New York Journal of Mathematics

New York J. Math. **25** (2019) 374–395.

One-sided approximation in affine function spaces

David Handelman and **Damien Roy**

Abstract. Let H be a subgroup of a partially ordered abelian group G with order unit u, and let $S(G,u)$ denote the convex subset of $\mathbb{R}^G$ consisting of all traces (states) τ on G with $\tau(u) = 1$. We say that H has property (B) if, for any integer $m \geq 2$, any $h \in H$ and any $\epsilon > 0$, there exists $h' \in H$ such that $\tau(h) - m\tau(h') \geq -\epsilon$ for each $\tau \in S(G,u)$. We show that, if $S(G,u)$ is finite-dimensional, this condition is equivalent to asking that $\tau(H)$ is $\{0\}$ or dense in $\mathbb{R}$ for all τ in the smallest face of $S(G,u)$ containing all traces that vanish identically on H. When G is a simple dimension group and H is a convex subgroup of G, we show that G/H is unperforated if and only if H has property (B). We apply both results to provide a criterion for a trace of G to be refinable when G is a simple dimension group with finitely many pure traces.

Contents

Received January 11, 2019.

2010 *Mathematics Subject Classification.* Primary 19K14; Secondary 11J25, 15A39, 06F20, 37A55.

Key words and phrases. partially ordered abelian groups, Choquet theory, approximation, trace, Gordan's theorem, Farkas' lemma, unperforation, refinable measure, diophantine inequalities, Kronecker theorem.

Both authors supported in part by NSERC Discovery grants.

ISSN 1076-9803/2019

1. Introduction

Throughout, n will stand for a positive integer. We denote by $(\mathbb{R}^n)^*$, the dual space to $\mathbb{R}^n$, consisting of the linear functionals on $\mathbb{R}^n$; by $(\mathbb{R}^n)^+$, the set of elements of $\mathbb{R}^n$ with non-negative coordinates; and by $(\mathbb{R}^n)^{++}$, the set of those with strictly positive coordinates. We also denote by $\|x\|$ the Euclidean norm of a point $x \in \mathbb{R}^n$.

Let H be a subgroup of $\mathbb{R}^n$, and let $m > 1$ be an integer. The condition

(A_m) for all $\epsilon > 0$ and $h \in H$, there exists $h' \in H$ such that $\|h - mh'\| \leq \epsilon$

is independent of m and is equivalent to H being dense in the vector space $\mathbb{R}\,H$ that it spans; so we may as well refer to it as property (A). By a theorem of Kronecker, this in turn implies the following.

Theorem A. *Let m and H be as above. Then H satisfies property (A_m) if and only if for all $\tau \in (\mathbb{R}^n)^*$, either $\tau(H) = \{0\}$ or $\tau(H)$ is dense in $\mathbb{R}$.*

Here, we are interested in the following one-sided approximation property:

(B_m) for all $\epsilon > 0$ and $h \in H$, there exists $h' \in H$ such that all coordinates of $h - mh'$ are bounded below by $-\epsilon$.

For example, the discrete subgroup $H = \mathbb{Z}^n$ of $\mathbb{R}^n$ has this property, while $H = \mathbb{Z}(1, -1)$ inside $\mathbb{R}^2$ does not.

Our principal result, in this context, characterizes subgroups H satisfying this property, in terms of *positive* linear functionals in a fashion analogous to that in Theorem A. Recall that a positive linear functional on $\mathbb{R}^n$ is a linear functional sending $(\mathbb{R}^n)^+$ to $\mathbb{R}^+$ or, equivalently, that sends the standard basis elements to nonnegative real numbers. Let K_n denote the usual standard $(n-1)$-simplex consisting of positive linear functionals τ in $(\mathbb{R}^n)^*$ such that $\tau(1, \ldots, 1) = 1$; its vertices are the n coordinate functions $\tau_1, \ldots, \tau_n$ where $\tau_i \colon \mathbb{R}^n \to \mathbb{R}$ is projection onto the ith coordinate.

Theorem B. *Let m and H be as above. Define F to be the smallest face of K_n containing the set*

$$Z(H) := \{\tau \in K_n \mid \tau(H) = \{0\}\}.$$

Then H satisfies property (B_m) if and only if for each $\tau \in F$, either we have $\tau(H) = \{0\}$ or $\tau(H)$ is dense in $\mathbb{R}$.

The set $Z(H)$ is a compact convex subset of K_n. If it is empty, then F is empty and the condition is vacuous, so H satisfies (B_m) in this case. We will see in section 5 that this happens if and only if $H \cap (\mathbb{R}^n)^{++} \neq \emptyset$. This can also be viewed as a consequence of Gordan's theorem (see Appendix A). In general, F is the convex hull of the set of projections τ_i for which $Z(H)$ contains at least one element of the form $a_1\tau_1 + \cdots + a_n\tau_n$ with $a_i > 0$.

When H is finitely generated with basis $\{h_1, \ldots, h_s\}$, property (B_m) for H amounts to the solvability of a system of Diophantine inequalities, namely the conditions that for all $\epsilon > 0$, all $\sigma \in \{-1, 1\}$ and all $i = 1, \ldots, s$, there exist integers $a_1, \dots, a_s \in \mathbb{Z}$ such that all coordinates of the point $\sigma h_i - a_1 h_1 - \cdots - a_s h_s$ are at least $-\epsilon$. In Section 9, we give a class of examples to which Theorem B applies, thereby solving the corresponding system of Diophantine inequalities.

It follows from Theorem B that property (B_m) is independent of m. In the next section, we extend this condition to an arbitrary subgroup H of a partially ordered abelian group G with order unit and we show that, in that context, it is again independent of m. From that point, we refer to it as property (B).

In section 3, we establish a necessary condition for H to have property (B) which, for $G = \mathbb{R}^n$, reduces to that of Theorem B. We show in section 5 that this condition is also sufficient when the trace space of G has finite dimension, thereby completing the proof of Theorem B. A construction in section 6 shows however that this is not true for a general group G. When G is a simple dimension group and H is a convex subgroup of G for which G/H is torsion-free, we prove in section 7 that G/H is unperforated if and only if H has property (B). This complements [BH, Proposition B.1] where the condition is shown to be sufficient.

Section 8 answers a question of [BH] by giving necessary and sufficient conditions for a trace of G to be refinable when G is a simple dimension group with finitely many pure traces. The notion of refinable trace arose from a property of measures on Cantor sets due to Akin [Ak], put in the context of invariant probability measures on Cantor dynamical systems, and subsequently translated to the setting of dimension groups by S. Bezuglyi and the first author via the ordered K_0 functor [BH]. Finally, Appendix A explains the connection between Gordan's theorem, Farkas's lemma, and some of our results.

2. Condition (B_m) in partially ordered abelian groups

By a *partially ordered abelian group*, we mean an abelian group G equipped with a translation invariant partial order $\leq$. The *positive cone* of such a pair $(G, \leq)$ is the set $G^+ = \{x \in G \mid 0 \leq x\}$. It satisfies

$$(G^+) + (G^+) \subseteq G^+ \quad \text{and} \quad (G^+) \cap (-G^+) = \{0\}.$$

Conversely, any subset G^+ of an abelian group G satisfying these conditions makes G into a partially ordered abelian group upon defining, for $x, y \in G$, that $x \leq y \iff y - x \in G^+$. As usual, we write $x < y$ when $x \leq y$ and $x \neq y$.

An *order unit* (or *strong unit*) of such a group G is a nonzero element, u, of G^+ such that for each $g \in G$, there exists a positive integer N with $-Nu \leq g \leq Nu$. The set of all order units of G is denoted G^{++}.

Let (G, u) be a partially ordered group with order unit u. A *trace* (or *state*) on G is a nonzero group homomorphism $\tau\colon G \to \mathbb{R}$ that is positive in the sense that $\tau(G^+) \subseteq \mathbb{R}^+$. We say that a trace τ is *normalized at u* if

$\tau(u) = 1$, and we denote by $S(G, u)$ the set of those traces. It is a compact and convex subset of $\mathbb{R}^G$ with respect to the product topology on $\mathbb{R}^G$, see [G; Proposition 6.2]. We denote by $\mathrm{Aff}\, S(G, u)$ the vector space consisting of all convex-linear continuous real-valued functions on $S(G, u)$, and we equip it with the supremum norm—so that it becomes a Banach space. It is also a partially ordered abelian group with respect to pointwise ordering where, for $\varphi, \psi \in \mathrm{Aff}\, S(G, u)$, we write $\varphi \le \psi$ when $\varphi(\tau) \le \psi(\tau)$ for all $\tau \in S(G, u)$.

The *affine representation* of (G, u) is the second dual map from G to $\mathrm{Aff}\, S(G, u)$ which, to each group element $g \in G$, associates the evaluation map $\widehat{g}\colon S(G, u) \to \mathbb{R}$ given by $\widehat{g}(\tau) = \tau(g)$ for all $\tau \in S(G, u)$. This is an order-preserving group homomorphism. Within $\mathrm{Aff}\, S(G, u)$, we identify $\mathbb{R}$ with the subspace of constant functions so that, for $g \in G$ and $a \in \mathbb{R}$, the condition $a \le \widehat{g}$ simply means $a \le \tau(g)$ for all $\tau \in S(G, u)$. For much more on this and other aspects of partially ordered abelian groups, the reader is referred to [Go].

