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Unbounded strongly irreducible operators and transitive representations of quivers on infinite-dimensional Hilbert spaces

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Abstract. We introduce unbounded strongly irreducible operators and transitive operators. These operators are related to a certain class of indecomposable Hilbert representations of quivers on infinite-dimensional Hilbert spaces. We regard the theory of Hilbert representations of quivers as a generalization of the theory of unbounded operators. A non-zero Hilbert representation of a quiver is said to be transitive if the endomorphism algebra is trivial. If a Hilbert representation of a quiver is transitive, then it is indecomposable. But the converse is not true. Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.

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1. Introduction

A bounded linear operator T on a Hilbert space H is called strongly irreducible if T cannot be decomposed to a non-trivial (not necessarily orthogonal) direct sum of two operators, that is, if there exist no non-trivial invariant closed subspaces M and N of T such that $M \cap N = 0$ and $M + N = H$. A strongly irreducible operator is an infinite-dimensional generalization of

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a Jordan block. F. Gilfeather [Gi] introduced the notion of strongly irreducible operators. We refer to excellent books [JiW1] and [JiW2] by Jiang and Wang on strongly irreducible operators.

In [EW1, EW2] we studied the relative positions of subspaces in a separable infinite-dimensional Hilbert space after Nazarova [Na1], Gelfand and Ponomarev [GeP]. We think that relative positions of subspaces have a close relation with subfactor theory [Jo, GoHJ].

Let H be a Hilbert space and $E_1, \dots E_n$ be n subspaces in H. Then it is said that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n subspaces in H or an n-subspace system in H. Two systems $\mathcal{S} = (H; E_1, \dots, E_n)$ and $\mathcal{T} = (K; F_1, \dots, F_n)$ are isomorphic if there exists an invertible operator $\varphi : H \to K$ such that $\varphi(E_i) = F_i$ for $i = 1, 2, \cdots, n$. A non-zero system $\mathcal{S} = (H; E_1, \dots, E_n)$ is said to be indecomposable if it cannot be decomposed to a non-trivial direct sum of two systems up to isomorphism. We recall that strongly irreducible operators contribute an important role to construct indecomposable systems of four subspaces [EW1].

On the other hand, Gabriel [Ga] introduced a finite-dimensional (linear) representation of quivers by attaching vector spaces and linear maps for vertices and edges of quivers respectively. A finite-dimensional indecomposable representation of a quiver is a direct graph generalization of a Jordan block. Historically, Kronecker [Kro] solved the indecomposable representations of $\tilde{A}_1$, the so called matrix pencils in 1890. Nazarova [Na1] and Gelfand-Ponomarev [GeP] treated the four-subspace situation $\tilde{D}_4$. Donovan-Freislich [DoF] and Nazarova [Na2] classified the indecomposable representations of the tame quivers. About these topics we also refer to Bernstein-Gelfand-Ponomarev [BGP], V. Dlab-Ringel [DlR], Ringel [Ri2], Gabriel-Roiter [GaR], Kac [Ka], and so on.

We recall infinite-dimensional representations in purely algebraic setting. In [Au] Auslander found that if a finite-dimensional algebra is not of finite representation type, then there exist indecomposable modules which are not of finite length. These are trivially infinite-dimensional. Several works about infinite-dimensional Kronecker modules have been done by N. Aronszjan, A. Dean, U. Fixman, F. Okoh and F. Zorzitto in [Ar, DeZ1, Fi, FiO, FiZ, Ok]. A. Dean and F. Zorzitto [DeZ2] constructed a family of infinite-dimensional indecomposable representations of $\tilde{D}_4$. K.Ringel [Ri1] founded a general theory of infinite-dimensional representations of tame, hereditary algebra (see also [Ri3, KrR]).

In [EW3, E] we started to investigate the representation theory of quivers on Hilbert spaces. We asked the existence of an indecomposable infinite-dimensional Hilbert representation for any quiver whose underlying undirected graph is one of extended Dynkin diagrams. And we solved it affirmatively using the unilateral shift S. The argument works even if we replace the unilateral shift S with any strongly irreducible operator. From this, it is suggested that strongly irreducible operators are useful to construct indecomposable Hilbert representations of quivers [EW4].

From the analogy of a transitive lattice (see P.R. Halmos [H] and K.J. Harrison, H. Radjavi and P. Rosenthal [HRR]), we called an indecomposable Hilbert representation (H, f) of a quiver such that $End(H, f) = \mathbb{C}I$ transitive. If a Hilbert representation of a quiver is transitive, then it is indecomposable. But the converse is not true. Therefore, it is important to investigate the existence problem of an transitive infinite-dimensional Hilbert representation for any quiver whose underlying undirected graph is one of extended Dynkin diagrams. In this direction, we [EW4] showed two kinds of constructions of quite non-trivial transitive Hilbert representations (H, f) of the Kronecker quiver.

In the purely algebraic setting, a representation of a quiver is called a brick if its endomorphism ring is a division ring. But for a Hilbert representation (H, f), $End(H, f)$ is a Banach algebra and not isomorphic to its purely algebraic endomorphism ring in general, because we only consider bounded endomorphisms. By the Gelfand-Mazur theorem, any Banach algebra over $\mathbb{C}$ which is a division ring must be isomorphic to $\mathbb{C}$.

We remark that locally scalar representations of quivers were introduced by Kruglyak and Roiter [KrRo]. But their subject is different from ours. We also refer to S. Kruglyak, V. Rabanovich, and Y. Samoilenko [KrRS] and Y. P. Moskaleva and Y. S. Samoilenko [MS].

We consider finite-dimensional indecomposable representations of quivers whose underlying graph is a Dynkin diagram. They are transitive (cf.[As]). But it is extremely difficult to solve the existence problem for infinite-dimensional indecomposable (also transitive) Hilbert representations of quivers whose underlying undirected graph is a Dynkin diagram. The existence is not known even for quivers whose underlying undirected graph is D_4.

In this paper we introduce unbounded strongly irreducible operators and transitive operators. It is known that any unbounded closed operator T on a Hilbert space can be realized as a quotient BA^{-1} of bounded operators A and B on H. This fact is related with operator ranges and intersections of domains of unbounded operators. See, for example, P. Fillmore and J. Williams [FiW], W.E. Kaufman [Kau] and H. Kosaki [Ko]. We point out that the study of an unbounded closed operator $T = BA^{-1}$ can be translated to the study of a Hilbert representation given by A and B of the Kronecker quiver. We show that some transitive operators are constructed by a certain transitive Hilbert representation of the Kronecker quiver. We regard the theory of Hilbert representations of quivers as a generalization of the theory of unbounded operators. We also solve completely the existence problem of infinite-dimensional transitive Hilbert representations of quivers whose underlying undirected graphs are the extended Dynkin diagrams.

Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. If the underlying undirected graph of Γ is not $\widetilde{A_n}$, then there exists an infinite-dimensional transitive Hilbert representation of Γ. If the underlying undirected graph of Γ is $\widetilde{A_n}$, then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver. We used unbounded transitive operators based on an idea of a transitive lattice by K.J. Harrison, H. Radjavi and P. Rosenthal ([HRR],[RR]).

2. Hilbert representations of quivers

A quiver $\Gamma = (V, E, s, r)$ is a quadruple consisting of the set V of vertices, the set E of arrows, and two maps $s, r : E \to V$ which associate with each arrow $\alpha \in E$ its support $s(\alpha)$ and range $r(\alpha)$. In this paper we assume that Γ is a finite quiver.

We denote by $\alpha : x \to y$ an arrow with $x = s(\alpha)$ and $y = r(\alpha)$. Thus a quiver is a directed graph. We denote by $|\Gamma|$ the underlying undirected graph of a quiver Γ. We say that a quiver Γ is connected if $|\Gamma|$ is a connected graph. A quiver Γ is called finite if both V and E are finite sets. A path of length m is a finite sequence $\alpha = (\alpha_1, \cdots, \alpha_m)$ of arrows such that $r(\alpha_k) = s(\alpha_{k+1})$ for $k = 1, \cdots, m-1$. Its support is $s(\alpha) = s(\alpha_1)$ and its range is $r(\alpha) = r(\alpha_m)$. A path of length $m \geq 1$ is called a cycle if its support and range coincide. A cycle of length one is called a loop. A quiver which is a loop is also called the Jordan quiver L. A quiver which is a cycle of length $m \geq 1$ is also called the oriented cyclic quiver C_m with length $m \geq 1$. A quiver is said to be acyclic if it contains no cycles.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. It is said that (H, f) is a Hilbert representation of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators with $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (H, f) and (K, g) be Hilbert representations of Γ. A homomorphism $T : (H, f) \to (K, g)$ is a family $T = (T_v)_{v \in V}$ of bounded operators $T_v : H_v \to K_v$ satisfying $T_{r(\alpha)} f_\alpha = g_\alpha T_{s(\alpha)}$ for any arrow $\alpha \in E$.

The composition $T \circ S$ of homomorphisms T and S is defined by $(T \circ S)_v = T_v \circ S_v$ for $v \in V$. In this way we have obtained a category HRep (Γ) of Hilbert representations of Γ. We denote by Hom $((H, f), (K, g))$ the set of homomorphisms $T : (H, f) \to (K, g)$. We denote by $End(H, f) := Hom((H, f), (H, f))$ the set of endomorphisms. We can regard $End(H, f)$ as a subalgebra of $\oplus_{v \in V} B(H_v)$.

In the paper we distinguish the following two classes of operators. A bounded operator A is said to be a projection(resp. an idempotent) if $A^2 =$

$A = A^*$ (resp. $A^2 = A$). We denote by

$$\begin{aligned} Idem(H,f) &:= \{T \in End(H,f) \mid T^2 = T\} \\ &= \{T = (T_v)_{v\in V} \in End(H,f) \mid T_v^2 = T_v (\text{for any } v \in V)\} \end{aligned}$$

the set of all idempotents of $End(H,f)$.

Let $0{=}(0_v)_{v\in V}$ be a family of zero endomorphisms and $I = (I_v)_{v\in V}$ be a family of identity endomorphisms. It is said that (H,f) and (K,g) are isomorphic, denoted by $(H,f) \cong (K,g)$, if there exists an isomorphism $\varphi : (H,f) \to (K,g)$, that is, there exists a family $\varphi = (\varphi_v)_{v\in V}$ of bounded invertible operators $\varphi_v \in B(H_v, K_v)$ such that $\varphi_{r(\alpha)} f_\alpha = g_\alpha \varphi_{s(\alpha)}$ for any arrow $\alpha \in E$. We say that (H,f) is a finite-dimensional representation if H_v is finite-dimensional for all $v \in V$. And (H,f) is an infinite-dimensional representation if H_v is infinite-dimensional for some $v \in V$.

We recall a notion of indecomposable representation in [EW3], that is, a representation which cannot be decomposed into a direct sum of smaller representations anymore.

Definition. Let $\Gamma = (V,E,s,r)$ be a finite quiver. Let (K,g) and $(K^{'}, g^{'})$ be Hilbert representations of Γ. We define the direct sum $(H,f) = (K,g) \oplus (K^{'}, g^{'})$ by $H_v = K_v \oplus K_v^{'}$ for $v \in V$ and $f_\alpha = g_\alpha \oplus g_\alpha^{'}$ for $\alpha \in E$. It is said that a Hilbert representation (H,f) is zero, denoted by $(H,f) = 0$ if $H_v = 0$ for any $v \in V$.

Definition. A Hilbert representation (H,f) of Γ is said to be decomposable if (H,f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H,f) of Γ is called indecomposable if it is not decomposable, that is, if $(H,f) \cong (K,g) \oplus (K',g')$ then $(K,g) \cong 0$ or $(K',g') \cong 0$.

The following proposition is useful to show the indecomposability in concrete examples.

Proposition 2.1. [EW3, Proposition 3.1.] *Let (H,f) be a Hilbert representation of a quiver Γ. Then the following conditions are equivalent:*

(1) (H,f) *is indecomposable.*
(2) $Idem(H,f) = \{0, I\}$.

Remark. The indecomposability of Hilbert representations of a quiver is an isomorphic invariant, but it is not a unitary invariant. Hence we cannot replace the set $Idem(H,f)$ of idempotents of endomorphisms by the subset of idempotents of endomorphisms which consists of projections to show the indecomposability.

Definition.([EW4, page 569]) A Hilbert representation (H,f) of a quiver Γ is said to be *transitive* if $End(H,f) = \mathbb{C}I$.

If a Hilbert representation (H,f) of Γ is transitive, then (H,f) is indecomposable. In fact, since $End(H,f) = \mathbb{C}I$, any idempotent endomorphism T is 0 or I. In purely algebraic setting, a representation of a quiver is said to be a *brick* if its endomorphism ring is a division ring (see for example, cf. [As]).

Let H be a Hilbert space and $E_1, \dots E_n$ be n subspaces in H. Then it is said that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n subspaces in H. Let $\mathcal{T} = (K; F_1, \dots, F_n)$ be another system of n subspaces in a Hilbert space K. Then we say that $\varphi : \mathcal{S} \to \mathcal{T}$ is a homomorphism if $\varphi : H \to K$ is a bounded linear operator satisfying that $\varphi(E_i) \subset F_i$ for $i = 1, \dots, n$. We say that $\varphi : \mathcal{S} \to \mathcal{T}$ is an isomorphism if $\varphi : H \to K$ is an invertible (i.e., bounded bijective) linear operator satisfying that $\varphi(E_i) = F_i$ for $i = 1, \dots, n$. It is said that systems $\mathcal{S}$ and $\mathcal{T}$ are *isomorphic* if there is an isomorphism $\varphi : \mathcal{S} \to \mathcal{T}$. This means that the relative positions of n subspaces $(E_1, \dots, E_n)$ in H and $(F_1, \dots, F_n)$ in K are same under disregarding angles. Let us denote by $Hom(\mathcal{S}, \mathcal{T})$ the set of homomorphisms of $\mathcal{S}$ to $\mathcal{T}$ and $End(\mathcal{S}) := Hom(\mathcal{S}, \mathcal{S})$ the set of endomorphisms on $\mathcal{S}$. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be systems of n subspaces in Hilbert spaces H and H'. Then their direct sum $\mathcal{S} \oplus \mathcal{S}'$ is defined by

$$\mathcal{S} \oplus \mathcal{S}' := (H \oplus H'; E_1 \oplus E'_1, \dots, E_n \oplus E'_n).$$

A system $\mathcal{S} = (H; E_1, \dots, E_n)$ of n subspaces is said to be *decomposable* if the system $\mathcal{S}$ is isomorphic to a direct sum of two non-zero systems. A non-zero system $\mathcal{S} = (H; E_1, \cdots, E_n)$ of n subspaces is called *indecomposable* if it is not decomposable.

We recall that strongly irreducible operators A play an extremely important role to construct indecomposable systems of four subspaces. Moreover the commutant $\{A\}'$ corresponds to the endomorphism ring.

For any single operator $A \in B(K)$ on a Hilbert space K, let $\mathcal{S}_A = (H; E_1, E_2, E_3, E_4)$ be the associated operator system such that $H = K \oplus K$ and

$$E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = \{(x, Ax); x \in K\}, E_4 = \{(y, y); y \in K\}.$$

It follows that

$$End(\mathcal{S}_A) = \{T \oplus T \in B(H); T \in B(K),\ AT = TA\}$$

is isomorphic to the commutant $\{A\}'$. The associated system $\mathcal{S}_A$ of four subspaces is indecomposable if and only if A is strongly irreducible. Moreover for any operators $A, B \in B(K)$ on a Hilbert space K, the associated systems $\mathcal{S}_A$ and $\mathcal{S}_B$ are isomorphic if and only if A and B are similar.

Following [H] and [HRR], we [EW1, page 272] introduced a transitive system of subspaces. A system $\mathcal{S} = (H; E_1, E_2, \cdots, E_n)$ of n subspaces in a Hilbert space is called transitive if the endomorphism algebra is trivial, that is,

$$End(\mathcal{S}) = \{A \in B(H);\ A(E_i) \subset E_i \text{ for any } i = 1, 2, \cdots, n\} = \mathbb{C}I.$$

3. Unbounded strongly irreducible operators

In this section we shall introduce unbounded strongly irreducible operators and transitive operators. These operators are related to a certain class of indecomposable Hilbert representations of quivers on infinite-dimensional Hilbert spaces and four- subspace systems. Let H be a Hilbert space and A a bounded linear operator on H. We denote the image of A by $\mathrm{Im}(A)$ and the graph of A by $G(A)$, that is, $G(A) = \{(x, Ax); x \in H\}$. For elements $x, y \in H$, we define a rank-one operator $\theta_{x,y}$ by $\theta_{x,y}(z) = (z|y)x$ for $z \in H$.

