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Test elements in solvable Baumslag-Solitar groups

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Abstract. In this paper, normal forms are established for group elements in the solvable Baumslag-Solitar groups to classify all test elements in these groups. These normal forms are used to identify two general types of endomorphisms and automorphisms are identified through these types. Test elements are then identified as elements whose total exponents on one of the generators is zero. Finally, we show that Turner's Retract Theorem does not hold for these groups by giving a specific counterexample.

Contents

1. Introduction

An element g of a group G is called a test element if it has the property that for every endomorphism $\varphi : G \to G$, $\varphi(g) = g$ implies φ is an automorphism. Test elements were introduced by Shpilrain in 1995 [9] and classified by Turner for free groups in 1996, using his Retract Theorem for free groups [11]. In this paper, the statement was a corollary to a theorem about the stable image of endomorphisms of free groups. The result was more directly restated in [5] as follows:

Theorem. *A word w in a free group F is a test word if and only if w is not in any proper retract.*

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Many of the works that have followed prove this theorem for test elements in different classes of groups. Test elements were classified in the free product of finite cyclic groups by Voce [12] who found similarly that group elements were test elements when they lay outside proper retracts. O'Neill and Turner proved that test elements for torsion-free stably hyperbolic groups were precisely those elements that lie outside proper retracts and showed that almost all surface groups have test elements [5]. The same authors described a method for finding test elements in the commutator subgroup of a direct product of groups with cyclic centralizers [6] and that work was developed further by Pan, Ma and Luo [7]. Test elements in finitely generated abelian groups were explored by Rocca and Turner [8].

More recently, Groves proved Turner's Retract Theorem more generally for all torsion-free hyperbolic groups [4], improving on the result by O'Neill and Turner. Snopce and Tanushevski proved that Turner's Retract Theorem held for finitely generated profinite groups [10] and the term 'Turner group', which is used to describe groups in which test elements are precisely those group elements that lie outside proper retracts, was established circa 2013 [3]. In short, Turner groups are groups for which Turner's Retract Theorem holds true.

The solvable Baumslag-Solitar Groups are a class of two-generator, one-relator groups which can be presented $BS_{1,n} = \langle a, t | tat^{-1} = a^n \rangle$ as in Farb and Mosher [2]. When $n = 1$, the group is the fundamental group of the Klein Bottle, which is well understood, so the remainder of the paper will be centered on classifying test elements in the solvable Baumslag-Solitar groups for which $n \geq 2$.

Using the single relator, one can see five useful relationships in establishing a normal form for group elements. The first pair, $ta = a^n t$ and $ta^{-1} = a^{-n}t$ demonstrate that t may be moved to the right of a power of the generator a at the expense of multiplying that power of a by n. More generally, if $k > 0$ then $t^k \cdot a^m = a^{m \cdot n^k} \cdot t^k$. Thus a positive power of t may be moved to the right of any power of a at the expense of multiplying the exponent of a by a power of n. The second pair, $at^{-1} = t^{-1}a^n$ and $a^{-1}t^{-1} = t^{-1}a^{-n}$, similarly demonstrate that a negative power of the generator t may be moved to the left of a power of the generator a at the expense of multiplying the exponent of a by a power of n. This suggests that any group element $g \in BS_{1,n}$ can be represented $g = t^{-k}a^x t^m$ for $k, m \geq 0$. The fifth useful relationship $t^{-1}a^n t = a$ implies that if x is a multiple of n that x may be reduced when both k and m are positive to arrive at the **normal form** $g = t^{-k}a^l t^m$ for integers k, l and m with $k, m \geq 0$, where if l is a multiple of n then k or m is zero.

We note that for any representation of $g \in BS_{1,n}$ that the total exponent in t, denoted $|g|_t = \left|t^{-k}a^l t^m\right|_t = m - k$ is a well-defined integer even though the total exponent in a is not. This fact is extremely useful when trying to understand the class of groups and the action of endomorphisms on it.

