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Composition operators on the Dirichlet space of the upper half-plane

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Abstract. It is well known that Hardy and weighted Bergman spaces of the upper half-plane do not support compact composition operators (see [M99] and [SS03]). In this paper, we prove that unlike Hardy and Bergman spaces, the Dirichlet space of the upper half-plane does support compact composition operators. Furthermore, bounded analytic symbols, which in the case of Hardy and weighted Bergman spaces of the upper half-plane do not even induce bounded composition operators, can induce compact composition operators on the Dirichlet space of the upper half-plane.

Contents

1. Introduction and preliminaries

Let Ω be a domain in the complex plane $\mathbb{C}$ and let φ be a holomorphic self-map of Ω. Then the equation $C_\varphi f = f \circ \varphi$, for f analytic in Ω, defines a composition operator C_φ with inducing map φ. During the past few decades, composition operators have been studied extensively on spaces of functions analytic on the open unit disk $\mathbb{D}$. As a consequence of the Littlewood subordination principle it is known that every analytic self-map φ of $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk. However, a self-map φ of $\mathbb{D}$ does not necessarily induce a bounded composition operator on the Dirichlet space of the open unit disk $\mathbb{D}$. An obvious necessary condition for it is that φ be in the Dirichlet space of the open unit disk. But this condition is not sufficient. A necessary and sufficient condition for φ to induce a bounded composition operator on

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the Dirichlet space of the open unit disk is given in terms of the counting function and Carleson measures (see [JM97] and [Z98] and the references therein). For more about composition operators, we refer to [CoM95] and [S93].

If we move to Hardy and weighted Bergman spaces of the upper half-plane

$$\Pi^+ = \{z \in \mathbb{C} : \Im z > 0\},$$

the situation is entirely different. There do exist analytic self-maps of the upper half-plane which do not induce bounded composition operators. Moreover, Hardy and Bergman spaces of the upper half-plane do not support compact composition operators (see [M99] and [SS03]). Recent work on composition operators on Hardy and weighted Bergman spaces of the upper half-plane can be found in [BT12],[CKS17],[EW11],[EJ12],[M89],[M99],[SS03], and [SiSh80].

Composition operators on the Dirichlet space $\mathcal{D}_{\Pi^+}$ of the upper half-plane remain untouched so far. In this paper, we characterize compact composition operators on the Dirichlet space of the upper half-plane. Recall that a function f that is analytic in the upper half-plane Π^+ belongs to the Dirichlet space $\mathcal{D}_{\Pi^+}$ if and only if

$$\int_{\Pi^+} |f'(z)|^2 dA(z) < \infty,$$

where $dA(z) = dxdy$ is ordinary area measure. The norm on $\mathcal{D}_{\Pi^+}$ is defined as

$$||f||^2_{\mathcal{D}_{\Pi^+}} = |f(i)|^2 + \int_{\Pi^+} |f'(z)|^2 dA(z),$$

For $z \in \Pi^+$ and $0 < r < 1$ we define

$$S(z, r\Im z) = \{w \in \Pi^+ : |w - z| < r\Im z\}.$$

Then for $0 < r < 1/3$, there exists a positive integer M and a sequence $\{z_n\}$ in Π^+ such that

$$\cup_{n=1}^{\infty} S(z_n, r\Im z_n) = \Pi^+$$

and every point in Π^+ belongs to at most M sets in $\{S(z_n, 3ry_n)\}_{n\in\mathbb{N}}$; see [KK01].

2. Boundedness and compactness of C_φ on $\mathcal{D}_{\Pi^+}$.

In this section we characterize bounded and compact composition operators on the Drichlet space of the upper half plane in terms of the counting function and Carleson measures.