P.R. Halmos [H] initiated the study of transitive lattices. A lattice $\mathcal{L}$ of subspaces of a Hilbert space H containing 0 and H is called a transitive lattice if

$$\{A \in B(H);\ AM \subset M \text{ for any } M \in \mathcal{L}\} = \mathbb{C}I.$$

K.J. Harrison, H. Radjavi and P. Rosenthal ([HRR]) constructed a transitive subspace lattice using an unbounded weighted shift as follows: Let $K = \ell^2(\mathbb{Z})$ be a Hilbert space with an orthogonal basis $\{e_i\}_{i=-\infty}^{+\infty}$. Let

$$w_n = 1 \quad (n \leq 0), \quad w_n = exp((-1)^n n!) \quad (n > 0).$$

Let T be the bilateral weighted shift defined by $Te_n = w_n e_{n+1}$, with the domain

$$D(T) = \{x = \sum_{i=-\infty}^{+\infty} \alpha_i e_i; \sum_{i=-\infty}^{+\infty} |\alpha_i w_i|^2 < +\infty\}.$$

Put $E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = G(T), E_4 = \{(x, x); x \in K\}$. Their transitive lattice is $\mathcal{L} = \{0, H = K \oplus K, E_1, E_2, E_3, E_4\}$. See also the book of Radjavi-Rosenthal [RR, 4.7. page 78].

We [EW4] considered a finite subspace lattice as a Hilbert representation of a quiver Γ as follows. Let $\mathcal{L} = \{0, M_1, M_2, \ldots, M_n, H\}$ be a finite lattice. Consider an n subspace quiver $R_n = (V, E, s, r)$, that is, $V = \{1, 2, \ldots, n, n+1\}$ and $E = \{\alpha_k;\ k = 1, \ldots, n\}$ with $s(\alpha_k) = k$ and $r(\alpha_k) = n + 1$ for $k = 1, \ldots, n$. Then there exists a Hilbert representation (K, f) of R_n such that $K_k = M_k$, $K_{n+1} = H$ and $f_{\alpha_k} : M_k \to H$ is an inclusion for $k = 1, \ldots, n$. The lattice $\mathcal{L}$ is transitive if and only if the corresponding Hilbert representation (K, f) is transitive. By this fact we may use the terminology "transitive" in the Hilbert representation case.

We recall some facts on strongly irreducible operators for convenience.

Lemma 3.1. *Let A be a bounded operator on a Hilbert space H. Then the following three conditions are equivalent:*

(0) *For any closed subspaces M and N of H with $H = M + N$ and $M \cap N = 0$, if $AM \subset M$ and $AN \subset N$, then $M = 0$ or $N = 0$.*
(1) *If $T \in B(H)$ is an idempotent in the commutant $\{A\}'$ of A, then $T = 0$ or $T = I$.*
(2) *If $T \in B(H)$ is an idempotent such that $(T \oplus T)(G(A)) \subset G(A)$, then $T = 0$ or $T = I$.*

Proof. Let M and N be closed subspaces of H such that $H = M + N$ and $M \cap N = 0$. Then there exists an idempotent E such that $M = E(H)$ and $N = (I - E)H$. Hence (0) is equivalent to (1).

We shall show that (1) is equivalent to (2). Assume that (1) holds. Let $T \in B(H)$ be an idempotent such that $(T \oplus T)(G(A)) \subset G(A)$. Then for any $x \in H$, there exists $y \in H$ such that $(T \oplus T)((x, Ax)) = (y, Ay)$. Hence $Tx = y$ and $TAx = Ay$. Thus $TA = AT$. Hence $T \in \{A\}'$. Since T is an idempotent, $T = 0$ or $T = I$. Hence (2) holds. Next we assume that (2) holds. Take an idempotent $T \in \{A\}' \cap B(H)$. Then

$$(T \oplus T)((x, Ax)) = (Tx, TAx) = (Tx, ATx).$$

Thus $(T \oplus T)(G(A)) \subset G(A)$. We have $T = 0$ or $T = I$. Hence (1) holds. □

Definition. A bounded operator $A \in B(H)$ is said to be strongly irreducible if A satisfies one of the three conditions of the above lemma.

Inspired by the example of K.J. Harrison, H. Radjavi and P. Rosenthal, we introduce unbounded strongly irreducible operators and unbounded transitive operators.

Definition. Let A be an unbounded closed operator on a Hilbert space H with the domain $D(A) \subset H$. We define the (bounded) commutant $\{A\}'$ of A by $\{A\}' = \{S \in B(H); S(D(A)) \subset D(A)$ and, for any $x \in D(A), ASx = SAx\}$. See for example [Ak, §17].

Let A and B be unbounded closed operators on H. We say that A and B are similar if there exists a bounded invertible operator $T \in B(H)$ such that $T(D(A)) = D(B)$ and $B = TAT^{-1}$. We say that A is an orthogonal direct sum $A_1 \oplus A_2$ of operators A_1 and A_2 on $H = H_1 \oplus H_2$ if $D(A) = \{(x_1, x_2); x_1 \in D(A_1), x_2 \in D(A_2)\}$ and $Ax = (A_1x_1, A_2x_2)$ for $x = (x_1, x_2) \in D(A)$.

Lemma 3.2. *Let A be an unbounded closed operator on a Hilbert space H with the domain $D(A) \subset H$. Then the following three conditions are equivalent:*

(0) *If A is similar to $A_1 \oplus A_2$ on $H = H_1 \oplus H_2$ for some unbounded closed operators A_1 and A_2, then $H_1 = 0$ or $H_2 = 0$.*
(1) *For any idempotent $E \in B(H)$, if E is in the commutant $\{A\}'$, then $E = 0$ or $E = I$.*
(2) *For any idempotent $E \in B(H)$, if $(E \oplus E)(G(A)) \subset G(A)$, then $E = 0$ or $E = I$.*

Proof. We shall show that (0)⇒(1). Let $E \in \{A\}'$ be an idempotent. We have $E(D(A)) \subset D(A)$ and $AEx = EAx$ for $x \in D(A)$. There exists an invertible operator $T \in B(H)$ such that $T(E(H)) = H_1$ and $T((I - E)H) = H_2$ and $H = H_1 \oplus H_2$. We define $A_1x = TAT^{-1}x = TAET^{-1}x$ for $x \in T(E(D(A))) \subset H_1$. Since $E(D(A)) \subset D(A)$, A_1 is well defined. And A_1 is an operator from $T(E(D(A)))$ to H_1 by $AEx = EAx$ for $x \in D(A)$. We define $A_2x = TAT^{-1}x = TA(I - E)T^{-1}x$ for $x \in T((I - E)(D(A))) \subset H_2$.

Since $E(D(A)) \subset D(A)$, A_2 is well defined. And A_2 is an operator from $T((I-E)(D(A)))$ to H_2 by $AEx = EAx$ for $x \in D(A)$. Hence we have

$$TAT^{-1} = TAET^{-1} + TA(I-E)T^{-1} = A_1 \oplus A_2.$$

Hence $A \cong A_1 \oplus A_2$ on $H_1 \oplus H_2$. Since (0) holds, we have $H_1 = 0$ or $H_2 = 0$. Hence $TE(H) = 0$ or $T(I-E)(H) = 0$. So $E = 0$ or $E = I$. Thus we have (0)⇒(1).

Conversely we shall show that (1)⇒(0). Assume that $A \cong A_1 \oplus A_2$ on $H_1 \oplus H_2$ for some unbounded closed operators A_1 and A_2. There exists an invertible operator $T \in B(H)$ such that $TAT^{-1}x = (A_1 \oplus A_2)x$ for $x \in D(A_1 \oplus A_2) = T(D(A))$. There exists an idempotent $E \in B(H)$ such that $T^{-1}H_1 = E(H)$ and $T^{-1}H_2 = (I-E)H$. We shall show that $E(D(A)) \subset D(A)$ and $AE = EA$ on $D(A)$. We have $T^{-1}D(A_1) \subset T^{-1}H_1 = EH$ and $T^{-1}D(A_2) \subset T^{-1}H_2 = (I-E)H$. $D(A) = T^{-1}D(A_1 \oplus A_2) = T^{-1}D(A_1) + T^{-1}D(A_2)$.

$$\begin{aligned} E(D(A)) &= E(T^{-1}D(A_1) + T^{-1}D(A_2)) = T^{-1}D(A_1) \\ &\subset T^{-1}D(A_1) + T^{-1}D(A_2) = D(A). \end{aligned}$$

For $x \in D(A)$, we can write $x = x_1 + x_2$ with $x_1 \in T^{-1}D(A_1)$ and $x_2 \in T^{-1}D(A_2)$. We have

$$\begin{aligned} AEx &= (T^{-1}(A_1 \oplus A_2)T)E(x_1 + x_2) \\ &= (T^{-1}(A_1 \oplus A_2)T)x_1 = T^{-1}A_1Tx_1 \end{aligned}$$

and

$$\begin{aligned} EAx &= E(T^{-1}(A_1 \oplus A_2)T)(x_1 + x_2) \\ &= E(T^{-1}A_1Tx_1 + T^{-1}A_2Tx_2) \\ &= T^{-1}A_1Tx_1. \end{aligned}$$

Thus we have $AE = EA$ on $D(A)$. Therefore $E = 0$ or $E = I$. Hence $H_1 = 0$ or $H_2 = 0$.

Next, we shall show that (1)⇒(2). Let $E \in B(H)$ be an idempotent such that $(E \oplus E)(G(A)) \subset G(A)$. Then for any $x \in D(A)$, there exists $y \in D(A)$ such that $(E \oplus E)(x, Ax) = (y, Ay)$. Hence

$$(Ex, EAx) = (y, Ay) = (Ex, AEx).$$

Thus $E \in \{A\}'$. By (1), then $E = 0$ or $E = I$.

Conversely, we shall show that (2)⇒(1). Let $E \in \{A\}'$ be an idempotent. Then $E(D(A)) \subset D(A)$, $EAx = AEx$ for $x \in D(A)$, and

$$(E \oplus E)((x, Ax)) = (Ex, EAx) = (Ex, AEx).$$

Hence $(E \oplus E)(G(A)) \subset G(A)$, and $E = 0$ or $E = I$. □

Definition. An unbounded closed operator A is said to be strongly irreducible if A satisfies one of the three conditions of the above lemma.

The next lemma is proved similarly.

Lemma 3.3. *Let A be an unbounded closed operator on a Hilbert space H with the domain $D(A) \subset H$. Then the following two conditions are equivalent:*

(1) *For any $T \in B(H)$, if T is in the commutant $\{A\}'$, then T is a scalar operator.*
(2) *For any $T \in B(H)$, if $(T \oplus T)(G(A)) \subset G(A)$, then T is a scalar operator.*

Definition. An unbounded closed operator A is said to be transitive if A satisfies one of the two conditions of the above lemma.

If an unbounded closed operator A is transitive, then A is strongly irreducible. Any bounded strongly irreducible operator A on a Hilbert space H with $\dim H \geq 2$ is not transitive, because $A \in \{A\}'$.

By the same argument we have the following lemma.

Lemma 3.4. *Let A be an unbounded closed operator on a Hilbert space K with the domain $D(A)$. Let $\mathcal{S}_A = (H; E_1, E_2, E_3, E_4)$ be a four-subspace system such that $H = K \oplus K$, $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Ax); x \in D(A)\}$, and $E_4 = \{(x, x); x \in K\}$. Then $\mathcal{S}_A$ is transitive if and only if A is transitive.*

We shall construct transitive operators using transitive Hilbert representations and quotients of operators.

Definition. Let A and B be bounded linear operators on a Hilbert space H. We say that $B(A|_{\mathrm{Ker}(A)^\perp})^{-1}$ is a quotient of B by A. We denote $(A|_{\mathrm{Ker}(A)^\perp})^{-1}$ briefly by A^{-1}. If we have an additional condition such that $\ker A \subset \ker B$, then the quotient is the mapping $Ax \mapsto Bx, x \in H$. In [Kau], Kaufman showed the following useful result about quotient operators.

Theorem 3.5. [Kau, Theorem 1, page 531] *Let T be an unbounded operator on a Hilbert space H. Then T is a closed operator if and only if $T = B(A|_{\mathrm{Ker}(A)^\perp})^{-1}$ for some $A, B \in B(H)$ such that $\mathrm{Im}(A^*) + \mathrm{Im}(B^*)$ is closed in H.*

We show that there is a non-zero surjective algebra homomorphism of the endomorphism algebra of a Hilbert representation of the Kronecker quiver to the endomorphism algebra of a four-subspace system. The Kronecker quiver Q is a quiver with two vertices $\{1, 2\}$ and two paralleled arrows $\{\alpha, \beta\}$:

$$Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

A Hilbert representation (H, f) of the Kronecker quiver is given by two Hilbert spaces H_1, H_2 and two bounded operators $f_\alpha, f_\beta : H_1 \to H_2$.

Proposition 3.6. *Let $K \neq 0$ be a Hilbert space and $A, B \in B(K)$. Let (H, f) be a Hilbert representation of the Kronecker quiver Q such that $H_1 = H_2 = K$, $f_\alpha = A$ and $f_\beta = B$. Let $\mathcal{S} = (E_0; E_1, E_2, E_3, E_4)$ be a four-subspace system such that $E_0 = K \oplus K$, $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$,*

$E_3 = \{(Ax, Bx); x \in K\}$, *and* $E_4 = \{(x, x); x \in K\}$. *Assume that* E_3 *is closed. Then there exists a non-zero surjective algebra homomorphism* Φ *of* $End(H, f)$ *to* $End(\mathcal{S})$. *Moreover, if* $\ker A \cap \ker B = 0$, *then* Φ *is one to one.*

Proof. Let (S, T) be in $End(H, f)$. We have $AS = TA$ and $BS = TB$. Since

$$(T \oplus T)(Ax, Bx) = (TAx, TBx) = (ASx, BSx),$$

we have $(T \oplus T)(E_3) \subset E_3$. Clearly $(T \oplus T)(E_i) \subset E_i$ for $i = 1, 2, 4$. Thus we have that $T \oplus T$ is in $End(\mathcal{S})$. We define a mapping Φ of $End(H, f)$ to $End(\mathcal{S})$ by $\Phi(S, T) = T \oplus T$. The map Φ is an algebra homomorphism. We shall show that the map Φ is onto.

Take $C \in End(\mathcal{S})$. Then there exists $T \in B(K)$ such that $C = (T \oplus T)$. We have that

$$(T \oplus T)\{(Ax, Bx); x \in K\} \subset \{(Ay, By); y \in K\}.$$

Hence, for any $x \in K$, there exists $y \in K$ such that $TAx = Ay$ and $TBx = By$. We put $L_0 = \ker A \cap \ker B$ and $L_1 = L_0^{\perp} \cap K$. By a decomposition of y such that $y = y_0 + y_1, y_0 \in L_0, y_1 \in L_1$, we have $TAx = Ay_1, TBx = By_1$. We define an operator S by $Sx = y_1$. We shall show that S is well defined. If there exists another $y' = y'_0 + y'_1 \in K$ for $y'_0 \in L_0$ and $y'_1 \in L_1$ such that $TAx = Ay' = Ay'_1$ and $TBx = By' = By'_1$. We have $Ay_1 = Ay'_1$ and $By_1 = By'_1$. Hence $y_1 - y'_1 \in (\ker A \cap \ker B) = L_0$. We also have $y_1 - y'_1 \in L_1$. Hence $y_1 - y'_1 \in L_0 \cap L_1 = (0)$. So $y_1 = y'_1$. Thus S is well defined.

Clearly S is linear. We shall show that S is a closed operator. Assume that $x_n \to x$ and $Sx_n = y_{n,1} \to y_1$, for $x_n, x \in K$ and $y_{n,1}, y_1 \in L_1$. Since $Sx_n = y_{n,1}$, we have that $TAx_n = Ay_{n,1} \to Ay_1$ and $TBx_n = By_{n,1} \to By_1$. If $n \to \infty$, then $TAx = Ay_1$ and $TBx = By_1$. It follows that $Sx = y_1$. Therefore S is closed. Hence S is bounded.

Since $TAx = Ay_1 = ASx$ and $TBx = By_1 = BSx$ for $x \in K$ and $y_1 \in L_1$, we have that $TA = AS$ and $TB = BS$. Hence $(S, T) \in End(H, g)$. And $\Phi(S, T) = T \oplus T$. Hence Φ is surjective. We shall show that if $\ker A \cap \ker B = 0$, then Φ is one-to-one. Suppose that $\Phi(S, T) = T \oplus T = 0$ for $(S, T) \in End(H, f)$. Then $T = 0$. We have that for any $x \in K$,

$$ASx = TAx = 0, \qquad BSx = TBx = 0.$$

Hence $Sx \in \ker A \cap \ker B = 0$. Since $Sx = 0$ for any $x \in K$, we have $S = 0$. Thus $(S, T) = 0$. Therefore Φ is one-to-one. $\square$

Remark. Let K be a Hilbert space and $A, B \in B(K)$. We consider

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} : K \to K \oplus K \text{ and } Zx = (Ax, Bx) \text{ for } x \in K.$$

We have

$$Z^* = (A^*, B^*) : K \oplus K \to K \text{ and } Z^* \begin{pmatrix} x \\ y \end{pmatrix} = A^*x + B^*y \text{ for } x, y \in K.$$

Since $\mathrm{Im}(Z)$ is closed if and only if $\mathrm{Im}(Z^*)$ is closed, we have that $\{(Ax, Bx); x \in K\}$ is closed if and only if $\mathrm{Im}(A^*) + \mathrm{Im}(B^*)$ is closed.

