

## On the relative $K$ -group in the ETNC

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ABSTRACT. We consider the Burns–Flach formulation of the equivariant Tamagawa number conjecture (ETNC). In their setup, a Tamagawa number is an element of a relative  $K$ -group. We show that this relative group agrees with an ordinary  $K$ -group, namely of the category of locally compact topological modules over the order. Its virtual objects are an equivariant Haar measure in a precise sense. We expect that all relative  $K$ -groups in the ETNC will have analogous interpretations. At present, we need to restrict to regular orders, e.g. hereditary.

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## 1. Introduction

**1.1. What is a Tamagawa number?** We assume basic familiarity with the Burns–Flach formulation of the ETNC [BurF01], at least with the introduction. We follow the same notation.

A Tamagawa number – most classically – is a ratio of volumes and as such a positive real number. This concept has undergone wide generalizations, and in the context of the Burns–Flach formulation, it is a value in the relative  $K$ -group  $K_0(\mathfrak{A}, \mathbb{R})$ . In the classical case,  $\mathfrak{A} = \mathbb{Z}$ , one has a canonical isomorphism

$$K_0(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}_{>0}^\times,$$

and by this identification the interpretation as a volume is still visible. A relative  $K$ -group, like the group  $K_0(\mathfrak{A}, \mathbb{R})$  here, is defined as the relevant term to complete a long exact sequence; in the case at hand this sequence is

$$\cdots \longrightarrow K_1(\mathfrak{A}) \longrightarrow K_1(A_{\mathbb{R}}) \longrightarrow K_0(\mathfrak{A}, \mathbb{R}) \longrightarrow K_0(\mathfrak{A}) \longrightarrow \cdots \quad (1.1)$$

Besides fitting in the right spot, no further interpretation comes with relative  $K$ -groups. In this paper, we propose a different viewpoint, where the volume interpretation of  $K_0(\mathfrak{A}, \mathbb{R})$  is strengthened. We introduce the category of locally compact topologized right  $\mathfrak{A}$ -modules, call it  $\text{LCA}_{\mathfrak{A}}$ . It turns out to be an exact category, so it comes with its own  $K$ -theory groups, its own determinant functors, etc. We prove:

**Theorem.** *Suppose  $\mathfrak{A}$  is a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then there is a canonical long exact sequence*

$$\cdots \longrightarrow K_n(\mathfrak{A}) \longrightarrow K_n(A_{\mathbb{R}}) \longrightarrow K_n(\text{LCA}_{\mathfrak{A}}) \longrightarrow K_{n-1}(\mathfrak{A}) \longrightarrow \cdots,$$

and for all  $n$ , there are canonical isomorphisms

$$K_n(\text{LCA}_{\mathfrak{A}}) \cong K_{n-1}(\mathfrak{A}, \mathbb{R}).$$

This will be Theorem 11.3. In particular, for  $n = 1$  we obtain the new interpretation  $K_1(\text{LCA}_{\mathfrak{A}}) \cong K_0(\mathfrak{A}, \mathbb{R})$ . So, when rephrasing the ETNC using this sequence instead, one can get rid of relative  $K$ -groups and every term is interpreted as an ‘absolute’  $K$ -group of a suitable category.

The idea for the above theorem can be explained as follows: By Deligne’s work, every exact category has a universal determinant functor to its virtual objects. We shall deduce from computations of Clausen [Cla17] that the universal determinant functor of the category of locally compact abelian (LCA) groups is precisely the Haar measure, see Theorem 12.8. Since classical Tamagawa numbers stem from the Haar measure, one may now speculate that the universal determinant functor of the *equivariant* counterpart, i.e. LCA groups with an action of  $\mathfrak{A}$  (which is exactly our category  $\text{LCA}_{\mathfrak{A}}!$ ), should be the right receptacle for equivariant Tamagawa measures. However, since so far the literature has used the relative  $K$ -group  $K_0(\mathfrak{A}, \mathbb{R})$  instead, there is no reasonable way to escape the hope that the groups must actually agree.

Our main theorem above shows that this speculation is spot on for regular orders.

We should clarify that our methods are by no means a formal construction transforming any sort of relative  $K$ -group into an absolute one. Indeed, our methods work very specifically for the relative  $K$ -group *with respect to the reals*, by exploiting that  $\mathbb{R}$  is a locally compact topological field. Although we do not touch on this topic in this paper, variations for  $\mathbb{Q}_p$  or more generally other locally compact topological fields should exist.

Our main theorem also yields an alternative description of  $K_0(\mathfrak{A}, \mathbb{R})$  by using Nenashev's generator and relator presentation for  $K_1$ -groups.

**Theorem.** *Let  $\mathfrak{A}$  be a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra. Then  $K_0(\mathfrak{A}, \mathbb{R})$  admits a presentation where generators are double short exact sequences*

$$A \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} B \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{s} \end{array} C,$$

for  $A, B, C$  locally compact right  $\mathfrak{A}$ -modules, modulo suitable relations coming from  $(3 \times 3)$ -diagrams (We explain this in detail in §2.1).

See Theorem 2.5. Regrettably, the identification of the image of  $K_0(\mathfrak{A}, \mathbb{Q})$  inside  $K_1(\mathrm{LCA}_{\mathfrak{A}})$  remains open at the moment. This would be important for the formulation of Rationality in the ETNC. Moreover, the restriction to regular orders is a nuisance and we hope to remove this assumption in the future. Maximal orders, and more generally hereditary orders are regular. Group rings  $\mathbb{Z}[G]$  are only Gorenstein, we cannot handle them at the moment. We can obtain the analogue of our main theorem for  $G$ -theory for all orders, with no extra conditions.

**Theorem.** *Suppose  $\mathfrak{A}$  is an arbitrary order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then there is a long exact sequence*

$$\cdots \longrightarrow G_n(\mathfrak{A}) \longrightarrow K_n(A_{\mathbb{R}}) \longrightarrow K_n(\mathrm{LCA}_{\mathfrak{A}}) \longrightarrow G_{n-1}(\mathfrak{A}) \longrightarrow \cdots,$$

where  $G_n(\mathfrak{A}) := K_n(\mathrm{Mod}_{\mathfrak{A}, fg})$  denotes the  $K$ -theory of the category of finitely generated right  $\mathfrak{A}$ -modules (this is often called “ $G$ -theory”).

See Theorem 11.4. So, we see that  $\mathrm{LCA}_{\mathfrak{A}}$  roughly speaking corresponds to the  $G$ -theory groups  $G_n(\mathfrak{A})$  in general. Really, we would want  $K$ -theory groups here. For  $K$ -groups of rings this amounts to restricting to *projective* modules among all modules, and similarly we will define a variant of  $\mathrm{LCA}_{\mathfrak{A}}$  in the future, fixing the issue in a similar way. For this category then all orders  $\mathfrak{A}$  will be fine, not just the regular ones. This still needs to be carefully worked out and written, but using a suitable full subcategory of  $\mathrm{LCA}_{\mathfrak{A}}$  should be enough.

To complete the overview, let us state the following result, Theorem 12.8, which is essentially due to Clausen, even though he has not spelled it out in this fashion in [Cla17].

**Theorem.** *The Haar functor  $Ha : \text{LCA}_{\mathbb{Z}}^{\times} \rightarrow \text{Tors}(\mathbb{R}_{>0}^{\times})$  is the universal determinant functor of the category  $\text{LCA}_{\mathbb{Z}}$ . Here*

- (1) *for any LCA group  $G$ ,  $Ha(G)$  denotes the  $\mathbb{R}_{>0}^{\times}$ -torsor of all Haar measures on  $G$ , and*
- (2) *Deligne’s Picard groupoid of virtual objects for  $\text{LCA}_{\mathbb{Z}}$  turns out to be isomorphic to the Picard groupoid of  $\mathbb{R}_{>0}^{\times}$ -torsors.*

Our central concept, the *equivariant Haar measure*, will generalize this theorem from  $\text{LCA}_{\mathbb{Z}}$  to  $\text{LCA}_{\mathfrak{A}}$ . However, this is in a way tautological since, guided by the above theorem, we simply define it as the universal determinant functor of the category  $\text{LCA}_{\mathfrak{A}}$ . Thus, studying the categorical properties of  $\text{LCA}_{\mathfrak{A}}$  will be the key tool in the paper.

We also prove the following reciprocity law. Write  $\mathbb{A}$  for the adèles of the rationals.

**Theorem (Reciprocity Law).** *Let  $\mathfrak{A}$  be an arbitrary order in a semisimple finite-dimensional  $\mathbb{Q}$ -algebra  $A$ . Then the composition of natural maps*

$$K(A) \longrightarrow K(A \otimes \mathbb{A}) \longrightarrow K(\text{LCA}_{\mathfrak{A}})$$

*is zero.*

See Theorem 13.1.

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## 2. Where does this picture come from?

We will now give a detailed exposition how the category  $\text{LCA}_{\mathfrak{A}}$  shows up. Those who are in a hurry may consider jumping to §3 right away. On the other hand, this section may serve as a survey.

Before Bloch and Kato proposed to interpret Tamagawa numbers in terms of realizations of motives, Tamagawa numbers would just stem from the evaluation of integrals on adelic points of algebraic groups; see the famous map of the ‘land of Tamagawa numbers’ [BloK90, bottom of p 336], and generally the introduction of their article.

Measures or volumes have since often been interpreted in terms of determinant functors in the sense of Deligne [Del87], [Knu02]. We refer the reader to §12 for a quick summary. The determinant line is a functor defined for all commutative rings. If, say,  $R$  happens to be an integral domain (or more generally if  $\text{Spec } R$  is connected), then it is

$$\text{PMod}(R)^{\times} \longrightarrow \text{Pic}_{R}^{\mathbb{Z}}, \quad M \longmapsto \left( \bigwedge^{\text{top}} M, \text{rk } M \right),$$

where  $\text{PMod}(R)$  is the exact category of finitely generated projective  $R$ -modules. Moreover,  $\text{PMod}(R)^\times$  denotes the same category but with all non-isomorphisms deleted from the collection of morphisms, and  $\text{Pic}_R^\mathbb{Z}$  is the Picard groupoid of  $\mathbb{Z}$ -graded lines: Its objects are pairs  $(L, n)$ , where  $L$  is a rank one projective  $R$ -module and  $n \in \mathbb{Z}$ . For a full definition and description of the Picard groupoid structure we refer to [BurF01, §2.5], [KM76, Chapter 1]. For  $R = \mathbb{R}$  one might hope that this would give the Haar measure, in the following sense:

The top exterior power  $\bigwedge^{\text{top}} M$  corresponds to a top form against which we can integrate. Nonetheless, this picture is slightly flawed. The issue is that in the category  $\text{PMod}(\mathbb{R})$  we have, vaguely speaking, orientation-reversing automorphisms like  $\mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto -x$ , changing the sign of the top form. This differs from the Haar measure, which always returns non-negative volumes. So there is a difference:

$$\begin{array}{ccc} \int_{\mathbb{R}^n} f \, d\omega & & \int_{\mathbb{R}^n} f \, |d\omega|. \\ \text{top exterior power} & & \text{Haar measure} \\ \text{(signed)} & & \text{(unsigned)} \end{array}$$

This might just seem like an innocent subtlety, but the homological implications of this difference run deeper. For example, the Picard groupoid  $\text{Pic}_R^\mathbb{Z}$  has the non-trivial symmetry constraint

$$(L, n) \otimes (L', n') \xrightarrow{(-1)^{nn' \cdot s}} (L', n') \otimes (L, n),$$

where  $s : L \otimes_R L' \cong L' \otimes_R L$  denotes the usual plain symmetry constraint of the symmetric monoidal structure of  $\text{PMod}(R)$ . This structure only stems from the signed nature and in the Haar setting there is no interplay between a sign and the ranks  $n, n'$ .

We will now be heading towards the viewpoint that once the Haar measure has been set up carefully as a determinant functor in the sense of Deligne, the corresponding  $\mathfrak{A}$ -equivariant analogue naturally takes values in  $K_0(\mathfrak{A}, \mathbb{R})$  instead of  $\mathbb{R}_{>0}^\times$  (and this aspect cannot be seen when using the top exterior power determinant line instead, because already in the non-equivariant setting there is a difference between the two when it comes to handling the signs).

This viewpoint supplies “*In our approach Tamagawa numbers are then elements of a relative algebraic  $K$ -group  $K_0(\mathfrak{A}, \mathbb{R})$* ” [BurF01, p. 502] with a very concrete philosophical interpretation.

If  $G$  is a locally compact abelian (LCA) group, it carries a Haar measure  $\mu_G$ , well-defined up to rescaling by a positive real constant, see [HN98, Ch. II, §5] for a review. Suppose

$$G' \hookrightarrow G \twoheadrightarrow G''$$

is an exact sequence of LCA groups. This means it is exact on the level of abelian groups and additionally the first arrow is a closed map, while the second is a quotient map in the sense of topology. Suppose we are given choices of Haar measures  $\mu_{G'}, \mu_{G''}$  on  $G'$  and  $G''$  respectively, then there is a unique choice of Haar measure  $\mu_G$  on  $G$  such that

$$\int_G f \, d\mu_G = \int_{G'} \int_{G''} f \, d\mu_{G''} d\mu_{G'} \tag{2.1}$$

holds for all  $f \in C_c(G)$  (compactly supported continuous functions). In particular,  $\mu_G$  induces  $\mu_{G'}$  on  $G'$ , and  $\mu_{G''}$  is the corresponding quotient measure. This is the fundamental compatibility construction for Haar measures in exact sequences.

(a) These constructions give rise to the *modulus*: Suppose  $f : G \xrightarrow{\sim} G$  is an automorphism of the LCA group. Then for any choice of the Haar measure  $\mu_G$ , the pullback  $f^*\mu_G$  is again a Haar measure. Being unique up to a positive scalar, this gives some unique  $c \in \mathbb{R}_{>0}^\times$  such that  $\mu_G = c \cdot f^*\mu_G$ . This construction can be promoted to a group homomorphism

$$|\cdot| : \text{Aut}(G) \longrightarrow \mathbb{R}_{>0}^\times.$$

This homomorphism is well-defined and not longer dependent on the initial choice of  $\mu_G$ . See [HN98, Ch. II, §5.6]

(b) If  $G$  is a discrete (resp. compact) LCA group, one still can choose a Haar measure, but a canonical choice is possible, namely  $\mu_G(\{g\}) := 1$  for each element, i.e.

$$\mu_G(U) := \#U \quad (\text{the counting measure})$$

if  $G$  is discrete, or the normalization  $\mu_G(G) = 1$  in the case if  $G$  is compact. So the cases of discrete or compact groups are special in the sense that there is a canonical normalization.

(c) The exact sequence compatibility of the Haar measure amounts to the following property: If

$$\begin{array}{ccccc} G' \hookrightarrow G & \twoheadrightarrow & G'' \\ f \downarrow \sim & & g \downarrow \sim & & h \downarrow \sim \\ G' \hookrightarrow G & \twoheadrightarrow & G'' \end{array}$$

is a commutative diagram, then the modulus satisfies

$$|g| = |f| \cdot |h|. \tag{2.2}$$

One can extract further laws describing the behaviour of Haar measures in complexes, or more complicated commutative diagrams, e.g., a  $(3 \times 3)$ -Lemma etc. Such rules have been studied to some extent by Oesterlé [Oes84, Appendice 4] and de Smit [dS96].

This concludes our quick summary of the Haar measure, from the viewpoint of picking an (inevitably non-canonical) choice of a normalization on each group. However, there is also a completely different viewpoint:

As was pointed out by Hoffmann and Spitzweck, the category  $\text{LCA}$  of locally compact groups is an exact category [HS07]. As a result, it comes with its own universal determinant functor in the sense of Deligne [Del87]:

$$\det : \text{LCA}^\times \longrightarrow V(\text{LCA}),$$

where  $V(\text{LCA})$  denotes the universal Picard groupoid of the category, also known as Deligne’s category of ‘virtual objects’.

Now, the properties of the Haar measure which we had recalled above can be used to construct a concrete determinant functor on  $\text{LCA}$  by hand: If  $A$  is an abelian group, recall that an  $A$ -torsor  $T$  is a simply transitive  $A$ -set. Hence, for every  $t \in T$  one gets an isomorphism of  $A$ -sets  $A \xrightarrow{\sim} T$ , but there is no canonical choice for such an identification. There is a tensor product  $T \otimes T'$ , defined by

$$\{(t, t') \in T \times T'\} / \langle (t, t') \sim (s, s') : \Leftrightarrow \exists a \in A \text{ s.t. } t = a + s, t' = a + s' \rangle.$$

The  $A$ -torsors form a category  $\text{Tors}(A)$ , and indeed a Picard groupoid. The group  $A$  itself with its natural  $A$ -set structure acts as the unit  $1_{\text{Tors}(A)}$ , and  $T^{-1}$  is defined by letting  $A$  act negatively.

Now, define the *Haar functor*

$$Ha(-) : \text{LCA}^\times \longrightarrow \text{Tors}(\mathbb{R}_{>0}^\times) \tag{2.3}$$

by sending each  $\text{LCA}$  group  $G$  to its set of Haar measures. Since the rescaling by a positive scalar acts simply transitively on them, this is an  $\mathbb{R}_{>0}^\times$ -torsor.

Let us recast the functoriality properties of the Haar measure in this language: (a) If  $f : G \xrightarrow{\sim} G$  denotes an automorphism, then any trivialization of the Haar torsor, i.e. an isomorphism of  $\mathbb{R}_{>0}^\times$ -sets  $t : \mathbb{R}_{>0}^\times \xrightarrow{\sim} Ha(G)$  gives a concrete measure  $d\mu_G \in Ha(G)$  attached to some real number, as does  $f^*d\mu_G$ . The modulus  $|f|$  is the fraction of these numbers. This corresponds to the canonical morphism

$$\text{Aut}(G) \longrightarrow \pi_1(\text{Tors}(\mathbb{R}_{>0}^\times)) \cong \mathbb{R}_{>0}^\times \tag{2.4}$$

coming from comparing trivializations. (c) For every short exact sequence  $G' \hookrightarrow G \twoheadrightarrow G''$  the compatibility of Equation 2.1 amounts to a canonical isomorphism of torsors,

$$Ha(G) \xrightarrow{\sim} Ha(G') \otimes_{\text{Tors}(\mathbb{R}_{>0}^\times)} Ha(G''). \tag{2.5}$$

Taken altogether, this gives the fundamental datum of a determinant functor, see Equation 12.1.

So far, we have skipped over the analogue of (b). Here the story differs a little: Write  $\text{LCA}_D$  (resp.  $\text{LCA}_C$ ) for the full subcategories of  $\text{LCA}$  of discrete (resp. compact)  $\text{LCA}$  groups. Restricted to these categories, the Haar torsor determinant functor admits a trivialization.

This can be seen by a variant of the Eilenberg swindle: Suppose  $C \in \text{LCA}_C$  is compact. Then there is an exact sequence

$$C \hookrightarrow \prod_{\mathbb{Z}} C \twoheadrightarrow \prod_{\mathbb{Z}} C, \tag{2.6}$$

using that the product of compact groups is again compact (Tychonoff's Theorem). Equation 2.5 provides us with a canonical isomorphism

$$Ha(\prod_{\mathbb{Z}} C) \xrightarrow{\sim} Ha(C) \otimes Ha(\prod_{\mathbb{Z}} C),$$

and  $\otimes$ -multiplying with the inverse of  $Ha(\prod_{\mathbb{Z}} C)$ , we get an isomorphism

$$t_C : 1_{\text{Tors}(A)} \xrightarrow{\sim} Ha(C),$$

i.e. a canonical trivialization of the torsor. This is also compatible with exact sequences, i.e. if  $C' \hookrightarrow C \twoheadrightarrow C''$  is an exact sequence of compact groups, the trivializations  $t_{(-)}$  for  $C', C, C''$  sit in a commutative diagram.

This argument also shows that the modulus is trivial: Firstly, the property of Equation 2.2 applied to

$$\begin{array}{ccccc} 0 & \hookrightarrow & \prod_{\mathbb{Z}} C & \xrightarrow{\sim} & \prod_{\mathbb{Z}} C \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ 0 & \hookrightarrow & \prod_{\mathbb{Z}} C & \xrightarrow{\sim} & \prod_{\mathbb{Z}} C \end{array} \tag{2.7}$$

shows that  $|g| = |h|$ , i.e. the modulus agrees when the automorphisms commute with an isomorphism. Every automorphism  $f : C \xrightarrow{\sim} C$  induces an automorphism factor-wise to  $\prod_{\mathbb{Z}} C$ , call it  $g$ . Hence, applied to the diagram

$$\begin{array}{ccccc} C & \hookrightarrow & \prod_{\mathbb{Z}} C & \twoheadrightarrow & \prod_{\mathbb{Z}} C \\ f \downarrow \sim & & g \downarrow \sim & & h \downarrow \sim \\ C & \hookrightarrow & \prod_{\mathbb{Z}} C & \twoheadrightarrow & \prod_{\mathbb{Z}} C \end{array}$$

whose rows are made from Equation 2.6, the same property shows that  $|g| = |f| \cdot |h|$ , where  $h$  is the induced map on the quotient. However, by our previous observation from Equation 2.7,  $|g| = |h|$ , so we deduce  $|f| = 1$ . The automorphism  $f$  was arbitrary, and it follows that the modulus is always 1 for every compact group.

For discrete groups  $D \in \text{LCA}_D$  instead take the short exact sequence  $D \hookrightarrow \coprod_{\mathbb{Z}} D \twoheadrightarrow \coprod_{\mathbb{Z}} D$  based on coproducts. These are still discrete. The same argument goes through.

While the argument here might look a little different, this is the torsor reformulation of property (b) saying that compact and discrete groups have a canonical normalization.

We summarize our observations as follows.



**Lemma 2.1.** *Let  $G \in \mathbf{LCA}$  be a compact group or a discrete group. Then for every determinant functor  $\mathcal{P} : \mathbf{C}^\times \rightarrow \mathbf{P}$ , with  $\mathbf{P}$  some Picard groupoid, the value  $\mathcal{P}(f) \in \pi_1(\mathbf{P})$  of any automorphism  $f : G \xrightarrow{\sim} G$  is trivial, i.e. the neutral element of the group.*

**Proof.** A careful reading of the previous arguments shows that they only use axioms of a determinant functor, as in Definition 12.1, so they hold in general. The construction of the modulus translates into mapping the isomorphism in the Picard groupoid  $\mathbf{P}$  to its class in  $\pi_1$ .  $\square$

Now let us compare the usual determinant line with the Haar measure:

*Example 2.2.* Firstly, we consider the category  $\mathbf{Vect}_f(\mathbb{R})$  of finite-dimensional real vector spaces. Deligne’s universal determinant functor is

$$\det : \mathbf{Vect}_f(\mathbb{R})^\times \rightarrow \mathbf{Pic}_{\mathbb{R}}^{\mathbb{Z}}$$

with values in the Picard groupoid of graded lines. In particular, for any automorphism  $f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ , the value lies in  $\pi_1(\mathbf{Pic}_{\mathbb{R}}^{\mathbb{Z}}) = \mathbb{R}^\times$ . This can be interpreted as a signed/oriented measure. Every finite-dimensional real vector space can also be regarded as an LCA group; the functor  $\psi_\infty : \mathbf{Vect}_f(\mathbb{R}) \rightarrow \mathbf{LCA}$  is exact. However, no determinant functor on  $\mathbf{LCA}$  can see the orientation. To see this, consider the map of multiplication with  $-1$ , which induces a map of exact sequences

$$\begin{array}{ccccc} \mathbb{Z} \hookrightarrow & \mathbb{R} & \twoheadrightarrow & \mathbb{T} & \\ -1 \downarrow \sim & -1 \downarrow \sim & & -1 \downarrow \sim & \\ \mathbb{Z} \hookrightarrow & \mathbb{R} & \twoheadrightarrow & \mathbb{T} & \end{array}$$

and by Equation 2.2 (or rather its analogue for general Picard groupoids) we obtain  $\mathcal{P}(-1_{\mathbb{R}}) = \mathcal{P}(-1_{\mathbb{Z}}) \cdot \mathcal{P}(-1_{\mathbb{T}})$ . However,  $\mathbb{Z}$  is discrete (resp.  $\mathbb{T}$  compact), so by Lemma 2.1, we must have  $\mathcal{P}(-1_{\mathbb{Z}}) = 1$ ,  $\mathcal{P}(-1_{\mathbb{T}}) = 1$ . Thus,  $\mathcal{P}(-1_{\mathbb{R}}) = 1$ . A more careful computation shows that the functor  $\psi_\infty$  induces the map  $\pi_1(\mathbf{Pic}_{\mathbb{R}}^{\mathbb{Z}}) \rightarrow \pi_1(V(\mathbf{LCA}))$ ,  $\alpha \mapsto |\alpha|$ , i.e. the Haar measure differs from the real determinant line just by forgetting the sign. We give a rigorous proof for this claim in Proposition 13.3 based on  $K$ -theory.