Let $\varphi : \Pi^+ \to \Pi^+$ be an analytic map and w be a point in Π^+. Let $\{z_k\}$ be the points in the upper half plane for which $\varphi(z_k) = w$ counting multiplicities. Define the counting function

$$n_\varphi : \Pi^+ \to \mathbb{N} \cup \{\infty\}$$

by

$$n_\varphi(w) = \mathrm{Card}\{z \in \Pi^+ : \varphi(z_k) = w\}$$

when the set $\{z \in \Pi^+ : \varphi(z_k) = w\}$ is finite, and $n_\varphi(w) = \infty$ otherwise. Also we set $n_\varphi(w) = 0$ if $w \notin \varphi(\Pi^+)$. Define

$$\mathfrak{N}_\varphi(w) = \begin{cases} n_\varphi(w) & (w \in \varphi(\Pi^+)) \\ 0 & \text{otherwise.} \end{cases}$$

The counting function $\mathfrak{N}_\varphi$ produces the following non-univalent change of variables formula. The proof is standard, but we include it for completeness.

Proposition 2.1. *Let $0 < r < 1/3$ be fixed. Let $f \in \mathcal{D}_{\Pi^+}$ and φ be a non-constant analytic self-map of Π^+. Then*

$$\int_{\Pi^+} |(f' \circ \varphi)(w)|^2 |\varphi'(w)|^2 dA(z) = \int_{\Pi^+} |f'(w)|^2 \mathfrak{N}_\varphi(w) dA(w). \tag{2.1}$$

Proof. Let

$$\Pi_0^+ = \{z \in \Pi^+ : \varphi'(z) \neq 0\}.$$

Then the set $\Pi^+ \setminus \Pi_0^+$ is at most countable. If $z \in \Pi_0^+$, then φ is one-one on $S(z, r\Im z)$ for some $r \in (0, 1/3)$. Thus we have

$$\int_{S(z,r\Im z)} |(f' \circ \varphi)(w)|^2 |\varphi'(w)|^2 dA(w) = \int_{\varphi(S(z,r\Im z))} |f'(w)|^2 dA(w).$$

The disks $S(z, r\Im z)$ form a cover for Π_0^+ and we can pick a countable subcover $\{S(z_n, r\Im z_n) : n \in \mathbb{N}\}$. Let $B_1 = S(z_1, ry_1)$ and

$$B_n = S(z_n, r\Im z_n,) \setminus \cup_{k=1}^{n-1} B_k$$

for all $n \geq 2$. Then $\{B_n : n \in \mathbb{N}\}$ is a pairwise disjoint cover of Π_0^+. Using (2.1), we have

$$\begin{aligned} \int_{\Pi^+} |(f' \circ \varphi)(w)|^2 |\varphi'(w)|^2 dA(w) &= \sum_{n=1}^{\infty} \int_{B_n} |(f' \circ \varphi)(w)|^2 |\varphi'(w)|^2 dA(w) \\ &= \sum_{n=1}^{\infty} \int_{\varphi(B_n)} |f'(w)|^2 dA(w) \\ &= \int_{\Pi^+} |f'(w)|^2 \sum_{n=1}^{\infty} \chi_{\varphi(B_n)}(w) dA(w) \\ &= \int_{\Pi^+} |f'(w)|^2 \mathfrak{N}_\varphi(w) dA(w). \end{aligned}$$

This completes the proof of the non-univalent change of variables formula. □

From now onwards, constants are denoted by C. They are positive and not necessarily the same at each occurrence.

Theorem 2.2. *Let $0 < r < 1/3$ be fixed and φ be a non-constant analytic self-map of Π^+. Then C_φ is bounded on $\mathcal{D}_{\Pi^+}$ if and only if*

$$\sup_{z\in\Pi^+} \frac{1}{(\Im z)^2} \int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w)dA(w) < \infty. \tag{2.2}$$

Proof. Suppose that (2.2) holds. By the closed graph theorem, we need to show that $C_\varphi f \in \mathcal{D}_{\Pi^+}$ whenever $f \in \mathcal{D}_{\Pi^+}$. By Proposition 2.1, we have

$$\int_{\Pi^+} |(C_\varphi f)'(w)|^2 dA(w) = \int_{\Pi^+} |f'(w)|^2 \mathfrak{N}_\varphi(w)dA(w). \tag{2.3}$$