Furthermore, the following equation also holds for this class of groups, and its justification is left to the reader:

$$(a^q t)^s = a^{q\left(1+n+\cdots+n^{s-1}\right)} t^s, s > 0 \tag{1.1}$$

This result and its inverse representation may be used many times in reducing the image of general group elements under endomorphism throughout the remainder of the paper.

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2. Endomorphisms and retractions of solvable Baumslag-Solitar groups

In this section, two types of endomorphisms of $\varphi : BS_{1,n} \to BS_{1,n}$ are classified based on the images of generators. The first type are shown to be monomorphisms and the second type is shown to contain the subclass of proper retracts.

Proposition 2.1. *Let $\varphi : BS_{1,n} \to BS_{1,n}$ be an endomorphism. Then it has one of two types defined by its action on the generators:*
Type 1: *$\varphi(a) = t^{-k}a^l t^k$; $\varphi(t) = t^{-p}a^q t^{p+1}$ for $k, p \geq 0$ and integers q and l.*
Type 2: *$\varphi(a) = 1$; $\varphi(t) = t^{-p}a^q t^r$ for $p, r \geq 0$ and if q is a multiple of n then $p = 0$ or $r = 0$.*

Proof. Suppose that $\varphi : BS_{1,n} \to BS_{1,n}$ for $n \geq 2$ is an endomorphism. Then the image of the generators have normal forms as stated in the previous section, say $\varphi(a) = t^{-k}a^l t^m$ and $\varphi(t) = t^{-p}a^q t^r$ with $\varphi(tat^{-1}) = \varphi(a^n)$. Note that the total exponents of the images are $\left|\varphi(tat^{-1})\right|_t = (m-k)$ and $|\varphi(a^n)|_t = n(m-k)$. Since the total exponents are well defined, $n(m-k) = (m-k)$ and since $n \geq 2$, we can conclude that $m - k = 0$ which implies $m = k$. Thus, any endomorphism of $BS_{1,n}$ for $n \geq 2$ must have the action $\varphi(a) = t^{-k}a^l t^k$ and $\varphi(t) = t^{-p}a^q t^r$.

If $l = 0$ then $\varphi(a) = 1$ and $\varphi(t) = t^{-p}a^q t^r$, which is an endomorphism of Type 2. Normal forms require that if q is a multiple of n that $p = 0$ or $r = 0$; otherwise, the relator may be applied and a reduced form will exist.

If $l \neq 0$ then further restrictions on the exponent of the image of t are shown below:
Case 1: Assuming $k = 0$,

$$a^{l\cdot n} = \varphi(a) = \varphi(tat^{-1}) = (t^{-p}a^q t^r) \cdot a^l \cdot (t^{-r}a^{-q}t^p)$$

Using the relator r times, the right side of the previous equation reduces so that

$$a^{l\cdot n} = t^{-p}a^{l\cdot n^r}t^p$$

and conjugating both sides of this equation by t^p yields

$$a^{l\cdot n^{p+1}} = a^{l\cdot n^r}$$

which implies $r = p+1$ since $n \geq 2$. This is an endomorphism of Type 1 with $\varphi(a) = a^l$ and $\varphi(t) = t^{-p}a^q t^{p+1}$.

Case 2: Assuming $k \neq 0$, the general form of a Type 1 endomorphism can be obtained most readily when following φ by the inner automorphism $\alpha_{t^k}(g) = t^k g t^{-k}$ and applying the argument in Case 1 to obtain the desired form. Alternatively, the reader could make a direct argument by examining 9 sub-cases, based on the relationships of k, p and r. □

It is useful to note that nontrivial endomorphisms of Type 2 have a cyclic image generated by the image of the generator t. It is also useful to note that endomorphisms of Type 1 respect the total exponent of t for each generator. For endomorphisms of this type, the image of the generator t is a conjugate of $a^q t$. More specifically, $\varphi(t) = t^{-p}(a^q t)t^p$. This is a useful fact for the following propositions.