*Example 2.3.* The analogous effect over the  $p$ -adics is more drastic. For any  $\alpha \in \mathbb{Z}_p^\times$  consider the map of exact sequences

$$\begin{array}{ccccc} \mathbb{Z}_p \hookrightarrow & \mathbb{Q}_p & \twoheadrightarrow & \mathbb{Q}_p/\mathbb{Z}_p & \\ \cdot \alpha \downarrow \sim & \cdot \alpha \downarrow \sim & & \cdot \alpha \downarrow \sim & \\ \mathbb{Z}_p \hookrightarrow & \mathbb{Q}_p & \twoheadrightarrow & \mathbb{Q}_p/\mathbb{Z}_p & \end{array}$$

This time  $\mathbb{Z}_p$  is compact and  $\mathbb{Q}_p/\mathbb{Z}_p$  discrete, but again the same Lemma 2.1 shows that we must have  $\mathcal{P}(\alpha_{\mathbb{Z}_p}) = 1$ ,  $\mathcal{P}(\alpha_{\mathbb{Q}_p/\mathbb{Z}_p}) = 1$ , and therefore

$\mathcal{P}(\alpha_{\mathbb{Q}_p}) = 1$ . Indeed, the difference between the  $p$ -adic determinant line and the Haar measure is taking the  $p$ -adic absolute value

$$\pi_1(\text{Pic}_{\mathbb{Q}_p}^{\mathbb{Z}}) \longrightarrow \pi_1(V(\text{LCA})), \quad \alpha \longmapsto |\alpha|_p.$$

Again, we see that the Haar measure is a lot coarser than the determinant line. See Proposition 13.3 for a rigorous proof of this claim.

The previous examples show that the Haar torsor, although related, is really a bit of a different invariant than the standard determinant line of a ring. The following result is more or less implicit in the work of Clausen [Cla17]:

**Theorem.** *The Haar functor  $Ha : \text{LCA}^\times \rightarrow \text{Tors}(\mathbb{R}_{>0}^\times)$  is the universal determinant functor of the category  $\text{LCA}$ . In particular, Deligne’s Picard groupoid of virtual objects for  $\text{LCA}$  is isomorphic to the Picard groupoid of  $\mathbb{R}_{>0}^\times$ -torsors.*

This will be Theorem 12.8 below. This result motivates to define an “equivariant Haar measure” simply by replacing  $\text{LCA}$  by the category  $\text{LCA}_{\mathfrak{A}}$  of locally compact  $\mathfrak{A}$ -modules and taking the universal determinant functor of this category. By the above theorem, it follows that for the non-equivariant setting  $\mathfrak{A} := \mathbb{Z}$  we get the classical Haar measure.

**Definition 2.4.** *Let  $\mathfrak{A}$  be an order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Define the equivariant Haar measure functor to be Deligne’s universal determinant functor of the exact category  $\text{LCA}_{\mathfrak{A}}$ , i.e.*

$$Ha_{\mathfrak{A}} : \text{LCA}_{\mathfrak{A}}^\times \longrightarrow V(\text{LCA}_{\mathfrak{A}}),$$

where  $V(\text{LCA}_{\mathfrak{A}})$  denotes Deligne’s virtual objects in this category. We may also call  $V(\text{LCA}_{\mathfrak{A}})$  the Picard groupoid of equivariant volumes.

Now, let us connect this back to the ETNC.

**Theorem.** *Let  $\mathfrak{A}$  be a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then*

$$\pi_1 V(\text{LCA}_{\mathfrak{A}}) \cong K_0(\mathfrak{A}, \mathbb{R}),$$

and, stronger, the Picard groupoid of equivariant volumes  $V(\text{LCA}_{\mathfrak{A}})$  is equivalent to the 1-truncation of the  $(-1)$ -shift of the fiber of the morphism  $K(\mathfrak{A}) \longrightarrow K(A_{\mathbb{R}})$ .

See Theorem 12.9.

The corresponding universal Picard categories  $V(\text{LCA}_D)$  resp.  $V(\text{LCA}_C)$  are the trivial Picard groupoid with one object and no non-trivial automorphisms.

**2.1. Nenashev’s presentation.** A double (short) exact sequence in  $\text{LCA}_{\mathfrak{A}}$  is the datum of two short exact sequences

$$\text{Yin} : A \xrightarrow{p} B \xrightarrow{r} C \quad \text{and} \quad \text{Yang} : A \xrightarrow{q} B \xrightarrow{s} C,$$

where (as we can see) only the maps may differ, but the three objects agree. We write

$$A \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} B \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{s} \end{array} C$$

as a convenient shorthand. The study of this concept was pioneered by Nenashev.

**Theorem 2.5.** *Let  $\mathfrak{A}$  be a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra. Then  $K_0(\mathfrak{A}, \mathbb{R})$ , or equivalently  $K_1(\text{LCA}_{\mathfrak{A}})$ , has the following presentation as an abelian group:*

- (1) Attach a generator to each double exact sequence

$$A \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} B \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{s} \end{array} C$$

with  $A, B, C$  locally compact right  $\mathfrak{A}$ -modules.

- (2) Whenever the yin and yang exact sequence agree, i.e.,

$$A \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{p} \end{array} B \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{r} \end{array} C,$$

we declare the class to be zero.

- (3) Suppose there is a (not necessarily commutative) diagram

$$\begin{array}{ccccc} A & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & B & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & C \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ D & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & E & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & F \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ G & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & H & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & I \end{array}$$

whose rows  $\text{Row}_i$  and columns  $\text{Col}_j$  are double exact sequences. Suppose after deleting all yin (resp. all yang) exact sequences, the remaining diagram commutes. Then impose the relation

$$\text{Row}_1 - \text{Row}_2 + \text{Row}_3 = \text{Col}_1 - \text{Col}_2 + \text{Col}_3. \tag{2.8}$$

**Proof.** This is precisely Nenashev’s presentation of  $K_1$ , see [Nen98, Theorem]. Hence, the validity of this presentation follows by our main theorem, which expresses  $K_0(\mathfrak{A}, \mathbb{R})$  as the  $K_1$ -group  $K_1(\text{LCA}_{\mathfrak{A}})$ .  $\square$

Instead of Nenashev’s presentation, one may also work with Grayson’s generalized formalism of binary complexes, which also works for higher  $K$ -groups [Gra12].

Let us demonstrate how to work with this presentation in two key cases.

*Example 2.6.* The map

$$K_1(A_{\mathbb{R}}) \longrightarrow K_0(\mathfrak{A}, \mathbb{R})$$

in Sequence 1.1 is given as follows: Write any matrix  $M \in \text{GL}(A_{\mathbb{R}})$  as an automorphism of  $X := A_{\mathbb{R}}^m$  with  $m$  sufficiently large. Then send

$$\left[ \varphi : X \xrightarrow{\sim} X \right] \mapsto \left[ \begin{array}{ccc} 0 & \xrightarrow{\quad 0 \quad} & X \\ \xrightarrow{\quad 0 \quad} & \searrow \varphi & \xrightarrow{\quad \varphi \quad} \\ & & X \end{array} \right],$$

where on the right  $X$  is now interpreted as a locally compact  $\mathfrak{A}$ -module, equipped with the topology coming from the real vector space structure. This is the map transforming automorphisms in  $\text{PMod}(A_{\mathbb{R}})$  to elements in the Nenashev presentation. This fact is standard and discussed in [Wei13, Ch. IV, Example 9.6.2], [Nen98, Equation (2.2)] (in these sources translate Weibel’s left sequence to the bottom sequence in Nenashev’s paper, as opposed to the top one, to get the signs to match).

*Example 2.7.* From the exactness of Sequence 1.1 we expect that the composition

$$K_1(\mathfrak{A}) \longrightarrow K_1(A_{\mathbb{R}}) \longrightarrow K_0(\mathfrak{A}, \mathbb{R})$$

is zero. Let us confirm this by a direct computation in the Nenashev presentation. We work in the category  $\text{LCA}_{\mathfrak{A}}$ . Let  $\mathfrak{X} \in \text{PMod}(\mathfrak{A})$ , which we tacitly equip with the discrete topology, while we equip  $X_{\mathbb{R}} := \mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{R}$  with the locally compact topology which comes from its real vector space structure. Define

$$T := X_{\mathbb{R}}/\mathfrak{X} \quad \text{in} \quad \text{LCA}_{\mathfrak{A}}.$$

Note that  $X_{\mathbb{R}}/\mathfrak{X}$ , as a cokernel in the category  $\text{LCA}_{\mathfrak{A}}$ , carries the quotient topology. Since  $\mathfrak{X}$  is a lattice of full rank in  $X_{\mathbb{R}}$  (see §3.1),  $T$  is topologically a compact torus of dimension  $\dim_{\mathbb{R}}(X_{\mathbb{R}})$ . We begin with the  $(3 \times 3)$ -diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{\quad} & \mathfrak{X} & \xrightarrow[\quad 1]{\quad \varphi \quad} & \mathfrak{X} \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ 0 & \xrightarrow{\quad} & X_{\mathbb{R}} & \xrightarrow[\quad 1]{\quad \varphi \otimes \mathbb{R} \quad} & X_{\mathbb{R}} \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ 0 & \xrightarrow{\quad} & T & \xrightarrow[\quad 1]{\quad \varphi \otimes \mathbb{R} \quad} & T, \end{array}$$

where we wrote  $\overline{\varphi \otimes \mathbb{R}}$  to refer to what the map  $\varphi \otimes \mathbb{R}$  induces on the torus quotient. As we see from Equation 2.8, this diagram induces the following relation.

$$\left[ \begin{array}{ccc} 0 & \hookrightarrow & \mathfrak{X} \\ & \searrow & \downarrow \varphi \\ & & \mathfrak{X} \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & \mathfrak{X} \end{array} \right] + \left[ \begin{array}{ccc} 0 & \hookrightarrow & T \\ & \searrow & \downarrow \overline{\varphi \otimes \mathbb{R}} \\ & & T \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & T \end{array} \right] = \left[ \begin{array}{ccc} 0 & \hookrightarrow & X_{\mathbb{R}} \\ & \searrow & \downarrow \varphi \otimes \mathbb{R} \\ & & X_{\mathbb{R}} \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & X_{\mathbb{R}} \end{array} \right] \tag{2.9}$$

Next, consider the  $(3 \times 3)$ -diagram

$$\begin{array}{ccccc} 0 & \hookrightarrow & \mathfrak{X} & \xrightarrow[\downarrow 1]{\downarrow \varphi} & \mathfrak{X} & \tag{2.10} \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow & \\ 0 & \hookrightarrow & \coprod_{\mathbb{N}} \mathfrak{X} & \xrightarrow[\downarrow 1]{\downarrow \varphi} & \coprod_{\mathbb{N}} \mathfrak{X} & \\ \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow & \\ 0 & \hookrightarrow & \coprod_{\mathbb{N}} \mathfrak{X} & \xrightarrow[\downarrow 1]{\downarrow \varphi} & \coprod_{\mathbb{N}} \mathfrak{X}, & \end{array}$$

where the middle and right downward exact sequence stem from the injection sending  $m$  to the sequence  $(m, 0, 0, 0, \dots)$  and the epic sending  $(m_1, m_2, \dots)$  to  $(m_2, m_3, \dots)$ . Here Equation 2.8 yields the relation

$$\left[ \begin{array}{ccc} 0 & \hookrightarrow & \mathfrak{X} \\ & \searrow & \downarrow \varphi \\ & & \mathfrak{X} \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & \mathfrak{X} \end{array} \right] = 0. \tag{2.11}$$

The analogous game can be played when replacing  $\mathfrak{X}$  and  $\coprod_{\mathbb{N}} \mathfrak{X}$  by the pair  $T$  and  $\coprod_{\mathbb{N}} T$ . We get

$$\left[ \begin{array}{ccc} 0 & \hookrightarrow & T \\ & \searrow & \downarrow \overline{\varphi \otimes \mathbb{R}} \\ & & T \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & T \end{array} \right] = 0. \tag{2.12}$$

All three relations of Equations 2.9, 2.11, 2.12 can be combined to give

$$\left[ \begin{array}{ccc} 0 & \hookrightarrow & X_{\mathbb{R}} \\ & \searrow & \downarrow \varphi \otimes \mathbb{R} \\ & & X_{\mathbb{R}} \\ & \swarrow & \uparrow 1 \\ 0 & \hookrightarrow & X_{\mathbb{R}} \end{array} \right] = 0.$$

In view of Example 2.6, this shows that if  $\varphi$  comes from an automorphism  $\varphi : \mathfrak{X} \xrightarrow{\sim} \mathfrak{X}$ , then its class in  $K_1(\text{LCA}_{\mathfrak{A}})$  and thus in  $K_0(\mathfrak{A}, \mathbb{R})$  vanishes. Of course this argument is just a variation of the Eilenberg swindle, performed in this explicit presentation, and corresponds under the bridge to the Haar measure in §2 to the fact that the Haar measure on compact resp. discrete LCA groups admits a canonical normalization.

*Example 2.8.* We note that there is nothing like Diagram 2.10 with  $A_{\mathbb{R}}$  instead of  $\mathfrak{X}$ , since both  $\coprod_{\mathbb{N}} A_{\mathbb{R}}$  as well as  $\coprod_{\mathbb{N}} A_{\mathbb{R}}$  fail to be locally compact.

**2.2. Strategy of proof.** In the special case  $\mathfrak{A} = \mathbb{Z}$  and  $A = \mathbb{Q}$ , our main result was proven by Clausen [Cla17], using  $\infty$ -categorical methods, and not in the context of the ETNC. We had already developed a generalization of Clausen’s result for maximal orders  $\mathcal{O}$  (i.e. the ring of integers) when  $A$  is a number field [Bra18], and in this paper we try to adapt the proof given *op. cit.* as far as possible. Let us list some of the obstacles where things become harder:

- The rings are now *non*-commutative and duality swaps left and right modules.
- We introduce a new method to create certain projective covers based on covering space theory, see Lemma 5.4. While this could certainly be replaced by an argument “by hand”, this is a very efficient new tool.
- We run into two separate issues when handling the compactly generated modules, see §10. We explain how to solve them there, and this fix even works for orders  $\mathfrak{A}$  of infinite global dimension. This might be important for the issue to remove the restriction to regular orders in our main theorem in the future. The principal trick is to “move” infinitely long resolutions into a piece of the category whose  $K$ -theory gets killed by an Eilenberg swindle. This way, that the resolution would not have been finite, does not need to bother us anymore.
- Most delicate: *Ibid.* all rings were Dedekind. Thus, if  $M \in \text{LCA}_{\mathcal{O}}$  was for example a torus  $\mathbb{T}^n$  topologically, then it followed that  $M$  was divisible and thus an injective  $\mathcal{O}$ -module (which is useful to split off direct summands). But over a general order  $\mathfrak{A}$ , more precisely over the non-hereditary ones, divisible modules need *not* be injective. In particular, some nice decomposition results in [Bra18] simply would be false in the setup of the present paper. This complicates all proofs which previously were relying on these decompositions.

### 3. Categorical framework

**3.1. Basics.** We recall that a module is called *semisimple* if it splits as a (possibly infinite) direct sum of simple modules. A unital associative (not necessarily commutative) ring  $A$  is called *semisimple* if one (then all) of the following properties hold: (a)  $A$  has trivial Jacobson radical and is left Artinian, (b) the category of left  $A$ -modules is split exact, (c) all left  $A$ -modules are semisimple, (d) all left  $A$ -modules are injective, (e) all left  $A$ -modules are projective, (f) its opposite algebra  $A^{op}$  is semisimple.

In particular, by (f) the conditions (b)-(e) could equivalently be demanded to hold for right  $A$ -modules.

Suppose  $A$  is a finite-dimensional semisimple  $\mathbb{Q}$ -algebra. By finite dimension we mean that  $A$  is finite-dimensional as a  $\mathbb{Q}$ -vector space. An *order*  $\mathfrak{A} \subseteq A$  is a subring of  $A$  which is a finitely generated  $\mathbb{Z}$ -module such that

$\mathbb{Q} \cdot \mathfrak{A} = A$ . More accurately, this should be called a  $\mathbb{Z}$ -order, but we will not consider any other types of orders.

As an abelian group, we have  $\mathfrak{A} \simeq \mathbb{Z}^n$ , where  $n := \dim_{\mathbb{Q}} A$ . To see this, tensor  $\mathfrak{A} \hookrightarrow A$  over  $\mathbb{Z}$  with  $\mathbb{Q}$ , giving the injection  $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow A$ , and the condition  $\mathbb{Q} \cdot \mathfrak{A} = A$  implies that this map must be surjective, and thus an isomorphism. However, as  $\mathfrak{A} \subseteq A$  forces  $\mathfrak{A}$  to be torsion-free, it can only be of the shape  $\mathbb{Z}^a$  for some  $a$  anyway, and then  $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^a$ , proving  $a = n$ .

Moreover, we see that  $\mathbb{Q} \cdot \mathfrak{A} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{A}$  and  $\mathfrak{A}$  is a full rank lattice in the finite-dimensional real vector space  $\mathbb{R} \cdot \mathfrak{A} \cong \mathbb{R} \otimes_{\mathbb{Q}} A$ . That is, the quotient  $(\mathbb{R} \cdot \mathfrak{A})/\mathfrak{A}$  is an  $n$ -dimensional real torus, topologically.

If  $R$  denotes a ring, we speak of an *algebraic  $R$ -module*  $M$  whenever we want to stress that we ignore any topological structures which are present on both  $R$  and  $M$ . Alternatively, regard both  $R$  and  $M$  as equipped with the discrete topology.

*Example 3.1.* For every finite group  $G$ , the group ring  $\mathbb{Z}[G]$  is an order in the rational group algebra  $\mathbb{Q}[G]$ .

While we will mostly be interested in  $K$ -theory, several of our results hold more generally for *localizing invariants* in the sense of Blumberg, Gepner and Tabuada, [BluGT13]. For any presentable stable  $\infty$ -category  $\mathbf{A}$  this roughly speaking amounts to a functor

$$L : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathbf{A} \tag{3.1}$$

which “has a localization sequence” as in  $K$ -theory. The precise definition is [BluGT13, Definition 8.1], and  $\text{Cat}_{\infty}^{\text{ex}}$  is also defined *loc. cit.*; we use exactly the same notation.

*Example 3.2.* Non-connective algebraic  $K$ -theory, frequently denoted by  $\mathbb{K}$  in the literature and also in this text, is a localizing invariant  $\mathbb{K} : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathbf{Sp}$ , where  $\mathbf{Sp}$  is the  $\infty$ -category of spectra.

Given an idempotent complete exact category  $\mathbf{C}$ , it comes with a canonical attached stable  $\infty$ -category. This can be done in two steps: (a) first, one forms the dg category  $\text{Ch}_{dg}^b(\mathbf{C})$  of bounded complexes in  $\mathbf{C}$  and its subcategory of acyclic complexes  $\text{Ac}_{dg}^b(\mathbf{C})$ . This is for example explained in [Büh10]. (b) Next, one forms their quotient, either as a dg quotient or already in the  $\infty$ -categorical setting. Following the latter path, by Lurie’s dg nerve construction [Lur, §1.3.1], each dg category gives rise to a stable  $\infty$ -category, [Lur, Prop. 1.3.2.10]. One then forms the cofiber in  $\text{Cat}_{\infty}^{\text{ex}}$  (or equivalently in  $\text{Cat}_{\infty}$ ),

$$D_{dg}^b(\mathbf{C}) := \text{Ch}_{dg}^b(\mathbf{C}) / \text{Ac}_{dg}^b(\mathbf{C}),$$

which is again a stable  $\infty$ -category [Lur, Prop. 1.1.4.6]. Its homotopy category is equivalent to the bounded derived category  $\mathcal{D}^b(\mathbf{C})$  of the exact category as set up in Bühler [Büh10]. This construction gives the stable  $\infty$ -category enhancement of the derived category (or a dg enhancement if one

goes the other path). This is explained in a little more detail in [AnGH19, Appendix, A.4-A.7]. If  $\mathcal{C}$  is an exact category which fails to be idempotent complete, do the same construction with the idempotent completion of  $\mathcal{C}$ . Whether one takes the quotient in the dg setting or stable  $\infty$ -setting, the key point is to quotient out the acyclic complexes, as this implements the data of the exact structure on  $\mathcal{C}$ .

**3.2. Quasi-abelian categories.** We will work a lot with exact categories. We shall use the conventions of Bühler [Büh10]. We write ‘ $\hookrightarrow$ ’ (resp. ‘ $\twoheadrightarrow$ ’) to denote an admissible monic (resp. admissible epic). The concept of a quasi-abelian category might be less well-known. It is also explained *loc. cit.*, but all we really need to know are the following facts: (a) Quasi-abelian categories  $\mathcal{C}$  are a particular type of exact category, (b) In a quasi-abelian category every morphism has a kernel and cokernel.

Unfortunately, the latter does not yet suffice to make  $\mathcal{C}$  an abelian category. By an insight of Hoffmann and Spitzweck [HS07] the category  $\text{LCA}$  of locally compact abelian (from now on:  $\text{LCA}$ ) groups is quasi-abelian. We shall recall this below in detail, but for the moment just note that if  $\mathbb{R}_\delta$  denotes the real numbers equipped with the discrete topology, then the identity  $f : \mathbb{R}_\delta \rightarrow \mathbb{R}$  is a continuous group homomorphism with zero kernel and zero cokernel, but  $f$  still fails to be an isomorphism. Such a behaviour would be impossible in an abelian category.

**3.3. Locally compact modules.** Suppose  $\mathfrak{A}$  is an order inside a semisimple  $\mathbb{Q}$ -algebra  $A$  of finite dimension. Note that it is sufficient to specify  $\mathfrak{A}$  since  $A = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{A}$  is uniquely determined by it.

**Definition 3.3.** *We define a category  $\text{LCA}_{\mathfrak{A}}$  such that*

- (1) *an object  $M$  is a right  $\mathfrak{A}$ -module along with the datum of an  $\text{LCA}$  group structure on its additive group  $(M; +)$  such that right multiplication by any  $\alpha \in \mathfrak{A}$  is a continuous endomorphism  $(M; +) \xrightarrow{\alpha} (M; +)$ ,*
- (2) *a morphism  $M \rightarrow M'$  is a continuous right  $\mathfrak{A}$ -module homomorphism.*

*We write  ${}_{\mathfrak{A}}\text{LCA}$  for the corresponding category of left modules. We shall write  $\text{LCA}$  as a shorthand for  $\text{LCA}_{\mathbb{Z}}$ .*

As an equivalent alternative description, we may view the ring  $\mathfrak{A}$  as equipped with the discrete topology and take locally compact topological right  $\mathfrak{A}$ -modules as objects for  $\text{LCA}_{\mathfrak{A}}$ .

*Remark 3.4.* Note that the opposite ring  $\mathfrak{A}^{op}$  is an order in the opposite algebra  $A^{op}$  (which is also semisimple, §3.1), so we have  ${}_{\mathfrak{A}}\text{LCA} = \text{LCA}_{\mathfrak{A}^{op}}$ . Hence, for most considerations it will be sufficient to discuss them for right modules and the left module case will be implied by switching to the opposite order in the opposite algebra.



Next, let  $\mathbb{T}$  denote the unit circle (or equivalently  $\mathbb{R}/\mathbb{Z}$ ) in the category  $\mathbf{LCA}$ . The following result naturally extends the observation of Hoffmann and Spitzweck that  $\mathbf{LCA}$  is a quasi-abelian category.

**Proposition 3.5.** *The category  $\mathbf{LCA}_{\mathfrak{A}}$  is a quasi-abelian exact category. There is an exact functor*

$$\begin{aligned} (-)^{\vee} : \mathbf{LCA}_{\mathfrak{A}}^{op} &\longrightarrow {}_{\mathfrak{A}}\mathbf{LCA} \\ M &\longmapsto \mathrm{Hom}(M, \mathbb{T}), \end{aligned}$$

where the continuous right  $\mathfrak{A}$ -module homomorphism group  $\mathrm{Hom}(M, \mathbb{T})$  is equipped with the compact-open topology (that is: on the level of the underlying  $\mathbf{LCA}$  group  $(M; +)$  this is the Pontryagin dual), and the left action

$$(\alpha \cdot \varphi)(m) := \varphi(m \cdot \alpha) \quad \text{for all } \alpha \in \mathfrak{A}, m \in M.$$

There is an analogous exact functor

$$(-)^{\vee} : {}_{\mathfrak{A}}\mathbf{LCA}^{op} \longrightarrow \mathbf{LCA}_{\mathfrak{A}}$$

such that there is a natural equivalence of functors from the identity functor to double dualization,

$$\eta : \mathrm{id} \longrightarrow (-)^{\vee} \circ [(-)^{\vee}]^{op}.$$

In less technical terms: For every object  $M \in \mathbf{LCA}_{\mathfrak{A}}$  there exists a reflexivity isomorphism  $\eta(M) : M \xrightarrow{\sim} M^{\vee\vee}$ , and the isomorphisms  $\eta(M)$  are natural in  $M$ . Under the forgetful functor  $\mathbf{LCA}_{\mathfrak{A}} \rightarrow \mathbf{LCA}$  (resp.  ${}_{\mathfrak{A}}\mathbf{LCA} \rightarrow \mathbf{LCA}$ ), both duality functors restrict to ordinary Pontryagin duality.

We warn the reader of the subtlety that  $\mathrm{Hom}(M, \mathbb{T})$  with the compact-open topology is again locally compact, but it is not true that  $\mathrm{Hom}(M, N)$  will be locally compact for other choices of  $N$ . See Moskowitz [Mos67, Theorem 4.3 (1')] for a discussion.

