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Degree product formula in the case of a finite group action

Piotr Bartłomiejczyk, Bartosz Kamedulski and Piotr Nowak-Przygodzki

Abstract. Let V, W be finite dimensional orthogonal representations of a finite group G. The equivariant degree with values in the Burnside ring of G has been studied extensively by many authors. We present a short proof of the degree product formula for local equivariant maps on V and W.

Contents

Introduction

One of the basic properties of the topological degree is the *product property*. Recall that a continuous map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ is

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called *local* if its set of zeros is compact. For such maps the classical Brouwer degree deg is well-defined and the product property holds. Namely,

Product property ([6, Prop. 8.7]). *Let $f\colon D_f \subset \mathbb{R}^m \to \mathbb{R}^m$ and $f'\colon D_{f'} \subset \mathbb{R}^n \to \mathbb{R}^n$ be local maps. Then $f \times f'\colon D_f \times D_{f'} \to \mathbb{R}^{m+n}$ is also a local map and*

$$\deg(f \times f') = \deg f \cdot \deg f'.$$

Our main goal is to present a short proof of an equivariant version of the product formula for equivariant local maps in the case of a finite group action. In that case the formula has an analogous form

$$\deg_G(f \times f') = \deg_G f \cdot \deg_G f',$$

but since the equivariant degree $\deg_G$ has its values in the *Burnside ring* of a finite group G, the multiplication on the right side of the formula takes place in this Burnside ring. It is worth pointing out that in [7] the authors proved the equivariant product formula in much more general setting i.e. in the case of a compact Lie group action. Unfortunately, this proof seems to be rather sketchy in some parts. We hope that our proof has the advantage of being straightforward and complete and can be seen as the first step towards proving the general case.

The paper is organized as follows. Section 1 contains preliminaries. In Section 2 we recall the concept of the equivariant degree $\deg_G$. Our main result is stated in Section 3. In Section 4 we introduce standard and polystandard maps and study their properties needed in the next section. Finally, Section 5 contains the proof of our main result.

1. Basic definitions

1.1. Local maps. The notation $A \Subset B$ means that A is a compact subset of B. For a topological space X, we denote by $\tau(X)$ the topology on X. For any topological spaces X and Y, let $\mathcal{M}(X,Y)$ be the set of all continuous maps $f\colon D_f \to Y$ such that D_f is an open subset of X. Let $\mathcal{R}$ be a family of subsets of Y. We define

$$\mathrm{Loc}(X,Y,\mathcal{R}) := \{\, f \in \mathcal{M}(X,Y) \mid f^{-1}(R) \Subset D_f \text{ for all } R \in \mathcal{R} \,\}.$$

We introduce a topology in $\mathrm{Loc}(X,Y,\mathcal{R})$ generated by the subbasis consisting of all sets of the form

- $H(C,U) := \{\, f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid C \subset D_f,\ f(C) \subset U \,\}$ for $C \Subset X$ and $U \in \tau(Y)$,
- $M(V,R) := \{\, f \in \mathrm{Loc}(X,Y,\mathcal{R}) \mid f^{-1}(R) \subset V \,\}$ for $V \in \tau(X)$ and $R \in \mathcal{R}$.

Elements of $\mathrm{Loc}(X,Y,\mathcal{R})$ are called *local maps*. The natural base point of $\mathrm{Loc}(X,Y,\mathcal{R})$ is the empty map. Let $\sqcup$ denote the union of two disjoint local maps. Moreover, in the case when $\mathcal{R} = \{\{y\}\}$ we will write $\mathrm{Loc}(X,Y,y)$ omitting double curly brackets. For more details we refer the reader to [4].

1.2. Equivariant maps. Assume that V is a real finite dimensional orthogonal representation of a finite group G. Let X be an arbitrary G-space. We say that $f\colon X \to V$ is *equivariant*, if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. We will denote by $\mathcal{C}_G(X, V)$ the space $\{f \in \mathrm{Loc}(X, V, 0) \mid f \text{ is equivariant}\}$ with the induced topology. Assume that Ω is an open invariant subset of V. Elements of $\mathcal{C}_G(\Omega, V)$ are called *equivariant local maps*.