Remark. The map Φ is not one-one in general. We shall give an example Φ which is not one to one. Let K be a Hilbert space and A, B be operators on $K \oplus K$ such that $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let S_1, T_1, S_2, T_2 be operators on $K \oplus K$ such that $S_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then (S_1, T_1) and (S_2, T_2) are in $End(H, f)$. And (S_1, T_1) and (S_2, T_2) give the same endomorphism $T_1 \oplus T_1$ of $\mathcal{S}$. Thus Φ is not one to one.

Under a certain condition we have a correspondence between transitive Hilbert representations of the Kronecker quiver and transitive operators.

Proposition 3.7. *Let K be a Hilbert space and $A, B \in B(K)$. Assume that $\ker A = 0$ and $\mathrm{Im}A^* + \mathrm{Im}B^*$ is closed in K. Let (H, f) be a Hilbert representation of the Kronecker quiver Q such that $H_1 = H_2 = K$, $f_\alpha = A$ and $f_\beta = B$. Then BA^{-1} is transitive if and only if (H, f) is transitive.*

Proof. At first we note that the graph $G(BA^{-1}) = \{(Ax, Bx); x \in K\}$, because $\ker(A) = 0$. Since $\mathrm{Im}A^* + \mathrm{Im}B^*$ is closed, the operator BA^{-1} is closed by the remark after Proposition 3.6 (or Theorem 3.5). Let $\mathcal{S}_{BA^{-1}} = (E_0; E_1, E_2, E_3, E_4)$ be a four-subspace system such that $E_0 = K \oplus K$, $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(Ax, Bx); x \in K\} = G(BA^{-1})$, and $E_4 = \{(x, x); x \in K\}$. Since $\ker(A) = 0$, there exists an algebra isomomorphism Φ of $End(H, f)$ onto $End(\mathcal{S}_{BA^{-1}})$ by Proposition 3.6. Therefore (H, f) is transitive if and only if $\mathcal{S}_{BA^{-1}}$ is transitive. Moreover $\mathcal{S}_{BA^{-1}}$ is transitive if and only if BA^{-1} is transitive by Lemma 3.4. This implies the conclusion. □

In the following we shall give some examples of transitive operators.

Proposition 3.8. *Let Q be the Kronecker quiver. Let S be the unilateral shift on $H = \ell^2(\mathbb{N})$ with a canonical basis $\{e_1, e_2, ...\}$. For a bounded weight vector $\lambda = (\lambda_1, \lambda_2, ...) \in \ell^\infty(\mathbb{N})$ we associate with a diagonal operator $D_\lambda = diag(\lambda_1, \lambda_2, ...)$, so that SD_λ is a weighted shift operator. We assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Take a vector $\overline{w} = (\overline{w_n})_n \in \ell^2(\mathbb{N})$ such that $w_n \neq 0$ for any $n \in \mathbb{N}$. Put $A = SD_\lambda + \theta_{e_1,\overline{w}}$ and $B = S$. Define a Hilbert representation (H^λ, f^λ) of the Kronecker quiver Q by $H^\lambda_1 = H^\lambda_2 = H$, $f^\lambda_\alpha = A$ and $f^\lambda_\beta = B$. Then $\ker A = 0$ and the quotient BA^{-1} is a transitive operator. Furthermore, the operator BA^{-1} is densely defined if and only if $\lambda_k \neq 0$ for each $k \in \mathbb{N}$ and $(\frac{w_k}{\lambda_k})_k \notin \ell^2(\mathbb{N})$.*

Proof. By [EW4, Theorem 3.7.], the Hilbert representation (H^λ, f^λ) is transitive.

For $x = (x_n)_n \in \ell^2(\mathbb{N})$, assume that

$$Ax = (SD_\lambda + \theta_{e_1,\overline{w}})x = (\sum_{n=1}^{\infty} x_n w_n, \lambda_1 x_1, \lambda_2 x_2, \cdots) = 0.$$

If $\lambda_k \neq 0$ for any $k \in \mathbb{N}$, then $x_k = 0$ for any $k \in \mathbb{N}$. If there exists a $k \in \mathbb{N}$ such that $\lambda_k = 0$, then $\lambda_i \neq 0$ for $i \neq k$. Hence $x_i = 0$ for $i \neq k$. Since $\sum_{n=1}^{\infty} x_n w_n = x_k w_k = 0$, $x_k = 0$ by $w_k \neq 0$. Thus we have that $x = 0$ and $\ker A = 0$. We note that $\mathrm{Im} B^* = \mathrm{Im} S^* = H$ and $\mathrm{Im} A^* + \mathrm{Im} B^* = H$ is closed in H. Hence BA^{-1} is a closed operator.

Next we shall consider the condition such that BA^{-1} is densely defined. We note that $\overline{D(BA^{-1})} = \overline{\mathrm{Im} A} = (\ker A^*)^{\perp}$. We shall show that $\ker A^* \neq 0$ if and only if (1) $\lambda_k = 0$ for some $k \in \mathbb{N}$ or (2) $\lambda_k \neq 0$ for any $k \in \mathbb{N}$ and $(\dfrac{w_k}{\lambda_k})_k \in \ell^2(\mathbb{N})$. We see that $A^* = D_\lambda^* S^* + \theta_{\overline{w},e_1}$ and $x = (x_n)_n$ is in $\ker A^*$ if and only if

$$(\overline{\lambda_1} x_2, \overline{\lambda_2} x_3, \cdots) = (-x_1 \overline{w_1}, -x_1 \overline{w_2}, \cdots).$$

Assume that (1) $\lambda_k = 0$ for some $k \in \mathbb{N}$. We put $x = (x_i)$ by

$$x_i = \begin{cases} 0 & (i \neq k+1), \\ 1 & (i = k+1). \end{cases}$$

We have that $x \in \ker A^*$ and $\ker A^* \neq 0$.

Assume that (2) $\lambda_k \neq 0$ for any $k \in \mathbb{N}$ and $(\dfrac{w_k}{\lambda_k})_k \in \ell^2(\mathbb{N})$. Take an element $x = (1, -\overline{\left(\dfrac{w_1}{\lambda_1}\right)}, -\overline{\left(\dfrac{w_2}{\lambda_2}\right)}, \cdots)$. We have $x \in \ker A^*$ and $\ker A^* \neq 0$. Conversely, assume that there exists $x (\neq 0) \in \ker A^*$. Assume that $x_1 \neq 0$. Since

$$(\overline{\lambda_1} x_2, \overline{\lambda_2} x_3, \cdots) = (-x_1 \overline{w_1}, -x_1 \overline{w_2}, \cdots),$$

and $\overline{w_k} \neq 0$ for any $k \in \mathbb{N}$, we have $\lambda_k \neq 0$ for any $k \in \mathbb{N}$.

Since $\left(-\dfrac{x_{k+1}}{x_1}\right)_k \in \ell^2(\mathbb{N})$ and $\left(-\dfrac{x_{k+1}}{x_1}\right)_k = \overline{\left(\dfrac{w_k}{\lambda_k}\right)_k}$, we have that $\overline{\left(\dfrac{w_k}{\lambda_k}\right)_k} \in \ell^2(\mathbb{N})$. Hence we have (2). Assume that $x_1 = 0$. Since $x \neq 0$, there exists $k \in \mathbb{N}$ such that $x_{k+1} \neq 0$. Hence $\lambda_k = 0$. Therefore we have (1). $\square$

Remark. The operator BA^{-1} is densely defined for $\lambda_n = 1/n, w_n = 1/n$ $(n \in \mathbb{N})$. The operator BA^{-1} is not densely defined for $(\lambda_n)_n$ defined by

$$\lambda_n = \begin{cases} 0 & (n = 1), \\ 1/n & (n \neq 1). \end{cases}$$

The operator BA^{-1} is not densely defined for $\lambda_n = 1-(1/2^n), w_n = 1/n$ $(n \in \mathbb{N})$.

We refer to [Sh] for weighted shifts.

Proposition 3.9. *Let Q be the Kronecker quiver and $H = \ell^2(\mathbb{Z})$. Let $a = (a(n))_{n\in\mathbb{Z}}, b = (b(n))_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ such that $a(n) \neq 0, b(n) \neq 0$ for any $n \in \mathbb{Z}$. We put $w_m = \dfrac{b(m)}{a(m)}, m \in \mathbb{Z}$. We put*

$$M_k(m,n) := \frac{w_m w_{m+1}\cdots w_{m+k-1}}{w_n w_{n+1}\cdots w_{n+k-1}} \text{ for } m,n \in \mathbb{Z}, k \geq 1.$$

Assume that for any $m \neq n, (M_k(m,n))_k$ is an unbounded sequence. Let D_a be a diagonal operator with $a = (a(n))_n$ as diagonal coefficients and D_b be a diagonal operator with $b = (b(n))_n$ as diagonal coefficients. Let U be the bilateral forward shift. Put $A = D_a$ and $B = UD_b$. Define a Hilbert representation (H,f) of the Kronecker quiver Q by $H_1 = H_2 = H$, $f_\alpha = A$ and $f_\beta = B$. Then the Hilbert representation (H,f) is transitive. We also have $\ker A = 0$ *and* $\ker B = 0$. *And the operator BA^{-1} is a densely defined transitive operator.*

Proof. As in [EW4, Theorem 3.8.], we can similarly prove that the Hilbert representation (H,f) is transitive. By Proposition 3.7, the operator BA^{-1} is transitive. ☐

Example. [EW4, Theorem 3.8.] Fix a positive constant $\lambda > 1$. Consider two sequences $a = (a(n))_{n\in\mathbb{Z}}$ and $b = (b(n))_{n\in\mathbb{Z}}$ by

$$a(n) = \begin{cases} e^{-\lambda^n} & (n \geq 1, n \text{ is even }), \\ 1 & (otherwise), \end{cases} \qquad b(n) = \begin{cases} e^{-\lambda^n} & (n \geq 1, n \text{ is odd }), \\ 1 & (otherwise). \end{cases}$$

These two sequences a and b satisfy the condition of the proposition above.

The concept of transitive operators are useful because certain transitive Hilbert representations of a quiver are given in terms of transitive operators in the next section.

4. Extended Dynkin diagrams and transitive Hilbert representations

We consider transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagra $\widetilde{A_n}$ $(n \geq 0)$. In the $\widetilde{A_0}$ case, the oriented cyclic quiver is also called Jordan quiver. Trivially we have no infinite-dimensional transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_0}$.

Next we consider transitive Hilbert representations of quivers whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}$ $(n \geq 1)$. The quiver C_n with $n \geq 2$ whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_{n-1}}$ is called the oriented cyclic quiver if the quiver has cyclic orientation. The set V of the vertices of C_n is $\{1, 2, \cdots, n\}$ and the set E of the arrows of C_n is $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that $s(\alpha_i) = i, r(\alpha_i) = i+1$ $(i = 1, \cdots n-1)$ and $s(\alpha_n) = n$, $r(\alpha_n) = 1$. For the $\tilde{A}_1$ case, the quivers are the oriented cyclic quiver C_2 and the Kronecker quiver Q.

Theorem 4.1. *Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}$, $n \geq 1$. If Γ is not an oriented cyclic quiver, then there exists an infinite-dimensional transitive Hilbert representation of Γ.*

Proof. Assume that Γ is not an oriented cyclic quiver. Then there exist vertices i and j and arrows α and β such that $s(\alpha) = i$, $r(\alpha) = i+1$ and $s(\beta) = j+1$, $r(\beta) = j \pmod n$. There exists a transitive Hilbert representation (H, f) of the Kronecker quiver Q given by $A, B \in B(H)$ in [EW4, Theorem 3.8.]. We construct a Hilbert representation (H', f') of $\Gamma = (V, E)$ such that $H'_k = H$ $(k \in V)$, $f'_\gamma = I_H$ for $\gamma \neq \alpha, \beta$ $(\gamma \in E)$, $f'_\alpha = A$, and $f'_\beta = B$. Then the representation (H', f') of $\Gamma = (V, E)$ is transitive. □

By Theorem 4.1, the remaining case of the problem for $\widetilde{A_n}$ $(n \geq 1)$ is an oriented cyclic quiver. It is enough to consider the case that $H_i \neq 0$ for any i by the following lemma.

Lemma 4.2. *Let (H, f) be a Hilbert representation of the oriented cyclic quiver C_n. Assume that there exists a vertex k such that $H_k = 0$ $(1 \leq k \leq n)$. Let (K, g) be a Hilbert representation of the oriented cyclic quiver C_{n-1} such that $K_i = H_i$ $(1 \leq i \leq k-1)$, $K_i = H_{i+1}$ $(k \leq i \leq n-1)$, $g_{\alpha_i} = f_{\alpha_i}$ $(1 \leq i \leq k-2)$, $g_{\alpha_{k-1}} = 0$, $g_{\alpha_i} = f_{\alpha_{i+1}}$ $(k \leq i \leq n-1)$. Then $End(H, f)$ is isomorphic to $End(K, g)$.*

Proof. Take $T = (T_i)_i \in End(H, f)$ for $i = 1, \cdots, n$. Since $H_k = 0$, $B(H_k) = 0$. Hence we can associate $T = (T_i)_i \in End(H, f)$ with $T' = (T'_i)_i \in End(K, g)$, by putting $T_i = T'_i$ for $1 \leq i \leq k-1$ and $T_{i+1} = T'_i$ for $k \leq i \leq n-1$. By this correspondence we have that $End(H, f)$ is isomorphic to $End(K, g)$. □

For the case that $H_i = \mathbb{C}$ or 0, we introduce a concept of an equivalence relation for vertices in terms of a Hilbert representation.

Definition. Let (H, f) be a Hilbert representation of the oriented cyclic quiver $C_n = (V, E)$ such that $H_i = \mathbb{C}$ or 0. We give an equivalence relation for the set of vertices $\{i \in V; H_i \neq 0\}$ as follows: Take vertices i, j such that $H_i \neq 0$ and $H_j \neq 0$. We say that vertices i and j are (H, f)-connected if (1) $i = j$ or (2) $i < j$ and $f_{\alpha_{j-1}} \neq 0, \cdots, f_{\alpha_{i+1}} \neq 0, f_{\alpha_i} \neq 0$ or (3) $i > j$ and $f_{\alpha_{i-1}} \neq 0, \cdots, f_{\alpha_{j+1}} \neq 0, f_{\alpha_j} \neq 0$.

Lemma 4.3. *Let (H, f) be a Hilbert representation of the oriented cyclic quiver C_n such that $H_i = \mathbb{C}$ or 0 $(i = 1, 2, \cdots, n)$. Then (H, f) is transitive if and only if there exists only one (H, f)-connected component.*

Proof. Assume that (H, f) is transitive. Assume that there exist two (H, f)-connected components D_1 and D_2 in the set $\{i \in V; H_i \neq 0\}$. Let $\lambda_1 \in \mathbb{C}$, $\lambda_2 \in \mathbb{C}$ such that $\lambda_1 \neq \lambda_2$. We define $T = (T_i)_{i \in V}$ by $T_i = \lambda_1$

for $i \in D_1$ and $T_j = \lambda_2$ for $j \in D_2$ and $T_k = 0$ for k (otherwise). Then $T = (T_i)_{i\in V}$ is in $End(H, f)$. This is a contradiction.

Conversely, assume that there exists only one (H, f)-connected component. Then there exist decomposition of V by D_3 and D_4 such that

$$D_3 \cup D_4 = V,\ D_3 \cap D_4 = \emptyset,\ D_3 = \{i; H_i \neq 0\},\ D_4 = \{j; H_j = 0\}$$

and D_3 is the (H, f)-connected component.

Let $T = (T_i)_{i\in V} \in End(H, f)$. Then $T_i = T_j$ for $i, j \in D_3$. In fact, if $i < j$, then $f_{\alpha_i} \neq 0, f_{\alpha_{i+1}} \neq 0, \cdots , f_{\alpha_{j-1}} \neq 0$ and $f_{\alpha_i}T_i = T_{i+1}f_{\alpha_i}, \cdots , f_{\alpha_{j-1}}T_{j-1} = T_jf_{\alpha_{j-1}}$. Since $f_{\alpha_i} \neq 0$ for $i \in D_3$, $T_i = T_{i+1} = \cdots = T_j$. Hence $T_i = T_j$ for all $i, j \in D_3$. And $T_i = T_j = 0$ for $i, j \in D_4$. Thus $End(H, f)$ is isomorphic to $\mathbb{C}$. Hence (H, f) is transitive. □

The next lemma guarantees that we may assume that $H_i \subset H_j$ if $\dim H_i \leq \dim H_j$.

Lemma 4.4. *Let $(H_i)_{i=1}^n$ be a family of nonzero Hilbert spaces. Then there exists a family $(K(i))_{i=1}^n$ of subspaces in a Hilbert space V, such that for any i $(1 \leq i \leq n)$, there exists a number $m(i)$ $(1 \leq m(i) \leq n)$ such that H_i is isomorphic to $\oplus_{j=1}^{m(i)} K(j)$.*

Proof. We arrange a family of Hilbert spaces $(H_i)_i$ in increasing order of dimension and as a result, we have $(H_{\ell(1)})$, $(H_{\ell(2)})$, $\cdots$, $(H_{\ell(n)})$ in increasing order of dimension. Construct an ambient space V and its increasing subspaces $H'_i \cong H_i$ such that $(H'_{\ell(1)}) \subset (H'_{\ell(2)}) \subset \cdots \subset (H'_{\ell(n)}) \subset V$. Put $K_1 = H'_{\ell(1)}, K_2 = H'_{\ell(2)} \cap (H'_{\ell(1)})^{\perp}, \cdots , K_n = H'_{\ell(n)} \cap (H'_{\ell(n-1)})^{\perp}$. Hence there exists a number $m(i)$ such that $H'_i = K(1) \oplus K(2) \oplus \cdots \oplus K(m(i))$. Thus we have that H_i is isomorphic to $K(1) \oplus K(2) \oplus \cdots \oplus K(m(i))$. □

Firstly we investigate transitive Hilbert representations of oriented cyclic quivers C_2 and C_3. Let (H, f) be a Hilbert representation of C_2. In what follows we denote $f_{\alpha_1}, f_{\alpha_2}$ by A_1, A_2 for short.

