + c\iint_{S(Q_R)}|Sf|^2dz \\
&\le c\left(\Big(\frac{\rho}{R}\Big)^{Q+2} + \theta_2\right)|S(Q_R)|^{\frac{p-2}{p}}\left(\iint_{S(Q_R)}\left(|SXu|^2+|Su|^\gamma\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}}\|Su\|^\gamma_{L^{\frac{p\gamma}{2}}} + c\iint_{S(Q_R)}|Sg|^{\tilde{q}}dz + c\iint_{S(Q_R)}|Sf|^2dz.
\end{aligned}
$$

By the above and (4.8), we have

$$
\begin{aligned}
&\iint_{S(Q_{2\rho})}\left(|SXu|^2+|Su|^\gamma\right)dz \\
&\le c\left(\Big(\frac{\rho}{R}\Big)^{Q+2} + \theta_2 + R^{\frac{4}{Q}}\sup_{I_R}\left(\int_{B_R}|SXu|^2dx\right)^{\frac{2}{Q}}\right) \\
&\qquad \cdot |S(Q_R)|^{\frac{p-2}{p}}\left(\iint_{S(Q_R)}\left(|SXu|^2+|Su|^\gamma\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}}\|Su\|^\gamma_{L^{\frac{p\gamma}{2}}} + c\iint_{S(Q_R)}|Sg|^{\tilde{q}}dz + c\iint_{S(Q_R)}|Sf|^2dz.
\end{aligned}
$$

From Theorem 3.4 and above inequality,

$$
\left(|S(Q_\rho)|^{\frac{p-2}{2}}\iint_{S(Q_\rho)}\left(|SXu|^2+|Su|^\gamma\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}}
$$

$$
\begin{aligned}
&\leq c \iint_{S(Q_{2\rho})} \left(|SXu|^2 + |Su|^\gamma\right) dz \\
&\quad + c|S(Q_{2\rho})|^{\frac{p-2}{p}} \left(\iint_{S(Q_{2\rho})} \left(|Sf|^2 + |Sg|^{\tilde{q}} + 1\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\leq c \left(\left(\frac{\rho}{R}\right)^{Q+2} + \theta_2 + R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx \right)^{\frac{2}{Q}} \right) \\
&\qquad \cdot |S(Q_R)|^{\frac{p-2}{p}} \left(\iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + c|S(Q_R)|^{\frac{p-2}{p}} \|Su\|^\gamma_{L^{\frac{p\gamma}{2}}} + c \iint_{S(Q_R)} |Sg|^{\tilde{q}} dz + c \iint_{S(Q_R)} |Sf|^2 dz \\
&\quad + c|S(Q_{2R})|^{\frac{p-2}{p}} \left(\iint_{S(Q_{2R})} \left(|Sf|^2 + |Sg|^{\tilde{q}} + 1\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\leq c \left(\left(\frac{\rho}{R}\right)^{Q+2} + \theta_2 + R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx \right)^{\frac{2}{Q}} \right) \\
&\qquad \cdot \left(|S(Q_R)|^{\frac{p-2}{2}} \iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + cR^{Q+2-\frac{2(Q+2)}{p}} \|Su\|^\gamma_{L^{\frac{p\gamma}{2}}} + cR^{Q+2-\frac{\tilde{q}(Q+2)}{\tau}} \|Sg\|^{\tilde{q}}_{L^\tau} \\
&\quad + cR^{Q+2-\frac{2(Q+2)}{\sigma}} \|Sf\|^2_{L^\sigma} + cR^{Q+2} \\
&\leq c \left(\left(\frac{\rho}{R}\right)^{Q+2} + \theta_2 + R^{\frac{4}{Q}} \sup_{I_R} \left(\int_{B_R} |SXu|^2 dx \right)^{\frac{2}{Q}} \right) \\
&\quad \cdot \left(|S(Q_R)|^{\frac{p-2}{2}} \iint_{S(Q_R)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}} \\
&\quad + CR^{Q+2\kappa},
\end{aligned}
$$

where $\kappa = \min\left\{1 - \frac{\tilde{q}(Q+2)}{2\tau}, 1 - \frac{Q+2}{\sigma}\right\}$,

$$
C = c\left(R^{2-2\kappa-\frac{2(Q+2)}{p}} \|Su\|^\gamma_{L^{\frac{p\gamma}{2}}} + \|Sg\|^{\tilde{q}}_{L^\tau} + \|Sf\|^2_{L^\sigma} + cR^{2-2\kappa} \right).
$$