The right side of (2.3) is dominated by

$$\begin{aligned}&\sum_{n=1}^{\infty} \int_{S(z_n,r\Im z_n)} |f'(w)|^2 \mathfrak{N}_\varphi(w)dA(w) \qquad (2.4)\\ &\quad\le \sum_{n=1}^{\infty} \sup\{|f'(w)|^2 : w\in S(z_n,r\Im z_n)\} \int_{S(z_n,r\Im z_n)} \mathfrak{N}_\varphi(w)dA(w).\end{aligned}$$

By the subharmonicity of $|f'(w)|^2$, (2.4) is further dominated by a constant multiple of

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\frac{1}{(\Im z_n)^2}\int_{S(z_n,2r\Im z_n)} |f'(w)|^2 dA(w)\Big)\int_{S(z_n,r\Im z_n)} \mathfrak{N}_\varphi(w)dA(w)\\ &\quad\le C\sum_{n=1}^{\infty}\int_{S(z_n,2r\Im z_n)} |f'(w)|^2 dA(w) \le CM||f||^2_{\mathcal{D}_{\Pi^+}}.\end{aligned}$$

Conversely, suppose that C_φ is bounded, then

$$||C_\varphi f||^2_{\mathcal{D}_{\Pi^+}} \le C||f||^2_{\mathcal{D}_{\Pi^+}}.$$

Let $f_z(w) = \Im z/(w-\overline{z})$. Then $f_z'(w) = \Im z/(w-\overline{z})^2$. If $z = x+iy$ and $w = t+iu$ are in Π^+, then

$$\begin{aligned}\int_{\Pi^+}\Big|\frac{1}{(w-\overline{z})^2}\Big|^2 dA(w) &= \int_0^\infty\int_{-\infty}^{\infty}\frac{1}{((x-t)^2+(y+u)^2)^2}\,dtdu\\ &\le \int_0^\infty\int_{-\infty}^{\infty}\frac{1}{(y+u)^3}\cdot\frac{y+u}{(x-t)^2+(y+u)^2}\,dtdu\\ &\le \frac{C}{y^2},\end{aligned}$$

where we have used the fact that

$$P_y(x,t) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2} \qquad (x,t\in\mathbb{R}, y>0)$$

is the Poisson kernel for Π^+ and so

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y+u}{(x-t)^2+(y+u)^2}dt = 1.$$

Thus $f_z \in \mathcal{D}_{\Pi^+}$. Moreover,

$$\sup_{z\in\Pi^+} ||f_z||_{\mathcal{D}_{\Pi^+}} \leq C.$$

So, from (2.3), we have

$$\begin{aligned}\frac{1}{(\Im z)^2}\int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w)dA(w) &\leq C\int_{S(z,r\Im z)} |f_z'(w)|^2\mathfrak{N}_\varphi(w)dA(w)\\ &\leq ||C_\varphi f||^2_{\mathcal{D}_{\Pi^+}} \leq C.\end{aligned}$$

Since $z \in \Pi^+$ is arbitrary, the desired result follows. □

It is well known (see [M89], [M99] and [SS03]) that a linear fractional map

$$\tau(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0, \tag{2.5}$$

induces a bounded composition operator on the Hardy or weighted Bergman spaces of the upper half plane if and only if $c = 0$. However, in view of Theorem 2.2, every linear fractional map τ defined in (2.5) induces a bounded composition operator on $\mathcal{D}_{\Pi^+}$.

As in [CKS17], let $\widehat{\Pi^+}$ denote the set $\overline{\Pi^+} \cup \{\infty\}$. For any function $F(z)$,

$$\lim_{z\to\partial\widehat{\Pi^+}} F(z) = 0$$

means that

$$\sup_{z\in\Pi^+\setminus K} |F(z)| \to 0$$

as the compact set $K \subset \Pi^+$ expands to the whole of Π^+, or equivalently, that $F(z) \to 0$ as $\Im z \to 0^+$ and $F(z) \to 0$ as $|z| \to \infty$.