We view $\mathbb{R}^n$ as a partially ordered abelian group with respect to the usual coordinatewise ordering. Its positive cone is $(\mathbb{R}^n)^+$ and its set of order units is $(\mathbb{R}^n)^{++}$, as defined in the introduction. In particular, the vector

$$\mathbf{1} = (1, \dots, 1),$$

with all coordinates equal to 1, is an order unit of $\mathbb{R}^n$. The corresponding trace space $S(\mathbb{R}^n, \mathbf{1})$ is the simplex K_n spanned by the coordinate functions $\tau_1, \dots, \tau_n$ in $(\mathbb{R}^n)^*$, and the affine representation from $\mathbb{R}^n$ to $\mathrm{Aff}\, K_n$ is an isomorphism of vector spaces over $\mathbb{R}$. For $g \in \mathbb{R}^n$ and $a \in \mathbb{R}$, the condition $a \le \widehat{g}$ reduces to $a \le \tau_i(g)$ for each $i = 1, \dots, n$. Thus, for a subgroup H of $\mathbb{R}^n$ and an integer $m \ge 2$, condition (B_m) can be restated as follows:

(B_m) for all $h \in H$ and $\epsilon > 0$, there exists $h' \in H$ such that $\widehat{h} - m\widehat{h'} \ge -\epsilon$.

We use this as a definition of the property (B_m) for a subgroup H of G. When it holds, then so do (B_n) for all divisors $n > 1$ of m, and so do (B_{m^j}) for all integers $j \ge 1$. More generally, the next result shows that the conditions (B_m) with $m > 1$ are mutually equivalent.

Proposition 2.1. *Let (G, u) be as above. Suppose that a subgroup H of G satisfies (B_m) for some integer $m > 1$. Then H satisfies (B_n) for all integers $n > 1$.*

In view of this, we simply say that H has property (B) if it satisfies (B_m) for some (and thus all) integers $m > 1$.

Proof. Let $h \in H$, let $n > 1$ be an integer and let $\epsilon > 0$. Since u is an order unit, there exists $\ell \in \mathbb{N}$ such that $h \le \ell u$. Choose an integer $j \ge 1$ such that $m^j \ge (2\ell + \epsilon)n/\epsilon$. Since H satisfies (B_m), it satisfies (B_{m^j}), and so there exists $h' \in H$ such that

$$\widehat{h} - m^j \widehat{h'} \ge -\epsilon/2.$$

Since $h \le \ell u$, we have $\widehat{h} \le \ell$ and so the above inequality yields

$$\widehat{h'} \le m^{-j}(\widehat{h} + \epsilon/2) \le m^{-j}(\ell + \epsilon/2) \le \epsilon/(2n).$$

Writing $m^j = qn - r$ with integers $q \ge 1$ and $0 \le r < n$, we conclude that

$$\widehat{h} - nq\widehat{h'} \ge -r\widehat{h'} - \epsilon/2 \ge -\epsilon,$$

thus $\widehat{h} - n\widehat{h''} \ge -\epsilon$ where $h'' = qh' \in H$. This shows that (B_n) is satisfied. $\square$

If G_0 is a subgroup of G containing u, then (G_0, u) is a partially ordered abelian group with order unit, with positive cone $G_0^+ = G^+ \cap G_0$. Importantly, by [GoH1, Theorem 3.2] (see also [Go, Corollary 4.3]), the map $\rho\colon S(G, u) \to S(G_0, u)$ sending a trace on G to its restriction to G_0 is a *surjective* affine (convex-linear) homomorphism. We deduce that the property (B) for a subgroup H of G depends only on the induced ordering on $H + \mathbb{Z}u$.

Proposition 2.2. *Let (G, u) be as above, let H be a subgroup of G, and let $G_0 = H + \mathbb{Z}u$. Then H has property (B) within (G, u) if and only if it has property (B) within (G_0, u).*

Proof. Let $m > 1$ be an integer. The condition (B_m) for H within (G, u) requests that, for each $h \in H$ and each $\epsilon > 0$, there exists $h' \in H$ such that $\tau(h - mh') \ge -\epsilon$ for all $\tau \in S(G, u)$. The condition within (G_0, u) is the same except that τ varies in $S(G_0, u)$. In view of the surjectivity of the restriction map from $S(G, u)$ to $S(G_0, u)$, the two conditions are thus the same. $\square$

3. A necessary condition for property (B)

Let H be a subgroup of a partially ordered abelian group with order unit (G, u). The set

$$Z_G(H) := \{\tau \in S(G, u) \mid \tau(h) = 0 \text{ for all } h \in H\}$$

is a compact convex subset of $S(G, u)$. When $G = \mathbb{R}^n$ and $u = \mathbf{1}$, this is the set denoted $Z(H)$ in Theorem B.

A *face* of $S(G, u)$ is a (possibly empty) subset F of $S(G, u)$ such that any line segment in $S(G, u)$ whose relative interior meets F, is contained in F. For any subset Z of $S(G, u)$, there is a smallest face containing Z. When Z is convex (such as the set $Z_G(H)$ defined above), it consists of all $\tau_1 \in S(G, u)$ for which there exist $\tau_2 \in S(G, u)$ and $\lambda \in (0, 1)$ such that $\lambda\tau_1 + (1 - \lambda)\tau_2 \in Z$, see [G, Proposition 5.7]. The *extreme boundary* of $S(G, u)$, denoted $\partial_e S(G, u)$, is the set of all traces $\tau \in S(G, u)$, called *pure traces*, which by themselves constitute faces $\{\tau\}$ of $S(G, u)$. For example, $\partial_e S(\mathbb{R}^n, \mathbf{1})$ is the set of coordinates functions $\{\tau_1, \dots, \tau_n\}$ in $(\mathbb{R}^n)^*$.

The next result provides a necessary condition for property (B) to hold. For $(G, u) = (\mathbb{R}^n, \mathbf{1})$, it is the same condition as in Theorem B. The proof that we provide below is an adaptation of the arguments from [BH, Proposition B.4].

Theorem 3.1. *Let (G,u) and H be as above, and let F be the smallest face of $S(G,u)$ containing $Z_G(H)$. Suppose that H satisfies condition (B). Then for any τ in F, either $\tau(H)$ is zero, or it is dense in $\mathbb{R}$.*

Proof. Pick τ_1 in F. Since $Z_G(H)$ is a convex subset of $S(G,u)$, there exist $\tau_2 \in S(G,u)$ and $\lambda \in (0,1)$ such that $\lambda\tau_1 + (1-\lambda)\tau_2 \in Z_G(H)$. Then, for any h in H, we have

$$\tau_1(h) = -\theta\tau_2(h), \quad \text{where } \theta = \frac{1-\lambda}{\lambda} > 0.$$

Suppose that $\tau_1(H) = \mathbb{Z}\delta$ for some real $\delta > 0$. For every h in H, it follows that

$$(\tau_1(h), \tau_2(h)) = \ell\delta(1, -\theta) \quad \text{for some } \ell \in \mathbb{Z}.$$

Setting $\epsilon = \delta\min\{1,\theta\}/2$, we deduce that $(\tau_1(h), \tau_2(h)) \geq (-\epsilon, -\epsilon)$ in $\mathbb{R}^2$ (with respect to the componentwise ordering) if and only if $\tau_1(h) = 0$. Choose h such that $\tau_1(h) = \delta$ generates $\tau_1(H)$, and let $m > 1$ be an integer. Then, for every h' in H, we have $\tau_1(h) \neq m\tau_1(h')$, so $\tau_1(h - mh') \neq 0$ and by the above we obtain

$$(\widehat{h} - m\widehat{h'})(\tau_i) = \tau_i(h - mh') < -\epsilon \quad \text{for some } i \in \{1,2\}.$$

This contradicts (B_m). Hence $\tau_1(H)$ is either zero or dense in $\mathbb{R}$. □

In section 5, we will show that the converse holds when $S(G,u)$ spans a finite-dimensional subspace of $\mathbb{R}^G$ or when H has finite rank. This will complete the proof of Theorem B. The next section provides the last tool that we need for this purpose.

4. An auxiliary result

Throughout this section we fix a Euclidean space E of finite dimension $n > 1$ with the scalar product of $\mathbf{x}, \mathbf{y} \in E$, denoted $\mathbf{x} \cdot \mathbf{y}$. We also fix a compact convex subset K of E containing 0. The notion of a face F of K and of the extreme boundary $\partial_e K$ of K is defined as in section 3, with $S(G,u)$ replaced by K. In particular, there exists a smallest face F of K containing 0. Our goal here is to prove the following result. Its relevance to property (B) will become clearer in the next section.

Proposition 4.1. *Let F be the smallest face of K containing 0 and let Y be a subgroup of E with $\mathbb{R}Y = E$. Suppose that $\{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{y} \in Y\}$ is dense in $\mathbb{R}$ for each $\mathbf{x} \in F \setminus \{0\}$. Fix an arbitrary choice of $\epsilon > 0$ and of $\mathbf{y} \in Y$. Then, there exists $\mathbf{y}_2 \in Y$ such that $\mathbf{x} \cdot (\mathbf{y} - 2\mathbf{y}_2) \geq -\epsilon$ for each $\mathbf{x} \in K$.*

For the rest of the section, we fix F and Y as in the statement of the proposition. We also define

$$E'' = \mathbb{R}F \quad \text{and} \quad E' = (E'')^{\perp},$$

so that $E = E' \oplus E''$ is an orthogonal sum decomposition. For the proof of the proposition, we will need the following intermediate results.