**Proof.** (1) The proof of Hoffmann and Spitzweck for  $\mathbf{LCA}$  carries over, [HS07, Proposition 1.2]. It shows that  $\mathbf{LCA}_{\mathfrak{A}}$  is quasi-abelian and therefore carries a natural exact structure, by a theorem of Schneiders, [Büh10, Proposition 4.4] or the discussion preceding [Sch99, Remark 1.1.11]. (2) We quickly check that for  $\alpha, \beta \in \mathfrak{A}$  and  $m \in M$  we have

$$(\beta\alpha \cdot \varphi)(m) = \varphi(m \cdot \beta\alpha) = \varphi((m \cdot \beta) \cdot \alpha) = (\alpha \cdot \varphi)(m \cdot \beta) = (\beta \cdot (\alpha \cdot \varphi))(m)$$

and that  $M^{\vee} = \mathrm{Hom}(M, \mathbb{T})$  is an algebraic left  $\mathfrak{A}$ -module follows literally from  $M$  being an algebraic right  $\mathfrak{A}$ -module. The continuity of the scalar action on  $M^{\vee}$  follows correspondingly from the one of  $M$ . For the double dualization, note that the standard reflexivity isomorphism of the underlying  $\mathbf{LCA}$  group  $\eta : \mathrm{id} \longrightarrow (-)^{\vee} \circ [(-)^{\vee}]^{op}$  has the required properties and additionally preserves the  $\mathfrak{A}$ -module action. We refer to [Mor77] for a proof of the duality statements for the underlying category  $\mathbf{LCA}$ .  $\square$

*Remark 3.6.* If  $\mathfrak{A}$  happens to be commutative, we have  $\mathfrak{A}^{op} = \mathfrak{A}$  and then we need not distinguish between left and right modules over  $\mathfrak{A}$ . Then Proposition 3.5 extends to show that  $LCA_{\mathfrak{A}}$  is an exact category with duality in the sense of [Sch10, Definition 2.1]. This remark applies for example if  $\mathfrak{A}$  is the ring of integers (or any order) in a number field. This generalizes [Bra18, Proposition 2.4].

**Lemma 3.7.** *A morphism  $f : G' \rightarrow G$  in  $LCA_{\mathfrak{A}}$  is*

- (1) *an admissible monic if and only if it is injective and a closed map,*
- (2) *an admissible epic if and only if it is surjective and an open map.*

**Proof.** This fact also carries over from LCA, see [HS07]. □

We recall a few basic concepts around topological groups.

**Definition 3.8.** *Suppose  $G \in LCA$  is an LCA group. A subset  $C \subseteq G$  is called symmetric if  $g \in C$  implies  $-g \in C$ .*

- (1) *We say  $G$  has no small subgroups if there exists a neighbourhood  $U$  of the zero element such that  $U$  does not contain any non-trivial subgroups of  $G$ . Write  $LCA_{nss}$  for the full subcategory of LCA of these groups.*
- (2) *We say  $G$  is compactly generated if there exists a symmetric compact subset  $C \subseteq G$  such that  $G = \bigcup_{n \geq 0} C^n$ . We write  $LCA_{cg}$  for the full subcategory of these groups.*
- (3) *We call  $G$  a vector group if it admits an isomorphism  $G \xrightarrow{\sim} \mathbb{R}^n$  for some  $n \in \mathbb{Z}_{\geq 0}$ . We write  $LCA_{\mathbb{R}}$  for these groups.*

The structure of these groups is discussed in Moskowitz [Mos67]. We recall that every  $G \in LCA_{cg}$  is of the shape  $\mathbb{R}^n \oplus \mathbb{Z}^m \oplus C$  for  $C$  compact and  $n, m$  finite; while every  $G \in LCA_{nss}$  is of the shape  $\mathbb{R}^n \oplus \mathbb{T}^m \oplus D$  with  $D$  discrete and again  $n, m$  finite. This is proven as Theorem 2.5 resp. Theorem 2.4 *op. cit.* respectively.

**Definition 3.9.** *Extending Definition 3.8, we now define further full subcategories  $LCA_{\mathfrak{A},cg} := LCA_{\mathfrak{A}} \cap LCA_{cg}$ , resp.  $LCA_{\mathfrak{A},nss} := LCA_{\mathfrak{A}} \cap LCA_{nss}$ .*

We shall later show that these subcategories are fully exact subcategories and in particular may be regarded as exact categories themselves (Remark 7.3). For the moment, we can only treat them as full additive subcategories.

**Lemma 3.10.** *The functor  $(-)^{\vee} : LCA_{\mathfrak{A}}^{op} \rightarrow {}_{\mathfrak{A}}LCA$*

- (1) *sends the full subcategory  $LCA_{\mathfrak{A},cg}^{op}$  to  ${}_{nss,\mathfrak{A}}LCA$  and conversely;*
- (2) *sends the full subcategory  $LCA_{\mathfrak{A},nss}^{op}$  to  ${}_{cg,\mathfrak{A}}LCA$  and conversely;*
- (3) *sends the full subcategory  $LCA_{\mathfrak{A},\mathbb{R}}^{op}$  to  ${}_{\mathbb{R},\mathfrak{A}}LCA$  and conversely;*
- (4) *sends projectives to injectives, and injectives to projectives.*

**Proof.** (1), (2), (3) depend only on qualities of the underlying LCA group, so they follow from [Mos67], Corollary 1 to Theorem 2.5 *loc. cit.* (4) follows

since going to the opposite category transitions the universal property of projectivity into the one for injectivity, and conversely.  $\square$

#### 4. Basic decompositions

**Theorem 4.1.** *Suppose  $M \in \text{LCA}_{\mathfrak{A}}$ .*

- (1) *Then there exists a clopen right  $\mathfrak{A}$ -submodule  $H$  and an exact sequence*

$$H \xrightarrow{i} M \rightarrow D$$

*in  $\text{LCA}_{\mathfrak{A}}$ , where  $H$  is compactly generated,  $D$  discrete, and  $i$  an open map.*

- (2) *Then there exists a compact right  $\mathfrak{A}$ -submodule  $C$  and an exact sequence*

$$C \hookrightarrow M \xrightarrow{q} N$$

*in  $\text{LCA}_{\mathfrak{A}}$ , where  $N$  has no small subgroups and  $q$  is an open and closed map.*

*The analogous claims for left modules hold for  $M \in {}_{\mathfrak{A}}\text{LCA}$ .*

**Proof.** (1) The first claim is proven fairly analogously to [Bra18, Lemma 2.14]. We work with right modules. If  $M = 0$ , take  $H := 0$ . Otherwise, let  $m \neq 0$  be any element of  $M$ . As discussed in §3.1,  $\mathfrak{A} \simeq \mathbb{Z}^n$  as abelian groups, so we may pick generators  $b_i$  and write  $\mathfrak{A} = \mathbb{Z}\langle b_1, \dots, b_n \rangle$ . Next, since  $M$  is locally compact, we find compact neighbourhoods of both  $0 \in M$  and  $m$ , so let  $U_0 \subseteq M$  be their union. Define

$$U_1 := \bigcup_{i=1}^n U_0 \cdot b_i$$

and the  $U_2 := U_1 \cup (-U_1)$ . We observe that (1)  $U_2$  is symmetric, (2)  $U_2$  is compact since it is a finite union of compacta, (3)  $U_2$  is open since it contains an open neighbourhood of  $0 \in M$ . Define  $H := \bigcup_{m \geq 1} U_2^m$ . Then  $H$  is compactly generated, and open since it is a union of opens. Thus,  $H$  is even clopen in  $M$ . We claim that  $H$  is a right  $\mathfrak{A}$ -module: Suppose  $h \in H$  and  $\alpha \in \mathfrak{A}$ . Then  $h \in U_2^m$  for some  $m$ , so  $h$  is a finite sum of terms of the shape  $\pm u \cdot b_i$  with  $u \in U_0$ . For each such term we have

$$(\pm u \cdot b_i) \cdot \alpha = \pm u \cdot (b_i \cdot \alpha) = \pm u \cdot \sum_{j=1}^n c_j b_j \quad \text{with} \quad c_i \in \mathbb{Z}$$

since the  $b_i$  form a  $\mathbb{Z}$ -basis of  $\mathfrak{A}$ . Unravelling each  $c_j \in \mathbb{Z}$  as a finite sum of “ $\pm 1$ ”, we note that this expression again is a finite sum of terms of the form  $\pm u \cdot b_j$ , i.e. lies in  $H$ . Thus,  $H \in \text{LCA}_{\mathfrak{A}}$  and the quotient  $D := M/H$  is discrete since  $H$  was open. The claim for left modules can be proven symmetrically. (2) We apply (1) to the Pontryagin dual, giving us an exact sequence

$$D^\vee \hookrightarrow M \rightarrow H^\vee$$

and the dual  $D^\vee$  of a discrete module is compact, giving  $C$ , and the quotient map  $q$  is open. By [Mor77, Proposition 11] it follows that  $q$  is also a closed map. Finally, the dual of a compactly generated module has no small subgroups, see Lemma 3.10.  $\square$

### 5. Vector $\mathfrak{A}$ -modules

Let  $A$  be a finite-dimensional semisimple  $\mathbb{Q}$ -algebra and  $\mathfrak{A} \subset A$  an order.

**Lemma 5.1.**  *$A$  is a left (and right) flat  $\mathfrak{A}$ -algebra.*

**Proof.** We have  $\mathbb{Q}\mathfrak{A} = A$ , so we can also write  $A = \operatorname{colim}_{\frac{1}{n}} \mathfrak{A}$ , where the colimit runs over all integers, partially ordered by divisibility. This presents  $A$  as a filtering colimit of (clearly flat!)  $\mathfrak{A}$ -algebras since  $\frac{1}{n}\mathfrak{A} \cong \mathfrak{A}$  since there cannot be any non-trivial torsion inside a  $\mathbb{Q}$ -vector space. However, a filtering colimit of flat algebras is still flat. For this, see [Lam99, (4.4), Proposition]; it just reduces to tensor products commuting with filtering colimits. Since multiplication with  $\frac{1}{n}$  is central, this argument works both for the left and the right  $\mathfrak{A}$ -module structure.  $\square$

**Lemma 5.2.** *Suppose  $M \in \operatorname{LCA}_{\mathfrak{A}, \mathbb{R}}$ . Then as an algebraic right  $\mathfrak{A}$ -module,  $M$  is injective and projective; i.e.,  $M$  is injective and projective in  $\operatorname{Mod}_{\mathfrak{A}}$ . The corresponding statement for left  $\mathfrak{A}$ -modules is also true.*

**Proof.** (Step 1) Firstly, we show injectivity for right  $\mathfrak{A}$ -modules. We define a map of right  $\mathfrak{A}$ -modules

$$\Phi : M \longrightarrow \operatorname{Hom}_A(A, M), \quad m \longmapsto (a \mapsto ma),$$

where  $\operatorname{Hom}_A(A, M)$  carries the right  $\mathfrak{A}$ -module structure given by  $(f \cdot r)(a) := f(r \cdot a)$  for  $r \in \mathfrak{A}$ . It is easy to check that  $\Phi$  respects the right  $\mathfrak{A}$ -module structure and the map is an isomorphism since  $f \mapsto f(1_A)$  is an inverse map. Now,  $M$  is an injective right  $\mathfrak{A}$ -module if and only if the functor  $N \mapsto \operatorname{Hom}_{\mathfrak{A}}(N, M)$  is exact. We express this functor in a different way by using the functorial isomorphisms

$$\operatorname{Hom}_{\mathfrak{A}}(N, M) \cong_{\Phi} \operatorname{Hom}_{\mathfrak{A}}(N, \operatorname{Hom}_A(A, M)) \cong \operatorname{Hom}_A(N \otimes_{\mathfrak{A}} A, M),$$

stemming from  $\Phi$  and the Hom-tensor adjunction. Since  $A$  is a flat left  $\mathfrak{A}$ -module (Lemma 5.1), the functor  $N \mapsto N \otimes_{\mathfrak{A}} A$  is exact. Moreover, the functor  $\operatorname{Hom}_A(-, M)$  is exact since  $A$  is semisimple, so every right  $A$ -module is injective (see §3.1). Hence, the composed functor is also exact and it follows that  $M$  is an injective algebraic right  $\mathfrak{A}$ -module. (Step 2) The argument to show injectivity for algebraic left  $\mathfrak{A}$ -modules is analogous. We only need to replace the isomorphism  $\Phi$  by

$$\Phi' : M \longrightarrow \operatorname{Hom}_A(A, M), \quad m \longmapsto (a \mapsto am)$$

and the left  $\mathfrak{A}$ -module structure  $(r \cdot f)(a) := f(a \cdot r)$ . (Step 3) Next, we show projectivity for right  $\mathfrak{A}$ -modules  $M$ . To this end, note that the Pontryagin dual  $M^\vee$  is a left  $\mathfrak{A}$ -module. As the Pontryagin dual of a vector group is

again a vector group, we get  $M^\vee \in \mathbb{R}_{\mathfrak{A}}\text{LCA}$ . By Step 2 it follows that  $M^\vee$  is an injective algebraic left  $\mathfrak{A}$ -module. We take Pontryagin duals again, transforming the universal property of injectivity into the universal property for projectivity. Hence,  $M^{\vee\vee} \cong M$  is a projective algebraic right  $\mathfrak{A}$ -module (this generalizes the corresponding argument of Moskowitz, [Mos67, Theorem 3.1]). (Step 4) Projectivity for left  $\mathfrak{A}$ -modules now follows analogously from Step 1: The dual  $M^\vee$  is a topological right  $\mathfrak{A}$ -module, injective as an algebraic right  $\mathfrak{A}$ -module by Step 1, and by double dualization  $M^{\vee\vee} \cong M$  is projective as an algebraic left  $\mathfrak{A}$ -module.  $\square$

**Lemma 5.3.** *Every object  $M \in \text{LCA}_{\mathfrak{A},cg}$  sits in a canonically determined exact sequence*

$$C \hookrightarrow M \twoheadrightarrow W$$

such that  $C$  is a compact submodule, and as an LCA group  $W$  is isomorphic to  $\mathbb{R}^n \oplus \mathbb{Z}^m$  for suitable  $n, m \in \mathbb{Z}_{>0}$ .

**Proof.** Firstly, we shall work with  $M$  viewed as an object of the category LCA alone. Being compactly generated, there exists an isomorphism  $M \simeq \mathbb{R}^n \oplus \mathbb{Z}^m \oplus C$  in LCA with  $C$  compact, [Mos67, Theorem 2.5]. For every  $\alpha \in \mathfrak{A}$  the right multiplication map is continuous, giving the solid arrows in the commutative diagram

$$\begin{array}{ccc} C & \overset{\quad}{\dashrightarrow} & C \\ \downarrow & & \downarrow \\ M & \xrightarrow{\cdot\alpha} & M \\ \downarrow & & \downarrow \\ M/C & & M/C. \end{array}$$

As the image of the upper left compactum  $C$  in the lower right quotient  $M/C$  is compact, but  $M/C \simeq \mathbb{R}^n \oplus \mathbb{Z}^m$  has no non-trivial compact subgroups, the universal property of kernels exhibits the dashed horizontal arrow. As this holds for all  $\alpha \in \mathfrak{A}$ , it follows that  $C$  is (1) closed in  $M$  since  $C$  is compact, and (2) closed under the right action of  $\mathfrak{A}$ . In particular,  $C$  is a closed submodule. This also implies that the quotient  $W := M/C$  also makes sense in  $\text{LCA}_{\mathfrak{A}}$ , with the same underlying LCA group, proving our claim.  $\square$

**Lemma 5.4.** *Suppose  $M \in \text{LCA}_{\mathfrak{A}}$ . If the underlying LCA group of  $M$  is*  
 (1) *isomorphic to  $\mathbb{R}^n \oplus \mathbb{Z}^m$ , then there exists an admissible monic  $M \hookrightarrow \tilde{M}$ , where  $\tilde{M}$  is a vector  $\mathfrak{A}$ -module and any morphism  $f$  as in*

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \tilde{M} \\ & \searrow f & \swarrow f' \\ & & V \end{array}$$

admits a lift  $f'$ . The cokernel of  $M \hookrightarrow \tilde{M}$  has underlying LCA group  $\mathbb{T}^m$ .

- (2) isomorphic to  $\mathbb{R}^n \oplus \mathbb{T}^m$ , then there exists an admissible epic  $\tilde{M} \twoheadrightarrow M$ , where  $\tilde{M}$  is a vector  $\mathfrak{A}$ -module and any morphism  $f$  as in

$$\begin{array}{ccc}
 & V & \\
 f' \swarrow & & \searrow f \\
 \tilde{M} & \xrightarrow{\quad} & M
 \end{array}$$

admits a lift  $f'$ . The kernel of  $\tilde{M} \twoheadrightarrow M$  has underlying LCA group  $\mathbb{Z}^m$ .

We prove this by using covering space theory. This might appear like overkill, but it gives a quick and clean solution to the problem.

**Proof.** We prove (2) for right  $\mathfrak{A}$ -modules: We first work in the category LCA alone. Since  $M$  is connected, locally path-connected and semi-locally simply connected,  $M$  admits a universal covering space  $\tilde{M}$ , see e.g. [GH81, (6.7) Theorem]. Take  $0 \in M$  to regard it as a pointed space. The fundamental group  $\pi_1(M, 0_M) \simeq \mathbb{Z}^m$  acts properly discontinuously on  $\tilde{M}$  via deck transformations. In particular, the covering map  $q : \tilde{M} \rightarrow M$  is a topological quotient map. Now, we shall use the lifting property of continuous maps along covering maps, [GH81, (6.1) Theorem]. Firstly, lift addition to  $\tilde{M}$  by lifting the map

$$\tilde{M} \times \tilde{M} \xrightarrow{q \times q} M \times M \xrightarrow{+} M$$

to  $\tilde{M}$ . Similarly, lift the negation map. All such lifts are unique, [GH81, (5.1) Theorem]. A standard computation confirms that this equips  $\tilde{M}$  with the structure of a topological abelian group, see for example [GH81, (6.11) Theorem]. Moreover,  $\tilde{M}$  is Hausdorff and locally compact since the covering map  $q$  is a local homeomorphism (or just use that we know that  $\tilde{M} \simeq \mathbb{R}^{n+m}$ ). It follows that  $\tilde{M} \in \text{LCA}$  and  $q$ , being surjective and open, is an admissible epic. Next, for each  $\alpha \in \mathfrak{A}$  lift the scalar action of  $M$  by lifting

$$\tilde{M} \xrightarrow{q} M \xrightarrow{\cdot \alpha} M,$$

again using [GH81, (6.1) Theorem]. Since the lifts of these maps are again continuous, we obtain  $\tilde{M} \in \text{LCA}_{\mathfrak{A}}$ , and since these were lifts of the scalar action on  $M$ , the morphism  $q$  is even a morphism in  $\text{LCA}_{\mathfrak{A}}$ . Thus, we have constructed an admissible epic  $\tilde{M} \twoheadrightarrow M$  in  $\text{LCA}_{\mathfrak{A}}$ , and now, finally, using that  $\tilde{M} \simeq \mathbb{R}^{n+m}$ , we see that  $\tilde{M}$  is indeed a vector  $\mathfrak{A}$ -module. The lifting property from  $f$  to  $f'$  again is nothing but the topological lifting, [GH81, (6.1) Theorem]. We quickly check that  $f'$  is a right  $\mathfrak{A}$ -module homomorphism: Since  $q$  is a right  $\mathfrak{A}$ -module homomorphism, we compute

$$q(f'(v) \cdot_{\tilde{M}} \alpha) = q(f'(v)) \cdot_M \alpha = f(v) \cdot_M \alpha = f(v \cdot_V \alpha) \quad \text{for all } v \in V,$$

since  $f$  is a right  $\mathfrak{A}$ -module homomorphism, but of course  $qf' = f$  also implies

$$q(f'(v \cdot_V \alpha)) = f(v \cdot_V \alpha)$$

and therefore  $f'(-) \cdot_{\tilde{M}} \alpha$  and  $f'(- \cdot_V \alpha)$  lift the same map, so by the uniqueness of lifts they must agree. But this is just the statement that  $f'$  is a module homomorphism. It is clear that a symmetric argument works for left  $\mathfrak{A}$ -modules. Now, we prove (1) for right modules by observing that the Pontryagin dual of  $\mathbb{R}^n \oplus \mathbb{Z}^m$  is  $\mathbb{R}^n \oplus \mathbb{T}^m$ , so by part (2) we get an epic  $\tilde{M} \twoheadrightarrow M^\vee$  in  $\mathfrak{A}\text{LCA}$ . Dualizing again, this becomes an admissible monic  $M \hookrightarrow \tilde{M}^\vee$ , but since the dual of a vector  $\mathfrak{A}$ -module is again a vector  $\mathfrak{A}$ -module, this proves the claim. Analogously for left modules  $M$ .  $\square$

*Remark 5.5.* The previous lemma can also be proved without using any covering spaces. Nonetheless, we would hope that this type of argument could be extended to a much wider class of topological groups. Even for the purposes of this text it would be nice to handle all groups of the shape  $G = \mathbb{R}^n \oplus \mathbb{T}^m \oplus \mathbb{Z}^d$ . The first problem is that classical covering space theory is only developed for connected base spaces. However, regarding applications to topological groups, a formalism for covering spaces of non-connected groups was developed by Taylor [Tay54] and extended by Brown and Mucuk [BroM94]. The first most drastic difference of this situation in comparison to the connected case is that the existence of a universal covering space hinges not just on local constraints (like local path-connectedness), but on a global topological obstruction, a class in  $H^3(\pi_0(G), \pi_1(G, e))$ , for  $e$  the neutral element of the group. In the case at hand, this causes no concern, this obstruction class vanishes if there is a section for the map  $G \twoheadrightarrow \pi_0(G)$ ; use [Tay54, (7.1)] or [BroS76, Theorem 3 and Proposition 4]. Such a section exists in our situation,

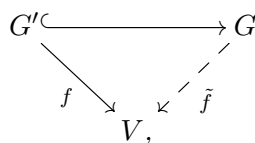
$$\mathbb{R}^n \oplus \mathbb{T}^m \oplus \mathbb{Z}^d \twoheadrightarrow \mathbb{Z}^d,$$

but as we shall see in Example 6.2 such sections cannot be expected to exist as  $\mathfrak{A}$ -module homomorphisms in the case of non-hereditary orders. Besides this issue, the lifting properties of module structures along such coverings appear not to be completely understood. We thank R. Brown for interesting correspondence.

*Remark 5.6.* Extending the previous remark, one could of course dream to work with even more general LCA groups  $G$ . Besides the complications from lack of connectedness, there are also issues around local path-connectedness. A theorem of Dixmier shows that  $G \in \text{LCA}$  is locally path-connected if and only if  $G \cong \mathbb{R}^n \oplus C \oplus D$  with  $D$  discrete and  $C$  compact and path-connected, [Arm81, (8.38)]. This restricts the consideration to compact groups, and for those Dixmier showed that a compact  $G \in \text{LCA}$  is path-connected if and only if its dual  $G^\vee$ , which is discrete, satisfies  $\text{Ext}_{\mathbb{Z}}^1(G^\vee, \mathbb{Z}) = 0$ , [Arm81, (8.25)]

Remarks, (a)]. Such abelian groups  $G^\vee$  are called ‘Whitehead groups’<sup>1</sup>. Free abelian groups clearly have this property, so  $G = \mathbb{T}^\omega$  is path-connected for all cardinals  $\omega$ . Any other path-connected example  $G$  would yield a non-free Whitehead group  $G^\vee$ . It was shown by Shelah that it is undecidable in ZFC set theory whether such groups exist or not. If we restrict to second countable LCA groups, the  $\mathbb{T}^\omega$  are the only compact groups which are path-connected. We are now remote from the subject of this paper, but we feel it might be worth to see that the assumptions of Lemma 5.4 are clearly much more restrictive than required, yet formulating a good generalization will be delicate.

**Lemma 5.7.** *Suppose  $G' \in \text{LCA}_{\mathfrak{A}}$  such that its underlying LCA group is isomorphic to  $\mathbb{R}^n \oplus \mathbb{Z}^m$  for suitable  $n, m \in \mathbb{Z}_{\geq 0}$ . Suppose  $G$  is a second such object, for possibly different values of  $n$  and  $m$ . Suppose we are given the solid arrows in the diagram*



where  $V$  is a vector  $\mathfrak{A}$ -module. Then a lift  $\tilde{f}$  exists.