Lemma 4.5. *Let (H, f) be a transitive Hilbert representation of C_2. Assume that $H_1 = H_2 = K \neq 0$, $A_1 \in \mathbb{C}$ and $A_2 \in \mathbb{C}$. If $A_1 \neq 0$ or $A_2 \neq 0$, then $K = \mathbb{C}$.*

Proof. Let $T \in B(K)$. Then $(T, T) \in End(H, f)$. In fact, $A_1T = TA_1$ and $A_2T = TA_2$. If $\dim K > 1$, $B(K) \neq \mathbb{C}I$. Since (H, f) is transitive, this is a contradiction. Thus $\dim K = 1$. □

Lemma 4.6. *Let (H, f) be a Hilbert representation of C_2. Then (H, f) is transitive if and only if one of the following conditions holds.*

(1) $H_1 = \mathbb{C}, H_2 = 0, A_1 = 0$ *and* $A_2 = 0$,
(2) $H_1 = 0, H_2 = \mathbb{C}, A_1 = 0$ *and* $A_2 = 0$,
(3) $H_1 = \mathbb{C}$ *and* $H_2 = \mathbb{C}$ *and* ($A_1 \neq 0$ *or* $A_2 \neq 0$).

Proof. If (1), (2) or (3) holds, then (H, f) is clearly transitive. Conversely, assume that (H, f) is transitive. Assume that $\dim H_1 \neq 0$ and $\dim H_2 = 0$. If $\dim H_1 > 1$, then there exists a non-scalar operator in $B(H_1)$. Since $B(H_1) = End(H, f)$, this contradicts the transitivity of (H, f). Hence $\dim H_1 = 1$. This is the case (1).

Similarly we have the case (2).

Therefore it is sufficient to assume that $\dim H_1 \neq 0$ and $\dim H_2 \neq 0$. By Lemma 4.4 we may assume that $\dim H_1 \leq \dim H_2$ and H_1 is a subspace of H_2. We define $T = (T_1, T_2) = (A_2A_1, A_1A_2)$. Then $T \in End(H, f)$. In fact,

$$A_1T_1 = A_1(A_2A_1) = (A_1A_2)A_1 = T_2A_1$$

and

$$T_1A_2 = (A_2A_1)A_2 = A_2(A_1A_2) = A_2T_2.$$

By the assumption of transitivity for (H, f),

$$(T_1, T_2) \in \{(\mu I_{H_1}, \mu I_{H_2}) | \mu \in \mathbb{C}\}.$$

Hence

$$T_1 = A_2A_1 = \mu I_{H_1}, T_2 = A_1A_2 = \mu I_{H_2} \text{ for some } \mu \in \mathbb{C}.$$

We denote by $E_1 \in B(H_1, H_2)$ the embedding map of H_1 into H_2 and $E_2 \in B(H_2, H_1)$ the projection map of H_2 onto H_1. We define

$$T^{\{1\}} = (T_1^{\{1\}}, T_2^{\{1\}}) = (A_2E_1, E_1A_2).$$

Then $T^{\{1\}} \in End(H, f)$. In fact,

$$\begin{aligned} A_1T_1^{\{1\}} &= A_1(A_2E_1) = (A_1A_2)E_1 = \mu I_{H_2}E_1 \\ &= \mu E_1 = E_1\mu I_{H_1} = (E_1A_2)A_1 = T_2^{\{1\}}A_1 \end{aligned}$$

and

$$T_1^{\{1\}}A_2 = (A_2E_1)A_2 = A_2(E_1A_2) = A_2T_2^{\{1\}}.$$

Thus $T^{\{1\}} \in End(H, f)$. Since (H, f) is transitive, there exists a constant $\mu^{\{1\}} \in \mathbb{C}$ such that $A_2E_1 = \mu^{\{1\}}I_{H_1}$ and $E_1A_2 = \mu^{\{1\}}I_{H_2}$. We define

$$T^{\{2\}} = (T_1^{\{2\}}, T_2^{\{2\}}) = (E_2A_1, A_1E_2).$$

Then $T^{\{2\}} \in End(H, f)$. In fact,

$$A_1T_1^{\{2\}} = A_1(E_2A_1) = (A_1E_2)A_1 = T_2^{\{2\}}A_1$$

and

$$\begin{aligned} T_1^{\{2\}}A_2 &= (E_2A_1)A_2 = E_2(\mu I_{H_2}) = \mu E_2 \\ &= \mu I_{H_1}E_2 = A_2(A_1E_2) = A_2T_2^{\{2\}}. \end{aligned}$$

Since (H, f) is transitive, there exists a constant $\mu^{\{2\}} \in \mathbb{C}$ such that $E_2A_1 = \mu^{\{2\}}I_{H_1}$ and $A_1E_2 = \mu^{\{2\}}I_{H_2}$.

We define

$$T^{\{1,2\}} = (T_1^{\{1,2\}}, T_2^{\{1,2\}}) = (E_2E_1, E_1E_2).$$

Then $T^{\{1,2\}} \in End(H,f)$. In fact,

$$\begin{aligned} A_1T_1^{\{1,2\}} &= A_1(E_2E_1) = (A_1E_2)E_1 = \mu^{\{2\}}I_{H_2}E_1 = \mu^{\{2\}}E_1 \\ &= \mu^{\{2\}}E_1 = E_1(\mu^{\{2\}}I_{H_1}) = E_1(E_2A_1) = T_2^{\{1,2\}}A_1, \end{aligned}$$

and

$$\begin{aligned} T_1^{\{1,2\}}A_2 &= (E_2E_1)A_2 = E_2(\mu^{\{1\}}I_{H_2}) = \mu^{\{1\}}E_2 \\ &= \mu^{\{1\}}E_2 = (\mu^{\{1\}}I_{H_1})E_2 = A_2(E_1E_2) = A_2T_2^{\{1,2\}}. \end{aligned}$$

Since (H,f) is transitive, there exists a constant $\mu^{\{1,2\}} \in \mathbb{C}$ such that

$$E_2E_1 = \mu^{\{1,2\}}I_{H_1}, \qquad E_1E_2 = \mu^{\{1,2\}}I_{H_2}.$$

For $x(\neq 0) \in H_1$, we have $x = E_2E_1x = \mu^{\{1,2\}}I_{H_1}x = \mu^{\{1,2\}}x$. Hence $\mu^{\{1,2\}} = 1$. If $H_1 \neq H_2$, then $H_1^{\perp} \cap H_2 \neq 0$. Take $x(\neq 0) \in H_1^{\perp} \cap H_2$. Then $E_1E_2x = \mu^{\{1,2\}}I_{H_2}x$. Hence $0 = x$. This is a contradiction. Thus $H_1 = H_2$ and $E_1 = E_2$. Since $A_1E_2 = \mu^{\{2\}}I_{H_2}$, $A_1 = \mu^{\{2\}}I_{H_1}$. And we also have $E_1A_2 = A_2 = \mu^{\{1\}}I_{H_1}$. Since (H,f) is transitive, $A_1 \neq 0$ or $A_2 \neq 0$. By Lemma 4.5, we have $H_1 = H_2 = \mathbb{C}$. Thus (H,f) is in the case (3). □

Let (H,f) be a Hilbert representation of the oriented cyclic quiver C_3. In the below we denote f_{α_1}, f_{α_2}, f_{α_3} by A_1, A_2, A_3 for short.

Lemma 4.7. *Let (H,f) be a transitive Hilbert representation of C_3. Assume that $H_i = \mathbb{C}$ $(i = 1,2,3)$. Then $A_iA_j \neq 0$ for some $i \neq j$.*

Proof. Assume that $A_i = A_j = 0$ for some $i \neq j$. We may and do assume $i = 1, j = 2$. Let $T = (T_1,T_2,T_3)$ such that $T_1 = T_3$,$T_2 \neq T_1$, $T_1 \neq 0$ and $T_2 \neq 0$. Then $T = (T_1,T_2,T_3)$ is in $End(H,f)$. Since (H,f) is transitive, $T_1 = T_2 = T_3 \in \mathbb{C}$. This is a contradiction. Hence this lemma holds. □

Lemma 4.8. *Let (H,f) be a Hilbert representation of C_3. Then (H,f) is transitive if and only if one of the following holds.*

(1) $H_1 = \mathbb{C}$ *and* $H_i = 0$ $(i = 2,3)$.
(2) $H_2 = \mathbb{C}$ *and* $H_i = 0$ $(i = 1,3)$.
(3) $H_3 = \mathbb{C}$ *and* $H_i = 0$ $(i = 1,2)$.
(4) $H_i = \mathbb{C}$ $(i = 1,2)$,$H_3 = 0$ *and* $A_1 \neq 0$.
(5) $H_i = \mathbb{C}$ $(i = 2,3)$,$H_1 = 0$ *and* $A_2 \neq 0$.
(6) $H_i = \mathbb{C}$ $(i = 1,3)$,$H_2 = 0$ *and* $A_3 \neq 0$.
(7) $H_i = \mathbb{C}$ $(i = 1,2,3)$ *and* $A_iA_j \neq 0$ *for some* $i \neq j$ $(i,j = 1,2,3)$.

Proof. If a Hilbert representations (H,f) satisfies (1), (2),$\cdots$ or (7), then the Hilbert representation is obviously transitive. Conversely assume that (H,f) is transitive. At first we assume that all Hilbert spaces $H_i \neq 0$ $(1 \leq i \leq 3)$ and by Lemma 4.4 a totally ordered set by inclusion order and $H_1 \subset H_i$ $(i = 2,3)$. We define

$$T_1 = A_3A_2A_1,\ T_2 = A_1A_3A_2,\ T_3 = A_2A_1A_3,\ T = (T_1,T_2,T_3).$$

We define a mapping $E_i \in B(H_i, H_{i+1})$ by

$$E_i = \begin{cases} \text{the inclusion map of } H_i \text{ into } H_{i+1}, & H_i \subset H_{i+1}, \\ \text{the projection map of } H_i \text{ onto } H_{i+1}, & H_{i+1} \subset H_i. \end{cases}$$

For a subset S of $\{1,2,3\}$, we define $B_i \in B(H_i, H_{i+1})$ by

$$B_i = \begin{cases} A_i & \text{if } i \notin S, \\ E_i & \text{if } i \in S. \end{cases}$$

We also define

$$T_1^S = B_3B_2B_1,\ T_2^S = B_1B_3B_2,\ T_3^S = B_2B_1B_3,\ T^S = (T_1^S, T_2^S, T_3^S).$$

We note that $T^S = (T_1^S, T_2^S, T_3^S)$ is obtained by replacing each word A_i in $T = (T_1, T_2, T_3)$ with E_i for all $i \in S$. We regard $T = (T_1, T_2, T_3)$ as $T^{\emptyset} = (T_1^{\emptyset}, T_2^{\emptyset}, T_3^{\emptyset})$. Since

$$A_1T_1 = A_1(A_3A_2A_1) = T_2A_1,$$

$$A_2T_2 = A_2(A_1A_3A_2) = T_3A_2,$$

$$A_3T_3 = A_3(A_2A_1A_3) = T_1A_3,$$

we have that T is in $End(H, f)$. Since (H, f) is transitive, there exists a constant $\mu \in \mathbb{C}$ such that

$$A_3A_2A_1 = \mu I_{H_1},\ A_1A_3A_2 = \mu I_{H_2},\ A_2A_1A_3 = \mu I_{H_3}.$$

For $S = \{1\}$, we define $T^S = T^{\{1\}} = (T_1^{\{1\}}, T_2^{\{1\}}, T_3^{\{1\}})$ by

$$T_1^{\{1\}} = A_3A_2E_1,\ T_2^{\{1\}} = E_1A_3A_2,\ T_3^{\{1\}} = A_2E_1A_3.$$

It follows that

$$\begin{aligned} A_1T_1^{\{1\}} &= A_1A_3A_2E_1 = \mu I_{H_2}E_1 = \mu E_1 \\ &= E_1\mu I_{H_1} = E_1(A_3A_2)A_1 = T_2^{\{1\}}A_1, \\ A_2T_2^{\{1\}} &= A_2E_1(A_3A_2) = T_3^{\{1\}}A_2, \\ A_3T_3^{\{1\}} &= A_3A_2E_1A_3 = T_1^{\{1\}}A_3. \end{aligned}$$

Thus $T^{\{1\}}$ is in $End(H, f)$. Since (H, f) is transitive, there exists a constant $\mu^{\{1\}} \in \mathbb{C}$ such that

$$A_3A_2E_1 = \mu^{\{1\}}I_{H_1},\ E_1A_3A_2 = \mu^{\{1\}}I_{H_2},\ A_2E_1A_3 = \mu^{\{1\}}I_{H_3}.$$

For $S = \{2\}$, we define $T^S = T^{\{2\}} = (T_1^{\{2\}}, T_2^{\{2\}}, T_3^{\{2\}})$ by

$$T_1^{\{2\}} = A_3E_2A_1,\ T_2^{\{2\}} = A_1A_3E_2,\ T_3^{\{2\}} = E_2A_1A_3.$$

It follows that

$$\begin{aligned} A_1T_1^{\{2\}} &= A_1A_3E_2A_1 = T_2^{\{2\}}A_1, \\ A_2T_2^{\{2\}} &= A_2A_1A_3E_2 = \mu I_{H_3}E_2 = \mu E_2 \\ &= E_2\mu I_{H_2} = E_2A_1A_3A_2 = T_3^{\{2\}}A_2, \\ A_3T_3^{\{2\}} &= A_3E_2A_1A_3 = T_1^{\{2\}}A_3. \end{aligned}$$

Thus $T^{\{2\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{2\}} \in \mathbb{C}$ such that

$$A_3E_2A_1 = \mu^{\{2\}}I_{H_1},\ A_1A_3E_2 = \mu^{\{2\}}I_{H_2},\ (E_2A_1)A_3 = \mu^{\{2\}}I_{H_3}.$$

For $S = \{3\}$, we define $T^S = T^{\{3\}} = (T_1^{\{3\}}, T_2^{\{3\}}, T_3^{\{3\}})$ by

$$T_1^{\{3\}} = E_3A_2A_1,\ T_2^{\{3\}} = A_1E_3A_2,\ T_3^{\{3\}} = A_2A_1E_3.$$

It follows that

$$\begin{aligned} &A_1T_1^{\{3\}} = A_1E_3A_2A_1 = T_2^{\{3\}}A_1, \\ &A_2T_2^{\{3\}} = A_2A_1E_3A_2 = T_3^{\{3\}}A_2, A_3T_3^{\{3\}} \qquad = A_3A_2A_1E_3 = \mu I_{H_1}E_3 = \mu E_3 \\ &\qquad = E_3\mu I_{H_3} = E_3A_2A_1A_3 = T_1^{\{3\}}A_3. \end{aligned}$$

Thus $T^{\{3\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{3\}} \in \mathbb{C}$ such that

$$E_3A_2A_1 = \mu^{\{3\}}I_{H_1},\ A_1E_3A_2 = \mu^{\{3\}}I_{H_2},\ A_2A_1E_3 = \mu^{\{3\}}I_{H_3}.$$

For $S = \{1,2\}$, we have

$$T^{\{1,2\}} = (T_1^{\{1,2\}}, T_2^{\{1,2\}}, T_3^{\{1,2\}}) = (A_3E_2E_1, E_1A_3E_2, E_2E_1A_3).$$

It follows that

$$\begin{aligned} A_1T_1^{\{1,2\}} &= A_1A_3E_2E_1 = \mu^{\{2\}}I_{H_2}E_1 = \mu^{\{2\}}E_1 \\ &= E_1\mu^{\{2\}}I_{H_1} = E_1A_3E_2A_1 = T_2^{\{1,2\}}A_1, \\ A_2T_2^{\{1,2\}} &= A_2E_1A_3E_2 = \mu^{\{1\}}I_{H_3}E_2 = \mu^{\{1\}}E_2 \\ &= E_2\mu^{\{1\}}I_{H_2} = E_2E_1A_3A_2 = T_3^{\{1,2\}}A_2, \\ A_3T_3^{\{1,2\}} &= A_3E_2E_1A_3 = T_1^{\{1,2\}}A_3. \end{aligned}$$

Thus $T^{\{1,2\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{1,2\}} \in \mathbb{C}$ such that

$$A_3E_2E_1 = \mu^{\{1,2\}}I_{H_1},\ E_1A_3E_2 = \mu^{\{1,2\}}I_{H_2},\ E_2E_1A_3 = \mu^{\{1,2\}}I_{H_3}.$$