Lemma 4.2. *The face F is a neighbourhood of 0 in E''.*

Proof. This is clear if $F = \{0\}$ because then $E'' = \{0\}$. Suppose that $s = \dim_{\mathbb{R}} E''$ is positive, and let $\{\mathbf{x}_1, \dots, \mathbf{x}_s\}$ be a basis of E'' contained in F. Since, for each $\mathbf{x} \in F$ there exists $\lambda > 0$ such that $-\lambda\mathbf{x} \in F$, we may assume that $-\mathbf{x}_1, \dots, -\mathbf{x}_s \in F$. Then F contains the convex hull of the $2s$ points $\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_s$ which is a neighbourhood of 0 in E''. □

Lemma 4.3. *Let $K' = \mathrm{proj}_{E'} K$ denote the image of K under the orthogonal projection on E'. Then K' is a compact convex subset of E' with $0 \in \partial_e K'$.*

Proof. Since the orthogonal projection on E' is linear, thus continuous, the image K' of K is convex, compact, and contains 0. If $0 \notin \partial_e K'$, there exists $\mathbf{x}' \in K' \setminus \{0\}$ such that $-\mathbf{x}' \in K'$. We can write $\mathbf{x}' = \mathbf{x}_1 + \mathbf{y}_1$ and $-\mathbf{x}' = \mathbf{x}_2 + \mathbf{y}_2$ for some $\mathbf{x}_1, \mathbf{x}_2 \in K$ and some $\mathbf{y}_1, \mathbf{y}_2 \in E''$. By Lemma 3, there exists $\delta \in (0, 1/2)$ such that $2\delta\mathbf{y}_i \in F \subseteq K$ for $i = 1, 2$. Since K is convex, containing 0 and $\mathbf{x}_i$, it also contains $2\delta\mathbf{x}_i$ for $i = 1, 2$. So it contains $\delta\mathbf{x}'$ and $-\delta\mathbf{x}'$, which in turn implies that $\delta\mathbf{x}' \in F$. However, this is impossible since $F \cap E' \subseteq E'' \cap E' = \{0\}$. This contradiction shows that $0 \in \partial_e K'$. □

Lemma 4.4. *The orthogonal projection $\mathrm{proj}_{E''} Y$ of Y on E'' is dense in E''.*

Proof. The group $\mathrm{proj}_{E''} Y$ is dense in E'' if and only if the orthogonal projection of Y on $\mathbb{R}\mathbf{x}$ is dense in $\mathbb{R}\mathbf{x}$ for each $\mathbf{x} \in E'' \setminus \{0\}$ or, equivalently, if and only if $\{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{y} \in Y\}$ is dense in $\mathbb{R}$ for each $\mathbf{x} \in E'' \setminus \{0\}$. Since, by Lemma 4.2, F is a neighbourhood of 0 in E'', this is equivalent to the hypothesis that $\{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{y} \in Y\}$ is dense in $\mathbb{R}$ for each $\mathbf{x} \in F \setminus \{0\}$. □

The next lemma is a basic tool of inhomogeneous Diophantine approximation.

Lemma 4.5. *Let C be a convex neighbourhood of 0 in a real vector space V of finite dimension $s \geq 1$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ be a basis of V contained in C. Then any translate of sC in V contains at least one element of the group $\mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_s$.*

Proof. Let $\mathbf{v} \in V$. Write $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_s\mathbf{v}_s$ with $a_1, \dots, a_s \in \mathbb{R}$. Then $\mathbf{v} + sC$ contains the point $\lceil a_1 \rceil \mathbf{v}_1 + \dots + \lceil a_s \rceil \mathbf{v}_s$, where $\lceil a \rceil$ stands for the least integer greater than or equal to a. □

It has the following consequence.

Lemma 4.6. *For each neighbourhood U'' of 0 in E'', there exists a compact neighbourhood U' of 0 in E' such that each translate of $U' + U''$ in E contains at least one element of Y.*

Proof. It suffices to prove this for a convex neighbourhood U'' of 0, assuming that $s = \dim_{\mathbb{R}}(E'') > 0$. Since, by Lemma 4.4, $\mathrm{proj}_{E''} Y$ is dense in

E'', there exist elements $\mathbf{y}_1, \ldots, \mathbf{y}_s$ of Y whose projections on E'' form a basis of E'' contained in $(ns)^{-1}U''$. Let $Y'' = \mathbb{Z}\mathbf{y}_1 + \cdots + \mathbb{Z}\mathbf{y}_s$. By Lemma 4.5, each translate of $n^{-1}U''$ contains at least one element of $\mathrm{proj}_{E''}Y''$. In particular, each element of Y is congruent modulo Y'' to a point $\mathbf{y}$ with $\mathrm{proj}_{E''}\mathbf{y} \in n^{-1}U''$. Since $\mathbb{R}Y = E$, we may therefore complete $\{\mathbf{y}_1, \ldots, \mathbf{y}_s\}$ to a basis $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ of E contained in Y, with $\mathrm{proj}_{E''}\mathbf{y}_j \in n^{-1}U''$ for $j = 1, \ldots, n$. Choose a compact neighbourhood U' of 0 in E' such that $n^{-1}U'$ contains the projections of $\mathbf{y}_1, \ldots, \mathbf{y}_n$ on E'. Then $n^{-1}(U' + U'')$ contains $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and so, by Lemma 4.5, each translate of $U' + U''$ in E contains at least one element of $\mathbb{Z}\mathbf{y}_1 + \cdots + \mathbb{Z}\mathbf{y}_n$. □

Lemma 4.7. *Let $\epsilon > 0$ and $\rho > 0$ be arbitrary. For each nonzero subspace V of E', there exists a closed ball B of V of radius ρ such that $\mathbf{x} \cdot \mathbf{y} \geq -\epsilon$ for all $\mathbf{x} \in K' \cap V$ and all $\mathbf{y} \in B$.*

Proof. Let V be a nonzero subspace of E'. By Lemma 4.3, the set K' is a compact convex subset of E' with $0 \in \partial_e K'$. Then, $K' \cap V$ is a compact convex subset of V with $0 \in \partial_e(K' \cap V)$. In particular, 0 belongs to the topological boundary of $K' \cap V$ in V. So, there exists a unit vector $\mathbf{u}$ of V such that $\mathbf{u} \cdot \mathbf{x} \geq 0$ for each $\mathbf{x} \in K' \cap V$, by [Go, Proposition 5.10]. We prove the existence of B by induction on $s = \dim_{\mathbb{R}} V$.

If $s = 1$, we have $K' \cap V \subseteq \mathbb{R}^+\mathbf{u}$. Then $B = [0, 2\rho]\mathbf{u}$ is a closed ball of V of radius ρ with $\mathbf{x} \cdot \mathbf{y} \geq 0$ for all $\mathbf{x} \in K' \cap V$ and $\mathbf{y} \in B$.

Suppose now that $s > 1$. Define $V_0 = \{\mathbf{x} \in V \mid \mathbf{u} \cdot \mathbf{x} = 0\}$ and set $K_0 = K' \cap V_0$. Then, V_0 is a subspace of V of dimension $s - 1$. So, we may assume the existence of a closed ball B_0 of V_0 of radius ρ such that $\mathbf{x} \cdot \mathbf{y} \geq -\epsilon/2$ for all $\mathbf{x} \in K_0$ and all $\mathbf{y} \in B_0$. Put

$$L = \sup\left\{\|\mathbf{x}\| \;\middle|\; \mathbf{x} \in K' \cap V\right\} \quad \text{and} \quad M = \sup\{\|\mathbf{y}\| \mid \mathbf{y} \in B_0\}.$$

Define also

$$U = \{\mathbf{x} \in V \mid \|\mathbf{x}\| < \epsilon/(2M)\} \quad \text{and} \quad F_\delta = \left\{\mathbf{x} \in K' \cap V \;\middle|\; \mathbf{u} \cdot \mathbf{x} \leq \delta\right\}$$

for each $\delta > 0$. Then $\{F_\delta \mid \delta > 0\}$ is a collection of closed subsets of $K' \cap V$, stable under finite intersections, whose intersection is K_0. Since $K' \cap V$ is compact and since $K_0 + U$ is an open subset of V containing K_0, there exists therefore $\delta > 0$ for which $F_\delta \subseteq K_0 + U$. Let B be any ball of V of radius ρ contained in $B_0 + [R, \infty)\mathbf{u}$, where $R = ML/\delta$. We claim that B has the required property.

To show this, choose any $\mathbf{x} \in K' \cap V$ and $\mathbf{y} \in B$. We may write $\mathbf{y} = \mathbf{y}_0 + t\mathbf{u}$ where $\mathbf{y}_0 \in B_0$ and $t \geq R$. If $\mathbf{x} \notin F_\delta$, we have $\mathbf{u} \cdot \mathbf{x} > \delta$ and so

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}_0 + t\mathbf{u} \cdot \mathbf{x} > -LM + R\delta = 0.$$

Otherwise, we have $\mathbf{x} \in F_\delta \subseteq K_0 + U$, so there exists $\mathbf{x}_0 \in K_0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \epsilon/(2M)$. Since $\mathbf{u} \cdot \mathbf{x} \geq 0$ and $\|\mathbf{y}_0\| \leq M$, we obtain

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot \mathbf{y}_0 + t\mathbf{u} \cdot \mathbf{x} \\ &\geq \mathbf{x} \cdot \mathbf{y}_0 \\ &= \mathbf{x}_0 \cdot \mathbf{y}_0 + (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{y}_0 \\ &\geq -\frac{\epsilon}{2} - \frac{\epsilon}{2M} M = -\epsilon. \end{aligned}$$

This finishes the proof of the lemma. □

Proof of Proposition 4.1. By Lemma 4.4, the group $\mathrm{proj}_{E''} Y$ is dense in E''. Thus, the same is true of $\mathrm{proj}_{E''}(2Y)$. If $E' = \{0\}$, this means that $2Y$ is dense in E and the conclusion follows. We thus assume that $E' \neq \{0\}$. Then, for each $\delta > 0$, Lemma 4.6 shows the existence of $\rho > 0$ such that each translate of $B'_\rho + B''_\delta$ contains at least one element of $2Y$, where $B'_\rho = \{\mathbf{y}' \in E' \mid \|\mathbf{y}'\| \leq \rho\}$ and $B''_\delta = \{\mathbf{y}'' \in E'' \mid \|\mathbf{y}''\| \leq \delta\}$. Moreover, applying Lemma 4.7 to the choice of $V = E'$, we obtain a translate B' of B'_ρ with the property that $\mathbf{x}' \cdot \mathbf{y}' \geq -\epsilon/2$ for any $\mathbf{x}' \in K'$ and any $\mathbf{y}' \in B'$.