**Proof.** We follow the strategy of Moskowitz [Mos67]. (Step 1) We work with right modules. Suppose  $G$  is a vector  $\mathfrak{A}$ -module. Then  $G'$  is a closed subgroup of a vector group  $\mathbb{R}^n$  on the level of LCA. It follows that there exists a basis  $b_1, \dots, b_n$  of  $G$  as a real vector space such that

$$G' = \mathbb{Z} \langle b_1, \dots, b_r \rangle \oplus \mathbb{R} \langle b_{r+1}, \dots, b_n \rangle$$

for some  $0 \leq r \leq n$ , cf. [Mor77, Theorem 6]. As  $f$  is an abelian group morphism, we have

$$f \left( \sum a_i b_i \right) = \sum a_i f(b_i)$$

for all  $a_1, \dots, a_r \in \mathbb{Z}$  and  $a_{r+1}, \dots, a_n \in \mathbb{Q}$ . By continuity, the same holds if instead  $a_{r+1}, \dots, a_n \in \mathbb{R}$ . Thus, if we define  $\tilde{f}(\sum a_i b_i) := \sum a_i f(b_i)$  for all  $a_i \in \mathbb{R}$ , this indeed extends  $f$ . Being a linear map on a finite-dimensional vector space,  $\tilde{f}$  is necessarily continuous. Moreover, it respects the  $\mathfrak{A}$ -module structure: As  $\tilde{f}$  is  $\mathbb{R}$ -linear, we can check this on the basis elements  $b_i$ , however they all lie in the subgroup  $G'$ , and so this follows since  $f$  respects the  $\mathfrak{A}$ -module structure. This proves our claim. (Step 2) Now, let  $G$  be arbitrary as in our claim. By Lemma 5.4, (1), we get

$$G' \hookrightarrow G \xrightarrow{i} \tilde{G},$$

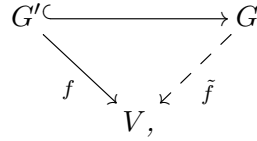
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<sup>1</sup>Of course some people just call  $K_1(R)$  the Whitehead group. What these groups have in common is that they are both abelian groups, but that’s about it.



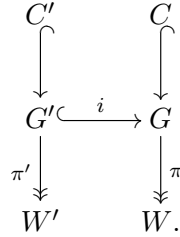
where  $\tilde{G}$  is a vector  $\mathfrak{A}$ -module. By Step 1 we can lift  $f : G' \rightarrow V$  to a map  $\tilde{G} \rightarrow V$ , and restricting to  $G$ , the map  $f' \circ i$  proves our claim. Finally, it is clear that all these steps also work for left modules.  $\square$

**Lemma 5.8.** *Suppose we are given the solid arrows in the diagram*



where  $V$  is a vector  $\mathfrak{A}$ -module and  $G', G \in \text{LCA}_{\mathfrak{A},cg}$ . Then a lift  $\tilde{f}$  exists.

**Proof.** We right away work in  $\text{LCA}_{\mathfrak{A}}$ . Apply Lemma 5.3 to both  $G'$  and  $G$ . We obtain a commutative diagram



The map from  $C'$  on the upper left to  $W$  on the lower right maps the compactum  $C'$  to a compact subgroup of  $W$ . But  $W$  is isomorphic to  $\mathbb{R}^n \oplus \mathbb{Z}^m$  as an LCA group, so this morphism must be zero. The universal property of kernels yields the lift  $i^\sharp$ , and then as a result the morphism  $j$  gets induced to the quotients:

$$\begin{array}{ccc}
 C' & \xrightarrow{i^\sharp} & C \\
 \downarrow & & \downarrow \\
 G' & \xrightarrow{i} & G \\
 \downarrow \pi' & & \downarrow \pi \\
 W' & \xrightarrow{j} & W.
 \end{array} \tag{5.1}$$

Observe that both  $\pi, \pi'$  are open morphisms (as they are admissible epics), but they are additionally closed morphisms since  $C', C$  are compact by [Mor77, Proposition 11] (beware: not every admissible epic in LCA is a closed map, see Example 5.10, neither does all open sets being clopen imply that all open maps are closed; not at all).

(Step 1) We claim that  $j$  is injective: Suppose  $w' \in W'$  and  $j(w') = 0$ . Since  $\pi'$  is surjective, we find some  $g' \in G'$  such that  $w' = \pi'(g')$  and thus  $j\pi'(g') = 0$ , i.e.  $\pi i(g') = 0$ . But  $i$  is injective, so we may just as well regard  $G'$  as a subset of  $G$ . We see that the kernel of  $\pi i$  is just  $G' \cap C$ ; it is those elements in  $C$  which come from the subset  $G'$ . However,  $C$  is

compact, so  $G' \cap C$  is a compact subgroup of  $G'$ . Its image  $\pi'(G' \cap C) \subseteq W'$  must also be compact, but since  $W'$  has no compact subgroups, we conclude that  $G' \cap C \subseteq C'$  (a different way to say this:  $C'$  is the maximal compact subgroup of  $G'$ ). Thus, our  $g'$  with  $\pi i(g') = 0$  must satisfy  $g' \in C'$ . Since  $g'$  was a preimage of  $w'$  in  $G'$ , it follows that  $w' = 0$ .

(Step 2) We claim that  $j$  is a closed map. Let  $T \subseteq W'$  be a closed set. As  $\pi'$  is continuous, the preimage

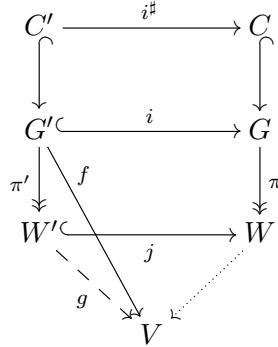
$$(\pi')^{-1}(T) = \{g' \in G' \mid \pi'(g') \in T\}$$

is closed in  $G'$ . Now, note that  $i$  is a closed map, and  $\pi$  is also a closed map (as we had recalled above), so  $(\pi \circ i)(\pi')^{-1}(T)$  is closed in  $W$ . However,  $\pi i = j\pi'$ , i.e.

$$(\pi \circ i)((\pi')^{-1}(T)) = (j \circ \pi')((\pi')^{-1}(T))$$

and  $\pi'((\pi')^{-1}(T)) = T$  (we always have ' $\subseteq$ ', and since  $\pi'$  is surjective, we even have equality). Thus, this set agrees with  $j(T)$ . As  $T$  was an arbitrary closed subset of  $W'$ , it follows that  $j$  maps closed sets to closed sets.

(Step 3) However, combining the previous steps, it follows that  $j$  is an admissible monic. We obtain the solid arrows of the diagram



and since  $V$  has no compact subgroups, but  $C'$  has compact image under  $f$ , we get the dashed lift  $g$  by the universal property of cokernels. The morphism  $j$  has all the properties required to apply Lemma 5.7, so we get the lift depicted by a dotted arrow. Composing with  $\pi$ , this yields the required lift and finishes the proof.  $\square$

*Remark 5.9.* At the beginning of the proof we have shown that  $C'$  maps to zero in  $W$ , so  $C' \subseteq C$  and trivially  $C' \subseteq G' \cap C$ , and in Step 1 we saw that  $G' \cap C \subseteq C'$ . It follows that the upper square in Diagram 5.1 is a pullback diagram. It remains a pullback in the category of abelian groups and then applying the Snake Lemma yields a different proof that  $j$  is injective. We cannot use the Snake Lemma right away in LCA itself, since at this step of the proof we do not yet know that  $j$  is an admissible morphism, compare [Büh10, Corollary 8.13].

*Example 5.10.* The hyperbola  $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$  is closed, but gets mapped to the open set  $\mathbb{R} \setminus \{0\}$  under the projection to (either) factor. These projection maps are therefore open, but not closed.

We recall the following elementary facts from topology.

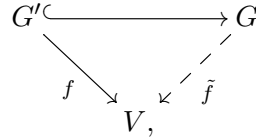
**Lemma 5.11.** *If  $f : G' \rightarrow G$  is an abelian group homomorphism of topological groups (not necessarily continuous), then  $f$  is continuous if and only if it is continuous in a neighbourhood of the neutral element of  $G'$ .*

**Lemma 5.12.** *Suppose  $G$  is a topological group. If  $H$  is an open subgroup and  $E$  an arbitrary subgroup, then  $H + E$  is an open subgroup.*

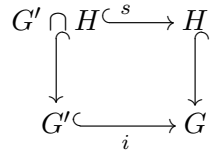
**Proof.** Just add an open neighbourhood of zero inside  $H$  to an arbitrary element of  $H + E$  to see this.  $\square$

**Theorem 5.13.** *Vector  $\mathfrak{A}$ -modules are injective and projective objects in  $\text{LCA}_{\mathfrak{A}}$ , as well as in  ${}_{\mathfrak{A}}\text{LCA}$ .*

**Proof.** We demonstrate injectivity in  $\text{LCA}_{\mathfrak{A}}$ . This means we have to show the following: Suppose we are given the solid arrows in the diagram



where  $V$  is a vector  $\mathfrak{A}$ -module and  $G', G \in \text{LCA}_{\mathfrak{A}}$ . Then a lift  $\tilde{f}$  exists. We use essentially the same reduction as Moskowitz [Mos67, Proposition 3.4], but in order to provide a complete proof, we repeat the argument: By Theorem 4.1 we find a clopen compactly generated  $H \hookrightarrow G$  with  $H \in \text{LCA}_{\mathfrak{A}}$ . We obtain the commutative diagram



and  $s$  is an admissible monic. Since  $H$  is compactly generated, so is its closed subgroup  $G' \cap H$  by [Mos67, Theorem 2.6]. We lift the restriction  $f|_{G' \cap H} : G' \cap H \rightarrow V$  to  $H$  using Lemma 5.8, call this  $f'$ . Thus, we have constructed a morphism  $(f, f') : G' \oplus H \rightarrow V$ . Since  $f$  and  $f'$  agree on  $G' \cap H$ , we see that  $(f, f')$  restricts to the zero map on this intersection, so by the universal property of cokernels applied to the exact sequence

$$G' \cap H \xrightarrow{\Delta} G' \oplus H \xrightarrow{+} G' + H,$$

where  $\Delta x := (x, -x)$ , the map descends to a morphism  $f'' : G' + H \rightarrow V$ . We note that on  $G'$  this map agrees with  $f$ , so  $f''$  is a partial lift. Next, since  $H$  is open in  $G$ , the sum  $G' + H$  is also open in  $G$  (Lemma 5.12). In

particular, it is a closed submodule, so we arrive at the solid arrows in a diagram

$$\begin{array}{ccccc}
 G' + H & \hookrightarrow & G & \twoheadrightarrow & D \\
 & \searrow f'' & \downarrow & & \\
 & & V & & 
 \end{array}$$

where  $D$  is discrete since  $G' + H$  was additionally open. Since  $V$  is injective as an algebraic right  $\mathfrak{A}$ -module by Lemma 5.2, we get the dashed arrow, as an algebraic  $\mathfrak{A}$ -module morphism. We need to check that it is continuous. However, for set-theoretic maps of topological groups it suffices to check continuity in a neighbourhood of the neutral element (Lemma 5.11) and since  $G' + H$  is open, it suffices to check it on  $G' + H$ , but there  $f$  agrees with  $f''$ , which we know to be continuous. Thus, the dashed arrow defines a morphism in  $\text{LCA}_{\mathfrak{A}}$ . The argument for left modules is analogous. Finally, projectivity follows by Pontryagin duality, see Lemma 3.10.  $\square$

### 6. Splitting off vector summands

**Lemma 6.1.** *Let  $M \in \text{LCA}_{\mathfrak{A}}$  be given. Suppose for the underlying LCA group there exists a direct sum decomposition*

$$M \cong V \oplus D \quad \text{in} \quad \text{LCA}$$

for  $V, D \in \text{LCA}$ , where  $D$  is discrete.

- (1) *If  $V$  is a vector group, then it additionally has a canonical structure as a topological right vector  $\mathfrak{A}$ -module. Hence,  $V \in \text{LCA}_{\mathfrak{A}}$ , and moreover there exists a direct sum decomposition*

$$M \cong V \oplus D \tag{6.1}$$

*also in  $\text{LCA}_{\mathfrak{A}}$ , where  $D \in \text{LCA}_{\mathfrak{A}}$  carries the discrete topology.*

- (2) *If  $\mathfrak{A}$  is a hereditary order and  $V \simeq \mathbb{R}^n \oplus \mathbb{T}^m$ , then  $V$  additionally has a canonical structure as an object in  $\text{LCA}_{\mathfrak{A}}$ . Moreover, the same direct sum decomposition of Equation 6.1 holds.*

This is the analogue of [Bra18, Lemma 2.11]. Note that we only obtain a weaker result if  $\mathfrak{A}$  fails to be hereditary. Indeed, the impossibility to split off torus summands in  $\text{LCA}_{\mathfrak{A}}$  is a key reason why the theory is more complicated than in [Bra18].

**Proof.** (Claim 1) We note that  $V \subseteq M$  can alternatively be described as the connected component of the neutral element since  $V$  is connected and  $D$  discrete. As the action of  $\mathfrak{A}$  is by continuous maps, the image of a connected set is connected, so it necessarily maps  $V$  into itself. It follows that  $V$  is an algebraic right  $\mathfrak{A}$ -submodule. Being a subgroup, the action of  $\mathfrak{A}$  is clearly still continuous, so it is also a topological right  $\mathfrak{A}$ -module, i.e.  $V \in \text{LCA}_{\mathfrak{A}}$ . In  $\text{LCA}$ , we know that the quotient  $M/V$  is discrete, so  $V$  is open (it is the

preimage of the open  $\{0\}$  under the quotient map) in  $M$  and thus clopen. Hence, we get a subobject

$$V \hookrightarrow M$$

in  $\text{LCA}_{\mathfrak{A}}$  and the inclusion is an admissible monic. From Lemma 5.2 we learn that  $V$  is algebraically an injective right  $\mathfrak{A}$ -module. Thus, as algebraic  $\mathfrak{A}$ -module maps, the injection

$$V \hookrightarrow M \quad (\text{in } \text{Mod}_{\mathfrak{A}})$$

splits and we get a direct sum decomposition  $M \simeq V \oplus D'$  for some complement  $D'$  in  $\text{Mod}_{\mathfrak{A}}$ . Quotienting out  $V$ , we see that  $D'$  is algebraically isomorphic to  $D$ , but since both carry the discrete topology, this is equivalent to them being isomorphic in  $\text{LCA}_{\mathfrak{A}}$ . Similarly, the algebraic right  $\mathfrak{A}$ -module section  $D' \hookrightarrow M$  is tautologically continuous as the preimage of any open is trivially open in the discrete topology. Thus, we also get a section in  $\text{LCA}_{\mathfrak{A}}$ , i.e. we have

$$M \simeq V \oplus D$$

in  $\text{LCA}_{\mathfrak{A}}$ . This proves the first claim.

(Claim 2) We adapt the proof of Claim 1. Again,  $V$  can be uniquely characterized as the connected component. The same argument shows  $V \in \text{LCA}_{\mathfrak{A}}$  and again we obtain that  $V \hookrightarrow M$  is a subobject in  $\text{LCA}_{\mathfrak{A}}$ . Next, apply Lemma 5.4 to  $V$ . We obtain a canonical admissible epic

$$q : \tilde{V} \twoheadrightarrow V,$$

where  $\tilde{V}$  is a vector  $\mathfrak{A}$ -module. By Lemma 5.2 we deduce that  $\tilde{V}$  is algebraically an injective  $\mathfrak{A}$ -module. However, being an epic in  $\text{LCA}_{\mathfrak{A}}$ ,  $q$  is also an epic in  $\text{Mod}_{\mathfrak{A}}$ . Since  $\mathfrak{A}$  is hereditary, all quotients of an injective module are again injective, cf. [Lam99, (3.22) Theorem]. Thus,  $V$  is also injective as an algebraic right module. Now, proceed as in the proof of Claim 1.  $\square$

*Example 6.2.* Let us demonstrate that the assumption of the order  $\mathfrak{A}$  being hereditary is truly necessary. Let  $G \simeq \mathbb{Z}/n$  be a cyclic group. We work with right  $\mathbb{Z}[G]$ -modules, where  $\mathbb{Z}[G] \simeq \mathbb{Z}[t]/(t^n - 1)$  is the group ring. We recall that the trivial  $G$ -module  $\mathbb{Z}$  has the standard projective resolution

$$\left[ \cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \right]_{-\infty,0} \xrightarrow{\Sigma} \mathbb{Z}, \quad (6.2)$$

where  $N := 1 + t + \cdots + t^{n-1}$  is the norm operator of  $G$  and  $\Sigma$  sends each  $g \in G$  to 1. Write  $N_G := \text{im}(N) \subseteq \mathbb{Z}[G]$  for the norm ideal, and  $I_G := \text{im}(1-t) \subseteq \mathbb{Z}[G]$  for the augmentation ideal. We have the short exact sequence

$$N_G \hookrightarrow \mathbb{R}[G] \twoheadrightarrow \mathbb{R}[G]/N_G$$

inducing the long exact sequence

$$\text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, \mathbb{R}[G]) \rightarrow \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, \mathbb{R}[G]/N_G) \xrightarrow{\partial} \text{Ext}_{\mathbb{Z}[G]}^{i+1}(\mathbb{Z}, N_G) \rightarrow \text{Ext}_{\mathbb{Z}[G]}^{i+1}(\mathbb{Z}, \mathbb{R}[G])$$

and the two outer terms vanish since  $\mathbb{R}[G]$  is an injective module (Lemma 5.2). Thus, we obtain an isomorphism

$$\text{Ext}_{\mathbb{Z}[G]}^1(\mathbb{Z}, \mathbb{R}[G]/N_G) \xrightarrow[\cong]{\partial} \text{Ext}_{\mathbb{Z}[G]}^2(\mathbb{Z}, N_G).$$

Next, observe that  $\mathbb{Z} \xrightarrow{\sim} N_G, 1 \mapsto N$ , is an isomorphism of  $\mathbb{Z}[G]$ -modules. Hence,  $\text{Ext}_{\mathbb{Z}[G]}^2(\mathbb{Z}, N_G) \cong H^2(G, \mathbb{Z})$  and the latter is known to be isomorphic to  $\mathbb{Z}/n$  (this computation can easily be done using the resolution in Equation 6.2). Instead of unwinding  $\partial$ , let us describe the  $\text{Ext}^1$ -group on the left explicitly. Using  $\text{Ext}_{\mathbb{Z}[G]}^1(\mathbb{Z}, \mathbb{R}[G]/N_G) \cong \mathbf{R}\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{R}[G]/N_G[1])$ , and the standard resolution, we learn that giving an element of  $\text{Ext}^1$  is equivalent to giving a morphism of complexes

$$\begin{array}{ccccccc} \cdots & \xrightarrow{N} & \mathbb{Z}[G] & \xrightarrow{1-t} & \mathbb{Z}[G] & \xrightarrow{N} & \mathbb{Z}[G] & \xrightarrow{1-t} & \mathbb{Z}[G] \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{R}[G]/N_G & \longrightarrow & 0 \end{array}$$

and since a  $\mathbb{Z}[G]$ -module morphism is defined on  $\mathbb{Z}[G]$  itself by assigning a value to  $1_G$ , the possible maps  $\alpha$  can only be multiplication  $1_G \mapsto 1_G \cdot \alpha_0$  with  $\alpha_0 \in \mathbb{Z}/n\mathbb{Z}$ . These maps are precisely those corresponding under  $\partial$  to  $H^2(G, \mathbb{Z}) \cong \mathbb{Z}/n$ . Next, let us concretely construct the extension corresponding to the element of the  $\text{Ext}^1$ -group. To this end, we need to form the pushout  $M_\alpha$  of

$$\begin{array}{ccccc} \mathbb{Z}[G] & \xrightarrow{1-t} & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{R}[G]/N_G & \longrightarrow & M_\alpha & \longrightarrow & \mathbb{Z}. \end{array}$$

Concretely, that is

$$M_\alpha := \frac{\{(a, b) \mid a \in \mathbb{Z}[G], b \in \mathbb{R}[G]/N_G\}}{\text{all pairs } (x(1-t), \bar{x}\alpha) \text{ with } x \in \mathbb{Z}[G]}.$$

We then get a short exact sequence

$$\mathbb{R}[G]/N_G \hookrightarrow M_\alpha \twoheadrightarrow \mathbb{Z}$$

in  $\text{LCA}_{\mathbb{Z}[G]}$ ; the maps are  $b \mapsto (0, b)$  and  $(a, b) \mapsto \Sigma a$ . On the level of the underlying LCA groups it looks like  $\mathbb{R}^{n-1} \oplus \mathbb{T} \hookrightarrow M_\alpha \twoheadrightarrow \mathbb{Z}$  in  $\text{LCA}$ . If this sequence splits in  $\text{LCA}_{\mathbb{Z}[G]}$ , then so it does in  $\text{Mod}_{\mathbb{Z}[G]}$ . However, unless  $\alpha = [0]$  in  $H^2(G, \mathbb{Z})$ , this is impossible.

*Example 6.3.* The same computation can be adapted to many other finite groups. For example, for  $G := D_{2n}$  the dihedral group of  $2n$  elements,  $H^2(D_{2n}, \mathbb{Z}) \cong \mathbb{Z}/2$  also gives rise to an analogous example.

**Theorem 6.4.** *Suppose  $\mathfrak{A}$  is a hereditary order.*

- (1) Every  $M \in \text{LCA}_{\mathfrak{A},nss}$  admits an isomorphism  $M \cong V \oplus T \oplus D$ , where  $V$  is a vector right  $\mathfrak{A}$ -module,  $T$  a right  $\mathfrak{A}$ -module and finite-dimensional real torus, and  $D$  a discrete right  $\mathfrak{A}$ -module.
- (2) Every  $M \in \text{LCA}_{\mathfrak{A},cg}$  admits an isomorphism  $M \cong V \oplus G \oplus C$ , where  $V$  is a vector right  $\mathfrak{A}$ -module,  $G$  a right  $\mathfrak{A}$ -module and finite rank free  $\mathbb{Z}$ -module, and  $C$  a compact right  $\mathfrak{A}$ -module.

The analogous claims hold for left modules. For  $\mathfrak{A}$  the maximal order of a Dedekind domain this is due to Levin [Lev73]. For non-hereditary orders we cannot expect such direct sum decompositions to exist.

**Proof.** (1) We work with right modules. The proof of [Bra18, Proposition 2.14] can be adapted with the following change: As  $M$  has no small subgroups, its underlying LCA group is isomorphic to  $\mathbb{R}^n \oplus \mathbb{T}^m \oplus D$  for  $D$  some discrete group. Now by Lemma 6.1 (2) this can be strengthened to a direct sum decomposition  $M \cong E \oplus D$  in  $\text{LCA}_{\mathfrak{A}}$  with  $D \in \text{LCA}_{\mathfrak{A}}$  discrete and  $E \in \text{LCA}_{\mathfrak{A}}$  having underlying LCA group  $\mathbb{R}^n \oplus \mathbb{T}^m$  (as witnessed by Example 6.2 this step required  $\mathfrak{A}$  to be hereditary). Now,  $E^\vee \in {}_{\mathfrak{A}}\text{LCA}$  has underlying LCA group  $\mathbb{R}^n \oplus \mathbb{Z}^m$ . Use the left module version of Lemma 6.1 to promote this to  $E^\vee \cong V \oplus G$  in  ${}_{\mathfrak{A}}\text{LCA}$  for  $V$  a left vector module and  $G$  a discrete left module with underlying group  $\mathbb{Z}^n$ . Dualizing again, we get  $E \cong V^\vee \oplus G^\vee$  in  $\text{LCA}_{\mathfrak{A}}$ , and  $G^\vee$  is a torus as required. (2) This argument is Pontryagin dual, using that duality exchanges compactly generated modules with those without small subgroups, Lemma 3.10. □

Combining all these results, we obtain some further flexibility with regards to decompositions in the style of Theorem 4.1, even without having to assume that  $\mathfrak{A}$  be hereditary.

**Lemma 6.5.** *Suppose  $M \in \text{LCA}_{\mathfrak{A}}$ . Then there exist exact sequences*

$$C \hookrightarrow M \twoheadrightarrow V \oplus D \quad \text{and} \quad V' \oplus C' \hookrightarrow M \twoheadrightarrow D' \tag{6.3}$$

with  $C, C'$  compact,  $V, V'$  vector  $\mathfrak{A}$ -modules and  $D, D'$  discrete.