1.3. Otopies. Let $I = [0, 1]$. We assume that the action of G on I is trivial. Any element of $\mathcal{C}_G(I \times \Omega, V)$ is called an *otopy*. Each otopy corresponds to a path in $\mathcal{C}_G(\Omega, V)$ and vice versa. Given an otopy $h\colon \Lambda \subset I \times \Omega \to V$ we can define for each $t \in I$:

- sets $\Lambda_t = \{x \in \Omega \mid (t, x) \in \Lambda\}$,
- maps $h_t\colon \Lambda_t \to V$ with $h_t(x) = h(t, x)$.

In this situation we say that h_0 and h_1 are *otopic*. Otopy gives an equivalence relation on $\mathcal{C}_G(\Omega, V)$. The set of otopy classes will be denoted by $\mathcal{C}_G[\Omega, V]$.

Remark 1.1. Observe that if $f \in \mathcal{C}_G(\Omega, V)$ and U is an open invariant subset of D_f such that $f^{-1}(0) \subset U$, then f and $f\restriction_U$ are otopic. In particular, if $f^{-1}(0) = \emptyset$ then f is otopic to the empty map.

1.4. G-actions. If H is a subgroup of G then

- (H) stands for the conjugacy class of H,
- NH is the normalizer of H in G,
- WH is the Weyl group of H i.e. $WH = NH/H$.

Recall that $G_x = \{g \in G \mid gx = x\}$. We define the following subsets of V:

$$V^H = \{x \in V \mid H \subset G_x\},$$
$$\Omega_H = \{x \in \Omega \mid H = G_x\},$$
$$\Omega_{(H)} = \{x \in \Omega \mid (H) = (G_x)\}.$$

The set $\mathrm{Iso}(\Omega) := \{(H) \mid H \text{ is a closed subgroup of } G \text{ and } \Omega_H \neq \emptyset\}$ is partially ordered. Namely, $(H) \leq (K)$ if H is conjugate to a subgroup of K.

We will make use of the following well-known facts:

- V^H is a linear subspace of V and an orthogonal representation of WH,
- Ω_H is open in V^H,
- the action of WH on Ω_H is free,
- $\Omega_{(H)}$ is a G-invariant submanifold of Ω,
- if (H) is maximal in $\mathrm{Iso}(\Omega)$ then $\Omega_{(H)}$ is closed in Ω.

1.5. Splitting of $\mathcal{C}_G[\Omega, V]$. Let Ω be an open invariant subset of a real finite dimensional orthogonal representation of a compact Lie group G. Assume that orbit types appearing in Ω are indexed (according to the partial order) by natural numbers $1, 2, \ldots, m$. Recall two splitting results concerning the set $\mathcal{C}_G[\Omega, V]$.

Theorem 1.2 ([1, Thm 5.4])**.** *There is a natural bijection*

$$\mathcal{C}_G[\Omega, V] \approx \mathcal{C}_{WH_m}\left[\Omega_{H_m}, V^{H_m}\right] \times \mathcal{C}_G\left[\Omega \setminus \Omega_{(H_m)}, V\right]. \tag{1.1}$$

Naively, it would seem that it is enough to define the above bijection by taking the otopy classes of the respective restrictions

$$[f] \mapsto \left(\left[f\restriction_{D_f \cap \Omega_{H_m}}\right], \left[f\restriction_{D_f \setminus \Omega_{(H_m)}}\right]\right).$$

Unfortunately, in general, it is not true that $f\restriction_{D_f \setminus \Omega_{(H_m)}} \in \mathcal{C}_G\left(\Omega \setminus \Omega_{(H_m)}, V\right)$. For that reason, we first need to perturbate f within its otopy class to guarantee that the restriction of the perturbation to the set $D_f \setminus \Omega_{(H_m)}$ is an element of $\mathcal{C}_G\left(\Omega \setminus \Omega_{(H_m)}, V\right)$. Moreover, our perturbation does not change f on $\Omega_{(H_m)}$. This procedure is described in detail in [1, Sec. 5].

If we apply induction to (1.1), we get immediately the following result.