We apply the above with $\delta = \epsilon/(2L)$ where $L = \sup\{\|\mathbf{x}\|\,;\, \mathbf{x} \in K\}$. Put $B = B' + B''_\delta$ for the corresponding choice of B'. Then any translate of B in E contains at least one element of $2Y$. In particular, for the given $\mathbf{y} \in Y$, there exists $\mathbf{y}_2 \in Y$ such that $2(-\mathbf{y}_2) \in (-\mathbf{y}) + B$, and thus $\mathbf{y} - 2\mathbf{y}_2 \in B$. Write $\mathbf{y} - 2\mathbf{y}_2 = \mathbf{y}' + \mathbf{y}''$ with $\mathbf{y}' \in B'$ and $\mathbf{y}'' \in B''_\delta$. Then, for any $\mathbf{x} \in K$, we find

$$\mathbf{x} \cdot (\mathbf{y} - 2\mathbf{y}_2) = \mathrm{proj}_{E'}(\mathbf{x}) \cdot \mathbf{y}' + \mathrm{proj}_{E''}(\mathbf{x}) \cdot \mathbf{y}'' \geq (-\epsilon/2) - L\delta \geq -\epsilon\,,$$

as required. □

5. Sufficient conditions for property (B)

We begin with the following observation.

Proposition 5.1. *Let H be a subgroup of a partially ordered abelian group with order unit (G, u). We have*

$$Z_G(H) = \emptyset \iff H \cap G^{++} \neq \emptyset.$$

When this happens, the group H has property (B).

The first assertion generalizes Gordan's theorem [Gor], see Appendix A. When $G = \mathbb{R}^n$ and $u = \mathbf{1}$, the second proves Theorem B in the case that $Z_G(H) = Z(H)$ is empty.

Proof. If H contains no order units, then $G_0 = H \oplus \mathbb{Z}u$ is a direct sum and the map $\tau_0 \colon G_0 \to \mathbb{R}$ given by $\tau_0(h + \ell u) = \ell$ for each pair $(h, \ell) \in H \times \mathbb{Z}$ is a positive homomorphism for the restriction to G_0 of the partial order on G. Since $u \in G_0$ and $\tau_0(u) = 1$, this maps extends to a trace $\tau \in S(G, u)$. Then $Z_G(H)$ is not empty, as it contains τ.

Conversely, if H contains an order unit v, then $u \le kv$ for some positive integer k, so $\widehat{v} \ge k^{-1}\widehat{u} = k^{-1}$, and thus $Z_G(H) = \emptyset$. Moreover, let $m > 1$ be an integer and let $h \in H$. Since mv is an order unit, we have $-h \le \ell mv$ for some $\ell \in \mathbb{N}$, thus $\widehat{h} - m\widehat{h'} \ge 0$ for $h' = -\ell v \in H$. Thus H satisfies (B_m) as well. $\square$

Theorem 5.2. *Let (G,u) be a partially ordered abelian group with order unit, let H be a subgroup of G, and let F be the smallest face of $S(G,u)$ containing $Z_G(H)$. Let $\widehat{H}$ denote the image of H in $\mathrm{Aff}\,(S(G,u))$ under the affine representation of (G,u). Suppose that $\mathbb{R}\widehat{H}$ is finite-dimensional and that $\tau(H)$ is dense in $\mathbb{R}$ for each $\tau \in F \setminus Z_G(H)$. Then, H has property (B).*

In particular, H has property (B) if $Z_G(H)$ is a face of $S(G,u)$ and $\mathbb{R}\widehat{H}$ has finite dimension. We will show in the next section that property (B) may fail if finite-dimensionality of $\mathbb{R}\widehat{H}$ is dropped.

Proof. We may assume that $Z_G(H) \ne \emptyset$ since otherwise we already know, by Proposition 5.1, that H has property (B). From this, we proceed in two steps. We first prove the statement when $G = H + \mathbb{Z}u$. Then, we show that the general case reduces to this special case.

So, assume first that $G = H + \mathbb{Z}u$. Then, as $Z_G(H)$ is not empty, we have $H \cap \mathbb{Z}u = \{0\}$ and $Z_G(H)$ consists of the single group homomorphism $\tau_0 \colon G \to \mathbb{Z}$ given by $\tau_0(h + nu) = n$ for each $h \in H$ and each $n \in \mathbb{Z}$. Choose $h_1, \dots, h_n \in H$ such that $\{\widehat{h}_1, \dots, \widehat{h}_n\}$ is a basis of $\mathbb{R}\widehat{H}$ over $\mathbb{R}$. We denote by $\varphi \colon H \to \mathbb{R}^n$ the group homomorphism that sends an element of h to the coordinates of $\widehat{h}$ in that basis, namely the n-tuple $\varphi(h) = (y_1, \dots, y_n) \in \mathbb{R}^n$ such that

$$\widehat{h} = y_1\widehat{h}_1 + \cdots + y_n\widehat{h}_n.$$

We also form the linear map $\psi \colon \mathbb{R}^G \to \mathbb{R}^n$ given by

$$\psi(\tau) = (\tau(h_1), \dots, \tau(h_n))$$

for each $\tau \in R^G$. Then, for each $\tau \in S(G,u)$ and each $h \in H$, we have

$$\tau(h) = \widehat{h}(\tau) = \psi(\tau) \cdot \varphi(h)$$

using the standard scalar product on $\mathbb{R}^n$. We deduce from this that ψ is one-to-one on $S(G,u)$, because if elements τ_1, τ_2 of $S(G,u)$ have the same image under ψ then, by the formula above, they coincide on H and therefore they coincide on G (since they take the value 1 at u).

Let $K = \psi(S(G,u))$ and $Y = \varphi(H)$. Then, K is a convex subset of $\mathbb{R}^n$ containing $\psi(\tau_0) = 0$ and ψ induces an affine homeomorphism from $S(G,u)$ to K. Accordingly, $F_0 := \psi(F)$ is the smallest face of K containing 0. Moreover, Y is a subgroup of $\mathbb{R}^n$ which contains $\mathbb{Z}^n$, the image of the group $\mathbb{Z}h_1 + \cdots + \mathbb{Z}h_n$ under φ. So, we have $\mathbb{R}Y = \mathbb{R}^n$. The hypothesis also implies that $\{\mathbf{x} \cdot \mathbf{y}\,;\, \mathbf{y} \in Y\}$ is dense in $\mathbb{R}$ for each $\mathbf{x} \in F_0 \setminus \{0\}$. So, K and Y fulfil all the hypotheses of Proposition 2 within the Euclidean space $\mathbb{R}^n$.

Let $\epsilon > 0$ and $h \in H$. Putting $\mathbf{y} = \varphi(h)$, Proposition 2 yields a point $\mathbf{y}' \in Y$ such that $\mathbf{x}\cdot(\mathbf{y}-2\mathbf{y}') \geq -\epsilon$ for each $\mathbf{x} \in K$. Choose any $h' \in H$ such that $\varphi(h') = \mathbf{y}'$. Then, the above property translates into $\widehat{h}(\tau)-2\widehat{h'}(\tau) \geq -\epsilon$ for each $\tau \in S(G,u)$, showing that H has property (B_2).

Now, we turn to the general case. Consider the group $G_0 = H + \mathbb{Z}u$ with the induced ordering from G. Since the map $\rho\colon S(G,u) \to S(G_0,u)$ sending a trace on (G,u) to its restriction on (G_0,u) is a surjective affine order-preserving homomorphism (see section 2), we have

$$Z_G(H) = \rho^{-1}(Z_{G_0}(H)) \quad \text{and} \quad F = \rho^{-1}(F_0),$$

where F_0 denotes the smallest face of $S(G_0,u)$ containing $Z_{G_0}(H)$. In particular $Z_{G_0}(H)$ is not empty, and the hypothesis that $\tau(H)$ be dense in $\mathbb{R}$ for each $\tau \in F \setminus Z_G(H)$ implies the same for each $\tau \in F_0 \setminus Z_{G_0}(H)$. Let $\widehat{H}_0$ denote the image of H in $\mathrm{Aff}\,(S(G_0,u))$. The dual map ρ^* from $\mathrm{Aff}\,(S(G_0,u))$ to $\mathrm{Aff}\,(S(G,u))$ given by composition with ρ is $\mathbb{R}$-linear and restricts to an isomorphism of $\mathbb{R}$-vector spaces from $\mathbb{R}\widehat{H}_0$ to $\mathbb{R}\widehat{H}$. In particular, $\mathbb{R}\widehat{H}_0$ is finite-dimensional. Thus the hypotheses of the theorem hold for H as a subgroup of G_0 and therefore, by the special case proved above, the group H has property (B) within G_0. By Proposition 2.1, we conclude that it also has property (B) within G. □

6. A class of counter-examples

The criterion for property (B) given in Theorem 5.2 requires that $\mathbb{R}\widehat{H}$ have finite dimension. The goal of this section is to exhibit examples showing that the theorem is false without this hypothesis. Our construction is based on the following observation.

Proposition 6.1. *Let X be any infinite set, and let H be the subgroup of $\mathbb{R}^X$ generated by an infinite $\mathbb{R}$-linearly independent sequence of bounded functions $(h_i\colon X \to \mathbb{R})_{i\in\mathbb{N}}$. Suppose that, for each nonzero f in $\mathbb{R}H$, there exists $x \in X$ for which $f(x) > 0$. Then there is a sequence of positive integers $(m_i)_{i\in\mathbb{N}}$ such that each nonzero element h of $H' = \sum_{i\geq 1}\mathbb{Z}m_ih_i$ satisfies $\sup_X h \geq 1$.*

Proof. We construct inductively integers $m_1, m_2, \dots$ so that $\sup_X h \geq 1$ for each $n \geq 1$ and each nonzero h in $H_n = \mathbb{Z}m_1h_1 + \cdots + \mathbb{Z}m_nh_n$.