**Proof.** We construct the sequence on the left first: By Theorem 4.1 (1) there exists an exact sequence  $H \hookrightarrow M \twoheadrightarrow D'$  with  $H \in \text{LCA}_{\mathfrak{A},cg}$  and  $D'$  discrete. Further, by Lemma 5.3 there is an exact sequence  $C \hookrightarrow H \twoheadrightarrow W$  with  $C$  compact and  $W \in \text{LCA}_{\mathfrak{A}}$  having underlying LCA group isomorphic to  $\mathbb{R}^n \oplus \mathbb{Z}^m$  for suitable  $n, m$ . This yields a filtration  $C \hookrightarrow H \hookrightarrow M$  and Noether’s Lemma for exact categories ([Büh10, Lemma 3.5]) tells us that

$$H/C \hookrightarrow M/C \twoheadrightarrow M/H$$

is exact in  $\text{LCA}_{\mathfrak{A}}$ . Unravelling these quotients, we learn that

$$W \hookrightarrow M/C \twoheadrightarrow D'$$

is exact. Now, by Lemma 6.1 (1) the module  $W$  splits as a direct sum  $W \cong V \oplus \tilde{D}$  with  $V$  a vector  $\mathfrak{A}$ -module and  $\tilde{D}$  discrete in  $\text{LCA}_{\mathfrak{A}}$ . Thus, by the above exact sequence,  $V$  is a subobject of  $M/C$ . However, by Theorem

5.13 vector modules are injective objects, so  $V$  is a direct summand of  $M/C$ . The complementary summand, call it  $J$ , then is an extension  $\tilde{D} \hookrightarrow J \twoheadrightarrow D'$  of discrete modules, so  $J$  is itself discrete; and  $M/C \cong V \oplus J$ . Combining these results, we arrive at the exact sequence  $C \hookrightarrow M \twoheadrightarrow V \oplus J$  with  $J$  discrete. Letting  $D := J$  proves our first claim. All arguments so far work for both left and right  $\mathfrak{A}$ -modules. Thus, we obtain the second exact sequence, i.e. Equation 6.3 on the right, by using the first exact sequence for the left module  $M^\vee$  and dualizing back.  $\square$

We record an elementary fact:

**Lemma 6.6.** *A module  $M \in \text{Mod}_{\mathfrak{A}}$  is finitely generated over  $\mathfrak{A}$  if and only if it is finitely generated as a  $\mathbb{Z}$ -module.*

**Lemma 6.7.** *Let  $\mathfrak{A}$  be an arbitrary order in  $A$ .*

- (1) *Every discrete module  $M \in \text{LCA}_{\mathfrak{A}}$  has a discrete projective cover  $P$  in  $\text{LCA}_{\mathfrak{A}}$ . If  $M$  is compactly generated,  $P$  is also compactly generated.*
- (2) *Every compact module  $M \in \text{LCA}_{\mathfrak{A}}$  has a compact injective hull  $I$  in  $\text{LCA}_{\mathfrak{A}}$ . If  $M$  has no small subgroups,  $I$  also has no small subgroups.*

**Proof.** (1) The category  $\text{Mod}_{\mathfrak{A}}$  has enough projectives, so we find a projective cover  $P \twoheadrightarrow M$ . Equipping  $P$  with the discrete topology, the epic is tautologically continuous and even an admissible epic in  $\text{LCA}_{\mathfrak{A}}$  because in the case of the discrete topology there is nothing to check topologically. For the second statement, note that a discrete module is compactly generated if and only if its underlying additive group is finitely generated by [Mos67, Theorem 2.5], so  $M$  is a finitely generated  $\mathfrak{A}$ -module by Lemma 6.6, and thus its projective cover is also finitely generated over  $\mathfrak{A}$ , and using the converse of Lemma 6.6, and [Mos67, Theorem 2.5] we deduce that  $P$  is compactly generated. (2) Pontryagin dual to (1), using that duality swaps compactly generated modules with those without small subgroups, Lemma 3.10.  $\square$

**Corollary 6.8.** *Let  $\mathfrak{A}$  be an arbitrary order in  $A$ .*

- (1) *Every discrete module  $M \in \text{LCA}_{\mathfrak{A}}$  has a projective resolution by discrete modules in  $\text{LCA}_{\mathfrak{A}}$  of length at most the projective dimension of  $M$  in  $\text{Mod}_{\mathfrak{A}}$ .*
- (2) *Every compact module  $M \in \text{LCA}_{\mathfrak{A}}$  has an injective resolution by compact modules in  $\text{LCA}_{\mathfrak{A}}$  of length at most the projective dimension of  $M^\vee$  in  ${}_{\mathfrak{A}}\text{Mod}$ .*

**Proof.** Inductive usage of Lemma 6.7.  $\square$

*Example 6.9.* For any finite group  $G \neq 1$ , the global dimension of  $\mathbb{Z}[G]$  is infinite. To see this, use that  $G$  contains a non-trivial cyclic subgroup  $C \simeq \mathbb{Z}/m$  with  $m \geq 2$ . The group cohomology of a cyclic group is periodic in the strictly positive degree range (compare Example 6.2, where we recall this) with  $H^{2n}(C, \mathbb{Z}) \cong \mathbb{Z}/m$  for all  $n \geq 1$ . Thus,

$$\text{Ext}_{\mathbb{Z}[G]}^{2n}(\mathbb{Z}, \text{Ind}_C^G(\mathbb{Z})) = H^{2n}(G, \text{Ind}_C^G(\mathbb{Z})) \cong H^{2n}(C, \mathbb{Z}) \cong \mathbb{Z}/m \neq 0.$$



The following result complements Corollary 6.8 regarding resolutions of opposite nature.

**Corollary 6.10.** *Let  $\mathfrak{A}$  be an arbitrary order in  $A$ .*

- (1) *Suppose  $T \in \text{LCA}_{\mathfrak{A}}$  has underlying LCA group  $\mathbb{T}^n$  for some  $n \in \mathbb{Z}_{\geq 0}$ . Then it admits a (possibly infinitely long) projective resolution in  $\text{LCA}_{\mathfrak{A}}$  starting in a vector  $\mathfrak{A}$ -module  $V$ ,*

$$\dots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow V \twoheadrightarrow T$$

*and if  $\mathfrak{A}$  has finite global dimension  $s$ , then the length of this resolution is at most  $s + 1$ .*

- (2) *Suppose  $G \in \text{LCA}_{\mathfrak{A}}$  has underlying LCA group  $\mathbb{Z}^n$  for some  $n \in \mathbb{Z}_{\geq 0}$ . Then it admits a (possibly infinitely long) injective resolution in  $\text{LCA}_{\mathfrak{A}}$  starting in a vector  $\mathfrak{A}$ -module  $V$ ,*

$$G \hookrightarrow V \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots,$$

*and if  $\mathfrak{A}$  has finite global dimension  $s$ , then the length of this resolution is at most  $s + 1$ .*

**Proof.** (1) By Lemma 5.4 (2) there exists an exact sequence  $D \hookrightarrow P \twoheadrightarrow T$  with  $D$  discrete. By Lemma 6.7 discrete modules permit projective covers in  $\text{LCA}_{\mathfrak{A}}$ . Inductively, this produces a projective resolution of  $T$ , the initial step being a vector module (which is projective by Theorem 5.13) and all further terms discrete. In particular, since the projective resolution of  $D$  can be carried out in  $\text{Mod}_{\mathfrak{A}}$ , and then just be imported to  $\text{LCA}_{\mathfrak{A}}$  by equipping every term with the discrete topology, the claim about the projective dimension follows. (2) Pontryagin dual to (1). □

*Example 6.11.* If  $T$  has the torus  $\mathbb{T}^\omega$  for some strictly infinite cardinal  $\omega$  as its underlying LCA group, then  $\mathbb{T}^\omega$  has small subgroups (since its Pontryagin dual is not compactly generated), and Moskowitz proves that  $T$  does not even admit a projective resolution only in LCA, [Mos67, Theorem 3.6 (2)]. Hence, the finite-dimensionality is crucial in Corollary 6.10. Similarly, if  $D$  is any discrete module in  $\text{LCA}_{\mathfrak{A}}$ , then if we take an injective resolution of  $D$  in  $\text{Mod}_{\mathfrak{A}}$ , say  $D \hookrightarrow I^\bullet$  and view this as an exact complex in  $\text{LCA}_{\mathfrak{A}}$ , each  $I^n$  equipped with the discrete topology, then in general there is no hope that the modules  $I^n$  are injective objects also in the category  $\text{LCA}_{\mathfrak{A}}$ .

We remind the reader that all global dimensions, including  $+\infty$ , can occur, see Remark 10.4.

### 7. Structure theorems

Below, we shall make frequent use of Schlichting’s concepts of left and right  $s$ -filtering categories, see [Bra18, Definition 3.1] or [BGW16, Appendix A].

**Proposition 7.1.** *The full subcategory  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is left  $s$ -filtering.*

**Proof.** Firstly, it is left filtering. This amounts to the purely topological fact that the closure of a compactly generated LCA group inside a locally compact group is again compactly generated. The proof of [Bra18, Lemma 3.2] works verbatim. It remains to prove that the inclusion  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is left special. If  $G'' \hookrightarrow G \xrightarrow{q} G'$  is an exact sequence with  $G' \in \text{LCA}_{\mathfrak{A},cg}$ , apply Lemma 6.5 to  $G$ . We arrive at the solid arrows in the following commutative diagram

$$\begin{array}{ccccc}
 C & & & & \\
 \downarrow i & \searrow h & & \searrow 0 & \\
 G & \xrightarrow{q} & G' & \xrightarrow{r} & G'/\text{im}(h) \\
 \downarrow & & & \nearrow w & \\
 V \oplus D & & & & 
 \end{array} \tag{7.1}$$

We explain how to set this up: (1) The map  $h$  is defined as the composition  $q \circ i$ . (2) Since  $C$  is compact, its set-theoretic image  $\text{im}_{\text{set}}(h)$  is also compact, and thus in particular closed in  $G'$ . As a result, the set-theoretic image agrees with the image in the sense of the category  $\text{LCA}_{\mathfrak{A}}$  (this is a subtle issue with terminology, see [Bra18, §2, Notation]), and moreover the quotient  $G'/\text{im}(h)$  exists in  $\text{LCA}_{\mathfrak{A}}$ . (3) Proceeding along the resulting quotient map in the diagram above, the dashed arrow  $w$  exists by the universal property of cokernels. Since  $\text{LCA}_{\mathfrak{A}}$  is quasi-abelian, it is idempotent complete and thus by [Büh10, Proposition 7.6] the morphism  $w$  must be an admissible epic. Let us try to characterize the underlying LCA group of  $X := G'/\text{im}(h)$ . As  $G'$  is compactly generated, so are all its quotients by [Mos67, Theorem 2.6]. Thus, as an LCA group, we have  $X \simeq \mathbb{R}^A \oplus \mathbb{Z}^B \oplus C$  for suitable  $A, B \in \mathbb{Z}_{\geq 0}$  and  $C$  compact. On the other hand,  $V \oplus D$  has the underlying LCA group  $\mathbb{R}^n \oplus D$  with  $D$  discrete and by [Mor77, Corollary 2 to Theorem 7] all closed subgroups can be classified, and as a result all admissible quotients can only have the underlying LCA group  $\mathbb{R}^N \oplus \mathbb{T}^M \oplus D'$  for suitable  $N, M \in \mathbb{Z}_{\geq 0}$  and  $D'$  discrete. Having these two contrasting structural results, we learn that  $X$  can only have  $\mathbb{R}^N \oplus \mathbb{T}^M \oplus \mathbb{Z}^B$  as its underlying LCA group for suitable  $N, M, B \in \mathbb{Z}_{\geq 0}$  (a very clean way to see this is to use the canonical filtration of [HS07, Proposition 2.2] in LCA since it canonically identifies the three direct summands as graded pieces of the filtration). Apply Lemma 5.3 to  $X$ , giving us an exact sequence

$$T \hookrightarrow X \twoheadrightarrow W \quad \text{in} \quad \text{LCA}_{\mathfrak{A}}$$

such that the underlying LCA groups of  $T$  are  $\mathbb{T}^M$ , and  $\mathbb{R}^N \oplus \mathbb{Z}^B$  for  $W$ . Apply Lemma 6.1 (1) to  $W$  to promote this exact sequence to

$$T \hookrightarrow X \twoheadrightarrow \tilde{V} \oplus \tilde{D} \quad \text{in} \quad \text{LCA}_{\mathfrak{A}}$$

with  $\tilde{V}$  a vector  $\mathfrak{A}$ -module and  $\tilde{D}$  having underlying LCA group  $\mathbb{Z}^B$ . We pick admissible epics from projective objects  $P_i \in \text{LCA}_{\mathfrak{A}}$  to these objects as follows: (1)  $c_1 : P_1 \twoheadrightarrow T$  can be produced by Lemma 5.4 (2), in this case  $P_1$  is a vector  $\mathfrak{A}$ -module; (2)  $c_2 : P_2 \twoheadrightarrow \tilde{V}$  is just the identity since by Theorem 5.13 the object  $\tilde{V}$  is itself already projective, and (3)  $c_3 : P_3 \twoheadrightarrow \tilde{D}$  is taken as a projective cover, which exists by Lemma 6.7; in this case  $P_3$  is a compactly generated discrete  $\mathfrak{A}$ -module. Since morphisms from projectives can be lifted along admissible epics, we obtain the lift  $\tilde{c}_{23}$  of

$$\begin{array}{ccc}
 P_1 & & P_2 \oplus P_3 \\
 c_1 \downarrow & \searrow & \downarrow c_2+c_3 \\
 T & \twoheadrightarrow X & \xleftarrow{\tilde{c}_{23}} \tilde{V} \oplus \tilde{D}
 \end{array} \tag{7.2}$$

Recalling that  $X = G'/\text{im}(h)$ , take this morphism  $P_1 \oplus P_2 \oplus P_3 \rightarrow G'/\text{im}(h)$ , which arises as the sum  $c_1 + \tilde{c}_{23}$ , and lift it also along the admissible epic  $r$  in Diagram 7.1. Call this lift  $\hat{c}$ . Then we arrive at a commutative diagram

$$\begin{array}{ccc}
 C \oplus P_1 \oplus P_2 \oplus P_3 & & \\
 i+\hat{c}' \downarrow & \searrow h+\hat{c} & \\
 G & \xrightarrow{q} & G'
 \end{array}$$

where  $\hat{c}'$  is a further lift of the morphism  $\hat{c}$  along the admissible epic  $q$ . Let us study the morphism  $h + \hat{c}$ : Firstly, it is a morphism in  $\text{LCA}_{\mathfrak{A}}$ . Next, tracing through the construction, it is surjective: all elements in  $\text{im}(h) \subseteq G'$  are of course surjected on by  $C$ , and  $P_1 \oplus P_2 \oplus P_3$  were constructed just in such a way to surject onto the quotient  $X = G'/\text{im}(h)$ ; see Diagram 7.2. Finally, since  $C$  is compact,  $P_1, P_2$  are vector  $\mathfrak{A}$ -modules, and  $P_3$  is compactly generated,  $C \oplus P_1 \oplus P_2 \oplus P_3$  is compactly generated in total. In particular, it is  $\sigma$ -compact (i.e. a countable union of compact sets), and thus by Pontryagin’s Open Mapping Theorem,  $h + \hat{c}$  is an open map, see [Mor77, Theorem 3]. Being surjective and open,  $h + \hat{c}$  is an admissible epic in  $\text{LCA}_{\mathfrak{A}}$ , Lemma 3.7. Thus, we can set up the following commutative diagram

$$\begin{array}{ccccc}
 K & \longrightarrow & C \oplus P_1 \oplus P_2 \oplus P_3 & \xrightarrow{h+\hat{c}} & G' \\
 \downarrow & & \downarrow & & \parallel \\
 G'' & \longrightarrow & G & \xrightarrow{q} & G'
 \end{array}$$

where  $K := \ker(h + \hat{c})$ , and the dashed arrow stems from the universal property of kernels. Since  $C \oplus P_1 \oplus P_2 \oplus P_3$  is compactly generated, so is its subobject  $K$  by [Mos67, Theorem 2.6 (2)], and thus the entire top row is an exact sequence with objects in  $\text{LCA}_{\mathfrak{A},cg}$ . This confirms that  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is left special.  $\square$

**Corollary 7.2.** *For an exact sequence  $G' \hookrightarrow G \twoheadrightarrow G''$  of objects in  $\text{LCA}_{\mathfrak{A}}$  we have  $G \in \text{LCA}_{\mathfrak{A},cg}$  if and only if  $G', G'' \in \text{LCA}_{\mathfrak{A},cg}$ .*

**Proof.** [BGW16, Proposition A.2 and Definition A.1 (1)]. □

*Remark 7.3.* In particular,  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is closed under extensions and thus carries a natural exact structure itself, [Büh10, Lemma 10.20].

*Remark 7.4.* By duality (Proposition 3.5) and the exchange properties of Lemma 3.10, Proposition 7.1 literally yields that

$${}_{nss, \mathfrak{A}}\text{LCA}^{op} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}^{op}$$

is left  $s$ -filtering. However, by the symmetry of left and right  $s$ -filtering, this just means that  ${}_{nss, \mathfrak{A}}\text{LCA} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}$  is right  $s$ -filtering. Applying this observation to the opposite order  $\mathfrak{A}$ , we get that  $\text{LCA}_{\mathfrak{A},nss} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is also right  $s$ -filtering. Using duality on this statement and Lemma 3.10 once more, we learn that  ${}_{cg, \mathfrak{A}}\text{LCA}^{op} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}^{op}$  is right  $s$ -filtering, and thus  ${}_{cg, \mathfrak{A}}\text{LCA} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}$  is left  $s$ -filtering. In particular, Corollary 7.2 generalizes to  $\text{LCA}_{\mathfrak{A},nss}$ , and all full subcategories

$\text{LCA}_{\mathfrak{A},nss} \hookrightarrow \text{LCA}_{\mathfrak{A}}, \text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}, {}_{cg, \mathfrak{A}}\text{LCA} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}, {}_{nss, \mathfrak{A}}\text{LCA} \hookrightarrow {}_{\mathfrak{A}}\text{LCA}$  are fully exact subcategories and hence carry the structure of an exact category themselves.

### 8. Injectives and projectives

We determine the projective objects in  $\text{LCA}_{\mathfrak{A}}$ . To this end, we follow the strategy of Moskowitz in [Mos67], who deals with the case of  $\text{LCA}$  itself.

**Proposition 8.1.** *Let  $\mathfrak{A}$  be an arbitrary order.*

- (1) *Every injective object in  $\text{LCA}_{\mathfrak{A}}$  is isomorphic to  $V \oplus I$ , where  $V$  is a vector  $\mathfrak{A}$ -module and  $I$  compact connected such that  $I^{\vee}$  is projective as an algebraic left  $\mathfrak{A}$ -module.*
- (2) *Every projective object in  $\text{LCA}_{\mathfrak{A}}$  is isomorphic to  $V \oplus P$ , where  $V$  is a vector  $\mathfrak{A}$ -module and  $P$  discrete such that  $P$  is projective as an algebraic right  $\mathfrak{A}$ -module.*

*Conversely, all objects of this shape are injective resp. projective.*

**Proof.** (1) Suppose  $G \in \text{LCA}_{\mathfrak{A}}$  is injective. Firstly, we claim that  $G$  must be injective in  $\text{Mod}_{\mathfrak{A}}$  as well. To see this, let  $M' \hookrightarrow M \twoheadrightarrow M''$  be any exact sequence in  $\text{Mod}_{\mathfrak{A}}$ . Equipped with the discrete topology, this is exact in  $\text{LCA}_{\mathfrak{A}}$ . Let  $f : M \rightarrow G$  be any morphism in  $\text{Mod}_{\mathfrak{A}}$ . As  $M$  carries the discrete topology,  $f$  is actually continuous, tautologically. Since  $G$  is injective in  $\text{LCA}_{\mathfrak{A}}$ , there exists a lift  $f'$  as in

$$\begin{array}{ccc}
 M' & \hookrightarrow & M \\
 & \searrow f & \swarrow f' \\
 & & G
 \end{array}$$

and if we forget the topology, the underlying algebraic morphism of  $f'$  is of course also a lift of  $f$  in  $\text{Mod}_{\mathfrak{A}}$ . Thus,  $G$  is injective in  $\text{Mod}_{\mathfrak{A}}$ . Next, we claim that  $G$  must be connected. To this end, consider the exact sequence  $\mathfrak{A} \hookrightarrow A \otimes_{\mathbb{Q}} \mathbb{R} \twoheadrightarrow T$ , where the middle term is viewed with the standard real topology, and  $T$  is plainly defined as the corresponding quotient. For any element  $x \in G$  we define a morphism  $f : \mathfrak{A} \rightarrow G$  by  $f(\alpha) := x \cdot \alpha$ . As  $G$  is injective, we can lift  $f$  along

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\quad} & A \otimes_{\mathbb{Q}} \mathbb{R} \\
 \searrow f & & \nearrow \tilde{f} \\
 & G &
 \end{array} \tag{8.1}$$

and (following Moskowitz’s idea) note that since  $A \otimes_{\mathbb{Q}} \mathbb{R}$  is connected (it has underlying LCA group  $\mathbb{R}^d$  for  $d := \dim_{\mathbb{Q}} A$ ), the image of the continuous map  $\tilde{f}$  must also be connected. However,  $\tilde{f}(1_{\mathfrak{A}}) = f(1_{\mathfrak{A}}) = x$  was arbitrary, so we deduce that all elements of  $G$  lie in the connected component of the neutral element. Thus,  $G$  is connected. Next, apply Lemma 6.5 to  $G$ , giving an exact sequence

$$V \oplus C \hookrightarrow G \xrightarrow{q} D \quad \text{in} \quad \text{LCA}_{\mathfrak{A}}$$

with  $V$  a vector  $\mathfrak{A}$ -module,  $C$  compact and  $D$  discrete. As  $G$  is connected, its image under the quotient map  $q$  is also connected, but  $D$  is discrete, so we must have  $D = 0$  as  $q$  is also surjective. Next, let  $C^0$  denote the connected component of  $C$ ; at first this only makes sense in LCA. However, for every  $\alpha \in \mathfrak{A}$  we get the solid arrows of the commutative diagram (of LCA groups)

$$\begin{array}{ccc}
 C^0 & \dashrightarrow & C^0 \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{\cdot \alpha} & C \\
 \downarrow & & \downarrow \\
 C/C^0 & & C/C^0.
 \end{array}$$

The composition of maps from the connected  $C^0$  on the upper left to the totally disconnected  $C/C^0$  on the lower right is necessarily zero by continuity. Thus, the dashed arrow exists by the universal property of kernels. As this holds for all  $\alpha \in \mathfrak{A}$ , it follows that the closed subgroup  $C^0$  actually defines a subobject of  $C$  in  $\text{LCA}_{\mathfrak{A}}$ , and the quotient  $C/C^0$  makes sense in  $\text{LCA}_{\mathfrak{A}}$  accordingly. Thus, we obtain a new exact sequence

$$V \oplus C^0 \hookrightarrow V \oplus \underset{=G}{C} \xrightarrow{q'} C/C^0.$$

Again, since  $G$  is connected, but  $C/C^0$  totally disconnected, we must have  $C/C^0 = 0$  because  $q'$  is surjective. It follows that  $G \cong V \oplus C^0$ , where

$V$  is a vector  $\mathfrak{A}$ -module. Recall that if a product of objects is injective, then so are its factors. Hence,  $C^0$  is compact and injective in  $\text{LCA}_{\mathfrak{A}}$ . By Pontryagin duality, it follows that  $C^{0\vee}$  is a discrete projective left  $\mathfrak{A}$ -module. This proves (1) for right modules. Note that the same argument works for left modules: just change that  $f$  needs to be defined as  $f(\alpha) := \alpha \cdot x$  in the context of Diagram 8.1. Finally, (2) follows by Pontryagin duality, noting that it exchanges injectives with projectives, compact with discrete, and duals of vector  $\mathfrak{A}$ -modules remain vector  $\mathfrak{A}$ -modules. (3) We conclude by showing that all such objects are indeed injective resp. projective. For the vector  $\mathfrak{A}$ -module summand, this is just Theorem 5.13. Next, if  $P$  is projective in  $\text{Mod}_{\mathfrak{A}}$  and discrete, then any lift as algebraic module maps is continuous, because any map originating from a discrete space is continuous. For injectives, argue by duality.  $\square$

### 9. Isolating the real part

We recall from [Bra18] that  $\text{LCA}_{\mathbb{R}C}$  is our short-hand for  $\text{LCA}_{\mathbb{Z},\mathbb{R}C}$ , i.e. the category of LCA groups which admit an isomorphism to  $\mathbb{R}^n \oplus C$  for some  $n$  and some compact  $C$ . As was shown *loc. cit.* this is an idempotent complete exact category, [Bra18, Lemma 3.10]. These results turn out to generalize to topological  $\mathfrak{A}$ -modules. Let us quickly set this up to the extent which we shall need later:

**Definition 9.1.** *Let  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  denote the full subcategory of  $\text{LCA}_{\mathfrak{A}}$  whose objects have underlying LCA group in  $\text{LCA}_{\mathbb{R}C}$ .*

So far, this only equips  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  with the structure of an additive category. Better though, let us show that the definition agrees with another ostensibly more restrictive definition:

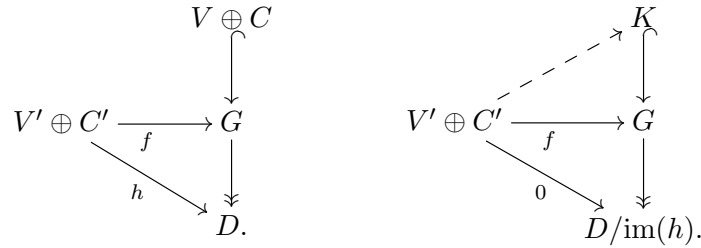
**Lemma 9.2.** *Equivalently,  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  can be defined as the full subcategory of  $\text{LCA}_{\mathfrak{A}}$  whose objects admit a direct sum decomposition  $G \cong V \oplus C$  with  $V$  a vector  $\mathfrak{A}$ -module and  $C$  compact in  $\text{LCA}_{\mathfrak{A}}$ .*

**Proof.** Suppose  $G \in \text{LCA}_{\mathfrak{A}}$  is as in Definition 9.1. The dual  $G^\vee \in {}_{\mathfrak{A}}\text{LCA}$  has underlying LCA group of the shape  $\mathbb{R}^n \oplus \tilde{D}$  for  $\tilde{D}$  discrete, so Lemma 6.1 (1) applies, allowing us to decompose  $G^\vee$  in  ${}_{\mathfrak{A}}\text{LCA}$  as a direct sum of a vector left  $\mathfrak{A}$ -module and a discrete module, and dualizing back, we get an isomorphism  $G \cong V \oplus C$  with  $V$  a vector module and  $C$  compact in  $\text{LCA}_{\mathfrak{A}}$ . The converse is clear.  $\square$

**Proposition 9.3.** *Let  $\mathfrak{A}$  be an arbitrary order.*

- (1) *The full subcategory  $\text{LCA}_{\mathfrak{A},\mathbb{R}C} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is closed under extensions.*
- (2) *The full subcategory  $\text{LCA}_{\mathfrak{A},\mathbb{R}C} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is left filtering; indeed filtering by a clopen subobject.*
- (3) *The category  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is idempotent complete.*

**Proof.** This can be proven largely as in [Bra18], just by replacing some ingredients by their analogues in our setting. We give a sketch: (1) By our definition of  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ , this is only a condition on the underlying LCA groups. In particular, it follows from  $\text{LCA}_{\mathbb{R}C}$  being closed under extensions in  $\text{LCA}$ , which is [Bra18, Lemma 3.8] for  $\mathcal{O} = \mathbb{Z}$ . (2) Suppose  $f : G' \rightarrow G$  is any morphism with  $G' \in \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ . Apply Lemma 9.2 to  $G'$ . Next, applying the right hand side sequence of Lemma 6.5 to  $G$ , we then arrive at the diagram below on the left:



As  $h$  is continuous, it necessarily maps the connected  $V'$  to zero in the discrete  $D$ , and maps the compact  $C'$  to a compact subgroup of  $D$ , which is necessarily finite. Now, we obtain from this the diagram above on the right, where  $K$  is simply defined as the kernel. As  $D/\text{im}(h)$  is discrete,  $K$  is open. The dashed lift exists by the universal property of kernels, and moreover  $K$  can be unraveled to arise as the extension  $V \oplus C \hookrightarrow K \twoheadrightarrow \text{im}(h)$ . Since  $\text{im}(h)$  is finite discrete, it is trivially compact, and thus by part (1) of our claim,  $K \in \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  and the dashed arrow is the required factorization. For more details, see the proof of [Bra18, Lemma 3.9], which uses the same pattern. (3) Just as [Bra18, Lemma 3.10] this follows from that in part (2) we actually showed that any morphism from  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  to an object  $G \in \text{LCA}_{\mathfrak{A}}$  factors over a clopen subobject  $K \in \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ , i.e. one with discrete quotient. The argument *loc. cit.* then also works in our generality.  $\square$

**Corollary 9.4.**  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  is an idempotent complete fully exact subcategory of  $\text{LCA}_{\mathfrak{A}}$ .