Corollary 1.3 ([1, Thm 6.1])**.** *There is a natural bijection*

$$\Psi\colon \mathcal{C}_G[\Omega, V] \to \prod_{i=1}^{m} \mathcal{C}_{WH_i}\left[\Omega_{H_i}, V^{H_i}\right]. \tag{1.2}$$

For $k = 1, 2, \ldots, m$, let

$$\pi_k\colon \prod_{i=1}^{m} \mathcal{C}_{WH_i}\left[\Omega_{H_i}, V^{H_i}\right] \to \mathcal{C}_{WH_k}\left[\Omega_{H_k}, V^{H_k}\right]$$

denote the natural projection.

1.6. Burnside ring. Assume again that G is finite. Let $\mathcal{A}^+(G)$ be the set of isomorphism classes of finite G-sets. While disjoint union of finite G-sets induces addition on $\mathcal{A}^+(G)$, cartesian product with diagonal action induces multiplication, i.e.

$$[X] + [Y] = [X \sqcup Y], \quad [X] \cdot [Y] = [X \times Y],$$

where $[X]$, $[Y]$ are isomorphism classes of finite G-sets. The resulting structure is a commutative semi-ring with identity.

Since every finite G-set is a disjoint union of its orbits, each element of the semi-ring can be presented uniquely as $\sum d_{(H)}[{}^{G}\!/_{H}]$, where each $d_{(H)}$ is a non-negative integer and $[{}^{G}\!/_{H}]$ is the isomorphism class of ${}^{G}\!/_{H}$, which depends only on the conjugacy class of H. The problem of decomposing ${}^{G}\!/_{H} \times {}^{G}\!/_{K}$ into orbits makes multiplication in $\mathcal{A}^+(G)$ non-trivial.

The Grothendieck ring constructed from $\mathcal{A}^+(G)$ is denoted by $\mathcal{A}(G)$ and called the *Burnside ring* of G. Additively, it is a free abelian group generated by isomorphism classes $[{}^{G}\!/_{H}]$ of ${}^{G}\!/_{H}$. $\mathcal{A}(G)$ is a commutative ring with the unit $[{}^{G}\!/_{G}]$.

1.7. Local cross sections of a vector bundle. All manifolds considered are without boundary. Assume $p\colon E \to M$ is a smooth (i.e., C^1) vector bundle. We will identify M with the zero section of E. A *local cross section* of a bundle $p\colon E \to M$ is a continuous map $s\colon U \to E$, where U is open in M, $s^{-1}(M)$ is compact and $p \circ s = \mathrm{Id}_U$. Let $\Gamma(M, E)$ denote the set of all local cross sections of E over M.

Assume that $\operatorname{rank} E = \dim M$ and E is orientable as a manifold. Let us denote by $\mathrm{I}(s)$ the oriented intersection number of a local cross section s (see for instance [9, 10]), which is an integer. The intersection number is otopy invariant i.e., if two local cross sections are otopic then they have the same intersection number. Moreover, the following result holds. We write here $\Gamma[M, E]$ for the set of otopy classes of local cross sections of E over M.

Theorem 1.4 ([2, Thm 5.2])**.** *If M is connected then the intersection number $\mathrm{I}\colon \Gamma[M, E] \to \mathbb{Z}$ is a bijection.*

2. Degree $\deg_G$

In papers [2, 3, 7] the authors introduce the equivariant degree $\deg_G : \mathcal{C}_G(V, V) \to \mathcal{A}(G)$ for the action of a compact Lie group G and prove that the degree has the following expected properties.

Additivity property. *If $f, f' \in \mathcal{C}_G(V, V)$ and $D_f \cap D_{f'} = \emptyset$ then*

$$\deg_G(f \sqcup f') = \deg_G f + \deg_G f'.$$

Otopy invariance. *Let $f, f' \in \mathcal{C}_G(V, V)$. If f and f' are otopic then*

$$\deg_G f = \deg_G f'.$$

Solution property. *If $\deg_G f \neq 0$ then $f(x) = 0$ for some $x \in D_f$.*

In order to formulate the next property, it is necessary to introduce some notation and make some assumptions. Let $B(p, r)$ denote the open r-ball in V around p. Assume that G is finite, $x \in V$ and $f\colon \cup_{y \in Gx} B(y, \epsilon) \to V$, where $\epsilon < \frac{1}{2} \min\{|a - b| \mid a, b \in Gx, a \neq b\}$.