For $S = \{1,3\}$, we have

$$T^{\{1,3\}} = (T_1^{\{1,3\}}, T_2^{\{1,3\}}, T_3^{\{1,3\}}) = (E_3A_2E_1, E_1E_3A_2, A_2E_1E_3).$$

It follows that

$$\begin{aligned}
A_1T_1^{\{1,3\}} &= A_1E_3A_2E_1 = \mu^{\{3\}}I_{H_2}E_1 = \mu^{\{3\}}E_1 \\
&= E_1\mu^{\{3\}}I_{H_1} = E_1E_3A_2A_1 = T_2^{\{1,3\}}A_1, \\
A_2T_2^{\{1,3\}} &= A_2E_1E_3A_2 = T_3^{\{1,3\}}A_2, \\
A_3T_3^{\{1,3\}} &= A_3A_2E_1E_3 = \mu^{\{1\}}I_{H_1}E_3 = \mu^{\{1\}}E_3 \\
&= E_3\mu^{\{1\}}I_{H_3} = E_3A_2E_1A_3 = T_1^{\{1,3\}}A_3.
\end{aligned}$$

Thus $T^{\{1,3\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{1,3\}} \in \mathbb{C}$ such that

$$E_3A_2E_1 = \mu^{\{1,3\}}I_{H_1},\ E_1E_3A_2 = \mu^{\{1,3\}}I_{H_2},\ A_2E_1E_3 = \mu^{\{1,3\}}I_{H_3}.$$

For $S = \{2,3\}$, we have

$$T^{\{2,3\}} = (T_1^{\{2,3\}}, T_2^{\{2,3\}}, T_3^{\{2,3\}}) = (E_3E_2A_1, A_1E_3E_2, E_2A_1E_3).$$

It follows that

$$\begin{aligned}
A_1T_1^{\{2,3\}} &= A_1E_3E_2A_1 = T_2^{\{2,3\}}A_1, \\
A_2T_2^{\{2,3\}} &= A_2A_1E_3E_2 = \mu^{\{3\}}I_{H_3}E_2 = \mu^{\{3\}}E_2 \\
&= E_2\mu^{\{3\}}I_{H_2} = E_2A_1E_3A_2 = T_3^{\{2,3\}}A_2, \\
A_3T_3^{\{2,3\}} &= A_3E_2A_1E_3 = \mu^{\{2\}}I_{H_1}E_3 = \mu^{\{2\}}E_3 \\
&= E_3\mu^{\{2\}}I_{H_3} = E_3E_2A_1A_3 = T_1^{\{2,3\}}A_3.
\end{aligned}$$

Thus $T^{\{2,3\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{2,3\}} \in \mathbb{C}$ such that

$$E_3E_2A_1 = \mu^{\{2,3\}}I_{H_1},\ A_1E_3E_2 = \mu^{\{2,3\}}I_{H_2},\ E_2A_1E_3 = \mu^{\{2,3\}}I_{H_3}.$$

For $S = \{1,2,3\}$, we have

$$T^{\{1,2,3\}} = (T_1^{\{1,2,3\}}, T_2^{\{1,2,3\}}, T_3^{\{1,2,3\}}) = (E_3E_2E_1, E_1E_3E_2, E_2E_1E_3).$$

It follows that

$$\begin{aligned}
A_1T_1^{\{1,2,3\}} &= A_1E_3E_2E_1 = \mu^{\{2,3\}}I_{H_1}E_1 = \mu^{\{2,3\}}E_1 \\
&= E_1\mu^{\{2,3\}}I_{H_1} = E_1E_3E_2A_1 = T_2^{\{1,2,3\}}A_1, \\
A_2T_2^{\{1,2,3\}} &= A_2E_1E_3E_2 = \mu^{\{1,3\}}I_{H_3}E_2 = \mu^{\{1,3\}}E_2 \\
&= E_2\mu^{\{1,3\}}I_{H_2} = E_2E_1E_3A_2 = T_3^{\{1,2,3\}}A_2, \\
A_3T_3^{\{1,2,3\}} &= A_3E_2E_1E_3 = \mu^{\{1,2\}}I_{H_1}E_3 = \mu^{\{1,2\}}E_3 \\
&= E_3\mu^{\{1,2\}}I_{H_3} = E_3E_2E_1A_3 = T_1^{\{1,2,3\}}A_3.
\end{aligned}$$

Thus $T^{\{1,2,3\}}$ is in $End(H,f)$. Since (H,f) is transitive, there exists a constant $\mu^{\{1,2,3\}} \in \mathbb{C}$ such that

$$E_3E_2E_1 = \mu^{\{1,2,3\}} I_{H_1},\ E_1E_3E_2 = \mu^{\{1,2,3\}} I_{H_2},\ E_2E_1E_3 = \mu^{\{1,2,3\}} I_{H_3}.$$

Take $x\ (\neq 0) \in H_1$. Since $H_1 \subset H_i$ $(1 \leq i \leq 3)$, $E_3E_2E_1x = x = \mu^{\{1,2,3\}} I_{H_1}x$. Hence $\mu^{\{1,2,3\}} = 1$. By Lemma 4.4, we can represent H_i by $H_i = K_1 \oplus K_2 \oplus \cdots \oplus K_{m(i)}$ $(1 \leq i \leq 3)$. We shall show that $H_1 = H_2 = H_3$.

Now, $m(1) = 1$ and assume that $m(2) \neq 1$. We compare $m(3)$ with $m(2)$. Assume that $m(3) < m(2)$. Take $x\ (\neq 0) \in K_{m(2)} \subset H_2$. Then $E_2x = 0$. This contradicts that $E_1E_3E_2 = I_{H_2}$. Assume that $m(3) \geq m(2)$. Take $x\ (\neq 0) \in K_{m(2)} \subset H_2$. Then $E_2x = x$ and $E_3x = 0$. This contradicts that $E_1E_3E_2 = I_{H_2}$.

Hence $m(1) = m(2)$ and $H_1 = H_2$. Next assume that $H_3 \neq H_1$ (hence $m(3) \neq 1$). Take $x\ (\neq 0) \in K_{m(3)} \subset H_3$. Then $E_3x = 0$. This contradicts that $E_2E_1E_3 = I_{H_3}$. Hence we have that $H_1 = H_2 = H_3 := M$. Therefore, $E_1 = E_2 = E_3 = I_M$ and $T_i^{\{1,2,3\}} = I_M \in \mathbb{C}$ Since

$$E_3E_2A_1 = \mu^{\{2,3\}} I_{H_1},\ E_1E_3A_2 = \mu^{\{1,3\}} I_{H_2},\ E_2E_1A_3 = \mu^{\{1,2\}} I_{H_3},$$

we have

$$A_1 = \mu^{\{2,3\}} I_M,\ A_2 = \mu^{\{1,3\}} I_M,\ A_3 = \mu^{\{1,2\}} I_M.$$

If $\dim M > 1$, there is a non-scalar operator $B \in B(M)$. Since A_1, A_2, A_3 are scalar operators, $(B,B,B) \in End(H,f)$. This contradicts that (H,f) is transitive. Hence we have $\dim M = 1$. By Lemma 4.7,$A_iA_j \neq 0$ for some $i \neq j$ $(i,j = 1,2,3)$. Thus (H,f) is in the case (7).

Next we consider other cases. Assume that there exists $H_i = 0$ for some i. Since (H,f) is transitive, the number $|\{i; H_i \neq 0\}|$ is 1 or 2. If $|\{i; H_i \neq 0\}| = 1 = |\{k\}|$, then $\dim H_k = 1$ because (H,f) is transitive. Hence these are in the cases (1), (2), (3). If $|\{i; H_i \neq 0\}| = 2 = |\{k,\ell\}|, (k < \ell \mod 3)$, then we consider the reduction C_2 of the quiver C_3 as it is shown in Lemma 4.2. Let (K,g) be the reduced Hilbert representation of C_2 from the Hilbert representation (H,f) of C_2 by Lemma 4.2. We have $End(H,f) \cong End(K,g)$. Hence $End(K,g)$ is transitive. By the same argument in the case (7), we have $\dim H_k = \dim H_\ell = 1$. Since (H,f) is transitive, $A_k \neq 0$. Thus these are in the cases (4), (5), (6). All these cases are summarized as the existence of unique (H,f)-connected component by Lemma 4.3. □

Let (H,f) be a Hilbert representation of C_n. In the below we denote $f_{\alpha_1}, f_{\alpha_2}, \cdots, f_{\alpha_n}$ by $A_1, A_2, \cdots, A_n$ for short.

Lemma 4.9. *Let (H,f) be a Hilbert representation of the oriented cyclic quiver C_n. Then (H,f) is transitive if and only if $H_i = \mathbb{C}$ or 0 and there exists only one (H,f)-connected component in $\{i \in V; H_i \neq 0\}$.*

Proof. Assume that $H_i = \mathbb{C}$ or 0 and there exists only one (H,f)-connected component in $\{i \in V; H_i \neq 0\}$. Then (H,f) is transitive by Lemma 4.3.

Conversely assume that (H, f) is transitive. At first we consider the case that $H_i \neq 0$ for any i. By lemma 4.4, we may and do assume that the family (H_i) of Hilbert spaces are totally ordered under the inclusion order. We also assume that $\dim H_1$ is the smallest dimension among $\{\dim H_i; i = 1, \cdots, n\}$.

We define $T = (T_1, T_2, \cdots, T_n)$ by

$$T_1 = A_n \cdots A_3 A_2 A_1, T_2 = A_1 A_n \cdots A_4 A_3 A_2, \cdots, T_n = A_{n-1} \cdots A_3 A_2 A_1 A_n.$$

Then $T = (T_1, T_2, \cdots, T_n)$ is clearly in $End(H, f)$.

We denote by E_i the following operator $E_i : H_i \to H_{i+1}$:

$$E_i = \begin{cases} \text{the inclusion map from } H_i \text{ into } H_{i+1}, & H_i \subset H_{i+1}, \\ \text{the projection map from} H_i \text{ onto } H_{i+1}, & H_{i+1} \subset H_i. \end{cases}$$

For $S \subseteq \{1, 2, \cdots, n\}$, we define $B_i \in B(H_i, H_{i+1})$, which depends on S, by

$$B_i = \begin{cases} A_i, & \text{if } i \notin S \\ E_i, & \text{if } i \in S \end{cases}$$

We also define $T_i^S \in B(H_i)$ and $T^S \in B(H_1 \oplus ... \oplus H_n)$ by

$$T_i^S = B_{i-1} B_{i-2} \cdots B_2 B_1 B_n B_{n-1} \cdots B_{i+1} B_i \quad \text{ for } 1 \leq i \leq n,$$

and $T^S = (T_1^S, T_2^S, \cdots, T_n^S)$. That is, $T^S = (T_1^S, T_2^S, \cdots, T_n^S)$ is obtained by replacing each word A_i in $T = (T_1, T_2, \cdots, T_n)$ with E_i for all $i \in S$.

For example, $T^{\{1\}} = (T_1^{\{1\}}, T_2^{\{1\}}, \cdots, T_n^{\{1\}})$ is given by

$$\begin{aligned} T_1^{\{1\}} &= A_n A_{n-1} \cdots A_2 E_1, \\ T_2^{\{1\}} &= E_1 A_n A_{n-1} \cdots A_3 A_2, \\ T_3^{\{1\}} &= A_2 E_1 A_n A_{n-1} \cdots A_4 A_3, \\ &\cdots \\ T_n^{\{1\}} &= A_{n-1} A_{n-2} \cdots A_2 E_1 A_n. \end{aligned}$$

We regard T as $T^{\emptyset}$.

In the following we shall show that $T^S = (T_1^S, T_2^S, \cdots, T_n^S)$ is in $End(H, f)$ for any $S \subseteq \{1, 2, \cdots, n\}$. We shall prove it by the induction on the number $k = |S|$. First consider the case $k = |S| = 0$, that is, $S = \emptyset$. Then $T^{\emptyset} = T = (T_1, T_2, \cdots, T_n)$ is clearly in $End(H, f)$.

Next, we assume that T^S is in $End(H, f)$ for $|S| = k$. Since (H, f) is transitive, there exists a constant $\mu^S \in \mathbb{C}$ such that $T_i^S = \mu^S I_{H_i}$ for any $i = 1, \ldots, n$. Take S such that $|S| = k + 1$. We shall show that T^S is in $End(H, f)$. It is enough to show that, for any $i = 1, \ldots, n$, we have $A_i T_i^S = T_{i+1}^S A_i$. First we consider the case that $i = 1$. We need to show the validity of the relation $A_1 T_1^S = T_2^S A_1$, that is, $A_1 B_n \cdots B_2 B_1 = B_1 B_n \cdots B_2 A_1$. Assume that 1 is in S. Then $B_1 = E_1$ and $T_i^{S \setminus \{1\}}$ is in $End(H, f)$ by the assumption of the induction. Since $A_1B_nB_{n-1}\cdots B_2$ and $B_nB_{n-1}\cdots B_2A_1$ have k changed letters, we have

$$T_2^{S\setminus\{1\}} = A_1B_nB_{n-1}\cdots B_2 = \mu^{S\setminus\{1\}}I_{H_2}$$

and

$$T_1^{S\setminus\{1\}} = B_nB_{n-1}\cdots B_2A_1 = \mu^{S\setminus\{1\}}I_{H_1}.$$

Therefore, we have

$$A_1T_1^S = A_1B_n\cdots B_1 = \mu^{S\setminus\{1\}}I_{H_2}B_1 = \mu^{S\setminus\{1\}}E_1$$

and

$$T_2^SA_1 = B_1B_nB_{n-1}\cdots B_2A_1 = B_1\mu^{S\setminus\{1\}}I_{H_1} = \mu^{S\setminus\{1\}}E_1.$$

Thus $A_1T_1^S = T_2^SA_1$.

Assume that 1 is not in S. Then $B_1 = A_1$. Hence

$$A_1T_1^S = A_1B_n\cdots B_1 = A_1B_n\cdots B_2A_1$$

and

$$T_2^SA_1 = B_1B_n\cdots B_2A_1 = A_1B_n\cdots B_2A_1.$$

Thus $A_1T_1^S = T_2^SA_1$.

For other cases that $i = 2, 3, \ldots n$, we also have that $A_iT_i^S = T_{i+1}^SA_i$. Hence, by induction, we have that T^S is in $End(H, f)$ for any $S \subset \{1, 2, \cdots n\}$.

In particular, put $S = \{1, 2, \cdots, n\}$. Since

$$T^{\{1,2,\cdots,n\}} = (T_1^{\{1,2,\cdots,n\}}, T_2^{\{1,2,\cdots,n\}}, \cdots, T_n^{\{1,2,\cdots,n\}})$$

is in $End(H, f)$ and (H, f) is transitive, there exits a constant $\mu^{\{1,2,\cdots,n\}} \in \mathbb{C}$ such that

$$T_i^{\{1,2,\cdots,n\}} = E_{i-1}\cdots E_1E_nE_{n-1}\cdots E_{i+1}E_i = \mu^{\{1,2,\cdots,n\}}I_{H_i}.$$

Take $x\ (\neq 0) \in H_1$. Since $H_1 \subset H_j$ for any $1 \leq j \leq n$ and $E_n\cdots E_1 = \mu^{\{1,2,\cdots,n\}}I_{H_1}$, we have that $x = \mu^{\{1,2,\cdots,n\}}x$. Hence $\mu^{\{1,2,\cdots,n\}} = 1$.

We shall show that $H_1 = H_2 = \cdots = H_n$. On the contrary we assume that $H_k \neq H_\ell$ for some $k \neq \ell$. Using Lemma 4.4, we can represent H_i as $H_i = K_1 \oplus K_2 \oplus \cdots \oplus K_{m(i)}$ and $m(1) = 1$. Then there exists the smallest i such that $m(i) > 1$. We compare $m(j)$ and $m(i)$. If there exists $m(j)$ such that $m(j) < m(i)$ $(i \leq j \leq n)$. Take $x\ (\neq 0) \in K_{m(i)} \subset H_i$. Then $E_{j-1}E_{j-2}\cdots E_{i+1}E_ix = 0$. This contradicts that $E_{i-1}\cdots E_1E_nE_{n-1}\cdots E_{i+1}E_i = I_{H_i}$.

If there exists no $m(j)$ such that $m(j) < m(i)$ $(i \leq j \leq n)$. Take $x\ (\neq 0) \in K_{m(i)} \subset H_i$. Then $E_{n-1}E_{n-2}\cdots E_{i+1}E_ix = x$, and $E_nx = 0$. This also contradicts that

$$E_{i-1}\cdots E_1E_nE_{n-1}\cdots E_{i+1}E_i = I_{H_i}.$$

Therefore we have that $H_1 = H_2 = \cdots = H_n =: M$. Moreover we also have that $E_1 = E_2 = \cdots = E_n = I_M$. In particular, $T_i^{\{1,2,\cdots,n\}} = I_M$ for any i and

$$A_i = E_{i-1}\cdots E_1 E_n E_{n-1}\cdots E_{i+1} A_i = T_i^{\{1,2,\cdots,n\}\setminus k} = \mu^{\{1,2,\cdots,n\}\setminus k} I_{H_k}.$$

We shall show that $\dim M = 1$. On the contrary, assume that $\dim M \geq 2$. Then there exists a non-scalar operator $B \in B(M)$. Since each A_k is a scalar operator for any k, $(B,\ldots,B)$ is in $End(H,f)$. This contradicts to that (H,f) is transitive. Therefore $\dim M = 1$. Hence we may assume that $H_i = \mathbb{C}$ for any i. Since (H,f) is transitive, there exists only one (H,f)-connected component on $V = \{1,2,\cdots,n\}$ by Lemma 4.3.