**Proof.** By Proposition 9.3 (1) it is an extension-closed subcategory of  $\text{LCA}_{\mathfrak{A}}$ , so by [Büh10, Lemma 10.20] this induces a canonical exact structure on  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ , and moreover renders it a fully exact subcategory [Büh10, Definition 10.21]. Proposition 9.3 (3) settles being idempotent complete.  $\square$

From now on, we may therefore regard  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  as an exact category.

**Definition 9.5.** Let  $\text{LCA}_{\mathfrak{A}, \mathbb{R}D}$  denote the full subcategory of  $\text{LCA}_{\mathfrak{A}}$  whose objects have underlying LCA group  $\mathbb{R}^n \oplus D$  for some  $n$  and some discrete  $D$ .

We will only work with this category in passing later (namely in the formulation of Lemma 10.2), but note that under duality it gets sent to  ${}_{\mathbb{R}C, \mathfrak{A}}\text{LCA}^{op}$ , so the above results can all be transported to  $\text{LCA}_{\mathfrak{A}, \mathbb{R}D}$  by dualization, in the

spirit of Remark 7.4. In particular, it is an idempotent complete fully exact subcategory of  $\text{LCA}_{\mathfrak{A}}$ , and the full subcategory  $\text{LCA}_{\mathfrak{A},\mathbb{R}C} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is right filtering. We have no use for these facts however, so we leave the details to the reader.

**Definition 9.6.** *Let  $\text{LCA}_{\mathfrak{A},C}$  (resp.  $\text{LCA}_{\mathfrak{A},D}$ ) be the full subcategory of  $\text{LCA}_{\mathfrak{A}}$  of compact (resp. discrete) objects.*

All these considerations carry over to left modules with the obvious modifications. Clearly  $\text{LCA}_{\mathfrak{A},D} \xrightarrow{\sim} \text{Mod}_{\mathfrak{A}}$  is an equivalence of categories, by simply forgetting the topology and conversely equipping all objects with the discrete topology. In particular,  $\text{LCA}_{\mathfrak{A},D}$  is an abelian category. Pontryagin duality induces an equivalence  $\text{LCA}_{\mathfrak{A},C}^{op} \xrightarrow{\sim} {}_{D,\mathfrak{A}}\text{LCA}$ , so the compact counterpart is also naturally an abelian category.

*Example 9.7.* The category  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is not abelian. Indeed, already in  $\text{LCA}_{\mathbb{R}C}$  the morphism  $\mathbb{R} \rightarrow \mathbb{T}$  has no kernel. It is an admissible epic in  $\text{LCA}$ , but not an admissible epic in  $\text{LCA}_{\mathbb{R}C}$ .

**Proposition 9.8.** *The subcategory  $\text{LCA}_{\mathfrak{A},C} \hookrightarrow \text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is left  $s$ -filtering.*

**Proof.** This is proven exactly as in [Bra18, Lemma 3.11]. Left filtering: Given  $V \oplus C \in \text{LCA}_{\mathfrak{A},\mathbb{R}C}$  with  $V$  a vector module and  $C$  compact and  $a : C' \rightarrow V \oplus C$  is any morphism with  $C'$  compact, then the set-theoretic image  $\text{im}_{\text{Set}}(a)$  is again a compact right  $\mathfrak{A}$ -module. Since  $V$  has no non-trivial compact subgroups, we obtain  $\text{im}_{\text{Set}}(a) \subseteq C$  and therefore the exact sequence

$$\text{im}_{\text{Set}}(a) \hookrightarrow V \oplus C \twoheadrightarrow V \oplus C / \text{im}_{\text{Set}}$$

in  $\text{LCA}_{\mathfrak{A}}$ . As all objects in this sequence lie in  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$ , it follows that it is also exact in  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  since  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is fully exact in  $\text{LCA}_{\mathfrak{A}}$ . Thus, the initial arrow  $\text{im}_{\text{Set}}(a) \hookrightarrow V \oplus C$  is an admissible monic in the category  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$ . Thus,

$$C \longrightarrow \text{im}_{\text{Set}}(a) \hookrightarrow V \oplus C$$

is the required factorization which shows that  $\text{LCA}_{\mathfrak{A},\mathbb{R}C} \hookrightarrow \text{LCA}_{\mathfrak{A}}$  is left filtering. Left special: Briefly, if  $V \oplus C \twoheadrightarrow C'$  is an admissible epic with  $C' \in \text{LCA}_{\mathfrak{A},C}$ , then we obtain a commutative diagram

$$\begin{array}{ccccc}
 C & & & & \\
 \downarrow & \searrow & & \searrow & \\
 & & 0 & & \\
 V \oplus C & \xrightarrow{h} & C' & \twoheadrightarrow & C'/\text{im}(h) \\
 \downarrow & \searrow & \twoheadrightarrow & & \\
 & & f & & \\
 V & \dashrightarrow & & & \\
 & & & & b
 \end{array}$$

and since  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is idempotent complete, [Büh10, Proposition 7.6] implies that  $b$  must be an admissible epic. A topological consideration (relying



only on the underlying LCA groups) can now be carried out, exactly as in [Bra18, Lemma 3.11], contrasting that  $V \simeq \mathbb{R}^n$  as an LCA group, but  $C'/\text{im}(h)$  being compact. The idea is that  $C'/\text{im}(h)$  can only be some torus, but this again is only possible if the kernel of  $b$  contains some summand of the shape  $\mathbb{Z}^i$  with  $i > 0$ , but this is impossible in  $\text{LCA}_{\mathbb{R}C}$ . As in *loc. cit.*, we deduce that  $h$  must have been an admissible epic to start with. This proves left special.  $\square$

**Proposition 9.9.** *Let  $\mathfrak{A}$  be an order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Define  $A_{\mathbb{R}} := A \otimes_{\mathbb{Z}} \mathbb{R}$ . For every localizing invariant  $K : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathcal{A}$  (as in Equation 3.1), there is a canonical equivalence  $K(\text{LCA}_{\mathfrak{A}, \mathbb{R}C}) \xrightarrow{\sim} K(A_{\mathbb{R}})$ .*

**Proof.** Again, we follow the pattern of [Bra18]. The steps are as follows: (1) Since  $\text{LCA}_{\mathfrak{A}, C} \hookrightarrow \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  is left  $s$ -filtering by Proposition 9.8, we get an exact sequence of exact categories

$$\text{LCA}_{\mathfrak{A}, C} \hookrightarrow \text{LCA}_{\mathfrak{A}, \mathbb{R}C} \rightarrow \text{LCA}_{\mathfrak{A}, \mathbb{R}C} / \text{LCA}_{\mathfrak{A}, C}.$$

We recall that the quotient exact category  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C} / \text{LCA}_{\mathfrak{A}, C}$  arises as the localization  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}[\Sigma_e^{-1}]$ , where  $\Sigma_e$  is the collection of admissible epics with kernel in the subcategory  $\text{LCA}_{\mathfrak{A}, C}$ . Following Schlichting [Sch04] this is again an exact category.

(2) We claim that there is an exact equivalence of exact categories

$$\Psi : \text{Mod}_{A_{\mathbb{R}}, fg} \longrightarrow \text{LCA}_{\mathfrak{A}, \mathbb{R}C} / \text{LCA}_{\mathfrak{A}, C}.$$

We define the functor  $\Psi$  by sending a finitely generated  $A_{\mathbb{R}}$ -module to its underlying real vector space with its natural real vector space topology. To this end, note that  $A$  is a finite-dimensional  $\mathbb{Q}$ -algebra, so  $A_{\mathbb{R}}$  is a finite-dimensional  $\mathbb{R}$ -algebra, and hence all finitely generated modules over it are also finite-dimensional over the reals. Any such vector space has a canonical real topology and it is of course locally compact. Moreover, every  $\mathbb{R}$ -linear map is automatically continuous with respect to this topology. This shows that  $\Psi$  is a functor. It is now easy to check that it is additive and moreover exact. Next, we claim that  $\Psi$  is fully faithful. This is also easy: The essential image in  $\text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  (i.e. without quotienting out  $\text{LCA}_{\mathfrak{A}, C}$ ) consists only of vector modules, so any continuous morphism between them is (a) an abelian group map, (b) by divisibility then also a  $\mathbb{Q}$ -vector space map, and (c) since it is also continuous, a density argument shows that it is even an  $\mathbb{R}$ -vector space map. Thus, the functor  $\text{Mod}_{A_{\mathbb{R}}, fg} \rightarrow \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$  is fully faithful. Quotienting out  $\text{LCA}_{\mathfrak{A}, C}$  does not harm this as we invert admissible epics with kernel in  $\text{LCA}_{\mathfrak{A}, C}$ , but a real vector space has no non-trivial compact subgroups. Hence,  $\Psi$  is also fully faithful. Finally, we claim that  $\Psi$  is essentially surjective: Given any  $G \in \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ , Lemma 9.2 yields the exact sequence,  $C \hookrightarrow G \xrightarrow{q} V$ , and after inverting  $\Sigma_e$ ,  $q$  becomes an isomorphism, and  $V$  lies in the image of  $\Psi$ , so  $G$  lies in the essential image. Being exact, fully faithful and essentially surjective,  $\Psi$  is an exact equivalence of exact

categories.

(3) As  $K$  is localizing, Schlichting’s Localization Theorem (in the formulation of [Bra18, Theorem 4.1]) yields a fiber sequence in  $\mathbf{A}$  of the shape

$$K(\mathrm{LCA}_{\mathfrak{A},C}) \longrightarrow K(\mathrm{LCA}_{\mathfrak{A},\mathbb{R}C}) \xrightarrow{a} K(\mathrm{Mod}_{A_{\mathbb{R}},fg}),$$

where we have used Step (2) to identify the third term. On the other hand, by the Eilenberg swindle,  $K(\mathrm{LCA}_{\mathfrak{A},C}) = 0$ , cf. [Bra18, Lemma 4.2], since infinite products of compact spaces are compact by Tychonoff’s Theorem. Thus,  $a$  must be an equivalence in  $\mathbf{A}$ . Finally, since  $A_{\mathbb{R}}$  is a semisimple algebra, every right  $A_{\mathbb{R}}$ -module is projective (see §3.1). Hence,  $K(\mathrm{Mod}_{A_{\mathbb{R}},fg}) \xrightarrow{\sim} K(A_{\mathbb{R}})$ . This finishes the proof.  $\square$

### 10. The compactly generated piece

**Proposition 10.1.** *For every localizing invariant  $K : \mathrm{Cat}_{\infty}^{\mathrm{ex}} \rightarrow \mathbf{A}$ , there is a canonical equivalence  $K(\mathrm{LCA}_{\mathfrak{A},\mathbb{R}C}) \xrightarrow{\sim} K(\mathrm{LCA}_{\mathfrak{A},cg})$ .*

This generalizes a corresponding result for  $\mathfrak{A} := \mathcal{O}$  the maximal order in a number field, which appears as the first step in the proof of [Bra18, Lemma 4.3].

**Lemma 10.2.** *Suppose  $\mathfrak{A}$  is an arbitrary order.*

- (1) *Suppose  $G \in \mathrm{LCA}_{\mathfrak{A},nss}$ . Then there exists an exact sequence*

$$Y_2 \hookrightarrow Y_1 \twoheadrightarrow G$$

*with  $Y_1, Y_2 \in \mathrm{LCA}_{\mathfrak{A},\mathbb{R}D}$ .*

- (2) *Suppose*

$$X \hookrightarrow Y \twoheadrightarrow Y'$$

*is an exact sequence in  $\mathrm{LCA}_{\mathfrak{A},nss}$  with  $Y, Y' \in \mathrm{LCA}_{\mathfrak{A},\mathbb{R}D}$ . Then it follows that  $X \in \mathrm{LCA}_{\mathfrak{A},\mathbb{R}D}$ .*

Note that since in (1) all terms of the sequence lie in  $\mathrm{LCA}_{\mathfrak{A},nss}$ , it makes no difference whether we speak of the sequence being exact in the category  $\mathrm{LCA}_{\mathfrak{A}}$  or in  $\mathrm{LCA}_{\mathfrak{A},nss}$ ; both concepts agree since  $\mathrm{LCA}_{\mathfrak{A},nss}$  is fully exact in  $\mathrm{LCA}_{\mathfrak{A}}$ .

**Proof.** (Step 1) We begin by addressing the first claim. Since  $G \in \mathrm{LCA}_{\mathfrak{A},nss}$ , its dual is compactly generated, i.e.  $G^{\vee} \in {}_{cg,\mathfrak{A}}\mathrm{LCA}$ , Lemma 3.10. By Lemma 5.3 there is an exact sequence  $C \hookrightarrow G^{\vee} \twoheadrightarrow W$  in  ${}_{\mathfrak{A}}\mathrm{LCA}$  with  $C$  compact and  $W$  having underlying LCA group  $\mathbb{R}^n \oplus \mathbb{Z}^m$ . Dualizing back, we get an exact sequence  $H \hookrightarrow G \twoheadrightarrow D$  with  $H$  having underlying LCA group  $\mathbb{R}^n \oplus \mathbb{T}^m$  and  $D$  discrete. We will set up a commutative diagram

$$\begin{array}{ccc} V & & P \\ q \downarrow & & \swarrow f' \quad \searrow f \\ H \hookrightarrow G & \xrightarrow{j} & G \xrightarrow{w} D \end{array} \tag{10.1}$$

as follows: (1) Since  $D$  is discrete, we may pick a projective cover  $P$ , which is still discrete, by Lemma 6.7, so  $P \in \text{LCA}_{\mathfrak{A}, \mathbb{R}D}$ . Being projective, the lift  $f'$  along the admissible epic  $w$  exists. (2) As  $H$  has underlying LCA group  $\mathbb{R}^n \oplus \mathbb{T}^m$ , by Lemma 5.4 (2) there exists a vector  $\mathfrak{A}$ -module  $V$  mapping to  $H$  through the admissible epic  $q$ . This sets up the diagram.

From this, we obtain a morphism  $h := f' + jq$ ,

$$V \oplus P \xrightarrow{h} G. \tag{10.2}$$

(Step 2) A simple diagram chase shows that  $h$  is a surjective map. Since  $f'$  and  $jq$  are continuous  $\mathfrak{A}$ -module homomorphisms, so is their sum. Next, we claim that  $h$  is an open map. The underlying topological space of  $V \oplus P$  is just the product  $V \times P$  in  $\text{Top}$ , so the Cartesian opens  $\{U_1 \times U_2\}_{U_1, U_2}$  with  $U_1 \subseteq V$  open and  $U_2 \subseteq P$  open form a basis of the topology. Hence, if  $U \subseteq V \times P$  is an arbitrary open, we may write it as

$$U = \bigcup_{i \in \mathcal{I}} U_{1,i} \times U_{2,i} \quad \text{for suitable opens} \quad U_{1,i} \subseteq V, U_{2,i} \subseteq P,$$

and  $\mathcal{I}$  some index set. Then

$$h(U) = h\left(\bigcup_{i \in \mathcal{I}} U_{1,i} \times U_{2,i}\right) = \bigcup_{i \in \mathcal{I}} h(U_{1,i} \times U_{2,i}) = \bigcup_{i \in \mathcal{I}} (jq(U_{1,i}) + f'(U_{2,i})).$$

Now, the map  $j$  is open (since the quotient  $D$  by  $j$  in Diagram 10.1 is discrete, see [Mor77, Proposition 14]) and  $q$  is an admissible epic and therefore also open. Thus,  $jq$  is an open map, and thus  $jq(U_{1,i})$  is an open set. It follows from Lemma 5.12 that all sets  $jq(U_{1,i}) + f'(U_{2,i})$  are open, and thus  $h(U)$ , being a union of open sets. As  $U$  was arbitrary, it follows that  $h$  is an open map. As  $h = f' + jq$  is open and surjective, it is an admissible epic. Hence, we can promote Equation 10.2 to an exact sequence

$$K \hookrightarrow V \oplus P \xrightarrow{h} G$$

in  $\text{LCA}_{\mathfrak{A}}$ , where  $K$  is plainly defined as the kernel, so  $K \in \text{LCA}_{\mathfrak{A}}$ . As the underlying LCA group of  $V \oplus P$  is  $\mathbb{R}^\ell \oplus D'$  for some  $\ell \in \mathbb{Z}_{\geq 0}$  and  $D'$  discrete (i.e.  $V \oplus P \in \text{LCA}_{\mathfrak{A}, \mathbb{R}D}$ ), the closed subgroup  $K$  must also lie in  $\text{LCA}_{\mathfrak{A}, \mathbb{R}D}$  by [Mor77], Corollary 2 to Theorem 7, combined with the following Remark *loc. cit.* to exclude the possibility of a torus factor. Hence, letting  $Y_2 := K$  and  $Y_1 := V \oplus P$  proves our claim. Note that the same argument which showed  $K \in \text{LCA}_{\mathfrak{A}, \mathbb{R}D}$  also settles our second claim.  $\square$

We actually need the dual formulation:

**Lemma 10.3.** *Suppose  $\mathfrak{A}$  is an arbitrary order.*

(1) *Suppose  $G \in \text{LCA}_{\mathfrak{A}, cg}$ . Then there exists an exact sequence*

$$G \hookrightarrow Y^1 \rightarrow Y^2$$

*with  $Y^1, Y^2 \in \text{LCA}_{\mathfrak{A}, \mathbb{R}C}$ .*

(2) *Suppose*

$$Y' \hookrightarrow Y \twoheadrightarrow X$$

is an exact sequence in  $\text{LCA}_{\mathfrak{A},cg}$  with  $Y, Y' \in \text{LCA}_{\mathfrak{A},\mathbb{R}C}$ . Then it follows that  $X \in \text{LCA}_{\mathfrak{A},\mathbb{R}C}$ . In other words: The full subcategory  $\text{LCA}_{\mathfrak{A},\mathbb{R}C}$  is closed under cokernels in  $\text{LCA}_{\mathfrak{A},cg}$ .

**Proof.** For  $G \in \text{LCA}_{\mathfrak{A},cg}$  we have  $G^\vee \in \text{LCA}_{\mathfrak{A}^{op},nss}$ . Apply Lemma 10.2 to  $G^\vee$ , then dualize back and use Lemma 3.10.  $\square$

*Elaboration 1.* The above proof is quick, but of course one can also prove this directly by dualizing the proof of Lemma 10.2. The topological argument is a little different in this case, and in some ways more involved. We sketch it: Firstly, given  $G \in \text{LCA}_{\mathfrak{A},cg}$ , by Lemma 5.3 there is an exact sequence  $C \hookrightarrow G \twoheadrightarrow W$  in  $\text{LCA}_{\mathfrak{A}}$  with compact and  $W$  having underlying LCA group of the shape  $\mathbb{R}^n \oplus \mathbb{Z}^m$ . By Lemma 6.1 (1) one may isolate a vector  $\mathfrak{A}$ -module summand  $V$  in  $W$ , and being projective (Theorem 5.13), this quotient object of  $G$  splits off as a direct summand. Thus, we may write  $G \cong G' \oplus V$ . This decomposes  $G'$  as  $C \hookrightarrow G' \twoheadrightarrow W'$  with  $W'$  having underlying group  $\mathbb{Z}^m$ . By Lemma 6.7 the compact  $C$  has a compact injective hull  $I$ , and by Lemma 5.4 there exists an admissible monic  $W' \hookrightarrow V'$  with  $V'$  a vector module. We obtain the diagram

$$\begin{array}{ccccc} C & \hookrightarrow & G' & \xrightarrow{q} & W' \\ \downarrow i & \swarrow e & & & \downarrow f \\ I & & & & V' \end{array}$$

analogous to Diagram 10.1. This time we wish to prove that the sum  $h := e + fq : G' \rightarrow I \oplus V'$  is a closed map. To this end, note that, as a topological space,  $G'$  is the disjoint union  $G' = \coprod_{w \in W'} (C + \tilde{w})$ , where  $\tilde{w}$  denotes an arbitrary (but fixed) lift of  $w \in W'$  to  $G'$ . Thus, any closed subset  $T \subseteq G'$  has the shape  $T = \coprod_w (T_w + \tilde{w})$  with  $T_w \subseteq C$  closed. Hence,

$$\begin{aligned} h(T) &= \bigcup_{w \in W'} h(T_w + \tilde{w}) = \bigcup_w e(T_w) + e(\tilde{w}) + fq(T_w) + fq(\tilde{w}) \\ &= \bigcup_{w \in W'} e(T_w) + e(\tilde{w}) + f(w) = \coprod_{w \in W'} (e(T_w + \tilde{w})) \times \{f(w)\} \end{aligned}$$

We have  $T_w \subseteq C$ , but  $e|_C = i$ , which is a closed map since it is an admissible monic, so  $e(T_w)$  is closed, and therefore also  $e(T_w + \tilde{w})$  as translation is a homeomorphism. Note that, topologically,  $f$  embeds the lattice  $W' \simeq \mathbb{Z}^m$  into a  $\mathbb{R}^m$ , so each value of  $f(w)$  is attained only once, explaining why the possibly countable union in the second line is a *disjoint* union. Each  $e(T_w + \tilde{w})$  being closed, this means that  $h(T)$  is closed in  $\coprod_{w \in W'} I \times \{w\} = I \times W'$ , which in turn is closed in  $I \times V'$ , which is the underlying topological space of  $I \oplus V'$ .

Armed with the previous lemma, we can prove the proposition.