Theorem 2.3. *Let φ be a non-constant holomorphic self-map of Π^+ such that C_φ is bounded on $\mathcal{D}_{\Pi^+}$. Then C_φ is compact on $\mathcal{D}_{\Pi^+}$ if and only if there exists $r \in (0,1)$ such that*

$$\lim_{z\to\partial\widehat{\Pi^+}} \frac{1}{(\Im z)^2}\int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w)dA(w) = 0. \tag{2.6}$$

Proof. Arguing by contradiction, first assume that C_φ is compact on $\mathcal{D}_{\Pi^+}$ but (2.6) does not hold. Then there is a $\delta > 0$ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ in Π^+ such that $\Im z_n \to 0$ or $|z_n| \to \infty$ and

$$\frac{1}{(\Im z)^2}\int_{S(z_n,r\Im z_n)} \mathfrak{N}_\varphi(w)dA(w) > \delta$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ consider the function

$$f_n(w) = \frac{\Im z_n}{w - \overline{z}_n}.$$

It is clear that f_n is norm bounded and $f_n \to 0$ uniformly on compact subsets of Π^+ as $\Im z_n \to 0$ or $|z_n| \to \infty$. Thus $\|C_\varphi f_n\|_{\mathcal{D}_{\Pi^+}} \to 0$ as $\Im z_n \to 0$ or $|z_n| \to \infty$. On the other hand, by (2.3), we have

$$\begin{aligned}\|C_\varphi f_n\|_{\mathcal{D}_{\Pi^+}} &\geq \int_{S(z_n, r\Im z_n)} |f_n'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) \\ &\geq \frac{C}{(\Im z_n)^2} \int_{S(z_n, r\Im z_n)} \mathfrak{N}_\varphi(w) dA(w) > C\delta,\end{aligned}$$

which is a contradiction.

Next, assume that (2.6) holds. Then for each $\epsilon > 0$ there is a compact subset K of Π^+ such that

$$\int_{S(z, r\Im z)} \mathfrak{N}_\varphi(w) dA(w) < \epsilon (\Im z)^2 \tag{2.7}$$

whenever $z \in \Pi^+ \setminus K$. Let $\{f_m\}$ be a sequence in $\mathcal{D}_{\Pi^+}$ such that

$$\sup_m ||f_m||_{\mathcal{D}_{\Pi^+}} \leq M_1$$

and $f_m \to 0$ uniformly on compact subsets of Π^+ as $m \to \infty$. Then

$$\begin{aligned}||C_\varphi f_m||^2_{\mathcal{D}_{\Pi^+}} &= |f_m(\varphi(i))|^2 + \int_{\Pi^+} |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) \\ &= |f_m(\varphi(i))|^2 + \int_K |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) \\ &+ \int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w).\end{aligned}$$

Note that $|f_m(\varphi(i))|^2 \to 0$ as $m \to \infty$ and

$$\int_K |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) \to 0 \quad \text{ as } \; m \to \infty. \tag{2.8}$$

Thus to prove that C_φ is compact on $\mathcal{D}_{\Pi^+}$ we just need to show that

$$\int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) \to 0 \quad \text{ as } \; m \to \infty.$$

As in the proof of Theorem 2.2, the above term is dominated by

$$\sum_{n=1}^\infty \Big(\frac{1}{(\Im z_n)^2} \int_{S(z_n, 2r\Im z_n)} |f_m'(w)|^2 dA(w) \Big) \int_{S(z_n, r\Im z_n) \cap (\Pi^+ \setminus K)} \mathfrak{N}_\varphi(w) dA(w).$$