Next we consider the case that there exists $H_i = 0$ for some i. If there exists only one vertex i such that $H_i \neq 0$, then $\dim H_i = 1$ because (H,f) is transitive. Therefore we may assume that there exists more than two vertices i such that $H_i \neq 0$.

We consider the reduction of the quiver C_n to the set of vertices i with $H_i \neq 0$ to get another quiver C_m $(2 \leq m \leq n)$.

Let (K,g) be the reduced Hilbert representation of C_m from the Hilbert representation (H,f) of C_n by Lemma 4.2. Then $End(H,f)$ is isomorphic to $End(K,g)$. Since (H,f) is transitive, (K,g) is also transitive.

Since we can adapt the above consideration to (K,g), we have that $H_i = \mathbb{C}$ for all i such that $H_i \neq 0$. Therefore in (H,f), we may and do have that $H_i = \mathbb{C}$ or 0 for $1 \leq i \leq n$. Since (H,f) is transitive, by Lemma 4.3, there exists only one (H,f)-connected component $\{i \in V; H_i \neq 0\}$. $\square$

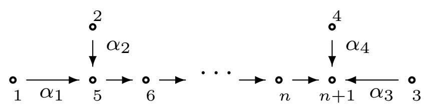
Theorem 4.10. *Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\widetilde{A_n}$, $n \geq 0$. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.*

Proof. Assume that Γ is not an oriented cyclic quiver. By Theorem 4.1, there exists an infinite-dimensional transitive Hilbert representation of Γ. Conversely, assume that Γ is an oriented cyclic quiver. Then transitive Hilbert representations of Γ are finite-dimensional by Lemma 4.9. Hence there exist no infinite-dimensional transitive Hilbert representations of Γ. $\square$

Gabriel's theorem states that a finite, connected quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8. In [EW3], we constructed some examples of indecomposable, infinite-dimensional representations of quivers with the underlying undirected graphs being extended Dynkin diagrams $\tilde{D}_n$ $(n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$. We used the quivers whose vertices are represented by a family of subspaces and whose arrows are represented by natural inclusion maps. Replacing the unilateral shift S

with a transitive operator in the construction of examples of indecomposable, infinite-dimensional representations of quivers in [EW3], we shall give some examples of infinite-dimensional transitive representations of quivers with the underlying undirected graphs being extended Dynkin diagrams $\tilde{D}_n$ $(n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$. Our construction of examples is considered as a modification of an unbounded operator used by Harrison, Radjavi and Rosenthal [HRR] to provide a transitive lattice.

Lemma 4.11. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{D}_n$ for $n \geq 4$:*

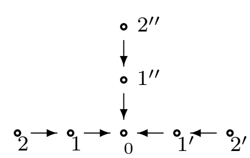


Then there exists an infinite-dimensional, transitive Hilbert representation (H, f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a transitive operator on K with the domain $D(S)$. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V},\ (f_\alpha)_{\alpha \in E})$ of Γ as follows: $H_1 = K \oplus 0$, $H_2 = 0 \oplus K$, $H_3 = \{(x, Sx) \in K \oplus K; x \in D(S)\}$, $H_4 = \{(x, x) \in K \oplus K; x \in K\}$, and $H_5 = H_6 = \cdots = H_{n+1} = K \oplus K$.

Let $f_{\alpha_k} : H_{s(\alpha_k)} \to H_{r(\alpha_k)}$ be the inclusion map for any $\alpha_k \in E$ for $k = 1, 2, 3, 4$, and $f_\beta = id$ for other arrows $\beta \in E$. Take $T = (T_v)_{v \in V} \in End(H, f)$. Since $T \in End(H, f)$ and any arrow is represented by the inclusion map, we have $T_i = T_j$ $(i = 5, \cdots, n+1), T_5 x = T_v x$ for any $v \in \{1, 2, 4\}$, any $x \in H_v$. In particular, $T_5 H_v \subset H_v$ $(v \in \{1, 2, 4\})$. Hence T_5 is written as $T_5 = A \oplus A$ as in [EW3, Lemma 6.1, Example 3]. Moreover H_3 is also invariant under T_5. Since S is transitive, we have that A is a scalar by Lemma 3.3. Thus T is a scalar, that is, $End(H, f) = \mathbb{C}$. Therefore (H, f) is transitive. $\square$

Lemma 4.12. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_6$:*

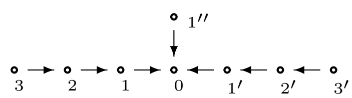


Then there exists an infinite-dimensional, transitive Hilbert representation (H, f) of Γ.

Proof. Let $(H, f) = ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ be the following Hilbert representation of Γ: Let $K = \ell^2(\mathbb{N})$ and S a transitive operator on K with the domain $D(S)$. Define $H_0 = K \oplus K \oplus K$, $H_1 = 0 \oplus K \oplus K$, $H_2 = 0 \oplus \{(y, Sy) \in K^2;\ y \in D(S)\}$, $H_{1'} = K \oplus K \oplus 0$, $H_{2'} = \{(x, x) \in K^2;\ x \in K\} \oplus 0$,

$H_{1''} = K \oplus 0 \oplus K$, and $H_{2''} = \{(x, 0, x) \in K^3;\ x \in K\}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. Take $T = (T_v)_{v \in V} \in End(H, f)$. Since any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in \{1, 1', 2', 1'', 2''\}$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Hence T_0 is written as $T_0 = A \oplus A \oplus A$. Moreover $H_2 = \{(0, x, Sx) \in K^3;\ x \in D(S)\}$ is also invariant under T_0. Hence for any $x \in D(S)$, there exists $y \in D(S)$ such that $(0, Ax, ASx) = (0, y, Sy)$ as in [EW3, Example 4]. Since S is transitive, we have that A is a scalar by Lemma 3.3. Thus T is a scalar, that is, $End(H, f) = \mathbb{C}$. Therefore (H, f) is transitive. $\square$

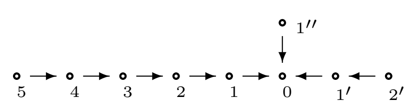
Lemma 4.13. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_7$:*



Then there exists an infinite-dimensional, transitive Hilbert representation (H, f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a transitive operator on K with the domain $D(S)$. Define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows: Let $H_0 = K \oplus K \oplus K \oplus K$, $H_1 = K \oplus 0 \oplus K \oplus K$, $H_2 = K \oplus 0 \oplus \{(x, x); x \in K\}$, $H_3 = K \oplus 0 \oplus 0 \oplus 0$, $H_{1'} = 0 \oplus K \oplus K \oplus K$, $H_{2'} = 0 \oplus K \oplus \{(y, Sy) \in K^2; y \in D(S)\}$, $H_{3'} = 0 \oplus K \oplus 0 \oplus 0$, and $H_{1''} = \{(x, y, x, y) \in K^4;\ x, y \in K\}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. Take $T = (T_v)_{v \in V} \in End(H, f)$. Since any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in \{1, 2, 3, 1', 2', 3', 1''\}$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Hence T_0 is written as $T_0 = A \oplus A \oplus A \oplus A$. Moreover $H_1 \cap H_{2'} = \{(0, 0, x, Sx) \in K^4;\ x \in D(S)\}$ is also invariant under T_0. Hence for any $x \in D(S)$, there exists $y \in D(S)$ such that $(0, 0, Ax, ASx) = (0, 0, y, Sy)$ as in [EW3, Lemma 6.2]. Since S is transitive , we have that A is a scalar by Lemma 3.3. Thus T is a scalar, that is, $End(H, f) = \mathbb{C}$. Therefore (H, f) is transitive. $\square$

Lemma 4.14. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_8$:*



Then there exists an infinite-dimensional, transitive Hilbert representation (H, f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a transitive operator on K with the domain $D(S)$. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows: Let $H_0 = K \oplus K \oplus K \oplus K \oplus K \oplus K$, $H_1 = \{(x, x) \in K^2;\ x \in$

$K\} \oplus K \oplus K \oplus K \oplus K$, $H_2 = 0 \oplus 0 \oplus K \oplus K \oplus K \oplus K$, $H_3 = 0 \oplus 0 \oplus 0 \oplus K \oplus K \oplus K$, $H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2;\ y \in D(S)\}$, $H_5 = 0 \oplus 0 \oplus 0 \oplus K \oplus 0 \oplus 0$, $H_{1'} = K \oplus K \oplus \{(x, y, x, y) \in K^4;\ x, y \in K\}$, $H_{2'} = K \oplus K \oplus 0 \oplus 0 \oplus 0 \oplus 0$, and $H_{1''} = \{(y, z, x, 0, y, z) \in K^6;\ x, y, z \in K\}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map.

Take $T = (T_v)_{v \in V} \in End(H, f)$. Since any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in V$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Since T_0 preserves subspaces $H_v, v = 1, 1', 1", 2, 2', 3, 5$, T_0 is written as

$$T_0 = A \oplus A \oplus A \oplus A \oplus A \oplus A.$$

Finally, $H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2;\ y \in K\}$ is invariant under T_0. Then for any $x \in K$ and $y \in D(S)$, there exist $x' \in K$ and $y' \in D(S)$ such that

$$T_0(0, 0, 0, x, y, Sy) = (0, 0, 0, Ax, Ay, ASy) = (0, 0, 0, x', y', Sy').$$

Hence $ASy = Sy' = SAy$ as in [EW3, Lemma 6.3]. Since S is transitive, we have that A is a scalar by Lemma 3.3.

Thus $T = (T_v)_{v \in V}$ is a scalar, that is, $End(H, f) = \mathbb{C}$. Therefore, (H, f) is transitive. □

Next, we shall investigate the endomorphism algebras of Hilbert representations. At first we recall some facts about reflection functors from [EW3].

Reflection functors are crucially used in the proof of the classification of finite-dimensional, indecomposable representations of tame quivers (cf.[As], [BGP], [DlR],[DoF], [GaR], [GeP]). As a matter of fact, many indecomposable representations of tame quivers can be reconstructed by iterating reflection functors on simple indecomposable representations. We can not expect such a best position in infinite-dimensional Hilbert representations. But reflection functors are still valuable to show that some property of representations of quivers on infinite-dimensional Hilbert spaces does not depend on the choice of orientations and does depend on the fact underlying undirected graphs are (extended) Dynkin diagrams or not.

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that a vertex $v \in V$ is a *sink* if $v \neq s(\alpha)$ for any $\alpha \in E$. Put $E^v = \{\alpha \in E;\ r(\alpha) = v\}$. We denote by $\overline{E}$ the set of all formally reversed new arrows $\overline{\alpha}$ for $\alpha \in E$. In this way if $\alpha : x \to y$ is an arrow, then $\overline{\alpha} : x \leftarrow y$.

Definition.[EW3] Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we construct a new quiver $\sigma_v^+(\Gamma) = (\sigma_v^+(V), \sigma_v^+(E), s, r)$ as follows: All the arrows of Γ having v as range are reversed and all the other arrows remain unchanged. That is,

$$\sigma_v^+(V) = V \quad \sigma_v^+(E) = (E \setminus E^v) \cup \overline{E^v},$$

where $\overline{E^v} = \{\overline{\alpha};\ \alpha \in E^v\}$.

Definition. [EW3] (reflection functor Φ_v^+.) Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we define a *reflection functor* at v

$$\Phi_v^+ : HRep(\Gamma) \to HRep(\sigma_v^+(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^+(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ, we define a Hilbert representation $(K, g) = \Phi_v^+(H, f)$ of $\sigma_v^+(\Gamma)$. Let

$$h_v : \oplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v$$

be a bounded linear operator defined by

$$h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha).$$

We shall define

$$K_v := \operatorname{Ker} h_v = \{(x_\alpha)_{\alpha \in E^v} \in \oplus_{\alpha \in E^v} H_{s(\alpha)};\ \sum_{\alpha \in E^v} f_\alpha(x_\alpha) = 0\}.$$

We also consider the canonical inclusion map $i_v : K_v \to \oplus_{\alpha \in E^v} H_{s(\alpha)}$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

For $\beta \in E^v$, let

$$P_\beta : \oplus_{\alpha \in E^v} H_{s(\alpha)} \to H_{s(\beta)}$$

be the canonical projection. Then we shall define

$$g_{\overline{\beta}} : K_{s(\overline{\beta})} = K_v \to K_{r(\overline{\beta})} = H_{s(\beta)} \quad \text{by} \quad g_{\overline{\beta}} = P_\beta \circ i_v$$

that is, $g_{\overline{\beta}}((x_\alpha)_{\alpha \in E^v}) = x_\beta$.

For $\beta \notin E^v$, let $g_\beta = f_\beta$. For a homomorphism $T : (H, f) \to (H', f')$, we define a homomorphism

$$S = (S_u)_{u \in V} = \Phi_v^+(T) : (K, g) = \Phi_v^+(H, f) \to (K', g') = \Phi_v^+(H', f')$$

If $u = v$, a bounded operator $S_v : K_v \to K'_v$ is given by

$$S_v((x_\alpha)_{\alpha \in E^v}) = (T_{s(\alpha)}(x_\alpha))_{\alpha \in E^v}.$$

It is easily seen that S_v is well-defined and we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_v & \xrightarrow{i_v} & \oplus_{\alpha \in E^v} H_{s(\alpha)} & \xrightarrow{h_v} & H_v \\
 & & {\scriptstyle S_v}\downarrow & & {\scriptstyle (T_{s(\alpha)})_{\alpha \in E^v}}\downarrow & & {\scriptstyle T_v}\downarrow \\
0 & \longrightarrow & {K'}_v & \xrightarrow{{i'}_v} & \oplus_{\alpha \in E^v} {H'}_{s(\alpha)} & \xrightarrow{h'_v} & {H'}_v
\end{array}$$

For other $u \in V$ with $u \neq v$, put

$$S_u = T_u : K_u = H_u \to K'_u = H'_u.$$

We also consider a dual of the above construction. We say that a vertex $v \in V$ is a *source* if $v \neq r(\alpha)$ for any $\alpha \in E$. Put $E_v = \{\alpha \in E;\ s(\alpha) = v\}$.

Definition.[EW3] Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we shall construct a new quiver $\sigma_v^-(\Gamma) = (\sigma_v^-(V), \sigma_v^-(E), s, r)$ as follows: All the arrows of Γ having v as source are reversed and all the other arrows remain unchanged. That is,

$$\sigma_v^-(V) = V \quad \sigma_v^-(E) = (E \setminus E_v) \cup \overline{E_v},$$

where $\overline{E_v} = \{\overline{\alpha};\ \alpha \in E_v\}$.

Definition.[EW3] (reflection functor Φ_v^-.) Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we shall define a *reflection functor* at v

$$\Phi_v^- : HRep(\Gamma) \to HRep(\sigma_v^-(\Gamma))$$

between the categories of Hilbert representations of Γ and $\sigma_v^-(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ, we define a Hilbert representation $(K, g) = \Phi_v^-(H, f)$ of $\sigma_v^-(\Gamma)$. Let

$$\hat{h}_v : H_v \to \oplus_{\alpha \in E_v} H_{r(\alpha)}$$

be a bounded linear operator defined by

$$\hat{h}_v(x) = (f_\alpha(x))_{\alpha \in E_v} \quad \text{for } x \in H_v.$$

We shall define

$$K_v := (\mathrm{Im}\hat{h}_v)^\perp = \mathrm{Ker}\,\hat{h}_v^* \subset \oplus_{\alpha \in E_v} H_{r(\alpha)},$$

where $\hat{h}_v^* : \oplus_{\alpha \in E_v} H_{r(\alpha)} \to H_v$ is given $\hat{h}_v^*((x_\alpha)_{\alpha \in E_v}) = \sum f_\alpha^*(x_\alpha)$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

Let $Q_v : \oplus_{\alpha \in E_v} H_{r(\alpha)} \to K_v$ be the canonical projection. For $\beta \in E_v$, let

$$j_\beta : H_{r(\beta)} \to \oplus_{\alpha \in E_v} H_{r(\alpha)}$$

be the canonical inclusion. We shall define

$$g_{\overline{\beta}} : K_{s(\overline{\beta})} = H_{r(\beta)} \to K_{r(\overline{\beta})} = K_v \quad \text{by } g_{\overline{\beta}} = Q_v \circ j_\beta.$$

For $\beta \notin E_v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \to (H', f')$, we shall define a homomorphism

$$S = (S_u)_{u \in V} = \Phi_v^-(T) : (K, g) = \Phi_v^-(H, f) \to (K', g') = \Phi_v^-(H', f').$$

For $u = v$, a bounded operator $S_v : K_v \to K'_v$ is given by

$$S_v((x_\alpha)_{\alpha \in E_v}) = Q'_v((T_{r(\alpha)}(x_\alpha))_{\alpha \in E_v}),$$

where $Q'_v : \oplus_{\alpha \in E_v} H'_{r(\alpha)} \to K'_v$ be the canonical projection.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
H_v & \xrightarrow{\hat{h}_v} & \oplus_{\alpha \in E_v} H_{r(\alpha)} & \xrightarrow{Q_v} & K_v & \longrightarrow & 0 \\
{\scriptstyle T_v}\downarrow & & \oplus_{\alpha \in E_v} T_{r(\alpha)}\downarrow & & {\scriptstyle S_v}\downarrow & & \\
H'_v & \xrightarrow{\hat{h}'_v} & \oplus_{\alpha \in E_v} H'_{r(\alpha)} & \xrightarrow{Q'_v} & K'_v & \longrightarrow & 0
\end{array}$$

For other $u \in V$ with $u \neq v$, put

$$S_u = T_u : K_u = H_u \to K'_u = H'_u.$$

We shall describe a relation between two (covariant) functors Φ_v^+ and Φ_v^-. We shall define another (contravariant) functor Φ^* at the begining.