For $n = 1$, we choose $x_1, x_2 \in X$ such that $h_1(x_1) > 0$ and $(-h_1)(x_2) > 0$, and we select $m_1 \in \mathbb{N}$ large enough so that $m_1h_1(x_1) \geq 1$ and $-m_1h_1(x_2) \geq 1$. Then, for each nonzero $h \in H_1 = \mathbb{Z}m_1h_1$, we obtain

$$\sup_X h \geq \max\{h(x_1), h(x_2)\} \geq 1.$$

Suppose that $m_1, \dots, m_n$ have been constructed for some integer $n \geq 1$. For each point $\mathbf{u} = (u_1, \dots, u_{n+1})$ in the unit sphere S of $\mathbb{R}^{n+1}$, the function

$f = u_1 h_1 + \cdots + u_{n+1} h_{n+1}$ is a nonzero element of $\mathbb{R}H$, and so there exists $x_{\mathbf{u}} \in X$ such that $f(x_{\mathbf{u}}) > 0$. Define

$$V_{\mathbf{u}} = \{(v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} \,;\, (v_1 h_1 + \cdots + v_{n+1} h_{n+1})(x_{\mathbf{u}}) > 0\}.$$

Then $(V_{\mathbf{u}})_{\mathbf{u} \in S}$ is an open covering of S. Since S is compact, it admits a finite subcover by open sets corresponding to points $x_1, \ldots, x_k \in X$. This means that the function $g\colon S \to \mathbb{R}$ given by

$$g(u_1, \ldots, u_{n+1}) = \max_{1 \le i \le k} (u_1 h_1 + \cdots + u_{n+1} h_{n+1})(x_i)$$

is strictly positive on S. Since it is continuous, it is therefore bounded below by some positive constant $\delta > 0$. We chose $m_{n+1} \in \mathbb{N}$ so that $m_{n+1}\delta \ge 1$, and claim that this integer has the desired property.

To show this, choose any nonzero element h of $H_{n+1} = H_n + \mathbb{Z} m_{n+1} h_{n+1}$. If $h \in H_n$, then by hypothesis we have $\sup_X h \ge 1$. Otherwise, we can write $h = \ell_1 m_1 h_1 + \cdots + \ell_{n+1} m_{n+1} h_{n+1}$ for some integers $\ell_1, \ldots, \ell_{n+1} \in \mathbb{Z}$ with $\ell_{n+1} \neq 0$. Put $\mathbf{v} = (\ell_1 m_1, \ldots, \ell_{n+1} m_{n+1})$ and $\mathbf{u} = \|\mathbf{v}\|^{-1}\mathbf{v}$. Since $\mathbf{u} \in S$ and $\|\mathbf{v}\| \ge m_{n+1}$, we obtain

$$\sup_X h \ge \max_{1 \le i \le k} h(x_i) = \|\mathbf{v}\| g(\mathbf{u}) \ge m_{n+1}\delta \ge 1,$$

as desired. □

Consider the vector space $G = \mathcal{C}(X, \mathbb{R})$ consisting of continuous real-valued functions on a compact Hausdorff topological space X. This is a partially ordered abelian group with respect to the pointwise ordering and the constant function $\mathbf{1}$ is an order unit. Each trace τ in $S(G, \mathbf{1})$ is given by $\tau(f) = \int_X f \, d\mu$ for a unique Borel probability measure μ on X and this identifies $S(G, \mathbf{1})$ to the set $M_1^+(X)$ of probability measures on X with the weak* topology (see [Go, Chapter 5]). Moreover, the extreme boundary of $S(G, \mathbf{1})$ is homeomorphic to X under the map that sends a point x in X to the evaluation ϵ_x at x given by

$$\epsilon_x(f) = f(x) \quad (f \in G)$$

corresponding to the point-mass measure δ_x at x [Go, Proposition 5.24]. In particular $\{\epsilon_x\}$ is a face of $S(G, \mathbf{1})$ for any $x \in X$.

Proposition 6.2. *Let X be an infinite compact metrizable topological space. Choose $x_0 \in X$ such that $X_0 := X \setminus \{x_0\}$ is dense in X, and consider $G = \mathcal{C}(X, \mathbb{R})$ as a partially ordered abelian group as above. Then there exists a subgroup H of G with the following properties:*

(i) $Z_G(H) = \{\epsilon_{x_0}\}$,
(ii) $\sup_X h \ge 1$ *for each* $h \in H \setminus \{0\}$.

In particular, $Z_G(H)$ is a face of $S(G, \mathbf{1})$, and H does not satisfy property (B).

The existence of x_0 follows from the fact that X is not discrete, since a discrete compact Hausdorff space is finite.

Proof. Let $G_0 = \ker(\epsilon_{x_0}) = \{g \in G\,;\, g(x_0) = 0\}$. Since X is compact metrizable, $\mathcal{C}(X,\mathbb{R})$ is a separable topologial space; so the group G_0 contains a dense countable sequence $(f_n)_{n\in\mathbb{N}}$ consisting of continuous functions with compact support in X_0. Choose a dense sequence $(x_n)_{n\in\mathbb{N}}$ in X_0 and a subsequence $(y_n)_{n\in\mathbb{N}}$ converging to x_0. Let

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{x_n} + \sum_{n=1}^{\infty} \delta_{y_n}.$$

This defines a finite positive measure on each compact subset K of X_0 since $X \setminus K$ contains y_n for all but finitely many n. Since $\mu(X_0) = \infty$, there is also, for each $m \in \mathbb{N}$, a function $g_m \in G_0$ with compact support in X_0 such that $\|g_m\|_\infty \le 1/m$ and $\int_X g_m d\mu = 1$. Let H_0 be the subgroup of G_0 generated by the functions

$$f_n - \left(\int_X f_n\, d\mu\right) g_m, \qquad (m,n) \in \mathbb{N}^2.$$

Since the sequence $(g_m)_{m\in\mathbb{N}}$ converges uniformly to 0 on X, the topological closure $\overline{H}_0$ of H_0 in G contains f_n for each $n \in \mathbb{N}$. We conclude that $\overline{H}_0 = G_0$, and so $Z_G(H_0) = Z_G(G_0) = \{\epsilon_{x_0}\}$. Moreover, we have $\int_X f\, d\mu = 0$ for each f in H_0. Therefore, the same is true for each f in $\mathbb{R}H_0$ and thus, for each non-zero $f \in \mathbb{R}H_0$, there exists $n \in \mathbb{N}$ for which $f(x_n) > 0$. Choose a basis $(h_i)_{i\in\mathbb{N}}$ of $\mathbb{R}H_0$ contained in H_0. By Proposition 6.1, there exists a sequence $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that the subgroup H of G spanned by $(m_ih_i)_{i\in\mathbb{N}}$ satisfies condition (ii). It also satisfies condition (i) since $Z_G(H) = Z_G(H_0)$.

Finally, let $h \in H \setminus 2H$ (for example $h = m_1h_1$), and let $h' \in H$. Then $2h' - h \neq 0$ satisfies $\sup_X(2h' - h) \ge 1$, so $h(x) - 2h'(x) \le -1$ for some $x \in X$, meaning that $\tau(h) - 2\tau(h') \le -1$ for the trace $\tau = \epsilon_x$. Thus H does not satisfy property (B_2). □

7. Link with unperforation of quotients

Let G be a partially ordered abelian group. We recall the following definitions:

- G is *directed* if any finite subset of G has an upper bound in G;
- G is *simple* if it is nonzero, directed, and $G^+ \setminus \{0\} = G^{++}$;
- G is *unperforated* if the condition $mg \ge 0$ with $g \in G$ and $m \in \mathbb{N}$ implies that $g \ge 0$;
- G has the *Riesz interpolation property* if, given $g_1, g_2, g_1', g_2' \in G$ with $g_i \le g_j'$ for each $i, j \in \{1,2\}$, there exists $g \in G$ such that $g_i \le g \le g_j'$ for each $i, j \in \{1,2\}$.

Clearly, G is directed if it admits an order unit. If G is unperforated, then it is torsion-free as an abelian group, and an element g of G is an order unit if and only if $\tau(g) > 0$ for all traces τ on G [EHS, Theorem 1.4] or [Go, Corollary 4.13]. The group G is called a *dimension group* if it is directed, unperforated, and has the Riesz interpolation property. For example, $\mathbb{R}^n$ is a simple dimension group for the *strict ordering* with positive cone $\{0\} \cup (\mathbb{R}^n)^{++}$, and so is any dense subgroup G of $\mathbb{R}^n$ with respect to the inherited ordering. This also applies to $\mathbb{Z}$ but not to $\mathbb{Z}^n$ if $n > 1$.

A subgroup H of G is said to be *convex* if whenever $h \leq g \leq h'$ with $h, h' \in H$ and $g \in G$, we have $g \in H$. Suppose that H is such a subgroup. Then the quotient G/H is a partially ordered abelian group with positive cone $(G/H)^+ = (G^+ + H)/H$; given $g, g' \in G$, we have $g + H \leq g' + H$ if and only if $g \leq g' + h$ for some $h \in H$. (This does not assume that H is directed.) The following result links unperforation of G/H to property (B) for H.

Proposition 7.1. *Suppose that (G, u) is a simple unperforated partially ordered abelian group with order unit, and let H be a convex subgroup of G for which G/H is torsion-free.*

(i) *If H has property (B), then G/H is unperforated.*
(ii) *If G/H is unperforated and if $\{\widehat{g} \mid g \in G\}$ is dense in* $\operatorname{Aff} S(G, u)$*, then H has property (B).*

The argument for (i) follows the proof of [BH, Proposition B.1], together with a simplification.