**Proof of Proposition 10.1.** We use [Kel96, Theorem 12.1]. We note that the conditions C1 and C2 *loc. cit.* hold (or more specifically the conditions stated right below the axioms *loc. cit.*), thanks to Lemma 10.3. Moreover, since the resolution provided by our lemma is finite, this gives a triangulated equivalence of the bounded derived categories  $\mathcal{D}^b(\text{LCA}_{\mathfrak{A},\mathbb{R}C}) \xrightarrow{\sim} \mathcal{D}^b(\text{LCA}_{\mathfrak{A},cg})$ . This suffices to know that the inclusion functor  $\text{LCA}_{\mathfrak{A},\mathbb{R}C} \hookrightarrow \text{LCA}_{\mathfrak{A},cg}$  induces an equivalence of stable  $\infty$ -categories, [BluGT13, Corollary 5.11].  $\square$

*Remark 10.4.* Since the topic of complications arising from high global dimension has come up, let us quickly recapitulate the situation:

- (1) Every hereditary order  $\mathfrak{A}$  has global dimension  $\leq 1$ , so these are regular.
- (2) Every maximal order  $\mathfrak{A}$  is hereditary, so these are covered by (1). See [CR90, (26.12) Theorem] for a proof.
- (3) Group rings  $\mathbb{Z}[G]$  unfortunately have infinite global dimension unless  $G = 1$ . See Example 6.9.
- (4) Between global dimension  $\leq 1$  and  $\infty$ , all other values are also possible. Jategaonkar [Jat73, Theorem 2, and Remark] gives (a) a construction of  $\mathbb{Z}_{(p)}$ -orders in the matrix ring  $M_n(\mathbb{Q})$  of global dimension  $n$  for any prescribed  $n \geq 2$ , and (b) on the last pages of the paper gives an example of orders  $\mathfrak{A}_1, \mathfrak{A}_2 \subset M_{2n+1}(\mathbb{Q})$  with

$$\text{gldim } \mathfrak{A}_1 - \text{gldim } \mathfrak{A}_2 = n$$

for any chosen  $n \geq 1$ . As a little elaboration: Firstly, observe that if  $\mathfrak{A} \subset A$  is an arbitrary order, then

$$\text{gldim } \mathfrak{A} = \sup_p (\text{gldim } \mathfrak{A} \otimes \mathbb{Z}_{(p)}), \tag{10.3}$$

where  $p$  runs through all primes, [AuG60, Corollary to Proposition 2.6]. For example, take

$$\Gamma_3 := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 5\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 5^2\mathbb{Z} & 5\mathbb{Z} & \mathbb{Z} \end{pmatrix} \subset M_3(\mathbb{Q}).$$

We have  $\text{gldim } \Gamma_3 \otimes \mathbb{Z}_{(5)} = 2$  (this is Jategaonkar’s example, [Jat73, Example after Theorem 2], over the discrete valuation ring  $\mathbb{Z}_{(5)}$ ). For all primes  $p \neq 5$ , we have  $\Gamma_3 \otimes \mathbb{Z}_{(p)} = M_3(\mathbb{Z}_{(p)})$ . By Morita invariance, the module category is equivalent to the one of  $\mathbb{Z}_{(p)}$ , so  $\text{gldim } \Gamma_3 \otimes \mathbb{Z}_{(p)} = 1$ . Finally, by Equation 10.3, this shows that  $\text{gldim } \mathfrak{A} = 2$ . In a similar way, all the examples in Jategaonkar’s paper can be promoted from the DVR situation to  $\mathbb{Z}$ -orders.

- (5) All orders of finite global dimension are of course regular, so the methods of this paper apply to them. However, we must admit that

we are not aware of any “real life” case study of the ETNC where orders of finite global dimension  $> 1$  have played a rôle.

### 11. Proof of the main theorem

**Proposition 11.1.** *Let  $A$  be any finite-dimensional semisimple  $\mathbb{Q}$ -algebra and  $\mathfrak{A} \subseteq A$  an order (not necessarily regular). Suppose  $K : \text{Cat}_\infty^{\text{ex}} \rightarrow \mathbf{A}$  is a localizing invariant with values in  $\mathbf{A}$  (as in Equation 3.1). Then there is a fiber sequence*

$$K(\text{Mod}_{\mathfrak{A},fg}) \xrightarrow{g} K(\text{LCA}_{\mathfrak{A},cg}) \xrightarrow{h} K(\text{LCA}_{\mathfrak{A}})$$

in  $\mathbf{A}$ . Here the map  $g$  is induced from the exact functor sending a finitely generated right  $\mathfrak{A}$ -module to itself, equipped with the discrete topology. The map  $h$  is induced from the inclusion  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$ .

**Proof.** (Step 1) We construct the following commutative diagram

$$\begin{CD} K(\text{Mod}_{\mathfrak{A},fg}) @>>> K(\text{Mod}_{\mathfrak{A}}) @>>> K(\text{Mod}_{\mathfrak{A}}/\text{Mod}_{\mathfrak{A},fg}) \\ @VgVV @VVV @VV\Phi V \\ K(\text{LCA}_{\mathfrak{A},cg}) @>>> K(\text{LCA}_{\mathfrak{A}}) @>>> K(\text{LCA}_{\mathfrak{A}}/\text{LCA}_{\mathfrak{A},cg}) \end{CD} \tag{11.1}$$

as follows: (a) We leave it to the reader to check that  $\text{Mod}_{\mathfrak{A},fg}$  is left  $s$ -filtering in  $\text{Mod}_{\mathfrak{A}}$ . This is indeed true for all associative unital rings, and amounts to showing that the image of each finitely generated right module is again finitely generated, as well as that every surjection onto a finitely generated module can be restricted to a surjection originating from a finitely generated submodule of the source. (b) Being left  $s$ -filtering, we get the exact sequence of exact categories

$$\text{Mod}_{\mathfrak{A},fg} \hookrightarrow \text{Mod}_{\mathfrak{A}} \twoheadrightarrow \text{Mod}_{\mathfrak{A}}/\text{Mod}_{\mathfrak{A},fg}. \tag{11.2}$$

(c) By Proposition 7.1 we also know that compactly generated  $\mathfrak{A}$ -modules are left  $s$ -filtering in  $\text{LCA}_{\mathfrak{A}}$ . Correspondingly, we get an exact sequence of exact categories

$$\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}} \twoheadrightarrow \text{LCA}_{\mathfrak{A}}/\text{LCA}_{\mathfrak{A},cg}. \tag{11.3}$$

Next, we construct a morphism from the sequence in Equation 11.2 to the one in Equation 11.3, given by exact functors. Firstly,  $\text{Mod}_{\mathfrak{A},fg} \rightarrow \text{LCA}_{\mathfrak{A},cg}$  just sends a finitely generated  $\mathfrak{A}$ -module to itself with the discrete topology. By Lemma 6.6 the underlying abelian group is then finitely generated, and thus indeed compactly generated as an LCA group. Since everything carries the discrete topology, we do not need to worry about continuity, and moreover the functor is clearly exact (in LCA a sequence of discrete groups is exact if and only if it is exact plainly as abelian groups). Secondly,  $\text{Mod}_{\mathfrak{A}} \rightarrow \text{LCA}_{\mathfrak{A}}$  is defined the same way, just without the finite generation

condition. Thirdly, we get an exact functor induced to the quotient exact categories

$$\Phi : \text{Mod}_{\mathfrak{A}}/\text{Mod}_{\mathfrak{A},fg} \longrightarrow \text{LCA}_{\mathfrak{A}}/\text{LCA}_{\mathfrak{A},cg}. \tag{11.4}$$

It is exact by construction. Now, we apply the invariant  $K$  and arrive at Diagram 11.1 as desired. Both rows are fiber sequences by Schlichting’s Localization Theorem, see [Bra18, Theorem 4.1].

(Step 2) Next, we claim that  $\Phi$  is an exact equivalence of exact categories. The functor is essentially surjective: Given any  $G$  in  $\text{LCA}_{\mathfrak{A}}$ , Lemma 6.5 produces an exact sequence

$$V \oplus C \hookrightarrow G \xrightarrow{q} D \quad \text{in} \quad \text{LCA}_{\mathfrak{A}}$$

with  $V$  a vector  $\mathfrak{A}$ -module,  $C$  compact and  $D$  discrete. We note that  $V \oplus C$  is compactly generated, so in the quotient category  $\text{LCA}_{\mathfrak{A}}/\text{LCA}_{\mathfrak{A},cg}$  the morphism  $q$  becomes an isomorphism. However,  $D$  clearly lies in the strict image of  $\Phi$  since it is discrete. Next, since all modules in the strict image of the functor are discrete, there is no difference between continuous or just algebraic right  $\mathfrak{A}$ -module homomorphisms. In particular, since a discrete module in  $\text{LCA}_{\mathfrak{A}}$  is compactly generated if and only if it is finitely generated as an abelian group, and thus by Lemma 6.6 if and only if it is finitely generated over  $\mathfrak{A}$ , we observe that the categorical quotients on the left and right in Equation 11.4 invert the same morphisms. Thus,  $\Phi$  is fully faithful. Being exact, essentially surjective, and fully faithful,  $\Phi$  is an exact equivalence of exact categories. It follows that  $K(\Phi)$  is an equivalence in  $\mathbf{A}$ . This shows that the right arrow in Diagram 11.1 is an equivalence and thus the left square is bi-Cartesian in  $\mathbf{A}$ .

(Step 3) Next,  $\text{Mod}_{\mathfrak{A}}$  is closed under countable coproducts, so  $K(\text{Mod}_{\mathfrak{A}}) = 0$  by the Eilenberg swindle, [Bra18, Lemma 4.2]. Thus, the left bi-Cartesian square in Diagram 11.1 itself pins down a contraction, and the lower left three terms give a fiber sequence

$$K(\text{Mod}_{\mathfrak{A},fg}) \xrightarrow{g} K(\text{LCA}_{\mathfrak{A},cg}) \longrightarrow K(\text{LCA}_{\mathfrak{A}}), \tag{11.5}$$

where  $g$  sends a finitely generated  $\mathfrak{A}$ -module to itself with the discrete topology. This is exactly the statement which we had claimed.  $\square$

We are ready to prove the main result of the paper.

**Theorem 11.2.** *Let  $A$  be any finite-dimensional semisimple  $\mathbb{Q}$ -algebra and  $\mathfrak{A} \subseteq A$  a regular order. Write  $A_{\mathbb{R}} := A \otimes_{\mathbb{Q}} \mathbb{R}$ . Suppose  $K : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathbf{A}$  is a localizing invariant with values in  $\mathbf{A}$ . Then there is a fiber sequence*

$$K(\mathfrak{A}) \xrightarrow{i} K(A_{\mathbb{R}}) \xrightarrow{j} K(\text{LCA}_{\mathfrak{A}})$$

in  $\mathbf{A}$ . The map  $i$  is induced from the exact functor  $M \mapsto M \otimes_{\mathfrak{A}} A_{\mathbb{R}}$ , and the map  $j$  is induced from the functor interpreting a finitely generated projective right  $A_{\mathbb{R}}$ -module as its underlying LCA group (using the real vector space topology), along with the right action coming from  $\mathfrak{A} \subset A_{\mathbb{R}}$ .

**Proof.** (Step 1) As in [BurF01], we write  $\text{PMod}(R)$  for the exact category of finitely generated projective right  $R$ -modules. Note that  $A_{\mathbb{R}}$  is a projective  $\mathfrak{A}$ -module (Lemma 5.2), and thus in particular flat over  $\mathfrak{A}$ . Hence, the tensor functor

$$\text{PMod}(\mathfrak{A}) \rightarrow \text{PMod}(A_{\mathbb{R}}), \quad M \mapsto M \otimes_{\mathfrak{A}} A_{\mathbb{R}}$$

is indeed exact. Moreover, the output is a finitely generated  $A_{\mathbb{R}}$ -module, but since  $A$  is semisimple over  $\mathbb{Q}$ , so is its base change  $A_{\mathbb{R}}$ . Hence, all modules are projective, so the functor really goes to  $\text{PMod}(A_{\mathbb{R}})$ . It follows that the morphism  $K(\mathfrak{A}) \xrightarrow{i} K(A_{\mathbb{R}})$  exists. The inclusion  $\text{PMod}(R) \subseteq \text{Mod}_{\mathfrak{A},fg}$  is an exact functor, and the induced map

$$K(\mathfrak{A}) =_{\text{def}} K(\text{PMod}(\mathfrak{A})) \xrightarrow{\sim} K(\text{Mod}_{\mathfrak{A},fg}) \tag{11.6}$$

is an equivalence. This can be seen as follows: By assumption  $\mathfrak{A}$  is a regular ring, i.e. every finitely generated right  $\mathfrak{A}$ -module admits a finite projective resolution. Thus, we may apply (the categorical dual of) [Kel96, Theorem 12.1]. The required Conditions C1, C2 *loc. cit.* are satisfied, see (the categorical dual of) [Kel96, Example 12.2] and  $\text{Mod}_{\mathfrak{A},fg}$  clearly has enough projectives. Invoking this theorem, we deduce that  $\mathcal{D}^b(\text{PMod}(\mathfrak{A})) \xrightarrow{\sim} \mathcal{D}^b(\text{Mod}_{\mathfrak{A},fg})$  is a triangulated equivalence, and thus the functor induces an equivalence of stable  $\infty$ -categories, [BluGT13, Corollary 5.11]. This confirms that the map in Equation 11.6 is indeed an equivalence. (Step 2) Next, we invoke Proposition 11.1 so that we have the fiber sequence

$$K(\text{Mod}_{\mathfrak{A},fg}) \xrightarrow{g} K(\text{LCA}_{\mathfrak{A},cg}) \longrightarrow K(\text{LCA}_{\mathfrak{A}})$$

in A. For the next step, we shall need an alternative description of  $g$ . For an exact category  $\mathcal{C}$ , we write  $\mathcal{E}\mathcal{C}$  for the exact category of exact sequences, [Büh10, Exercise 3.9]. We set up a functor

$$p : \text{PMod}(\mathfrak{A}) \longrightarrow \mathcal{E}\text{LCA}_{\mathfrak{A},cg} \tag{11.7}$$

by

$$M \mapsto [M \hookrightarrow M \otimes_{\mathfrak{A}} A_{\mathbb{R}} \twoheadrightarrow (M \otimes_{\mathfrak{A}} A_{\mathbb{R}})/M],$$

where (a) on the left  $M$  is regarded with the discrete topology, (b)  $M \otimes_{\mathfrak{A}} A_{\mathbb{R}}$  is equipped with the locally compact topology coming from the finite-dimensional real vector space structure, (c)  $(M \otimes_{\mathfrak{A}} A_{\mathbb{R}})/M$  is equipped with the quotient topology making the sequence exact. The functor  $p$  is exact. To see this, observe that  $\text{PMod}(\mathfrak{A})$  is split exact, so the obvious additivity of the functor suffices for its exactness. We write  $p_i : \text{PMod}(\mathfrak{A}) \rightarrow \text{LCA}_{\mathfrak{A},cg}$  with  $i = 1, 2, 3$  for the three individual functors to the left, middle resp. right term of the exact sequence. Note that for  $M := \mathfrak{A}$ , we have  $p_1(M) \simeq \mathbb{Z}^n$  on the level of the underlying LCA group (see §3.1), moreover  $p_2(M) \simeq \mathbb{R}^n$  and  $p_3(M) \simeq \mathbb{T}^n$ , and on the level of the underlying LCA groups the sequence  $p(M)$  is just  $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{T}^n$ . Since every localizing invariant is also additive in the sense of [BluGT13], the exact functor  $p$  induces a relation among the



induced maps  $p_{i*} : K(\mathfrak{A}) \rightarrow K(\text{LCA}_{\mathfrak{A},cg})$ , namely  $p_{2*} = p_{1*} + p_{3*}$ . The functor  $p_3$  admits a factorization

$$p_3 : \text{PMod}(\mathfrak{A}) \longrightarrow \text{LCA}_{\mathfrak{A},C} \longrightarrow \text{LCA}_{\mathfrak{A},cg},$$

with  $\text{LCA}_{\mathfrak{A},C}$  being the full subcategory of compact modules (Definition 9.6), because we have just seen that  $p_3$  maps every  $M \in \text{PMod}(\mathfrak{A})$  to a compact module. By the Eilenberg swindle,  $K(\text{LCA}_{\mathfrak{A},C}) = 0$ , see [Bra18, Lemma 4.2]. Thus,  $p_{3*}$  must be the zero map. We conclude that  $p_{1*} = p_{2*}$ . The diagram

$$\begin{array}{ccc} K(\text{PMod}(\mathfrak{A})) & \xrightarrow{\sim} & K(\text{Mod}_{\mathfrak{A},fg}) \\ p_1 \downarrow & \swarrow g & \\ K(\text{LCA}_{\mathfrak{A},cg}) & & \end{array}$$

commutes, where the top horizontal map is the equivalence of Equation 11.6, and  $g$  is the map of Equation 11.5. Combining these two facts, we deduce that upon (equivalently) replacing  $K(\text{Mod}_{\mathfrak{A},fg})$  by  $K(\mathfrak{A})$  in Equation 11.5, we obtain

$$K(\mathfrak{A}) \xrightarrow{p_2} K(\text{LCA}_{\mathfrak{A},cg}) \longrightarrow K(\text{LCA}_{\mathfrak{A}}). \tag{11.8}$$

(Step 3) Using Proposition 10.1, we have  $K(\text{LCA}_{\mathfrak{A},\mathbb{R}C}) \xrightarrow{\sim} K(\text{LCA}_{\mathfrak{A},cg})$ , and by Proposition 9.9 moreover  $K(\text{LCA}_{\mathfrak{A},\mathbb{R}C}) \xrightarrow{\sim} K(A_{\mathbb{R}})$ . Combining both equivalences, we may replace the middle term of the sequence in Equation 11.8 by  $K(A_{\mathbb{R}})$ . Even better, note from the construction of the maps in the cited propositions that this equivalence stems from the inclusion of categories  $\text{PMod}(A_{\mathbb{R}}) \subseteq \text{LCA}_{\mathfrak{A},cg}$  (where each module  $M \in \text{PMod}(A_{\mathbb{R}})$  is regarded as equipped with the locally compact topology coming from its real vector space structure), and the image of the functor  $p_2$  indeed lies in this subcategory, so we get a new fiber sequence

$$K(\mathfrak{A}) \xrightarrow{p_2} K(A_{\mathbb{R}}) \xrightarrow{j} K(\text{LCA}_{\mathfrak{A}}),$$

where the first map is still induced from the exact functor  $p_2$ . Finally, note that the functor underlying  $p_2$  is exactly the map  $i$  which we are discussing in the statement of the theorem, so this correctly identifies the first map. Similarly, the second map arises as the composition of  $\text{PMod}(A_{\mathbb{R}}) \subseteq \text{LCA}_{\mathfrak{A},cg}$  and the inclusion  $\text{LCA}_{\mathfrak{A},cg} \hookrightarrow \text{LCA}_{\mathfrak{A}}$ , see Equation 11.3, so we also see that the map  $j$  is exactly the one as we claim in the theorem. This finishes the proof.  $\square$

We obtain the formulation of the introduction:

**Theorem 11.3.** *Suppose that  $\mathfrak{A}$  is a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then there is a long exact sequence*

$$\cdots \longrightarrow K_n(\mathfrak{A}) \longrightarrow K_n(A_{\mathbb{R}}) \longrightarrow K_n(\text{LCA}_{\mathfrak{A}}) \longrightarrow K_{n-1}(\mathfrak{A}) \longrightarrow \cdots,$$

and for all  $n$ , there are canonical isomorphisms

$$K_n(\text{LCA}_{\mathfrak{A}}) \cong K_{n-1}(\mathfrak{A}, \mathbb{R}).$$

Here  $K_*(-)$  denotes ordinary (Quillen) algebraic  $K$ -theory.

**Proof.** We use that non-connective algebraic  $K$ -theory is a localizing invariant  $\mathbb{K} : \text{Cat}_\infty^{\text{ex}} \rightarrow \text{Sp}$  with values in spectra, [BluGT13]. Thus, Theorem 11.2 applies, and the long exact sequence of homotopy groups associated to the fiber sequence gives us a long exact sequence

$$\cdots \longrightarrow \mathbb{K}_n(\mathfrak{A}) \longrightarrow \mathbb{K}_n(A_{\mathbb{R}}) \longrightarrow \mathbb{K}_n(\text{LCA}_{\mathfrak{A}}) \longrightarrow \mathbb{K}_{n-1}(\mathfrak{A}) \longrightarrow \cdots$$

of non-connective  $K$ -groups. Around degree zero it reads

$$\cdots \longrightarrow \mathbb{K}_0(\mathfrak{A}) \longrightarrow \mathbb{K}_0(A_{\mathbb{R}}) \longrightarrow \mathbb{K}_0(\text{LCA}_{\mathfrak{A}}) \longrightarrow \mathbb{K}_{-1}(\mathfrak{A}) \longrightarrow \mathbb{K}_{-1}(A_{\mathbb{R}}) \longrightarrow \cdots.$$

Since  $\mathfrak{A}$  and  $A_{\mathbb{R}}$  are regular rings, their non-connective  $K$ -theory agrees with the ordinary  $K$ -theory, i.e.  $\mathbb{K}_i(\mathfrak{A}) = K_i(\mathfrak{A})$  for all  $i \geq 1$  anyway, they agree for  $K_0$  by idempotent completeness (same for  $A_{\mathbb{R}}$ ), and  $\mathbb{K}_{-i}(\mathfrak{A}) = \mathbb{K}_{-i}(A_{\mathbb{R}}) = 0$  for all  $i \geq 1$ , [Sch06, Remark 7]. Observe that the exactness of the sequence then implies  $\mathbb{K}_{-i}(\text{LCA}_{\mathfrak{A}}) = 0$  for  $i \geq 1$ . Next, the category  $\text{LCA}_{\mathfrak{A}}$  has all kernels, so it is idempotent complete. It follows that  $\mathbb{K}_0(\text{LCA}_{\mathfrak{A}}) = K_0(\text{LCA}_{\mathfrak{A}})$ , [Sch06, Remark 3]. Hence, all letters  $\mathbb{K}$  in the above sequence can be replaced by  $K$  without a change. We have proven the claim.  $\square$

With the tools which we have available, here is the best we can do at the moment regarding non-regular orders.

**Theorem 11.4.** *Suppose  $\mathfrak{A}$  is an arbitrary order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then there is a long exact sequence*

$$\cdots \longrightarrow G_n(\mathfrak{A}) \longrightarrow K_n(A_{\mathbb{R}}) \longrightarrow K_n(\text{LCA}_{\mathfrak{A}}) \longrightarrow G_{n-1}(\mathfrak{A}) \longrightarrow \cdots,$$

where  $G_n(\mathfrak{A}) := K_n(\text{Mod}_{\mathfrak{A},fg})$  denotes the  $K$ -theory of the category of finitely generated right  $\mathfrak{A}$ -modules (this is often called “ $G$ -theory”).

**Proof.** Use Proposition 11.1 for non-connective  $K$ -theory  $\mathbb{K} : \text{Cat}_\infty^{\text{ex}} \rightarrow \text{Sp}$  as in the proof of Theorem 11.3. The argument that  $\mathbb{K}_n(A_{\mathbb{R}}) = K_n(A_{\mathbb{R}})$  holds for all  $n$  is still valid since  $A_{\mathbb{R}}$  is regular. Since  $\text{Mod}_{\mathfrak{A},fg}$  is a Noetherian abelian category, [Sch06, Theorem 7] shows that  $\mathbb{K}_n(\text{Mod}_{\mathfrak{A},fg}) = 0$  for all  $n < 0$ , and  $\mathbb{K}_0(\text{Mod}_{\mathfrak{A},fg}) = K_0(\text{Mod}_{\mathfrak{A},fg})$  since the category is idempotent complete, [Sch06, Remark 3]. As in the proof of Theorem 11.3, we obtain  $\mathbb{K}_n(\text{LCA}_{\mathfrak{A}}) = K_n(\text{LCA}_{\mathfrak{A}})$ .  $\square$

## 12. Equivariant Haar measure

In this section we will supply some details regarding the interpretation of Haar measures as a determinant functor. For background on Picard groupoids we refer to [Del87, §4], [BurF01, §2.1] or [Bre11, §2]. We write  $\text{Picard}$  for the category of Picard groupoids, defined as in [JO12, 1.3. Definition].

**12.1. Determinant functors.** Let  $\mathcal{C}$  be an exact category. Let  $\mathcal{C}^\times$  denote its maximal inner groupoid, i.e. the category with the same objects but we only keep isomorphisms as morphisms.

**Definition 12.1** ([Del87, §4.3]). *Suppose  $(\mathcal{P}, \otimes)$  is a Picard groupoid. A determinant functor on  $\mathcal{C}$  is a functor*

$$\mathcal{P} : \mathcal{C}^\times \longrightarrow \mathcal{P}$$

with the following extra structure and axioms:

- (1) For every exact sequence  $\Sigma : G' \hookrightarrow G \twoheadrightarrow G''$  in  $\mathcal{C}$  we are given an isomorphism

$$\mathcal{P}(\Sigma) : \mathcal{P}(G) \xrightarrow{\sim} \mathcal{P}(G') \otimes_{\mathcal{P}} \mathcal{P}(G''). \tag{12.1}$$

Moreover, this isomorphism is functorial in morphisms of exact sequences.

- (2) For every zero object  $Z$  in  $\mathcal{C}$  we are given an isomorphism  $z : \mathcal{P}(Z) \xrightarrow{\sim} 1_{\mathcal{P}}$  to the neutral object of the Picard groupoid. Henceforth, we simply write  $0$  for a zero object.
- (3) If  $f : G \rightarrow G'$  is an isomorphism in  $\mathcal{C}$ , write

$$\Sigma_l : 0 \hookrightarrow G \twoheadrightarrow G' \quad \text{and} \quad \Sigma_r : G \hookrightarrow G' \twoheadrightarrow 0$$

for the depicted exact sequences. We demand that

$$\mathcal{P}(G) \xrightarrow[\mathcal{P}(\Sigma_l)]{\sim} \mathcal{P}(0) \otimes_{\mathcal{P}} \mathcal{P}(G') \xrightarrow[z \otimes \text{id}]{\sim} 1_{\mathcal{P}} \otimes_{\mathcal{P}} \mathcal{P}(G') \xrightarrow[\mathcal{P}]{\sim} \mathcal{P}(G') \tag{12.2}$$

agrees with the map  $\mathcal{P}(f) : \mathcal{P}(G) \xrightarrow{\sim} \mathcal{P}(G')$ . Analogously, we demand that  $\mathcal{P}(f^{-1})$  agrees with a variant of Equation 12.2 using  $\Sigma_r$  instead of  $\Sigma_l$ .