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We shall define the opposite quiver $\overline{\Gamma} = (\overline{V}, \overline{E}, s, r)$ by reversing all the arrows, more precisely, $\overline{V} = V$ and $\overline{E} = \{\overline{\alpha};\ \alpha \in E\}$.

Definition.[EW3] Let $\Gamma = (V, E, s, r)$ be a finite quiver and $\overline{\Gamma} = (\overline{V}, \overline{E}, s, r)$ its opposite quiver. We shall define a contravariant functor

$$\Phi^* : HRep(\Gamma) \to HRep(\overline{\Gamma})$$

between the categories of Hilbert representations of Γ and $\overline{\Gamma}$ as follows: For a Hilbert representation (H, f) of Γ, we define a Hilbert representation $(K, g) = \Phi^*(H, f)$ of $\overline{\Gamma}$ by $K_u = H_u$ for $u \in V$ and $g_{\overline{\alpha}} = f_\alpha^*$ for $\alpha \in E$. For a homomorphism $T : (H, f) \to (H', f')$, we define a homomorphism

$$S = (S_u)_{u \in V} = \Phi^*(T) : (K', g') = \Phi^*(H', f') \to (K, g) = \Phi^*(H, f),$$

by bounded operators $S_u : K'_u = H'_u \to K_u = H_u$ given by $S_u = T_u^*$.

Proposition 4.15. [EW3, Proposition 4.2.] *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a source of Γ, then v is a sink of $\overline{\Gamma}$, $\sigma_v^-(\Gamma) = \overline{\sigma_v^+(\overline{\Gamma})}$ and the following assertions hold:*

(1) *For a Hilbert representation (H, f) of Γ,*
$$\Phi_v^-(H, f) = \Phi^*(\Phi_v^+(\Phi^*(H, f))).$$
(2) *For a homomorphism $T : (H, f) \to (H', f')$,*
$$\Phi_v^-(T) = \Phi^*(\Phi_v^+(\Phi^*(T))).$$

Proposition 4.16. [EW3, Proposition 4.3.] *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a sink of Γ, then v is a source of $\overline{\Gamma}$, $\sigma_v^+(\Gamma) = \overline{\sigma_v^-(\overline{\Gamma})}$ and the following assertions hold:*

(1) *For a Hilbert representation (H, f) of Γ,*
$$\Phi_v^+(H, f) = \Phi^*(\Phi_v^-(\Phi^*(H, f))).$$
(2) *For a homomorphism $T : (H, f) \to (H', f')$,*
$$\Phi_v^+(T) = \Phi^*(\Phi_v^-(\Phi^*(T))).$$

We shall investigate endomorphisms of Hilbert representations and its images of reflection functors. In the case of infinite-dimensional Hilbert representations, we need to assume a certain closedness condition at a sink or a source.

Definition.[EW3] Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. We recall that $E^v = \{\alpha;\ r(\alpha) = v\}$. It is said that a Hilbert representation (H, f) of Γ is *closed* at v if $\sum_{\alpha \in E^v} \mathrm{Im} f_\alpha \subset H_v$ is a closed subspace. It is said that (H, f) is *full* at v if $\sum_{\alpha \in E^v} \mathrm{Im} f_\alpha = H_v$.

Definition.([EW3]) Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. We recall that $E_v = \{\alpha | s(\alpha) = v\}$. It is said that a Hilbert representation (H,f) of Γ is *co-closed* at v if $\sum_{\alpha\in E_v} \mathrm{Im} f_\alpha^* \subset H_v$ is a closed subspace. It is said that (H,f) is *co-full* at v if $\sum_{\alpha\in E_v} \mathrm{Im} f_\alpha^* = H_v$.

We note that the properties of fullness, co-fullness, closedness and co-closedness are preserved under isomorphism of Hilbert representations.

Lemma 4.17. *Let Γ be a finite quiver and $v \in \Gamma$ a sink. Let (H,f) and (K,g) be isomorphic Hilbert representations of Γ. If (H,f) is full (resp.closed) at v, then (K,g) is full (resp.closed) at v.*

Proof. Assume that (H,f) is full at v. Since (H,f) and (K,g) are isomorphic, there exists a family $S=(S_u)_{u\in V}$ of bounded invertible operators such that $S_{r(\alpha)} f_\alpha = g_\alpha S_{s(\alpha)}$ for $\alpha \in E$. Take an element $y \in K_v$. By the invertibility of S_v, there exists an element $x \in H_v$ such that $S_v(x) = y$. Since (H,f) is full at v, there exist $x_{s(\alpha)} \in H_{s(\alpha)}$ such that $\sum_{\alpha\in E^v} f_\alpha(x_{s(\alpha)}) = x$. We put $y_{s(\alpha)} := S_{s(\alpha)}(x_{s(\alpha)})$. Then

$$\begin{aligned}\sum_{\alpha\in E^v} g_\alpha(y_{s(\alpha)}) &= \sum_{\alpha\in E^v} g_\alpha S_{s(\alpha)}(x_{s(\alpha)}) = \sum_{\alpha\in E^v} S_v f_\alpha(x_{s(\alpha)}) \\ &= S_v \sum_{\alpha\in E^v} f_\alpha(x_{s(\alpha)}) = S_v(x) = y\end{aligned}$$

Hence (K,g) is full at v.

We can similarly prove that closedness property is preserved under isomorphism of Hilbert representations. $\square$

Lemma 4.18. *Let Γ be a finite quiver and $v \in V$ a source. Let (H,f) and (K,g) be isomorphic Hilbert representations of Γ. If (H,f) is co-full (resp.co-closed) at v, then (K,g) is co-full (resp.co-closed) at v.*

Proof. Since (H,f) and (K,g) are isomorphic, $\Phi^*(H,f)$ and $\Phi^*(K,g)$ are isomorphic. Hence the case of co-fullness is reduced to the case of fullness. We can similarly prove that co-closedness property is preserved under isomorphism of Hilbert representations. $\square$

The following theorem is well known for finite-dimensional Hilbert spaces ([As, page289, 5.7. Corollary] and [DlR, page16, Proposition 2.1]).

Theorem 4.19. *Let $\Gamma = (V,E,s,r)$ be a finite quiver and $v \in V$ a sink. If a Hilbert representation (H,f) of Γ is full at v, then the map $\Phi_v^+ : End(H,f) \to End(\Phi_v^+(H,f))$ is an isomorphism as $\mathbb{C}$-algebras.*

Proof. We put $(K,g) := \Phi_v^+(H,f)$. The mapping Φ_v^+ gives a mapping of $End(H,f)$ to $End(K,g)$. At first we shall show that Φ_v^+ is one to one. Assume that $S := \Phi_v^+(T) = 0$ for $T \in End(H,f)$. We have $S_u = T_u = 0$ $(u \neq v)$. From this we shall show that $T_v = 0$. Since $T \in End(H,f)$, $T_v f_\alpha = f_\alpha T_{s(\alpha)}$ for $\alpha \in E^v = \{\alpha \in E; r(\alpha) = v\}$. Hence, for $x_\alpha \in H_{s(\alpha)}$,

$$T_v(\sum_{\alpha\in E^v} f_\alpha(x_\alpha)) = \sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x_\alpha) = 0.$$

Since (H, f) is full at v, $T_v = 0$. Thus Φ_v^+ is one to one. Next we shall show that Φ_v^+ is onto. Take $S = (S_u)_{u\in V} \in End(K, g)$. We put $T_u = S_u$ for $u \neq v$. We shall define an operator $T_v : H_v \to H_v$ such that $T_v(\sum_{\alpha\in E^v} f_\alpha(x_\alpha)) = \sum_{\alpha\in E^v} f_\alpha(T_{s(\alpha)}(x_\alpha))$ for $x_\alpha \in H_{s(\alpha)}$. We need to show that T_v is well defined. If there exists an element $(x'_\alpha)_{\alpha\in E^v} \in \oplus_{\alpha\in E^v} H_{s(\alpha)}$ such that

$$\sum_{\alpha\in E^v} f_\alpha(x_\alpha) = \sum_{\alpha\in E^v} f_\alpha(x'_\alpha),$$

then we must show that

$$\sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x_\alpha) = \sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x'_\alpha).$$

Since $\sum_{\alpha\in E^v} f_\alpha(x_\alpha) = \sum_{\alpha\in E^v} f_\alpha(x'_\alpha)$, we have

$$h_v((x_\alpha - x'_\alpha)_{\alpha\in E^v}) = \sum_{\alpha\in E^v} f_\alpha(x_\alpha - x'_\alpha) = 0.$$

Hence $(x_\alpha - x'_\alpha)_{\alpha\in E^v} \in \ker h_v = K_v$. Since $S_v : K_v \to K_v$, we have $S_v((x_\alpha - x'_\alpha)_{\alpha\in E^v}) \in \ker h_v = K_v$. Hence $h_v(S_v((x_\alpha - x'_\alpha)_{\alpha\in E^v})) = 0$. Since $S \in End(K, g)$, we have $S_{s(\alpha)} g_{\bar\alpha} = g_{\bar\alpha} S_v$ for $\alpha \in E^v$,

$$S_{s(\alpha)} g_{\bar\alpha}((x_\beta - x'_\beta)_{\beta\in E^v})) = S_{s(\alpha)}(x_\alpha - x'_\alpha) = T_{s(\alpha)}(x_\alpha - x'_\alpha),$$

and

$$g_{\bar\alpha} S_v((x_\beta - x'_\beta)_{\beta\in E^v})) = P_\alpha S_v((x_\beta - x'_\beta)_{\beta\in E^v})).$$

Hence

$$T_{s(\alpha)}(x_\alpha - x'_\alpha) = P_\alpha S_v((x_\beta - x'_\beta)_{\beta\in E^v})).$$

Then

$$\sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x_\alpha - x'_\alpha) = \sum_{\alpha\in E^v} f_\alpha P_\alpha S_v((x_\beta - x'_\beta)_{\beta\in E^v})$$

and

$$\sum_{\alpha\in E^v} f_\alpha P_\alpha S_v((x_\beta - x'_\beta)_{\beta\in E^v})) = h_v(S_v((x_\beta - x'_\beta)_{\beta\in E^v})) = 0.$$

This gives

$$\sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x_\alpha) = \sum_{\alpha\in E^v} f_\alpha T_{s(\alpha)}(x'_\alpha).$$

Thus T_v is well defined.

Next we shall show that $T_v f_\alpha(x) = f_\alpha T_{s(\alpha)}(x)$ for $x \in H_{s(\alpha)}$. Take and fix $x \in H_{s(\alpha)}$ for $\alpha \in E^v$. For $\beta \in E^v$, we put

$$x_\beta = \begin{cases} x & (\beta = \alpha), \\ 0 & (\beta \neq \alpha). \end{cases}$$

Since $T_v(\sum_{\beta\in E^v} f_\beta(x_\beta)) = \sum_{\beta\in E^v} f_\beta(T_{s(\beta)}(x_\beta))$, we have

$$T_v f_\alpha(x) = \sum_\beta f_\beta T_{s(\beta)}(x_\beta) = f_\alpha T_{s(\alpha)}(x).$$

Next we shall show that $T_v : H_v \to H_v$ is bounded. We decompose

$$\oplus_{\alpha\in E^v} H_{s(\alpha)} = \ker h_v \oplus (\ker h_v)^\perp = K_v \oplus K_v^\perp.$$

By the Banach invertibility theorem, $h_v|_{(K_v)^\perp} : (K_v)^\perp \to H_v$ is a bounded invertible operator. We shall show that there exists a positive constant c such that $\| T_v x \| \leqq c \| x \|$ for any $x \in H_v$. Take $x = h((x_\alpha)_{\alpha\in E^v}) = \sum_{x\in E^v} f_\alpha(x_\alpha)$. We get

$$\begin{aligned}
&\| T_v(x) \| = \| \sum_{\alpha\in E^v} T_v(f_\alpha(x_\alpha)) \| = \| \sum_{x\in E^v} f_\alpha(T_{s(\alpha)}(x_\alpha)) \| \\
&= \| ((f_\alpha T_{s(\alpha)})_{\alpha\in E^v})((x_\alpha)_{\alpha\in E^v}) \| \leqq \| ((f_\alpha T_{s(\alpha)})_{\alpha\in E^v}) \| \| ((x_\alpha)_{\alpha\in E^v}) \| \\
&= \| ((f_\alpha T_{s(\alpha)})_{\alpha\in E^v}) \| \| (h|_{K_v^\perp})^{-1} \| \| x \| \leqq c \| x \|
\end{aligned}$$

where $((f_\alpha T_{s(\alpha)})_{\alpha\in E^v})$ is a row matrix and

$$c := \| ((f_\alpha T_{s(\alpha)})_{\alpha\in E^v}) \| \| (h|_{K_v^\perp})^{-1} \| .$$

Hence T_v is bounded. Next we shall show that $\Phi_v^+(T) = S$. Since $S \in End(K,g)$, $S_{s(\alpha)} P_\alpha i_v = S_{s(\alpha)} g_{\bar\alpha} = g_{\bar\alpha} S_v = P_\alpha i_v S_v$ for $\alpha \in E^v$. For $((x_\alpha)_{\alpha\in E^v}) \in K_v$, we have

$$S_v((x_\alpha)) = (P_\alpha i_v S_v((x_\alpha)))_{\alpha\in E^v} = (S_{s(\alpha)} P_\alpha i_v((x_\alpha)))_{\alpha\in E^v} = (S_{s(\alpha)}(x_\alpha)).$$

By the definition of $\Phi_v^+(T)$, $(\Phi_v^+(T))_u = S_u = T_u$ for $u \neq v$. For $u = v$ and $((x_\alpha)_{\alpha\in E^v}) \in K_v$,

$$\begin{aligned}
(\Phi_v^+(T))_v((x_\alpha)_{\alpha\in E^v}) &= ((T_{s(\alpha)}(x_\alpha))_{\alpha\in E^v}) \\
&= ((S_{s(\alpha)}(x_\alpha))_{\alpha\in E^v}) = S_v((x_\alpha)_{\alpha\in E^v}).
\end{aligned}$$

Thus $(\Phi_v^+(T))_v = S_v$. Hence $\Phi_v^+(T) = S$. Hence Φ_v^+ is onto. We conclude that $End(H,f) \cong End(\Phi_v^+(H,f))$ as $\mathbb{C}$-algebras. □

Corollary 4.20. *Let $\Gamma = (V,E,s,r)$ be a finite quiver and $v \in V$ a sink. Assume that a Hilbert representation (H,f) of Γ is full at v. If (H,f) is transitive (resp. indecomposable), then $\Phi_v^+(H,f)$ is transitive(resp. indecomposable).*

The following theorem is well known for finite-dimensional Hilbert spaces ([As, page289, 5.7. Corollary] and [DlR, page16, Proposition 2.1]).