By (2.7), we have

$$\int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathfrak{N}_\varphi(w) dA(w) < \epsilon \sum_{n=1}^\infty \int_{S(z_n, 2r\Im z_n)} |f_m'(w)|^2 dA(w) \leq \epsilon M M_1 \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\int_{\Pi^+} |f_m'(w)|^2 \mathfrak{N}_\varphi(z) dA(w) \to 0 \quad \text{as} \quad m \to \infty.$$

Hence C_φ is compact on $\mathcal{D}_{\Pi^+}$. □

In [M89], Matache proved that if φ is a bounded analytic self mapping of Π^+, then C_φ cannot be bounded on the Hardy space $H^p(\Pi^+)$. Also, bounded analytic self-maps of Π^+ cannot induce bounded composition operators on weighted Bergman spaces of the upper half-plane; see [SS03]. As an application of Theorem 2.3, we prove that there are non-trivial analytic self-maps of the upper half-plane that induce compact composition operators on $\mathcal{D}_{\Pi^+}$.

Corollary 2.4. *Let φ be a conformal mapping from Π^+ to a relatively compact subset of Π^+. Then φ induces a compact composition operator on $\mathcal{D}_{\Pi^+}$.*

Proof. Suppose that $K = \varphi(\Pi^+)$ is a relatively compact subset of Π^+. Then

$$\mathfrak{N}_\varphi(w) = \begin{cases} 1 & \text{if} \quad w \in K \\ 0 & \text{if} \quad w \notin K. \end{cases}$$

Therefore,

$$\begin{aligned} &\sup_{z\in\Pi^+} \frac{1}{(\Im z)^2} \int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w) dA(w) \\ &\quad= \sup_{z\in\Pi^+} \Bigg(\frac{1}{(\Im z)^2} \int_{S(z,r\Im z)\cap K} \mathfrak{N}_\varphi(w) dA(w) \\ &\qquad\quad + \frac{1}{(\Im z)^2} \int_{S(z,r\Im z)\cap(\Pi^+\setminus K)} \mathfrak{N}_\varphi(w) dA(w) \Bigg) \\ &\quad\le C \sup_{z\in\Pi^+} \frac{1}{(\Im z)^2} A(S(z,r\Im z)\cap\Pi^+) \le C. \end{aligned}$$

Thus by (2.2), φ induces a bounded composition operator on $\mathcal{D}_{\Pi^+}$.

Next, we prove that φ induces a compact composition operator on $\mathcal{D}_{\Pi^+}$. In view of Theorem 2.3, we need to show that

$$\lim_{z\to\partial\widehat{\Pi^+}} \frac{1}{(\Im z)^2} \int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w) dA(w) = 0. \tag{2.10}$$

Since $K = \varphi(\Pi^+)$ is a a relatively compact subset of Π^+ and

$$\sup_{z\in\Pi^+} \frac{1}{(\Im z)^2} \int_{S(z,r\Im z)} \mathfrak{N}_\varphi(w) dA(w)$$

is finite, so (2.10) is vacuously true. Hence φ induces a compact composition operator on $\mathcal{D}_{\Pi^+}$. □

Recall that the *valence* of an analytic self-mapping φ of Π^+ is

$$N = \sup_{w\in\Pi^+} n_\varphi(w).$$

The function φ is said to have *bounded valence* if $N < \infty$, that is, if there is a positive integer N such that φ takes every value at most N times in Π^+.

Corollary 2.5. *Let φ be of bounded valence and φ maps Π^+ to a relatively compact subset of Π^+. Then φ induces a compact composition operator on $\mathcal{D}_{\Pi^+}$.*

The proof follows on the same lines as the proof of Corollary 2.4. We omit the details.

Example 2.6. Let

$$\varphi(z) = 2i + \frac{1}{(z+i)\log(z+ei)} \quad (z \in \Pi^+).$$

Then $\varphi(\Pi^+)$ is a relatively compact subset of Π^+ (see Example 4.4 in [CKS17]). Thus φ induces a compact composition operator on $\mathcal{D}_{\Pi^+}$.

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