Normalization property. *If $f(y + v) = v$ for $y \in Gx$ and $|v| < \epsilon$, then $f \in \mathcal{C}_G(V, V)$, $f^{-1}(0) = Gx$ and $\deg_G f = [G/G_x]$.*

Remark 2.1. The equivariant degree, as an element of the Burnside ring, consists of multiple coefficients. Examination of these allows not only to find orbits of zeros, but also to analyze their orbit types.

Recall here that the main goal of this paper is to show that the degree $\deg_G$ has the *product property* as well.

3. Main result

Assume that

- G is a finite group,

- V and W are real finite dimensional orthogonal representations of G.

Recall that $\mathcal{C}_G(V,V)$ denotes the space of equivariant local maps in V.

Main Theorem. *If $f \in \mathcal{C}_G(V,V)$ and $f' \in \mathcal{C}_G(W,W)$, then $f \times f' \in \mathcal{C}_G(V \oplus W, V \oplus W)$ and*

$$\deg_G(f \times f') = \deg_G f \cdot \deg_G f',$$

where "$\cdot$" denotes the multiplication in the Burnside ring $\mathcal{A}(G)$.

4. Standard and polystandard maps

In this section we introduce standard and polystandard maps and study their basic properties. These maps will play the crucial role in the proof of Main Theorem in the next section. Let us start with recalling the definition of an ϵ-normal neighbourhood. Assume that:

- Y is a linear subspace of $\mathbb{R}^n$,
- U is an open subset of Y.

Let $Y^\perp$ denote the orthogonal complement of Y in $\mathbb{R}^n$. For $\epsilon > 0$ let us denote by U^ϵ the set $U^\epsilon = \{x + v \mid x \in U, v \in Y^\perp, |v| < \epsilon\}$. Any such set will be called an *ϵ-normal neighbourhood* of U.

Now we are ready to introduce two important classes of maps: standard and polystandard. A map $f \in \mathcal{C}_G(V,V)$ is called *standard* if

- $f^{-1}(0) = Gx_0$ for some $x_0 \in D_f$,
- there is an open subset U of V_H, where $H = G_{x_0}$, and $\epsilon > 0$ such that:
  - $f^{-1}(0) \cap U = \{x_0\}$,
  - $U^\epsilon \subset D_f$,
  - $f(x+v) = f(x) + v$ for all $x \in U, v \in (V^H)^\perp, |v| < \epsilon$.

In such situation we also say that f is standard with respect to x_0, U and ϵ. A map $f \in \mathcal{C}_G(V,V)$ is called *m-standard* if there are standard maps f_i $(i = 1, \ldots, m)$ with disjoint domains such that:

$$f^{-1}(0) \subset \sqcup_{i=1}^m D_{f_i} \subset D_f.$$

If a map is m-standard for some m, we call it *polystandard*. A finite disjoint union of standard maps is called *strictly polystandard*. By Remark 1.1, any polystandard map is otopic to a strictly polystandard one.

In what follows, we will need the notation that relates the classical topological and equivariant degrees. Let f be standard with respect to x, U and ϵ and let $\alpha = Gx$. Consider a map $f_x \colon U \subset V_H \to V^H$ given by $f_x = f \restriction_U$. Let $d_x = \deg(f_x, U)$.