Theorem 4.21. *Let $\Gamma = (V,E,s,r)$ be a finite quiver and $v \in V$ a source. If a Hilbert representation (H,f) of Γ is co-full at v. Then $\Phi_v^- : End(H,f) \to End(\Phi_v^-(H,f))$ is an isomorphism as $\mathbb{C}$-algebras.*

Proof. We put $(K,g) := \Phi_v^-(H,f)$. The mapping Φ_v^- gives a mapping of $End(H,f)$ to $End(K,g)$. At first we shall show that Φ_v^- is one to one. Assume that $S := \Phi_v^-(T) = 0$ for $T \in End(H,f)$. We shall show that $T_v = 0$. Since $T \in End(H,f)$, $f_\alpha T_v = T_{r(\alpha)} f_\alpha$ for $\alpha \in E_v$. For $(x_\alpha)_{\alpha\in E_v} \in \oplus_{\alpha\in E_v} H_{r(\alpha)}$, we have

$$T_v^*(\sum f_\alpha^*(x_\alpha)) = \sum f_\alpha^*(T_{r(\alpha)}^*(x_\alpha)) = \sum f_\alpha^*(S_{r(\alpha)}^*(x_\alpha)) = 0.$$

Since (H,f) is co-full at v, $T_v^* = 0$. Hence $T_v = 0$. Thus Φ_v^- is one to one. Next we shall show that Φ_v^- is onto. We put $T_u = S_u$ for $u \neq v$. And we shall define an operator $W_v : H_v \to H_v$ such that for $(x_\alpha)_{\alpha\in E_v} \in \oplus_{\alpha\in E_v} H_{r(\alpha)}$,

$$W_v(\sum_{\alpha\in E_v} f_\alpha^*(x_\alpha)) = \sum_{\alpha\in E_v} f_\alpha^*(T_{r(\alpha)}^*(x_\alpha)).$$

We need to show that W_v is well defined. Assume that there exists an element $(x'_\alpha)_{\alpha\in E_v} \in \oplus_{\alpha\in E_v} H_{r(\alpha)}$ such that

$$\sum_{\alpha\in E_v} f_\alpha^*(x_\alpha) = \sum_{\alpha\in E_v} f_\alpha^*(x'_\alpha).$$

We have

$$\hat{h_v}^*((x_\alpha - x'_\alpha)) = \sum_{\alpha\in E_v} f_\alpha^*(x_\alpha - x'_\alpha) = 0.$$

Hence $(x_\alpha - x'_\alpha)_{\alpha\in E_v} \in \ker \hat{h_v}^* = K_v$. Since $S_v^* : K_v \to K_v$, we have $S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v}) \in K_v$. Hence $\hat{h_v}^*(S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v})) = 0$. Since $S \in End(K,g)$, we have $S_v g_{\bar\beta} = g_{\bar\beta} S_{r(\beta)}$ and $g_{\bar\beta}^* S_v^* = S_{r(\beta)}^* g_{\bar\beta}^*$. Hence

$$g_{\bar\beta}^* S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v}) = P_{r(\beta)} i_v S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v})$$

and

$$S_{r(\beta)}^* g_{\bar\beta}^*((x_\alpha - x'_\alpha)_{\alpha\in E_v}) = S_{r(\beta)}^*(x_\beta - x'_\beta).$$

Thus we have

$$P_{r(\beta)} i_v S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v}) = S_{r(\beta)}^*(x_\beta - x'_\beta)$$

and

$$\sum f_\beta^* P_{r(\beta)} i_v (S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v})) = \sum f_\beta^* S_{r(\beta)}^*(x_\beta - x'_\beta).$$

Since

$$\sum f_\beta^* P_{r(\beta)} i_v (S_v^*((x_\alpha - x'_\alpha)_{\alpha\in E_v})) = \hat{h_v}^*(S_v^*((x_\beta - x'_\beta)_{\beta\in E_v})) = 0,$$

we have

$$\sum f_\beta^* S_{r(\beta)}^*(x_\beta - x'_\beta) = \sum_{\beta\in E_v} f_\beta^* T_{r(\beta)}^*(x_\beta - x'_\beta) = 0.$$

Hence

$$\sum f_\beta^* T_{r(\beta)}^*(x_\beta) = \sum f_\beta^* T_{r(\beta)}^*(x'_\beta).$$

Thus W_v is well defined. Put $T_v = W_v^*$. Next we shall show that $f_\alpha T_v = T_{s(\alpha)} f_\alpha$ and $T_v^* f_\alpha^* = f_\alpha^* T_{s(\alpha)}^*$. Take and fix $x \in H_{r(\alpha)}$. For $\beta \in E_v$, we put

$$x_\beta = \begin{cases} x & (\beta = \alpha), \\ 0 & (\beta \neq \alpha). \end{cases}$$

By the definition of $W_v = T_v^*$,

$$W_v(\sum_{\alpha\in E^v} f_\alpha^*(x_\alpha)) = \sum_{\alpha\in E^v} f_\alpha^*(T_{r(\alpha)}^*(x_\alpha)).$$

Hence
$$T_v^* f_\alpha^*(x) = \sum_\beta f_\beta^* T_{r(\beta)}^*(x_\beta) = f_\alpha^* T_{r(\alpha)}^*(x) \text{ for } x \in H_{r(\alpha)}.$$
Thus we proved it.

Next we shall show that $W_v = T_v^* : H_v \to H_v$ is bounded. By the Banach invertibility theorem, $\hat{h}_v^*|_{(K_v)^\perp} : (K_v)^\perp \to H_v$ is a bounded invertible operator. We shall show that there exists a positive constant c such that $\| T_v^* x \| \leqq c \| x \|$ for any $x \in H_v$. For $x \in H_v$, there exists $(x_\alpha)_{\alpha\in E_v} \in (K_v)^\perp$ such that $x = \hat{h}_v^*((x_\alpha)_{\alpha\in E_v}) = \sum_{\alpha\in E_v} f_\alpha^*(x_\alpha)$. We have
$$\begin{aligned} \| T_v^*(x) \| &= \| \sum_{\alpha\in E_v} T_v^*(f_\alpha^*(x_\alpha)) \| = \| \sum_{\alpha\in E_v} f_\alpha^*(T_{r(\alpha)}^*(x_\alpha)) \| \\ &= \| (f_\alpha^* T_{r(\alpha)}^*)_{\alpha\in E_v} (x_\alpha)_{\alpha\in E_v} \| \leqq \| (f_\alpha^* T_{r(\alpha)}^*)_{\alpha\in E_v} \| \| (x_\alpha)_{\alpha\in E_v} \| \\ &= \| (f_\alpha^* T_{r(\alpha)}^*)_{\alpha\in E_v} \| \| (\hat{h}_v^*|_{K_v^\perp})^{-1} \| \| x \| \leqq c \| x \|, \end{aligned}$$
where $(f_\alpha^* T_{r(\alpha)}^*)_{\alpha\in E_v}$ is a row matrix and
$$c := \| ((f_\alpha^* T_{r(\alpha)}^*))_{\alpha\in E_v} \| \| (\hat{h}_v^*|_{K_v^\perp})^{-1} \| .$$
Hence T_v is bounded. Next we shall show that $\Phi_v^-(T) = S$. By the definition of $\Phi_v^-(T)$, $(\Phi_v^-(T))_u = S_u = T_u$ for $u\ (\neq v) \in V$. Since $S \in End(K,g)$, we have
$$S_v Q_v j_\beta = S_v g_{\bar{\beta}} = g_{\bar{\beta}} S_{r(\beta)} = Q_v j_\beta S_{r(\beta)}$$
for $\beta \in E_v$. For $(x_\beta)_{\beta\in E_v} \in K_v$, we have
$$\begin{aligned} S_v((x_\beta)_{\beta\in E_v}) &= S_v Q_v(\sum_{\beta\in E_v} j_\beta(x_\beta)) = \sum_{\beta\in E_v} S_v Q_v j_\beta(x_\beta) \\ &= \sum_{\beta\in E_v} Q_v j_\beta(S_{r(\beta)} x_\beta) = Q_v \sum_{\beta\in E_v} j_\beta(S_{r(\beta)} x_\beta) \\ &= Q_v((S_{r(\beta)} x_\beta)_{\beta\in E_v}). \end{aligned}$$
Thus
$$S_v((x_\beta)_{\beta\in E_v}) = Q_v((S_{r(\beta)} x_\beta)_{\beta\in E_v}).$$
For $u = v$ and $((x_\alpha)_{\alpha\in E_v}) \in K_v$,
$$\begin{aligned} (\Phi_v^-(T))_v((x_\alpha)_{\alpha\in E_v}) &= Q_v((T_{r(\alpha)} x_\alpha)_{\alpha\in E_v})) = Q_v((S_{r(\alpha)} x_\alpha)_{\alpha\in E_v})) \\ &= S_v((x_\alpha)_{\alpha\in E_v}). \end{aligned}$$
Thus $(\Phi_v^-(T))_v = S_v$. Hence $\Phi_v^-(T) = S$ and
$$\Phi_v^- : End(H,f) \to End(\Phi_v^-(H,f))$$
is onto. Thus we have $End(H,f) \cong End(\Phi_v^-(H,f))$ as $\mathbb{C}$-algebras. $\square$

Corollary 4.22. *Let $\Gamma = (V,E,s,r)$ be a finite quiver and $v \in V$ a source. Assume that a Hilbert representation (H,f) of Γ is co-full at v. If (H,f) is transitive, then $\Phi_v^-(H,f)$ is transitive. Similarly, if (H,f) is indecomposable, then $\Phi_v^{-1}(H,f)$ is indecomposable.*

Next, we shall show the existence of infinite-dimensional transitive Hilbert representations of quivers with any orientation whose underlying undirected graphs are extended Dynkin diagrams $\tilde{D_n}$ $(n \geq 4), \tilde{E_6}, \tilde{E_7}$ and $\tilde{E_8}$.

We recall some definitions and lemmas in [EW3].

Definition.[EW3] Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n. We count the arrows from the left as $\alpha_k : s(\alpha_k) \to r(\alpha_k),\ (k = 1, \dots, n-1)$. Let (H, f) be a Hilbert representation of Γ. We denote f_{α_k} briefly by f_k. For example,

$$\circ_{H_1} \xrightarrow{f_1} \circ_{H_2} \xrightarrow{f_2} \circ_{H_3} \xleftarrow{f_3} \circ_{H_4} \xrightarrow{f_4} \circ_{H_5} \xleftarrow{f_5} \circ_{H_6}$$

It is said that (H, f) is *positive-unitary diagonal* if there exist $m \in \mathbb{N}$ and orthogonal decompositions (admitting zero components) of Hilbert spaces

$$H_k = \oplus_{i=1}^m H_{k,i} \quad (k = 1, \dots, n)$$

and decompositions of operators

$$f_k = \oplus_{i=1}^m f_{k,i} : \oplus_{i=1}^m H_{s(\alpha_k),i} \to \oplus_{i=1}^m H_{r(\alpha_k),i} \qquad (k = 1, \dots, n)$$

such that each $f_{k,i} : H_{s(\alpha_k),i} \to H_{r(\alpha_k),i}$ is written as $f_{k,i} = 0$ or $f_{k,i} = \lambda_{k,i} u_{k,i}$ for some positive scalar $\lambda_{k,i}$ and onto unitary $u_{k,i} \in B(H_{s(\alpha_k),i}, H_{r(\alpha_k),i})$.

It is easily seen that if (H, f) is positive-unitary diagonal, then $\Phi^*(H, f)$ is also positive-unitary diagonal.

Lemma 4.23. [EW3, Lemma 6.4.] *Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ. Suppose that (H, f) is positive-unitary diagonal. Then (H, f) is closed at any sink of Γ and co-closed at any source of Γ.*

Proposition 4.24. [EW3, Proposition 6.5.] *Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ. Let v be a source of Γ. Suppose that (H, f) is positive-unitary diagonal. Then $\Phi_v^-(H, f)$ is also positive-unitary diagonal.*

It is known that every orientation of Dynkin diagram A_n is obtained by an iteration of σ_v^- at sources v except the right end from a particular orientation as follows:

Lemma 4.25. [EW3, Lemma 6.6.] *Let Γ_0 and Γ be quivers whose underlying undirected graphs are the same Dynkin diagram A_n for $n \geq 2$. Assume that Γ_0 is the following:*

$$\circ_1 \longrightarrow \circ_2 \longrightarrow \circ_3 \cdots \circ_{n-1} \longrightarrow \circ_n$$

Then there exists a sequence $v_1, \dots, v_m$ of vertices in Γ_0 such that

(1) *for each $k = 1, \dots, m$, v_k is a source in $\sigma_{v_{k-1}}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0)$,*
(2) $\sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma$,
(3) *for each $k = 1, \dots, m$, $v_k \neq n$.*

Lemma 4.26. [EW3, Lemma 5.6.] *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Then for any Hilbert representation (H, f) of Γ, $\Phi_v^+(H, f)$ is co-full at v.*

Theorem 4.27. [EW3, Theorem 5.13.] *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Assume that a Hilbert representation (H, f) of Γ is indecomposable and co-closed at v. Then the following assertions hold:*

(1) *If $\Phi_v^-(H, f) = 0$, then $H_v = \mathbb{C}$, $H_u = 0$ for any $u \in V$ with $u \neq v$ and $f_\alpha = 0$ for any $\alpha \in E$.*
(2) *If $\Phi_v^-(H, f) \neq 0$, then $\Phi_v^-(H, f)$ is also indecomposable and $(H, f) \cong \Phi_v^+\Phi_v^-(H, f))$.*

The following is one of the main theorems in this paper.

Theorem 4.28. *Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.*

Proof. Suppose that Γ is an oriented cyclic quiver. Theorem 4.10 proves the nonexistence of infinite-dimensional transitive Hilbert representation of Γ. Suppose that Γ is not an oriented cyclic quiver. We shall prove the existence of infinite-dimensional transitive Hilbert representations of Γ. When $\widetilde{A_n}$ case, Theorem 4.10 proves the existence of infinite-dimensional transitive Hilbert representations of Γ. Next we consider the case that the $|\Gamma|$ is $\tilde{D_n}$. Let Γ_0 be the quiver of Lemma 4.11 and $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{D_n}$, but their orientations are different in general. Let Γ_1 be a quiver such that $|\Gamma_1| = \tilde{D_n}$ and the orientation is as same as Γ on the path between 5 and $n+1$ and as same as Γ_0 on the rest four "wings". We shall define a Hilbert representation $(H^{(1)}, f^{(1)})$ of Γ_1 modifying $(H^{(0)}, f^{(0)})$. We put $f_\beta^{(1)} = I$ for any arrow β in the path between 5 and $n+1$. and $f_\beta^{(0)} = f_\beta^{(1)}$ for other arrow β. The same proof for $(H^{(0)}, f^{(0)})$ shows that $(H^{(1)}, f^{(1)})$ is transitive. Since $f_{\alpha_i}^{(1)}$ $(i = 1, \cdots, 4)$ is an inclusion map, $(H^{(1)}, f^{(1)})$ is co-full at sources 1,2,3 and 4. By Theorem 4.21, a certain iteration of reflection functors at a source 1,2,3 or 4 on $(H^{(1)}, f^{(1)})$ gives an infinite-dimensional, transitive, Hilbert representation of Γ. We have proved this case.

Next we consider the case that the $|\Gamma|$ is $\tilde{E_6}$. Let Γ_0 be the quiver of Lemma 4.12, and we denote here by $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{E_6}$, but their orientations are different in general. Three "wings" of $|\Gamma_0|$ $2-1-0$, $2'-1'-0$, $2''-1''-0$ can be regarded as Dynkin diagrams A_3. Applying Lemma 4.25 for these wings locally, we can find a sequence $v_1, \ldots, v_m$ of vertices in Γ_0 such that

(1) for each $k = 1, \ldots, m$, v_k is a source in $\sigma_{v_{k-1}}^- \ldots \sigma_{v_2}^-\sigma_{v_1}^-(\Gamma_0)$,
(2) $\sigma_{v_m}^- \ldots \sigma_{v_2}^-\sigma_{v_1}^-(\Gamma_0) = \Gamma$,

(3) for each $k = 1, \dots, m$, $v_k \neq 0$.

We note that co-closedness of Hilbert representations at a source can be checked locally around the source. Since the restriction of the representation $(H^{(0)}, f^{(0)})$ to each "wing" is positive-unitary diagonal and the iteration of reflection functors does not move the vertex 0, we can apply Lemma 4.23 and Proposition 4.24 locally that $\Phi^-_{v_{k-1}} \dots \ \Phi^-_{v_2}\Phi^-_{v_1}(H^{(0)}, f^{(0)})$ is co-closed at v_k for $k = 1, \dots, m$. Since the particular Hilbert space $H^{(0)}_0$ associated with the vertex 0 is infinite-dimensional and remains unchanged under the iteration of the reflection functors above, $\Phi^-_{v_i} \cdots \Phi^-_{v_1}(H^{(0)}, f^{(0)})$ $(1 \leq i \leq m)$ is infinite-dimensional. Therefore Theorem 4.27 implies that

$$\Phi^-_{v_i} \cdots \Phi^-_{v_1}(H^{(0)}, f^{(0)}) \ (1 \leq i \leq m)$$

is an infinite-dimensional indecomposable Hilbert representation of

$$\sigma^-_{v_i} \dots \sigma^-_{v_2}\sigma^-_{v_1}(\Gamma).$$

By Theorem 4.27, for

$$(K, g) := \Phi^-_{v_i} \cdots \Phi^-_{v_1}(H^{(0)}, f^{(0)}) \ (1 \leq i \leq m),$$

we have

$$(K, g) \cong \Phi^+_{v_{i+1}}\Phi^-_{v_{i+1}}(K, g).$$

On the other hand, by Lemma 4.26, $\Phi^+_{v_{i+1}}\Phi^-_{v_{i+1}}(K, g)$ is co-full at v_{i+1}. Since $(K, g) \cong \Phi^+_{v_{i+1}}\Phi^-_{v_{i+1}}(K, g)$, by Lemma 4.18, we have that (K, g) is co-full at v_{i+1}. Hence Theorem 4.21 implies that $End(K, g) \cong End(\Phi^-_{v_{i+1}}(K, g))$. By induction, we have

$$End(H^{(0)}, f^{(0)}) \cong End(\Phi^-_{v_m} \cdots \Phi^-_{v_1}(H^{(0)}, f^{(0)})).$$

Since $(H^{(0)}, f^{(0)})$ is transitive, $(\Phi^-_{v_m} \cdots \Phi^-_{v_1}(H^{(0)}, f^{(0)}))$ is also transitive. Thus there exist infinite-dimensional transitive Hilbert representations for quivers with any orientation whose underlying undirected graphs is extended Dynkin diagram $\widetilde{E_6}$. The other cases $\tilde{E}_7$ and $\tilde{E}_8$ are proved similarly. □

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