By Lemma 2.8, we get

$$
\left(|S(Q_\rho)|^{\frac{p-2}{2}} \iint_{S(Q_\rho)} \left(|SXu|^2 + |Su|^\gamma\right)^{\frac{p}{2}} dz \right)^{\frac{2}{p}}
$$

$$\leq c\Big(\frac{\rho}{R}\Big)^{Q+2\kappa}\left(|S(Q_R)|^{\frac{p-2}{2}}\iint_{S(Q_R)}\Big(|SXu|^2+|Su|^\gamma\Big)^{\frac{p}{2}}dz\right)^{\frac{2}{p}}+C\rho^{Q+2\kappa}.$$

Then

$$\begin{aligned}(4.10)\quad &\iint_{S(Q_\rho)}\Big(|SXu|^2+|Su|^\gamma\Big)^{\frac{p}{2}}dz\\ &\leq c\Big(\frac{\rho}{R}\Big)^{\frac{pQ}{2}+p\kappa}\Big(\frac{|S(Q_\rho)|}{|S(Q_R)|}\Big)^{\frac{2-p}{2}}\iint_{S(Q_R)}\Big(|SXu|^2+|Su|^\gamma\Big)^{\frac{p}{2}}dz\\ &\quad+C\rho^{\frac{pQ}{2}+p\kappa}|S(Q_\rho)|^{\frac{2-p}{2}}\\ &\leq c\Big(\frac{\rho}{R}\Big)^{Q+2-p+p\kappa}\iint_{S(Q_R)}\Big(|SXu|^2+|Su|^\gamma\Big)^{\frac{p}{2}}dz+C\rho^{Q+2-p+p\kappa}.\end{aligned}$$

Hence

$$Su\in L^{\frac{p\gamma}{2},Q+2-p+p\kappa}\left(S(B_\rho),\mathbb{R}^N\right),SXu\in L^{p,Q+2-p+p\kappa}\left(S(B_\rho),\mathbb{R}^N\right),$$

where $\kappa=\min\Big\{1-\frac{\tilde q(Q+2)}{2\tau},1-\frac{Q+2}{\sigma}\Big\}$, and the proof is finished. □

Proof of Theorem 1.2. By (3.12) and (4.10)($p=2$), we get

$$\begin{aligned}&\iint_{S(Q_\rho)}\Big|Su-(Su)_{S(Q_\rho)}\Big|^2dz\leq c\iint_{S(Q_\rho)}|Su|^2dz\\ &\leq c\rho^2\iint_{S(Q_{2\rho})}|SXu|^2dz+c\rho^2\iint_{S(Q_{2\rho})}\Big(|Su|^\gamma+|Sf|^2+|Sg|^{\tilde q}\Big)dz\\ &\leq c\rho^2\iint_{S(Q_{2\rho})}\Big(|SXu|^2+|Su|^\gamma\Big)dz\\ &\quad+c\rho^2\iint_{S(Q_{2\rho})}\Big(|Su|^\gamma+|Sf|^2+|Sg|^{\tilde q}\Big)dz\\ &\leq c\rho^2\Big(\frac{\rho}{R}\Big)^{Q+2\kappa}\iint_{S(Q_R)}\Big(|SXu|^2+|Su|^\gamma\Big)dz+C\rho^{Q+2+2\kappa}.\end{aligned}$$

Then we have

$$Su\in\mathcal{L}^{2,Q+2+2\kappa}(S(Q_\rho),\mathbb{R}^N).$$

By Lemma 2.12, we get

$$Su\in C^\kappa\left(S(Q_\rho),\mathbb{R}^N\right),\quad\kappa=\min\left\{1-\frac{\tilde q(Q+2)}{2\tau},1-\frac{Q+2}{\sigma}\right\}.\qquad\square$$

We would like to thank the referees for their kind comments and suggestions.

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(Yan Dong) Department of Applied Mathematics, Hubei University of Economics, Wuhan, Hubei, 430205, China
dongyan@mail.nwpu.edu.cn

(Dongyan Li) Department of Applied Mathematics, Xi'an Polytechnic University, Xi'an, Shaanxi, 710048, China
w408867388w@126.com

This paper is available via http://nyjm.albany.edu/j/2018/24-4.html.