Proposition 4.1. *For each $g \in G$ the equality $d_x = d_{gx}$ holds.*

Proof. Let $g \in G$ and $K = G_{gx}$. Then $V^K = gV^H$. Consider a map $f_{gx} \colon gU \subset V_K \to V^K$ satisfying $f_{gx}(y) = gf_x(g^{-1}y)$. Since $g \colon V^H \to V^K$ is an isomorphism of linear spaces,

$$d_{gx} = \deg(f_{gx}, gU) = \deg(f_x, U) = d_x. \qquad \square$$

Define $d_\alpha = d_y$, where y is any element of α. Proposition 4.1 guarantees that the integer d_α is well-defined. The main advantage of standard and polystandard maps is that we can immediately compute their equivariant degree $\deg_G$ if we know the value of d_α. Namely, let f be a standard map and $\alpha = Gx = f^{-1}(0)$. Then

$$\deg_G f = d_\alpha[G/G_x] = d_\alpha[\alpha].$$

More generally, let f be a m-standard map, and let $\{\alpha_i\}_{i=1}^m$ denote the set of orbits of zeros of f. Then

$$\deg_G f = \sum_{i=1}^m d_{\alpha_i}[\alpha_i].$$

5. Proof of Main Theorem

To prove Main Theorem we will need two lemmas.

Lemma 5.1. *Let f, f' be standard maps and $\alpha = f^{-1}(0)$, $\beta = (f')^{-1}(0)$. Then $f \times f'$ is polystandard and for each orbit $\gamma \subset \alpha \times \beta$ we have:*

$$d_\gamma = d_\alpha \cdot d_\beta.$$

Moreover,

$$\deg_G(f \times f') = \deg_G f \cdot \deg_G f'.$$

Proof. First we show that $f \times f'$ is polystandard. Assume that f is standard with respect to x_0, U and ϵ and f' is standard with respect to x_0', U' and ϵ'. Let $H = G_{x_0}$ and $K = G_{x_0'}$. Note that $(f \times f')^{-1}(0) = Gx_0 \times Gx_0'$ is a finite union of orbits and $G_{(x_0,x_0')} = H \cap K$. Since $U^\epsilon \times U'^{\epsilon'}$ is open in $V \oplus W$ and

$$(x_0, x_0') \in V_H \times W_K \subset (V \oplus W)_{H\cap K},$$

there exists an open subset $U'' \subset (V \oplus W)_{H\cap K}$ and $\epsilon'' > 0$ such that $(f \times f')^{-1}(0) \cap U'' = \{(x_0, x_0')\}$ and $U''^{\epsilon''} \subset U^\epsilon \times U'^{\epsilon'}$.

Now let us check that $(f \times f')(x'' + w'') = (f \times f')(x'') + w''$ for $x'' \in U''$, $w'' \in ((V \oplus W)^{H\cap K})^\perp$, $|w''| < \epsilon''$. Note that since $U'' \subset U^\epsilon \times U'^{\epsilon'}$, x'' has the unique representation as $(x, x') + (v, v')$, where $(x, x') \in U \times U'$ and $(v, v') \in (V^H)^\perp \oplus (W^K)^\perp$. Moreover, since

$$((V \oplus W)^{H\cap K})^\perp \subset (V^H)^\perp \oplus (W^K)^\perp,$$

w'' can be uniquely written as (w, w') with $w \in (V^H)^\perp$ and $w' \in (W^K)^\perp$. Hence

$$\begin{aligned}(f \times f')(x'' + w'') &= (f \times f')(x + v + w, x' + v' + w') \\ &= \big(f(x + v + w), f'(x' + v' + w')\big) = \big(f(x) + v + w, f'(x') + v' + w'\big) \\ &= \big(f(x + v) + w, f'(x' + v') + w'\big) = (f \times f')(x'') + w'',\end{aligned}$$

which proves that $f \times f'$ is polystandard.

Next we show the formula $d_\gamma = d_\alpha \cdot d_\beta$. Let $(x_0, x_0') \in \gamma \subset \alpha \times \beta$. Observe that

$$d_\alpha = d_{x_0} = \deg(f_{x_0}, U) \stackrel{1}{=} \deg(f, U^\epsilon),$$
$$d_\beta = d_{x_0'} = \deg(f'_{x_0'}, U') \stackrel{1}{=} \deg(f', U'^{\epsilon'}).$$

Thus we get

$$d_\gamma = d_{(x_0,x_0')} = \deg((f \times f')_{(x_0,x_0')}, U'') \stackrel{1}{=} \deg(f \times f', U''^{\epsilon''})$$
$$\stackrel{2}{=} \deg(f \times f', U^\epsilon \times U'^{\epsilon'}) \stackrel{1}{=} \deg(f, U^\epsilon) \cdot \deg(f', U'^{\epsilon'}) = d_\alpha \cdot d_\beta.$$

In the above we used two properties of the classical topological degree: the product formula (1) and the localization of zeros (2).