Proof. (i) Suppose that H has property (B). Let $g \in G$ and $m \geq 2$ be an integer such that $m(g + H) \geq H$. We need to show that $g + H \geq H$. If $m(g + H) = H$ then $g + H = H$ because G/H is torsion-free, and we are done. Otherwise, we have $mg + h > 0$ for some $h \in H$ and so $mg + h \in G^{++}$ because G is simple. Choose $n \in \mathbb{N}$ such that $n(mg + h) \geq u$. Then, we have $m\widehat{g} + \widehat{h} \geq 1/n$. Moreover, since H satisfies (B_m), there exists $h' \in H$ with $-\widehat{h} + m\widehat{h}' \geq -1/(2n)$. Combining these inequalities, we deduce that $\widehat{g} + \widehat{h}' \geq 1/(2mn)$. This implies that $g + h' \in G^{++}$ since G is unperforated, and so $g + H \geq H$.

(ii) Suppose that G/H is unperforated and that $\{\widehat{g} \mid g \in G\}$ is dense in $\operatorname{Aff} S(G, u)$. Let $h \in H$ and $\epsilon > 0$. The set of $a \in \operatorname{Aff} S(G, u)$ with $\epsilon/4 < a - \widehat{h}/2 < \epsilon/2$ is open and not empty as it contains $\widehat{h}/2 + \epsilon/3$. Thus it contains $\widehat{g}$ for some $g \in G$. This means that $\epsilon/2 \leq 2\widehat{g} - \widehat{h} \leq \epsilon$. The first inequality implies that $2g - h \in G^{++}$ since G is unperforated. In particular, we have $2g - h \geq 0$, so $2(g + H) \geq H$, and unperforation of G/H yields $g + H \in (G/H)^+$. Thus there exists h' in H such that $g - h' \in G^+$. Then $\widehat{g} \geq \widehat{h}'$ and we obtain $-\epsilon \leq \widehat{h} - 2\widehat{g} \leq \widehat{h} - 2\widehat{h}'$, showing that H satisfies (B_2). □

Corollary 7.2. *Suppose that (G, u) is a simple dimension group with order unit, and let H be a convex subgroup of G for which G/H is torsion-free. Then G/H is unperforated if and only if H has property (B) inside G.*

Necessity of the condition is proved in [BH, Proposition B.1].

Proof. If G is cyclic as an abelian group, then H is $\{0\}$ or G. So G/H is unperforated and H has property (B). If G is noncyclic, then $\{\widehat{g} \mid g \in G\}$ is dense in $\operatorname{Aff} S(G,u)$ by [GoH2, Corollary 4.10] or [Go, Theorem 14.14], and the conclusion follows from the Proposition. □

An example of a convex subgroup H of a simple dimension group G such that G/H is torsion-free but *holey*, i.e., not unperforated, is given in [BH, Appendix B]. It suffices to take for G a rank 3 dense subgroup of $\mathbb{R}^2$ of the form $G = \mathbb{Z}^2 + \mathbb{Z}(\alpha,\beta)$ with the strict ordering, where $\{1,\alpha,\beta\}$ is linearly independent over $\mathbb{Q}$, and to take $H = \mathbb{Z}(-1,1)$ (it is convex since $H \cap G^+ = \{(0,0)\}$).

8. Application to refinable traces

The motivating reason to consider unperforation of quotients comes from the study of refinable traces (originally refinable measures on Cantor dynamical systems, see [Ak]). We first recall some definitions from [BH, Sections 1 and 7].

Let (G,u) be a dimension group with order unit, and let U be a nonempty subset of $S(G,u)$. We say that U is

- *refinable* if whenever $a_1, a_2, b \in G^+$ satisfy $\tau(a_1) + \tau(a_2) = \tau(b)$ for each $\tau \in U$, there exist $a_1', a_2' \in G^+$ such that $a_1' + a_2' = b$ and $\tau(a_i') = \tau(a_i)$ for each $i = 1, 2$ and each $\tau \in U$;
- *weakly good* if for any $a, b \in G^+ \setminus \{0\}$ satisfying
$$\inf_{\tau \in U} (\tau(b) - \tau(a)) > 0$$
there exists $a' \in G^+ \setminus \{0\}$ such that $a' < b$ and $\tau(a') = \tau(a)$ for each $\tau \in U$;
- *good* if for any $a \in G$ and $b \in G^+ \setminus \{0\}$ satisfying
$$\inf_{\tau \in U} \tau(a) > 0 \quad \text{and} \quad \inf_{\tau \in U} (\tau(b) - \tau(a)) > 0$$
there exists $a' \in G^+ \setminus \{0\}$ such that $a' < b$ and $\tau(a') = \tau(a)$ for each $\tau \in U$.

Note that when U is a compact subset of $S(G,u)$, for example when U is finite or when $U = Z_G(H)$ for a subgroup H of G, the condition $\inf_{\tau \in U} \tau(a) > 0$ for $a \in G$ is equivalent to $\tau(a) > 0$ for each $\tau \in U$; similarly the condition $\inf_{\tau \in U}(\tau(b) - \tau(a)) > 0$ for $a, b \in G$ is then equivalent to $\tau(a) < \tau(b)$ for each $\tau \in U$.

We say that a single trace $\tau \in S(G,u)$ is *refinable, weakly good, or good* if the singleton $\{\tau\}$ has the corresponding property.

There are generalizations, variations, and implications discussed in [BH]. For example, [BH, Lemma 1.1(b)] shows that a trace $\tau \in S(G,u)$ is good if and only if it is weakly good. Moreover, a good trace is refinable. Here we combine our previous results with [BH, Proposition 7.6] to prove the following.

Theorem 8.1. *Let (G, u) be a simple dimension group for which* $\operatorname{Aff} S(G, u)$ *is finite-dimensional, let $\tau \in S(G, u)$, and let*

$$Z := Z_G(\ker \tau) = \{\sigma \in S(G, u) \mid \ker \tau \subseteq \ker \sigma\}.$$

The following conditions are equivalent:

(i) *τ is refinable,*
(ii) *Z is refinable,*
(iii) *Z is good,*
(iv) *Z is weakly good.*

When they are satisfied, $G/\ker \tau$ is a simple dimension group; in particular, it is unperforated.

Proof. (i) $\Leftrightarrow$ (ii): This is because elements a_1, a_2, b of G satisfy the condition $\tau(a_1) + \tau(a_2) = \tau(b)$ if and only if $\sigma(a_1) + \sigma(a_2) = \sigma(b)$ for each $\sigma \in Z$; similarly $a, a' \in G$ satisfy $\tau(a) = \tau(a')$ if and only if $\sigma(a) = \sigma(a')$ for each $\sigma \in Z$.

(ii) $\Rightarrow$ (iii): Suppose that Z is refinable. Put $H = \ker \tau = \cap_{\sigma \in Z} \ker \sigma$. Then, by [BH, Proposition B.5], every trace in $S(G, u)$ maps H to $\{0\}$ or to a dense subgroup of $\mathbb{R}$. As $\operatorname{Aff} S(G, u)$ has finite dimension, it follows from Theorem 5.2 that H has property (B) inside G and so, by Corollary 7.2, the torsion-free group G/H is unperforated. By [BH, Proposition 7.6(a)], this quotient also has the Riesz interpolation property. So, G/H is a simple dimension group. To conclude that Z is a good set of traces, we need to modify slightly the argument of [BH, Proposition 7.6(f)] as follows.

Let $a \in G$ and $b \in G^+$ with $0 < \sigma(a) < \sigma(b)$ for each $\sigma \in Z$. As the traces in $S(G/H, u + H)$ are induced by the elements of $Z_G(H) = Z$, and as G/H is unperforated, we deduce that $a + H$ and $b - a + H$ belong to $(G/H)^{++}$ [Go, Corollary 4.13]. Thus, these classes contain elements a_1 and a_2 of G^+, respectively. We have $\sigma(a_1) + \sigma(a_2) = \sigma(b)$ for each $\sigma \in Z$. Since Z is refinable, there exist $a_1', a_2' \in G^+$ such that $a_1' + a_2' = b$ and $\sigma(a_i') = \sigma(a_i)$ for each $i = 1, 2$ and each $\sigma \in Z$. In particular, $a_1' \in G^+$ satisfies $a_1' \leq b$ and $\sigma(a_1') = \sigma(a)$ for each $\sigma \in Z$. We have $a_1' \neq 0$ and $a_1' \neq b$ since $0 < \tau(a_1') < \tau(b)$, thus $0 < a_1' < b$ since G is simple. Therefore Z is good.

(iv) $\Rightarrow$ (ii): Suppose that Z is weakly good, and that $a_1, a_2, b \in G^+$ satisfy $\sigma(a_1) + \sigma(a_2) = \sigma(b)$ for all $\sigma \in Z$. If $a_2 \neq 0$, then $a_2 \in G^{++}$, so for all $\sigma \in Z$ we have $\sigma(a_1) = \sigma(b) - \sigma(a_2) < \sigma(b)$. As Z is weakly good, there exists $a_1' \in G^+$ such that $a_1' \leq b$ and $\sigma(a_1') = \sigma(a_1)$ for all $\sigma \in Z$. Put $a_2' = b - a_1'$. Then $a_1', a_2' \in G^+$ satisfy $a_1' + a_2' = b$ and $\sigma(a_i') = \sigma(a_i)$ for all $i = 1, 2$ and all $\sigma \in Z$. If $a_2 = 0$, the same holds for the choice of $a_1' = b$ and $a_2' = 0$. Thus, Z is refinable.