Proposition 2.2. *If $\varphi : BS_{1,n} \to BS_{1,n}$ is an endomorphism of Type 1, then φ is a monomorphism.*

Proof. Let φ be any Type 1 endomorphism of $BS_{1,n}$. Then $\varphi(a) = t^{-k}a^l t^k$ and $\varphi(t) = t^{-p}a^q t^{p+1}$. Suppose that $g = t^{-i}a^s t^j$ and $\varphi(g) = 1$. Then $|\varphi(g)|_t = 0$. Reducing the image of g to its normal form reveals that $i = j$ because of this total exponent condition. Hence, $g = t^{-j}a^s t^j$. Thus:

$$\begin{aligned} 1 = \varphi(g) &= \varphi(t^{-j})\varphi(a^s)\varphi(t^j) \\ &= (t^{-p}(a^q t)t^p)^{-j}(t^{-k}a^l t^k)^s(t^{-p}(a^q t)t^p)^j \\ &= (t^{-p}(a^q t)^{-j}t^p)(t^{-k}(a^l)^s t^k)(t^{-p}(a^q t)^j t^p) \\ &= h \cdot a^{ls} \cdot h^{-1}, \end{aligned}$$

where $h = (t^{-p}(a^q t)^{-j}t^{p-k})$.

Conjugation on both sides of the equation yields $a^{l\cdot s} = 1$. This implies that $l = 0$ or $s = 0$, but if $l = 0$, then φ is not a Type 1 endomorphism. Therefore, $s = 0$ and $g = 1$, as desired. □

Recall that an endomorphism $\rho : G \to G$ is a **retraction** if it has the property that $\rho^2(g) = \rho(g)$ for all $g \in G$. The image of the endomorphism is called a **retract** and it is a **proper retract** [11] if the image is a proper subgroup.

Proposition 2.3. *Let $\rho : BS_{1,n} \to BS_{1,n}$ be a nontrivial proper retraction. Then ρ is an endomorphism of Type 2 such that $\rho(t) = t^{-p}a^q t^{p+1}$.*

Proof. If ρ is a nontrivial *proper* retraction, then it cannot be a monomorphism which implies ρ cannot be an endomorphism of Type 1. Therefore, ρ is of Type 2 with $\rho(a) = 1$ and $\rho(t) = t^{-p}a^q t^r$. Note that $\rho^2(t) =$

$\rho(\rho(t)) = \rho(t^{-p}a^q t^r) = (t^{-p}a^q t^r)^{r-p}$. However, since $\rho^2(t) = \rho(t)$, this implies $(t^{-p}a^q t^r)^{r-p} = (t^{-p}a^q t^r)^1$. Examination of the total exponents on t reveals that $(r-p)^2 = (r-p)$ and thus $r-p=0$ or $r-p=1$. If $r-p=0$, we obtain the trivial retraction which contradicts the assumption. Thus, $r-p=1$ which implies $r=p+1$, as desired. □

Proposition 2.4. *Let $\rho : BS_{1,n} \to BS_{1,n}$ be an endomorphism such that $\rho(a) = 1$ and $\rho(t) = t^{-p}a^q t^{p+1}$. Then ρ is a retraction.*

Proof. It is enough to demonstrate that $\rho^2(g) = \rho(g)$ for each of the generators. Note that with the assumptions, $\rho^2(a) = \rho(1) = 1 = \rho(a)$ and

$$\begin{aligned}\rho^2(t) &= \rho(\rho(t)) \\ &= \rho(t^{-p}a^q t^{p+1}) \\ &= \rho(t)^{-p} \cdot \rho(a)^q \cdot \rho(t)^{p+1} \\ &= \rho(t)^{-p} \cdot (1) \cdot \rho(t)^{p+1} \\ &= \rho(t).\end{aligned}$$

Since the retraction property holds on each of the generators, it must hold for all group elements, as desired. □

Propositions 2.3 and 2.4 demonstrate that proper retracts of the solvable Baumslag-Solitar groups are cyclic groups of the form $R = \langle t^{-p}a^q t^{p+1}\rangle$ for $p \geq 0$ and integer q. Furthermore, if $p \neq 0$, then due to normal forms, we may assume that $gcd(q,n) = 1$. This leads to the following proposition:

Proposition 2.5. *If $g \in BS_{1,n}$ is not the identity element and g lies in a proper retract, then $g = t^{-p}a^{q(1+n+\cdots+n^{i-1})}t^{p+i}$ for positive integer i or $g = t^{-p+j}a^{-q(1+n+\cdots+n^{-j-1})}t^p$, for negative integer j.*

Proof. If g is in a proper retraction of $BS_{1,n}$ then g can be obtained as a power of a cyclic generator of the form $t^{-p}a^q t^{p+1}$ for $p > 0$ and integer q or for $p = 0$ and integer q with $gcd(n,q) = 1$. If i is a positive integer then:

$$\begin{aligned}g &= (t^{-p}a^q t^{p+1})^i \\ &= t^{-p}(a^q t)^i t^p \\ &= t^{-p}a^{q(1+n+\cdots+n^{i-1})}t^{p+i}, &&\text{by equation 1.1.}\end{aligned}$$

If j is negative then:

$$\begin{aligned}g &= (t^{-p}a^q t^{p+1})^j \\ &= (t^{-p}(a^q t)t^p)^j \\ &= t^{-p+j}a^{-q(1+n+\cdots+n^{-j-1})}t^p, &&\text{by equation 1.1.}\end{aligned}$$

□

3. Automorphisms of solvable Baumslag-Solitar groups

In classifying automorphisms of the solvable Baumslag-Solitar groups, the work of Collins and Levin [1] should be duly noted. For the benefit of the reader, we classify all automorphisms of $BS_{1,n}$ using the terminology stated above to completely identify all test elements in $BS_{1,n}$.

Lemma 3.1. *An endomorphism $\varphi : BS_{1,n} \to BS_{1,n}$ is an automorphism if and only if $l \neq 0$ divides some power of n with $\varphi(a) = t^{-k}a^l t^k$ and $\varphi(t) = t^{-p}a^q t^{p+1}$.*

Proof. ($\Rightarrow$) First, let φ be an automorphism as stated above and consider the case where $k = 0$. Consider it's inverse, say ψ, which is an endomorphism of Type 1 with the following general form:

$$\psi(a) = t^{-\alpha}a^{\beta}t^{\alpha}$$
$$\psi(t) = t^{-\gamma}a^{\delta}t^{\gamma+1}$$

Since $(\psi \circ \varphi)(a) = a$, $t^{-\alpha}a^{\beta \cdot l}t^{\alpha} = a$ by applying the composition when $k = 0$. Conjugation on both sides of the equation by t^{α} and repeated use of the relator $tat^{-1} = a^n$ yields $a^{\beta \cdot l} = a^{n^{\alpha}}$. This implies that $\beta \cdot l = n^{\alpha}$ and therefore, l divides a power of n. If $k \neq 0$, one follows φ by the inner automorphism $\alpha_{t^k}(g) = t^k g t^{-k}$ and maintains a similar argument to obtain the desired conclusion.

($\Leftarrow$) Suppose that $\beta \cdot l = n^{\alpha}$ for the endomorphism $\varphi : BS_{1,n} \to BS_{1,n}$. We first consider the endomorphism with $k = 0$ so that

$$\varphi(a) = a^l$$
$$\varphi(t) = t^{-p}a^q t^{p+1}.$$

The inverse endomorphism $\psi = \varphi^{-1}$ is given as follows:

$$\psi(a) = t^{-\alpha}a^{\beta}t^{\alpha} \tag{3.1}$$
$$\psi(t) = t^{-p-\alpha}a^{-\beta \cdot q}t^{p+\alpha+1} \tag{3.2}$$

One can readily obtain that $\psi(\varphi(a)) = t^{-\alpha}a^{\beta \cdot l}t^{\alpha} = t^{-\alpha}a^{n^{\alpha}}t^{\alpha} = a$ by iteratively applying the alternative formulation of the relator, $t^{-1}a^n t = a$. To show $\psi(\varphi(t)) = t$ requires the reduction of $\psi(\varphi(t))$ to its normal form using two distinct applications of equation (1.1) and that $t^p a^{\beta \cdot q} t^{-p} = a^{\beta \cdot q \cdot n^p}$ through iterative application of the relator. In the case that $k \neq 0$, we follow φ by the inner automorphism $\alpha_{t^k}(g) = t^k g t^{-k}$ and provide a similar argument to obtain the desired conclusion. $\square$

Stated another way, the preceding argument demonstrates that, up to inner automorphism, an automorphism of $BS_{1,n}$ requires that the image of the first generator, a, be either itself or a properly chosen power of itself. Intuitively, this leads to the first example of a test element in $BS_{1,n}$, given below.