Finally, we prove the product formula for standard maps. As we have shown, $f \times f'$ is m-standard for some m. Decompose $\alpha \times \beta$ into the disjoint union of orbits $\bigsqcup_{i=1}^m \gamma_i$. We have $d_{\gamma_i} = d_\alpha \cdot d_\beta$ for each $i = 1, 2, \ldots, m$. Thus we get

$$\deg_G(f \times f') = \sum_{i=1}^m d_{\gamma_i}[\gamma_i] = \sum_{i=1}^m d_\alpha d_\beta [\gamma_i] = d_\alpha d_\beta \sum_{i=1}^m [\gamma_i]$$
$$= d_\alpha d_\beta [\alpha \times \beta] = d_\alpha[\alpha] \cdot d_\beta[\beta] = \deg_G f \cdot \deg_G f',$$

which establishes the desired formula. □

We precede the next lemma by recalling the following notation. Orbit types in V are indexed (according to the partial order) by natural numbers $i = 1, 2, \ldots, m$. In particular, $H_1 = G$. Write $M_i = V_{H_i}/WH_i$ and $E_i = \left(V_{H_i} \times V^{H_i}\right)/WH_i$. Recall that $p_i \colon E_i \to M_i$ is a vector bundle such that $\operatorname{rank} E_i = \dim M_i$ and E_i is orientable as a manifold. Moreover, the bundle $E_i \to M_i$ is naturally isomorphic to the tangent bundle $TM_i \to M_i$.

Recall that $\Gamma(M_i, E_i)$ ($\Gamma[M_i, E_i]$) stands for the set of (otopy classes of) local cross sections of the bundle p_i. Moreover, let us denote by

$$\Theta_i \colon \mathcal{C}_{WH_i}\left(V_{H_i}, V^{H_i}\right) \to \Gamma(M_i, E_i)$$

the function defined by the formula $\Theta_i(f)([x]) = [(x, f(x))]$, where $x \in V_{H_i}$ and by

$$\Xi_i \colon \mathcal{C}_{WH_i}\left[V_{H_i}, V^{H_i}\right] \to \Gamma[M_i, E_i]$$

the function given by $\Xi_i([f]) = [\Theta_i(f)]$. Since WH_i acts freely on V_{H_i}, both Θ_i and Ξ_i are bijections (see [2, Thm 4.1]).

Let $\{M_{ij}\}_j$ denote the set of connected components of the manifold M_i and $n(i)$ denote the number of these components, which is finite or countable. Observe that, by Theorem 1.4, the function

$$\mathrm{I}_i \colon \Gamma[M_i, E_i] \to \sum_{j=1}^{n(i)} \mathbb{Z}$$

defined by $\mathrm{I}_i([s]) = \{\mathrm{I}\left(s\restriction_{M_{ij}}\right)\}_{j=1}^{n(i)}$ is a bijection.

Now we are ready to define the function

$$\Phi\colon \mathcal{C}_G(V,V) \to \prod_{i=1}^{m}\left(\sum_{j=1}^{n(i)}\mathbb{Z}\right)$$

required in the formulation of the next lemma. Namely, let

$$\Phi(f) := \{(\mathrm{I}_i \circ \Xi_i \circ \pi_i \circ \Psi)([f])\}_{i=1}^{m}$$

with the notation Ψ and π_i introduced in Subsection 1.5. By [2, Thm 5.2] the function Φ has the following properties

- $\Phi(\mathcal{C}_G(V,V)) = \begin{cases} \prod_{i=1}^{m}\left(\sum_{j=1}^{n(i)}\mathbb{Z}\right) & \text{if } \dim V^G > 0, \\ \{0,1\}\times\prod_{i=2}^{m}\left(\sum_{j=1}^{n(i)}\mathbb{Z}\right) & \text{if } \dim V^G = 0, \end{cases}$
- the function induced by Φ on $\mathcal{C}_G[V,V]$, which will be denoted by the same letter, is an injection.