This completes the proof since the implication (iii) $\Rightarrow$ (iv) is immediate and the last assertion has been established in the course of proving (ii) $\Rightarrow$ (iii). $\square$

Corollary 8.2. *Under the same hypotheses, the following conditions are equivalent:*

(i) *τ is good;*
(ii) *τ is refinable and $Z_G(\ker\tau) = \{\tau\}$.*

Proof. Suppose first that τ is good, and let $\sigma \in Z_G(\ker\tau)$. Since we have $\ker\tau \subseteq \ker\sigma$, there exists a group homomorphism $\psi\colon \tau(G) \to \mathbb{R}$ such that $\sigma(a) = \psi(\tau(a))$ for each $a \in G$. We have $\psi(1) = 1$ since $\sigma(u) = \tau(u) = 1$. Moreover, ψ is order preserving: if $\tau(a) \geq 0$ for some $a \in G$, then, since τ is good and $a \leq nu$ for some $n \in \mathbb{N}$, there exists $a' \in G^+$ with $\tau(a') = \tau(a)$, and so $\psi(\tau(a)) = \sigma(a) = \sigma(a') \geq 0$. Thus, ψ is the inclusion of $\tau(G)$ in $\mathbb{R}$ and therefore $\sigma = \tau$. As τ is refinable, this proves that (i) implies (ii). The converse follows from Theorem 8.1. □

Suppose now that (G, u) is an arbitrary simple dimension group G with order unit, and let $\phi\colon G \to \operatorname{Aff} S(G, u)$ be the natural map. It is shown in [BH, Corollary 1.8] that a trace $\tau \in S(G, u)$ is good if and only if $\phi(\ker\tau)$ is dense in $\{h \in \operatorname{Aff} S(G, u) \mid h(\tau) = 0\}$, and this characterization is very useful. There is a similar necessary condition for τ to be refinable [BH, Proposition 7.7(e)], but this is far from sufficient, even when $\operatorname{Aff} S(G, u)$ has finite dimension. However, when G is $\mathbb{R}^n$ with the strict ordering, [H2, Appendix 2] provides a simple geometric description of the good subsets of $S(\mathbb{R}^n, \mathbf{1}) = K_n$ of the form $K_n \cap V$ for a subspace V of $(\mathbb{R}^n)^*$ (conjectured in [BH, p. 6295]). Together with Theorem 8.1, this yields a geometric description of the refinable traces of $\mathbb{R}^n$.

As shown in [BH, Lemma 7.3], sufficient for a trace $\tau \in S(G, u)$ to be refinable is that $\ker\tau = \operatorname{Inf} G$, where $\operatorname{Inf} G = \ker\phi$ is called the *infinitesimal subgroup* of G. If G is countable, it follows from [GiHH, Proposition 1.7] that the collection of such traces is a dense G_δ of $S(G, u)$. It is contained in the set of refinable traces of G. The next proposition provides a case of equality.

Proposition 8.3. *Let G be a dense subgroup of $\mathbb{R}^n$, free of rank $n+1$, equipped with the strict ordering inherited from $\mathbb{R}^n$, and let $\tau \in S(G, u)$ for some $u \in G^{++}$. Then τ is refinable if and only if $\ker\tau = \{0\}$.*

A partially ordered abelian group G as in the statement of the proposition is called a *critical dimension group of rank* $n+1$ (cf., [H1]).

Proof. Suppose that τ is refinable, and let $H = \ker\tau$. Since H is a proper subgroup of G, it is discrete in $\mathbb{R}^n$. Let S denote the set of all linear forms $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ for which $\sigma(H)$ is a nonzero discrete subgroup of $\mathbb{R}$. If $H \neq \{0\}$, then S is a dense subset of $(\mathbb{R}^n)^*$, stable under multiplication by positive real numbers; in particular S contains an element of $S(G, u)$, contradicting [BH, Proposition B.5]. Thus H must be $\{0\}$. The converse is clear. □

9. A class of examples

Recall that Theorem B (stated in the introduction) follows from Theorems 3.1 and 5.2. In this section, we apply the result to determine, among a class of subgroups of $\mathbb{R}^n$, those that have the one-sided approximation property (B). We will also use the following consequence of Theorem B, emphasizing the link between properties (A) and (B).

Theorem 9.1. *Let H be a subgroup of $\mathbb{R}^n$, let F be the smallest face of K_n containing $Z(H) = \{\tau \in K_n \mid \tau(H) = \{0\}\}$, let $\tau_{i_1}, \dots, \tau_{i_\ell}$ be the vertices of F, and let $\{\tau_{j_1}, \dots, \tau_{j_k}\}$ be a maximal set of vertices of F whose restrictions to $\mathbb{R}H$ are linearly independent over $\mathbb{R}$. Then the following conditions are equivalent:*

(i) *the group H has property (B) as a subgroup of $\mathbb{R}^n$,*
(ii) *its projection $H' := \{(\tau_{i_1}(h), \dots, \tau_{i_\ell}(h)) \mid h \in H\}$ has property (A) in $\mathbb{R}^\ell$,*
(iii) *its projection $H'' := \{(\tau_{j_1}(h), \dots, \tau_{j_k}(h)) \mid h \in H\}$ is dense in $\mathbb{R}^k$.*

Recall that $\{\tau_1, \dots, \tau_n\}$ denotes the basis of $(\mathbb{R}^n)^*$ dual to the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$.

Proof. By Theorem B, condition (i) is equivalent to requiring that $\tau(H)$ be $\{0\}$ or dense in $\mathbb{R}$ for each $\tau \in F$, while by Theorem A, condition (ii) is equivalent to asking that $\phi(H)$ is $\{0\}$ or dense in $\mathbb{R}$ for each $\phi \in \mathbb{R}F$. Thus the latter implies the former. To prove the converse, suppose that condition (i) holds and let $\phi \in \mathbb{R}F$. We have $\phi = \phi(e_{i_1})\tau_{i_1} + \cdots + \phi(e_{i_\ell})\tau_{i_\ell}$. By hypothesis, each $\nu \in Z(H)$ admits a similar decomposition with coefficients $\nu(e_{i_j}) \geq 0$ $(1 \leq j \leq \ell)$ of sum 1 and, for each $j = 1, \dots, \ell$, there is at least one element ν_j of $Z(H)$ with $\nu_j(e_{i_j}) > 0$. Then, $\nu = \ell^{-1}(\nu_1 + \cdots + \nu_\ell) \in Z(H)$ has $\nu(e_{i_j}) > 0$ for $j = 1, \dots, \ell$. In other words, ν belongs to the relative interior of F. Choose $a > 0$ such that $c_j := \phi(e_{i_j}) + a\nu(e_{i_j}) > 0$ for $j = 1, \dots, \ell$ and let $c = c_1 + \cdots + c_\ell$. Then $\tau := c^{-1}(\phi + a\nu)$ belongs to F and so $\tau(H)$ is zero or dense in $\mathbb{R}$. Therefore, the same applies to $\phi(H) = c\tau(H)$, showing that condition (ii) holds.

Finally, the projection map $\pi \colon \mathbb{R}^\ell \to \mathbb{R}^k$ given by

$$\pi(x_{i_1}, \dots, x_{i_\ell}) = (x_{j_1}, \dots, x_{j_k})$$

maps H' to H'' and induces an isomorphism of vector spaces from $\mathbb{R}H'$ to $\mathbb{R}H'' = \mathbb{R}^k$. By Kronecker's theorem, condition (ii) is equivalent to H' being dense in $\mathbb{R}H'$. Thus it is equivalent to H'' being dense in $\mathbb{R}^k$, which is condition (iii). $\square$

Suppose that $n \geq 2$ and let $\tau = (\tau_1 + \tau_2)/2 \in K_n$. We consider subgroups H of $\mathbb{R}^n$ contained in $\ker \tau$, or equivalently, for which $\tau \in Z(H)$. We first note the following simple consequence of the result above.

Corollary 9.2. *With the notation above, suppose that $Z(H) = \{\tau\}$. Then H has property (B) if and only if $\tau_1(H)$ is dense in $\mathbb{R}$.*

Proof. Here the smallest face F of K_n that contains $Z(H)$ is the convex hull of τ_1 and τ_2. As the restriction of τ_1 to $\mathbb{R}H$ is nonzero while τ_2 coincides with $-\tau_1$ on $\mathbb{R}H$, Theorem 9.1 applies with $k = 1$ and $j_1 = 1$, and the conclusion follows. □

We now turn to a more specific example.

Example 9.3. *Suppose that $n \geq 3$ and let $\alpha_1, \dots, \alpha_{n-1}, \eta_1, \dots, \eta_{n-2} \in \mathbb{R}$. Consider the subgroup H of $\mathbb{R}^n$ generated by the rows $h_1, \dots, h_{n-1}$ of the matrix*

$$C = \begin{pmatrix} \alpha_1 & -\alpha_1 & & & & \\ \vdots & \vdots & & & I_{n-2} & \\ \alpha_{n-2} & -\alpha_{n-2} & & & & \\ \alpha_{n-1} & -\alpha_{n-1} & \eta_1 & \eta_2 & \dots & \eta_{n-2} \end{pmatrix},$$

where I_{n-2} is the identity matrix of size $n-2$. Let S denote the set of indices j with $1 \leq j \leq n-2$ for which $\alpha_j \neq 0$. Then H has property (B) if and only if $\tau_1(H) = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{n-1}$ is zero or dense in $\mathbb{R}$ and one of the following mutually exclusive conditions holds.

(i) *The rank of C is $n-1$.*
(ii) *The rank of C is $n-2$ and not all α_j with $j \in S$ have the same sign.*
(iii) *The rank of C is $n-2$, all α_j with $j \in S$ have the same sign, and the set $\{1\} \cup \{\eta_j \mid j \in S\}$ is linearly independent over $\mathbb{Q}$.*

As the proof will show, $Z(H)$ is the singleton $\{\tau\}$ in cases (i) and (ii), while it is a line segment in case (iii).