Lemma 3.2. *If $BS_{1,n} = \langle a, t | tat^{-1} = a^n \rangle$ then the generator a is a test element.*

Proof. Suppose that $\varphi : BS_{1,n} \to BS_{1,n}$ is an endomorphism such that $\varphi(a) = a$. Then φ is an endomorphism of Type 1 with $k = 0$ as in Proposition 2.1. The inverse endomorphism ψ will require $\alpha = 0$ and $\beta = 1$ as in (3.1) and (3.2). Thus, if $\varphi(a) = a$ and $\varphi(t) = t^{-p}a^q t^{p+1}$ then the inverse endomorphism will be given by $\varphi^{-1}(a) = a$ and $\varphi^{-1}(t) = t^{-p}a^{-q}t^{p+1}$. This establishes that a is a test element. □

4. Classification of test elements in solvable Baumslag-Solitar groups

Theorem 4.1. *Suppose that $g \in BS_{1,n} = \langle a, t | tat^{-1} = a^n \rangle$ for $n \geq 2$ and that g is not the identity element. Then g is a test element if and only if the total exponent on t, $|g|_t = 0$.*

Proof. ($\Leftarrow$) Suppose that $g \in BS_{1,n}$ has the normal form $g = t^{-j}a^m t^j$ with $j \geq 0, m \neq 0$ and that $\varphi : BS_{1,n} \to BS_{1,n}$ with $\varphi(g) = g$. Then φ is an endomorphism of Type 1 (otherwise, $\varphi(g) = 1$) with $\varphi(a) = t^{-k}a^l t^k$ and $\varphi(t) = t^{-p}a^q t^{p+1}$ for $k, p \geq 0$ and integer q. Note that:

$$t^{-j}a^m t^j = g = \varphi(g) = \varphi\left(t^{-j}a^m t^j\right).$$

Reducing the image on the right hand side into its normal form, one obtains:

$$t^{-j}a^m t^j = t^{-j-k}a^{l \cdot m}t^{j+k}.$$

Conjugation by t^{j+k} on each side of this equation and simplifying yields

$$a^{m \cdot n^k} = a^{l \cdot m}.$$

Since $m \neq 0$, we have $l = n^k$, and by Lemma 3.1, φ is an automorphism.

($\Rightarrow$) We prove via the contrapositive and conversely, suppose that $g \in BS_{1,n}$ has a nonzero total exponent in t, say $g = t^{-p}a^q t^r$, with $p, r \geq 0$ and $p \neq r$. Then g is fixed by the endomorphism defined on the generators $\varphi(a) = a^{n^p - n^r + 1}$ and $\varphi(t) = a^{q \cdot (n-1)}t$. By Lemma 3.1, this endomorphism is not an automorphism because $\gcd(n^p - n^r + 1, n) = 1$. Thus, g is not a test element. □

5. Turner's retract theorem

The retract theorem was initially established by E. C. Turner for finitely generated free groups when Turner proved that the test elements of free groups were precisely those elements that were not contained in proper retracts [11]. More recently, the term Turner group has been used to describe those groups for which being a test element is equivalent to lying outside all proper retracts such as described by Fine, et. al. [3] and by Snopce and Tanushevski [10]. Based on the preceding results, we see that the retract theorem does not hold for the solvable Baumslag-Solitar groups because it is clear that the element $g = t^{-4}at^7$ is neither a test element (since $|g|_t = 3$) nor does it lie in any proper retract for $BS_{1,n}, n \geq 2$. In the parlance of the previously cited literature, the solvable Baumslag-Solitar Groups are not Turner Groups when $n \geq 2$.

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