Lemma 5.2. *For any system $\{c_{ij}\} \in \Phi(\mathcal{C}_G(V,V))$ there is a strictly polystandard map f such that $\Phi(f) = \{c_{ij}\}$.*

Proof. We need to consider two cases.

Case 1: $\dim V^G > 0$. Under that assumption we have $\dim M_i > 0$ for each $i = 1, 2, \ldots, m$. Fix $\{c_{ij}\} \in \Phi(\mathcal{C}_G(V,V))$, where $c_{ij} \in \mathbb{Z}$. On the component M_{ij} choose $|c_{ij}|$ points together with their disjoint disc neighbourhoods. Let us denote by P_{ij} the set of these points and by F_{ij} the union of their neighbourhoods. By Corollary A.2 from Appendix A, there is a local cross section $s_{ij}\colon F_{ij} \subset M_{ij} \to E_i$ such that

$$s_{ij}^{-1}(M_{ij}) = P_{ij} \quad \text{and} \quad I(s_{ij}) = c_{ij}.$$

Next we define a local cross section $s_i\colon F_i \subset M_i \to E_i$ as a disjoint union $s_i = \sqcup_{j=1}^{n(i)} s_{ij}$. Note that the set $\cup_{i,j} P_{ij}$ is finite, because only a finite number of c_{ij} are nonzero. Set $f_i = \Theta_i^{-1}(s_i)$. By the definition of Θ_i, $f_i \in \mathcal{C}_{WH_i}\left(V_{H_i}, V^{H_i}\right)$ and by the construction of s_i, the domain D_{f_i} is the union of disjoint discs around all points in $f_i^{-1}(0)$. Consequently, the set $D := \cup_{i=1}^{m} D_{f_i}$ is a finite disjoint union of discs R_k such that every disc contains one zero of a given f_i. Thus $D = \sqcup_k R_k$. Observe that there is $\epsilon > 0$ such that the sets R_k^ϵ are pairwise disjoint. Since any point of $\sqcup_k R_k^\epsilon$ can be uniquely represented in the form $gx + gv$, where $x \in D_{f_i}$, $v \in (V^{H_i})^\perp$, $|v| < \epsilon$, let us define $f\colon D_f := \sqcup_k R_k^\epsilon \to V$ by

$$f(gx + gv) = gf_i(x) + gv$$

for all $g \in G$, $x \in D_{f_i}$, $v \in (V^{H_i})^\perp$, $|v| < \epsilon$. Our construction guarantees that

- $f \in \mathcal{C}_G(V,V)$,
- f is strictly polystandard,
- $\Phi(f) = \{c_{ij}\}$.

CASE 2: $\dim V^G = 0$. In that situation, $M_1 = \{0\}$ and $\dim M_i > 0$ for $i > 1$. Analogously as in the previous case, we define $f\colon D_f \to V$, but now in the construction of f we take into account M_i only for $i > 1$. By choosing the disc neighbourhoods small enough, we can guarantee that $0 \notin \mathrm{cl}(D_f)$. Hence there is $\delta > 0$ such that $B(0,\delta) \cap D_f = \emptyset$, where $B(0,\delta)$ denotes the open δ-ball in V around the origin. Set

$$\widetilde{f} = \begin{cases} f & \text{if } c_{11} = 0, \\ f \sqcup \mathrm{Id}\restriction_{B(0,\delta)} & \text{if } c_{11} = 1. \end{cases}$$

It is easy to see that $\Phi\big(\widetilde{f}\,\big) = \{c_{ij}\}$. □

Corollary 5.3. *In each otopy class in $\mathcal{C}_G(V,V)$ there is a strictly polystandard map.*

Proof. Recall that $\Phi\colon \mathcal{C}_G[V,V] \to \prod\big(\sum \mathbb{Z}\big)$ is an injection. Let $[f] \in \mathcal{C}_G[V,V]$. By Lemma 5.2, there is a strictly polystandard map $f' \in \mathcal{C}_G(V,V)$ such that $\Phi([f']) = \Phi([f])$. From the injectivity of Φ, $f' \in [f]$. □

It occurs that Main Theorem is now a consequence of Lemma 5.1 and Corollary 5.3.