Proof. Since $\tau \in Z(H)$ and since τ_1 belongs to the smallest face of K_n containing τ, the condition that $\tau_1(H)$ is zero or dense in $\mathbb{R}$ is necessary by Theorem B. Suppose that it holds. If the rank of C is $n-1$, then $\mathbb{R}\,H = \ker\tau$, thus $Z(H) = K_n \cap \mathbb{R}\tau = \{\tau\}$ and so H has property (B) by Corollary 9.2. Otherwise, the rank of C is $n-2$ and we have

$$h_{n-1} = \eta_1 h_1 + \cdots + \eta_{n-2} h_{n-2}.$$

Moreover, the linear form ϕ given by

$$\phi = \tau_2 + \alpha_1\tau_3 + \cdots + \alpha_{n-2}\tau_n = \tau_2 + \sum_{j\in S} \alpha_j \tau_{j+2}$$

vanishes on $h_1, \dots, h_{n-2}$ and therefore on the whole of H. This means that the annihilator of H in $(\mathbb{R}^n)^*$ is $\mathbb{R}\tau + \mathbb{R}\phi$ and so

$$Z(H) = K_n \cap (\mathbb{R}\tau + \mathbb{R}\phi).$$

If the real numbers α_j with $j \in S$ do not all have the same sign, this implies that $Z(H) = K_n \cap \mathbb{R}\tau = \{\tau\}$ and again H has property (B) by Corollary 9.2. Otherwise ϕ or $2\tau - \phi$ is a positive linear functional and we find that $Z(H)$ is the line segment in K_n joining τ to either $c^{-1}\phi$ or $c^{-1}(2\tau - \phi)$, where $c = 1 + \sum_{j\in S} |\alpha_j|$. Then the smallest face F of K_n containing $Z(H)$ is the convex hull of the set $\{\tau_1, \tau_2\} \cup \{\tau_{j+2} \mid j \in S\}$ and $\{\tau_{j+2} \mid j \in S\}$ is a maximal set of vertices of F whose restriction to $\mathbb{R}H$ are linearly independent over $\mathbb{R}$. Letting $j_1, \dots, j_k$ denote the distinct elements of S, we deduce from Theorem 9.1, that H has property (B) if and only if its projection

$$\{(t_{j_1+2}(h), \dots, \tau_{j_k+2}(h)) \mid h \in H\} = \mathbb{Z}^k + \mathbb{Z}(\eta_{j_1}, \dots, \eta_{j_k}).$$

is dense in $\mathbb{R}^k$, that is if and only if $1, \eta_{j_1}, \dots, \eta_{j_k}$ are linearly independent over $\mathbb{Q}$. □

Appendix A. Gordan's theorem and Farkas' lemma

Gordan's theorem (not to be confused with Gordan's lemma, a result—by the same Gordan—concerning toric varieties) asserts the following [Gor].

Theorem A.1 (Gordan)**.** *Let A be an $m \times n$ real matrix. Exactly one of the following is true.*

(i) *There exists $\mathbf{y} \in (\mathbb{R}^n)^+ \setminus \{\mathbf{0}\}$ such that $A\mathbf{y}^t = \mathbf{0}$.*
(ii) *There exists $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x}A \in (\mathbb{R}^m)^{++}$.*

Here, we view the elements of $\mathbb{R}^m$ and of $\mathbb{R}^n$ as row vectors. If (ii) holds, then we may choose $\mathbf{x}$ in $\mathbb{Z}^m$, by a simple density argument. Thus Gordan's theorem is the special case of Proposition 5.1 applied to the group $G = \mathbb{R}^n$ with the usual coordinatewise ordering, and to the subgroup H of G generated by the rows of A: alternative (i) says that there exists a trace τ of G such that $\tau(H) = \{0\}$, while alternative (ii) with $\mathbf{x} \in \mathbb{Z}^m$ says that H contains an element of G^{++}.

Gordan's theorem was followed 29 years later by Farkas' lemma [F], which is now typically used to prove the former. We state Farkas' Lemma below in one of its numerous equivalent forms. It can be used to provide a direct proof of Theorem B.

Theorem A.2 (Farkas)**.** *Let A be a real $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^n$. Exactly one of the following is true.*

(i) *There exists $\mathbf{y} \in (\mathbb{R}^n)^+$ such that $A\mathbf{y}^t = \mathbf{0}$ and $\mathbf{b}\mathbf{y}^t < 0$.*
(ii) *There exists $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x}A \le \mathbf{b}$.*

Contrary to Gordan's theorem, this result has limited extension to partially ordered abelian groups. To explain, let $G = \mathbb{R}^n$ equipped with the coordinatewise ordering, and let H be the subspace of $\mathbb{R}^n$ generated, over $\mathbb{R}$, by the rows of A, where A is as in the statement of Theorem A.2. Then the result says that, for any $g \in G$, either (ii) there exists $h \in H$ such that $h \le g$, or (i) there exists a trace $\tau \in S(G, \mathbf{1})$ such that $\tau(H) = \{0\}$ and $\tau(g) < 0$. Equivalently, this means that, for any $g \in G$, we have

$$(g + H) \cap G^+ \neq \{0\} \iff \tau(g) \ge 0 \text{ for all } \tau \in Z_G(H). \tag{A.1}$$

This property is easy to characterize when $H \cap G^+ = \{0\}$, or more generally, when H is a convex subgroup of G.

Lemma A.3. *Let (G, u) be a partially ordered abelian group with order unit, and let H be a convex subgroup of G. Then (A.1) holds for each $g \in G$ if and only if the partially ordered group G/H is archimedean.*

Recall that a partially ordered abelian group G is *archimedean* if for $x, y \in G$, the condition $nx \leq y$ for all positive integers n entails that $-x \in G^+$ (a inequivalent definition, frequently seen, requires $x \in G^+$ at the outset). When G has an order unit u, it is archimedean if and only if the traces in $S(G, u)$ determine the ordering on G, that is, $G^+ = \{g \in G \mid \hat{g} \geq 0\}$ ([Go, Theorem 4.14]). Under the hypotheses of Lemma A.3, this is exactly what (A.1) means for the pair $(G/H, u + H)$.

Archimedeanness is a strong property, particularly if the partially ordered abelian group is simple. It was not known until 2013 that for every (infinite-dimensional) metrizable Choquet simplex K, there exists a simple archimedean dimension group (G, u) whose trace space is K [H3].

References

[Ak] Akin, Ethan. Measures on Cantor space. *Topology Proc.* **24** (1999), Summer, 1–34 (2001). MR1876365 (2002j:28002), Zbl 1025.28007. 376, 388

[BH] Bezuglyi, Sergey; Handelman, David. Measures on Cantor sets: the good, the ugly, the bad. *Trans. Amer. Math. Soc.* **366** (2014), no. 12, 6247–6311. MR3267010, Zbl 1328.37007, arXiv:1201.1953, doi: 10.1090/s0002-9947-2014-06035-2. 376, 378, 387, 388, 389, 390

[EHS] Effros, Edward G.; Handelman, David E.; Shen, Chao Liang. Dimension groups and their affine representations. *Amer. J. Math.* **102** (1980), no. 2, 385–407. MR0564479 (83g:46061), Zbl 0457.46047, doi: 10.2307/2374244. 386

[F] Farkas, Julius. Theorie der einfachen Ungleichungen. *J. Reine Angew. Math.* **124** (1902), 1–27. MR1580578, JFM 32.0169.02. 393

[GiHH] Giordano, Thierry; Handelman, David E.; Hosseini, Maryam. Orbit equivalence of Cantor minimal systems and their continuous spectra. *Math. Z.* **289** (2018), no. 3–4, 1199–1218. MR3830245, Zbl 1394.37015, doi: 10.1007/s00209-017-1994-9. 390

[Go] Goodearl, Kenneth R. Partially ordered abelian groups with interpolation. Mathematical Surveys and Monographs, 20. *American Mathematical Society, Providence RI,* 1986. xxii+336 pp. ISBN: 0-8218-1520-2. MR0845783 (88f:06013), Zbl 0589.06008, doi: 10.1090/surv/020. 377, 378, 381, 385, 386, 388, 389, 394

[GoH1] Goodearl, Kenneth R.; Handelman, David E. Rank functions and K_0 of regular rings. *J. Pure and Appl. Algebra* **7** (1976), no. 2, 195–216. MR0389965 (52#10794), Zbl 0321.16009, doi: 10.1016/0022-4049(76)90032-3. 378

[GoH2] Goodearl, Kenneth R.; Handelman, David E. Metric completions of partially ordered Abelian groups. *Indiana Univ. Math. J.* **29** (1980), no. 6, 861–895. MR0589651 (82b:06020), Zbl 0455.06012, doi: 10.1512/iumj.1980.29.29060. 388

[Gor] Gordan, Paul. Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten. *Math. Ann.* **6** (1873), no. 1, 23–28. MR1509805, JFM 05.0095.01, doi: 10.1007/bf01442864. 382, 393

[H1] Handelman, David. Free rank $n+1$ dense subgroups of $\mathbb{R}^n$ and their endomorphisms. *J. Funct. Anal.* **46** (1982), no. 1, 1–27. MR0654462 (84h:06019), Zbl 0489.20044, doi: 10.1016/0022-1236(82)90041-6. 390

[H2] Handelman, David. Good traces for not necessarily simple dimension groups. *Pacific J. Math.* **281** (2016), no. 2, 365–419. MR3463041, Zbl 1354.37018, arXiv:1309.7424, doi: 10.2140/pjm.2016.281.365. 390

[H3] Handelman, David. Simple Archimedean dimension groups. *Proc. Amer. Math. Soc.* **141** (2013), no. 11, 3787–3792. MR3091768, Zbl 1282.06017, arXiv:1102.2965, doi: 10.1090/s0002-9939-2013-11672-2. 394

(David Handelman) Department of Mathematics and Statistics, University of Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada
dehsg@uottawa.ca

(Damien Roy) Department of Mathematics and Statistics, University of Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada
droy@uottawa.ca

This paper is available via http://nyjm.albany.edu/j/2019/25-18.html.