Proof of Main Theorem. The fact that $f \times f' \in \mathcal{C}_G(V \oplus W, V \oplus W)$ is obvious. By Corollary 5.3, f and f' are otopic to strictly polystandard maps $\sqcup_k f_k$ and $\sqcup_l f'_l$ respectively, where f_k and f'_l are standard. In consequence, $f \times f'$ is otopic to $\sqcup_k f_k \times \sqcup_l f'_l = \sqcup_{k,l}(f_k \times f'_l)$. Hence

$$\begin{aligned} \deg_G(f \times f') &\overset{1}{=} \deg_G \sqcup_{k,l}(f_k \times f'_l) \overset{2}{=} \sum_{k,l} \deg_G(f_k \times f'_l) \\ &\overset{3}{=} \sum_{k,l} \deg_G f_k \cdot \deg_G f'_l = \Big(\sum_k \deg_G f_k\Big) \cdot \Big(\sum_l \deg_G f'_l\Big) \\ &\overset{2}{=} \Big(\deg_G \sqcup_k f_k\Big) \cdot \Big(\deg_G \sqcup_l f'_l\Big) \overset{1}{=} \deg_G f \cdot \deg_G f' \end{aligned}$$

from the otopy invariance property (1), the additivity property (2) and Lemma 5.1 (3). This completes the proof. □

Remark 5.4. Apart from the equivariant degree $\deg_G$, the equivariant gradient degree $\deg_G^\nabla$ with values in the Euler-tom Dieck ring $\mathcal{U}(G)$ is also considered, studied and applied (see for example [3, 5, 8, 11]). In many situations $\deg_G^\nabla$ gives more information than $\deg_G$. However, for a finite group the Burnside ring $\mathcal{A}(G)$ and the Euler-tom Dieck ring $\mathcal{U}(G)$ are identical. Moreover, $\deg_G = \deg_G^\nabla$ if we restrict ourselves to equivariant gradient local maps. In consequence, in the case of a finite group action the product formula holds also for $\deg_G^\nabla$.

Appendix A.

Assume that $p\colon E \to M$ is a vector bundle over a manifold M such that $\dim M > 0$, $\operatorname{rank} E = \dim M$ and E is orientable as a manifold.

Lemma A.1. *For any $q \in M$, any disc neighbourhood D of q and any $\alpha \in \{-1, 1\}$ there is a local cross section $s\colon D \subset M \to E$ such that $s^{-1}(M) = \{q\}$ and $I(s) = \alpha$.*

Proof. Let $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$ and $TB = B \times \mathbb{R}^n$. Consider a local cross section $s_A\colon B \to TB$ given by $s_A(x) = (x, Ax)$, where A is linear and $\det A = \alpha$. Observe that $I(s_A) = \alpha$.

Let us note that we can identify TD with $E\,|_D$. Take a diffeomorphism $\varphi : B \to D$ such that $\varphi(0) = q$. It induces a tangent map $T\varphi\colon TB \to TD$. Finally, define a local cross section $s\colon D \to E$ by formula $s(x) = T\varphi(s_A(\varphi^{-1}(x)))$. It is easy to see that $s^{-1}(M) = q$ and $I(s) = I(s_A) = \alpha$. □

An immediate consequence of the above lemma is the following result.

Corollary A.2. *Let $l \in \mathbb{Z}\setminus\{0\}$. For any set Q of $|l|$ distinct points in M and any set of disjoint disc neighbourhoods of these points there is a local cross section s defined on the union of these neighbourhoods such that $s^{-1}(M) = Q$ and $I(s) = l$.*

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(P. Bartłomiejczyk) Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland
piobartl@pg.edu.pl

(B. Kamedulski) Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, Wita Stwosza 57, 80-308 Gdańsk, Poland, and Gdynia Maritime University, Department of Mathematics, Morska 81-87, 81-225 Gdynia, Poland
bartosz.kamedulski@gmail.com

(P. Nowak-Przygodzki) Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland
piotrnp@wp.pl

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