- (4) If  $G_1 \hookrightarrow G_2 \hookrightarrow G_3$  is given, we demand that the diagram

$$\begin{array}{ccc} \mathcal{P}(G_3) & \xrightarrow{\sim} & \mathcal{P}(G_1) \otimes \mathcal{P}(G_3/G_1) \\ \sim \downarrow & & \downarrow \sim \\ \mathcal{P}(G_2) \otimes \mathcal{P}(G_3/G_2) & \xrightarrow{\sim} & \mathcal{P}(G_1) \otimes \mathcal{P}(G_2/G_1) \otimes \mathcal{P}(G_3/G_2) \end{array}$$

commutes.

**12.2. Picard groupoids as spectra.** Let us quickly recall that there are (at least) two different perspectives how to think about Picard groupoids. On the one hand, one can think of the algebraic definition in terms of a symmetric monoidal category as in [BurF01, §2], and on the other hand one can think of them as spectra whose possibly non-zero homotopy groups are confined to degrees 0 and 1. We denote the latter category by  $\text{Sp}^{0,1}$ , its objects are also known as “stable 1-types”.

**Theorem 12.2.** *There is an equivalence of homotopy categories*

$$\Psi : \text{Ho}(\text{Picard}) \xrightarrow{\sim} \text{Ho}(\text{Sp}^{0,1}). \tag{12.3}$$

*This correspondence preserves the notions of homotopy groups  $\pi_0, \pi_1$  on either side.*

Proofs are given in [Pat12, §5.1, Theorem 5.3] or [JO12, 1.5 Theorem], but this fact was known long before, certainly to Deligne and Drinfeld. Under this correspondence, the homotopy groups  $\pi_i$  for  $i = 0, 1$  of a Picard groupoid  $(\mathcal{P}, \otimes)$  which are discussed in Burns–Flach [BurF01, §2.1] match isomorphically with the homotopy groups of  $\Psi(\mathcal{P}, \otimes)$  as a spectrum. Let us point out that the spectra in  $\text{Sp}^{0,1}$  can also be modelled alternatively as genuine spaces (even in the format of CW complexes) under a further equivalence between the homotopy categories of connective spectra with infinite loop spaces. We will not make use of this at all, but it gives the homotopy groups  $\pi_i$  a very concrete meaning. Nonetheless, it is typically futile to try to visualize the actual topology of such spaces.

**Definition 12.3** (Deligne, [Del87, §4.2]). *Suppose  $\mathcal{C}$  is an exact category. The truncation  $\tau_{\leq 1}K(\mathcal{C})$  of the connective  $K$ -theory spectrum, viewed as a Picard groupoid (via Equation 12.3), is called the Picard groupoid of virtual objects. We denote it by*

$$V(\mathcal{C}) := \Psi^{-1}(\tau_{\leq 1}K(\mathcal{C})).$$

*Remark 12.4.* Picard groupoids can also be modelled in very different ways, e.g., the stable quadratic modules of Baues [MT07], see also [MTW15] for a broader discussion.

**12.3. Universal determinant.** Let us also recall a basic feature of  $K$ -theory. The following discussion depends a little on what definition the reader has in mind for  $K_1$ . If  $R$  denotes a unital associative ring, the first  $K$ -theory group is most frequently defined using the formula

$$K_1(R) := \frac{\text{GL}(R)}{[\text{GL}(R), \text{GL}(R)]}, \tag{12.4}$$

i.e.  $K_1$  is the abelianization of the group  $\text{GL}(R)$ , which in turn is the group of matrices of arbitrary large rank, viewed as a compatible system by embedding rank  $(n \times n)$ -matrices into  $(n' \times n')$ -matrices for  $n' \geq n$  as the top left minor and completing the diagonal by 1's, and all off-diagonal terms by 0's.

From this description one can attach to every finitely generated projective  $R$ -module  $M \in \text{PMod}(R)$  along with an automorphism  $f : M \xrightarrow{\sim} M$  an element of  $K_1$ . If  $M$  is free, this is clear: Just pick an  $R$ -module isomorphism  $\phi : M \simeq R^n$  for a suitable  $n$ ; then along this isomorphism  $f$  becomes an element  $\phi f \phi^{-1} \in \text{GL}_n(R) \subseteq \text{GL}(R)$ , and thanks to the abelianization, the choice of the isomorphism  $\phi$  was irrelevant. If  $M$  is projective, write  $M$  as a direct summand of a free module, say  $M \oplus M' \simeq R^n$ , and prolong

$f$  as  $f \oplus 1_{M'}$  to  $R^n$ . Then proceed as before. This construction gives a well-defined map

$$\mathrm{Aut}(M) \longrightarrow K_1(R),$$

for any  $M \in \mathrm{PMod}(R)$ . Of course, choosing  $M$  also gives a well-defined element in  $K_0(R)$ , namely the class  $[M]$  representing the module. Both of these constructions can be combined, giving a symmetric monoidal functor of Picard groupoids

$$\mathrm{PMod}(R)^\times \longrightarrow V(R) := V(\mathrm{PMod}(R)).$$

See also [BurF01, §2.5].

Of course, other authors right away define  $K_1(R)$  through generators and relations, where generators have the shape  $[M, f]$ , which  $M$  being a finitely generated projective  $R$ -module and  $f$  an automorphism, e.g., [FK06, §1.1]. A more detailed discussion of the variants of these constructions can be found in Weibel [Wei13], Chapter III, Lemma 1.6 and Corollary 1.6.3.

A key insight due to Deligne is that the universal determinant functor of an exact category arises through its  $K$ -theory, and essentially just amounts to truncating the  $K$ -theory spectrum to degrees  $[0, 1]$ . For an arbitrary exact category  $\mathcal{C}$ , one can set up a canonical symmetric monoidal functor

$$\mathcal{C}^\times \longrightarrow V(\mathcal{C}). \quad (*)$$

We have allowed ourselves the quick detour through projective  $R$ -modules and the explicit presentation of Equation 12.4 to motivate this construction in a fairly concrete case.

*Remark 12.5.* The construction of the functor in Equation  $*$  requires entering the simplicial details of the definition of  $K$ -theory. The  $K$ -theory of an exact category  $\mathcal{C}$  is a special case of the  $K$ -theory of a Waldhausen category (take the admissible monics as the cofibrations and isomorphisms as weak equivalences). Waldhausen constructs a map

$$|\mathcal{C}^\times| \longrightarrow \Omega |(S_\bullet \mathcal{C})^\times| = K(\mathcal{C}),$$

where  $|-|$  denotes geometric realization,  $S_\bullet$  is Waldhausen's  $S$ -construction, and  $\Omega$  is the loop space functor, see [Wal85, bottom of p. 329] or [Wei13, Chapter IV, Remark 8.3.2]. This induces a map to the truncation,

$$|\mathcal{C}^\times| \longrightarrow K(\mathcal{C}) \longrightarrow \tau_{\leq 1} K(\mathcal{C})$$

and under the identification of the spaces with Picard groupoids, this becomes the functor in Equation  $*$ .

The universality of  $\mathcal{C}^\times \rightarrow V(\mathcal{C})$  as a determinant functor now essentially reduces to the fact that the truncation  $\tau_{\leq 1}$  is the universal operation having a stable  $[0, 1]$ -type as output.

**Theorem 12.6** (Deligne). *Suppose  $\mathcal{C}$  is an exact category. Then the functor of Equation  $*$  is universal for all determinant functors: If  $\mathcal{P} : \mathcal{C}^\times \rightarrow \mathbf{P}$  is any determinant functor, then it factors*

$$\begin{array}{ccc} \mathcal{C}^\times & \xrightarrow{\mathcal{P}} & \mathbf{P} \\ \downarrow * & \nearrow & \\ \Psi^{-1}\tau_{\leq 1}K(\mathcal{C}), & & \end{array}$$

where  $(*)$  is the morphism from  $\mathcal{C}^\times$  to the virtual objects.

We refer to Deligne [Del87] for the proof.

Suppose  $A$  is an abelian group. The category of  $A$ -torsors is a Picard groupoid, denote it by  $\text{Tors}(A)$ , see [BBE02, Appendix A.1] for background. In this situation

$$\pi_0(\text{Tors}(A)) = 0 \quad \text{and} \quad \pi_1(\text{Tors}(A)) = A.$$

The other important Picard groupoid is the *determinant line*  $\text{Pic}_R^{\mathbb{Z}}$  for a commutative ring  $R$ . This formalism is developed by Knudsen and Mumford [KM76, Chapter 1]. In certain situations, this is universal:

**Theorem 12.7** (Knudsen–Mumford, Deligne). *Suppose  $R$  is (among other examples)*

- *a commutative local ring, or*
- *the ring of integers of a number field,*

*then the universal determinant functor of the category of finitely generated projective  $R$ -modules  $\text{PMod}(R)$  is equivalent to the determinant line,*

$$\begin{aligned} \text{PMod}(R) &\longrightarrow \text{Pic}_R^{\mathbb{Z}} \\ M &\longmapsto \left( \bigwedge^{\text{top}} M, \text{rk } M \right). \end{aligned}$$

Although not relevant for our purposes, Knudsen and Mumford develop the determinant line right away not just for commutative rings, but for an arbitrary scheme.

The determinant line plays a fundamental rôle for the ETNC with commutative coefficients, in the works predating [BurF01].

With these preparations, we are ready to identify the Haar torsor as the universal determinant functor on the exact category  $\text{LCA}$ . We do not know who made this observation first. In a perhaps not truly proven state it seems to have been folklore for a while, but a complete proof probably only became possible with the work of Clausen [Cla17].

**Theorem 12.8** (Universality of the Haar torsor). *The Haar functor  $H_a : \text{LCA}^\times \rightarrow \text{Tors}(\mathbb{R}_{>0}^\times)$  is the universal determinant functor of the category  $\text{LCA}$ . In particular, Deligne’s Picard groupoid of virtual objects for  $\text{LCA}$  is isomorphic to the Picard groupoid of  $\mathbb{R}_{>0}^\times$ -torsors.*

**Proof.** Let  $\det : \mathrm{LCA}^\times \rightarrow V(\mathrm{LCA})$  be the universal determinant functor in the sense of Deligne. Since  $Ha$  is a determinant functor (we checked this in §2), we obtain a canonical factorization  $F : V(\mathrm{LCA}) \rightarrow \mathrm{Tors}(\mathbb{R}_{>0}^\times)$ . Now, by using Clausen’s computation of the  $K$ -theory of  $\mathrm{LCA}$  [Cla17], there is a long exact sequence

$$K_1(\mathbb{Z}) \rightarrow K_1(\mathbb{R}) \rightarrow K_1(\mathrm{LCA}) \rightarrow K_0(\mathbb{Z}) \rightarrow K_0(\mathbb{R}) \rightarrow K_0(\mathrm{LCA}) \rightarrow 0.$$

While this result is originally due to Clausen, it of course also follows from our main result Theorem 11.3 in the special case of the order  $\mathfrak{A} := \mathbb{Z}$  in  $A := \mathbb{Q}$ . Unravelling the terms, we obtain

$$\mathbb{Z}^\times \rightarrow \mathbb{R}^\times \rightarrow K_1(\mathrm{LCA}) \rightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \rightarrow K_0(\mathrm{LCA}) \rightarrow 0, \quad (12.5)$$

and therefore  $K_0(\mathrm{LCA}) = 0$  and the absolute value isomorphism

$$K_1(\mathrm{LCA}) \cong \mathbb{R}^\times / \mathbb{Z}^\times \xrightarrow[\mathrm{abs}]{\sim} \mathbb{R}_{>0}^\times.$$

By Deligne’s Theorem 12.6 the Picard groupoid of virtual objects is, when regarded as a spectrum, the truncation of  $K(\mathrm{LCA})$  to a stable  $[0, 1]$ -type. Hence, on  $\pi_0, \pi_1$  the functor  $F$  induces the maps

$$\begin{aligned} 0 &= \pi_0(V(\mathrm{LCA})) \rightarrow \pi_0(\mathrm{Tors}(\mathbb{R}_{>0}^\times)) = 0 \\ \mathbb{R}_{>0}^\times &= \pi_1(V(\mathrm{LCA})) \rightarrow \pi_1(\mathrm{Tors}(\mathbb{R}_{>0}^\times)) \cong \mathbb{R}_{>0}^\times \end{aligned}$$

because the homotopy groups of  $V(\mathrm{LCA})$  in degree 0, 1 then agree with the ones of  $K(\mathrm{LCA})$ . Since the construction of the modulus, see Equation 2.4, can be recast as the comparison of a trivialization of the torsor with the pullback of that trivialization along an automorphism, it follows that the map on the level of  $\pi_1$  is an isomorphism. However, a map among stable  $[0, 1]$ -types, or equivalently a symmetric monoidal functor among Picard groupoids is an equivalence if it induces an isomorphism of  $\pi_0$  and  $\pi_1$ , [Pat12, Lemma 5.7]. Hence,  $F$  is an equivalence of Picard groupoids and via the isomorphism  $F$ ,

$$\det : \mathrm{LCA}^\times \rightarrow V(\mathrm{LCA}) \xrightarrow[F]{\sim} \mathrm{Tors}(\mathbb{R}_{>0}^\times),$$

we identify the Haar functor as the universal determinant functor of  $\mathrm{LCA}$ .  $\square$

**Theorem 12.9.** *Let  $\mathfrak{A}$  be a regular order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . Then*

$$\begin{aligned} \pi_0 V(\mathrm{LCA}_{\mathfrak{A}}) &\cong K_{-1}(\mathfrak{A}, \mathbb{R}) \\ \pi_1 V(\mathrm{LCA}_{\mathfrak{A}}) &\cong K_0(\mathfrak{A}, \mathbb{R}), \end{aligned}$$

*and, stronger, the Picard groupoid of equivariant volumes  $V(\mathrm{LCA}_{\mathfrak{A}})$  is equivalent to the 1-truncation of the  $(-1)$ -shift of the fiber of the morphism  $K(\mathfrak{A}) \rightarrow K(A_{\mathbb{R}})$ .*

**Proof.** The ring morphism  $c : \mathfrak{A} \rightarrow A_{\mathbb{R}}$ , which plainly arises from tensoring  $(-) \mapsto (-) \otimes_{\mathbb{Z}} \mathbb{R}$ , induces the relative  $K$ -theory fiber sequence

$$K(\mathfrak{A}, \mathbb{R}) \rightarrow K(\mathfrak{A}) \xrightarrow{c} K(A_{\mathbb{R}}),$$

where  $K(\mathfrak{A}, \mathbb{R}) := \text{hofib}(c : K(\mathfrak{A}) \rightarrow K(A_{\mathbb{R}}))$  is just defined as the homotopy fiber. This is equivalent to the conventions used in [BurF01]. By Theorem 11.2 it follows that there is an equivalence  $K(\text{LCA}_{\mathfrak{A}}) \rightarrow \Sigma K(\mathfrak{A}, \mathbb{R})$ , where  $\Sigma$  denotes the suspension. On the level of homotopy groups, this is a shift of degree by one, thus

$$K_i(\mathfrak{A}, \mathbb{R}) \cong K_{i+1}(\text{LCA}_{\mathfrak{A}})$$

holds for all integers  $i$ . □

### 13. Real and $p$ -adic comparison

Let  $\mathfrak{A}$  be an arbitrary order in a finite-dimensional semisimple  $\mathbb{Q}$ -algebra  $A$ . We recall that  $A := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q}$ . As in [BurF01, §2.7], we define

$$\widehat{A} := A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{A}_{fin},$$

where  $\mathbb{A}_{fin}$  denotes the *finite part* of the adèles  $\mathbb{A}$  of the rational number field  $\mathbb{Q}$ , i.e. the restricted product  $\prod'_p (\mathbb{Q}_p, \mathbb{Z}_p)$ , where  $p$  only runs over the primes (excluding the infinite place). When we speak of  $\widehat{A}$ , we regard it with the locally compact topology coming from  $\mathbb{A}$ . In more detail: As an abelian group, we have  $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{A}_{fin} \xrightarrow{\phi} \mathbb{A}_{fin}^n$  for  $n := \dim_{\mathbb{Q}} A$ , and if we equip the right side with the standard topology of the adèles, then this induces a topology on the left side. This topology is independent of the choice of the isomorphism  $\phi$ .

**Theorem 13.1** (Reciprocity Law). *Suppose  $\mathfrak{A}$  is an arbitrary order in a semisimple  $\mathbb{Q}$ -algebra  $A$  of finite dimension. Then the composition*

$$K(A) \rightarrow K(A \otimes \mathbb{A}) \rightarrow K(\text{LCA}_{\mathfrak{A}}) \tag{13.1}$$

*is zero.*

- (1) *The first arrow is induced from the exact functor  $M \mapsto M \otimes_A \mathbb{A}$  of tensoring with the adèles of the rationals.*
- (2) *The second arrow is induced from mapping  $A \otimes \mathbb{A}$  to itself, where now the tensor product is equipped with the locally compact topology induced from the adèles.*

This is a non-commutative version of the reciprocity-like statements of [Cla17]. Note in the proof below that the essential bits of the proof are not so much arithmetic, but rather topological. However, of course arithmetic and topology are intertwined concepts when it comes to the adèles.



**Proof.** Recall that

$$\mathbb{Q} \hookrightarrow \mathbb{A} \twoheadrightarrow \mathbb{A}/\mathbb{Q}$$

is an exact sequence in  $\mathbf{LCA}$ , where  $\mathbb{Q}$  is embedded diagonally as a closed subgroup. The quotient  $\mathbb{A}/\mathbb{Q}$  is compact connected. We define three exact functors

$$f_i : \mathbf{PMod}(A) \longrightarrow \mathbf{LCA}_{\mathfrak{A}}$$

$$f_1(A) := A, \quad f_2(A) := A \otimes_{\mathbb{Q}} \mathbb{A}, \quad f_3(A) := (A \otimes_{\mathbb{Q}} \mathbb{A})/A,$$

where  $A$  in  $f_1$  is equipped with the discrete topology,  $A \otimes_{\mathbb{Q}} \mathbb{A}$  in  $f_2$  (which is  $\simeq \mathbb{A}^n$  as an LCA group) with the natural topology of the adèles, and the values of  $f_3$  with the quotient topology: Since  $\mathbb{Q} \hookrightarrow \mathbb{A}$  is closed,  $A \hookrightarrow A \otimes_{\mathbb{Q}} \mathbb{A}$  is also closed. It suffices that we have given the values of  $f_i$  for  $i = 1, 2, 3$  only for  $A$  since  $A$  is a projective generator of the split exact category  $\mathbf{PMod}(A)$ , so in order to define an exact functor it suffices to specify its value on a projective generator. Since for every  $W \in \mathbf{PMod}(A)$  we obtain that  $f_1(W) \hookrightarrow f_2(W) \twoheadrightarrow f_3(W)$  is exact in  $\mathbf{LCA}_{\mathfrak{A}}$ , and functorially so, the Additivity Theorem implies that

$$f_{2*} = f_{1*} + f_{3*} : K(A) \longrightarrow K(\mathbf{LCA}_{\mathfrak{A}})$$

as maps of spectra (this is analogous to an idea in the proof of Theorem 11.2). As  $f_{1*} = f_{3*} = 0$ , because they factor over the fully exact subcategories  $\mathbf{LCA}_{\mathfrak{A},D}$  resp.  $\mathbf{LCA}_{\mathfrak{A},C}$  whose  $K$ -theory is contractible by [Bra18, Lemma 4.2], we deduce  $f_{2*} = 0$ . Finally, observe that the composition of the functors in the statement of our theorem literally give the functor  $f_2$ , finishing the proof.  $\square$

*Example 13.2.* Suppose  $F$  is a number field and  $\mathfrak{A} := \mathcal{O}$  its ring of integers. Then the reciprocity law, Theorem 13.1, states that the composition

$$K(F) \longrightarrow K(\mathbb{A}_F) \longrightarrow K(\mathbf{LCA}_{\mathcal{O}})$$

is zero. In degree one, this becomes  $F^\times \rightarrow K_1(\mathbb{A}_F) \rightarrow K_1(\mathbf{LCA}_{\mathcal{O}})$  being the zero map. This is quite close to the statement that

$$F^\times \hookrightarrow \mathbb{A}_F^\times \twoheadrightarrow K_1(\mathbf{LCA}_F)$$

is an exact sequence, proven in joint work with Arndt in [ArB19]. These reciprocity statements are modelled after Clausen's ideas in [Cla17].

At the beginning of the paper we had discussed the difference between the real determinant line and the Haar torsor, and we had made some claims regarding their precise relation which so far we have not proved, and we fill this gap now.

**Proposition 13.3.** *We compare the real/ $p$ -adic determinant line torsor with the Haar torsor in the non-equivariant setting:*

(1) In Example 2.2 the map

$$\pi_1(\text{Pic}_{\mathbb{R}}^{\mathbb{Z}}) \rightarrow \pi_1(V(\text{LCA}))$$

is given by  $\alpha \mapsto |\alpha|$ , i.e. in degree one the Haar torsor differs from the graded real determinant line precisely by forgetting the sign.

(2) In Example 2.3 the map

$$\pi_1(\text{Pic}_{\mathbb{Q}_p}^{\mathbb{Z}}) \longrightarrow \pi_1(V(\text{LCA}))$$

is given by  $\alpha \mapsto |\alpha|_p$ , i.e. in degree one the Haar torsor differs from the graded  $p$ -adic determinant line precisely by just keeping the  $p$ -adic valuation and nothing else (so it does not see the multiplication by any unit in  $\mathbb{Z}_p^\times$ ).

(3) In both examples,

$$\pi_0(\text{Pic}_{\mathbb{Z}}^{\mathbb{Z}}) \longrightarrow \pi_0(V(\text{LCA}))$$

is the map  $\mathbb{Z} \rightarrow 0$ , i.e. the Haar torsor does not retain any information about the grading.

**Proof.** (1) The map in question is induced from the map of  $K$ -theory spectra  $K(\mathbb{R}) \rightarrow K(\text{LCA})$  upon truncation  $\tau_{\leq 1}$  and  $\Psi^{-1}$ , so we just need to understand  $K_1(\mathbb{R}) \rightarrow K_1(\text{LCA})$ . This map already appears in the proof of Theorem 12.8, namely in Equation 12.5 and our discussion *ibid.* showed that it agrees with the absolute value homomorphism. (2) This is a little trickier, but very beautiful (as we believe). We use the Reciprocity Law, Theorem 13.1, and prove this by a global-to-local argument. We only need the case  $\mathfrak{A} := \mathbb{Z}$  and  $A := \mathbb{Q}$ . Then the Reciprocity Law in degree one tells us that the composition

$$a : K_1(\mathbb{Q}) \longrightarrow K_1(\mathbb{A}_{fin}) \oplus K_1(\mathbb{R}) \longrightarrow K_1(\text{LCA}) \tag{13.2}$$

is zero. Clearly multiplication by  $p$ ,  $\mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$ , is an isomorphism, so it defines an element  $[p] \in \mathbb{Q}^\times \cong K_1(\mathbb{Q})$ . Multiplication by  $p$  also defines an isomorphism of the adèles in  $\text{LCA}$ ,

$$\mathbb{A} \xrightarrow{\sim} \mathbb{A}, \quad \left( \text{or } \mathbb{A}_{fin} \oplus \mathbb{R} \xrightarrow{\sim} \mathbb{A}_{fin} \oplus \mathbb{R} \right)$$

and we can get this isomorphism by applying the functor underlying the map  $a$  in Equation 13.2, namely  $a(\mathbb{Q}) \xrightarrow{a(-p)} a(\mathbb{Q})$ . However, since  $a = 0$  on the level of  $K$ -theory by the Reciprocity Law, the class  $a([p]) \in K_1(\text{LCA})$  is zero. Now, writing the adèles as a product

$$\mathbb{A} \simeq \mathbb{Q}_p \oplus \prod'_{\ell \neq p, \infty} (\mathbb{Q}_\ell, \mathbb{Z}_\ell) \oplus \mathbb{R}, \tag{13.3}$$

where the restricted product runs over all places  $\ell \notin \{p, \infty\}$ , we also have the following: On each  $\mathbb{Q}_\ell$  with  $\ell \neq p$  and  $\ell$  a finite place, we have  $p \in \mathbb{Z}_\ell^\times$ , so by the argument in Example 2.3 (applied for the prime  $\ell$ ) the multiplication by  $p$  automorphism induces the trivial class in  $K_1(\text{LCA})$ . Since this applies

to all topological  $\ell$ -torsion factors of  $\prod'_{\ell \neq p, \infty} (\mathbb{Q}_\ell, \mathbb{Z}_\ell)$ , the induced map of multiplication with  $p$  on  $K$ -theory on this object is zero. Further, on  $\mathbb{R}$  we know by part (1) of this proof that we get  $p = |p| \in K_1(\mathbf{LCA})$ , and on  $\mathbb{Q}_p$  we do not yet know what class in  $K_1$  we get, but this is what we wanted to compute. We may call it  $\beta \in \mathbb{R}_{>0}^\times$ . However, since we know that on the entire adèles the  $K$ -theory class is the neutral element, we obtain by the additivity along direct sums in Equation 13.3 the identity

$$1 = |p| \cdot \beta \cdot 1,$$

where  $\beta$  is the class of multiplication by  $p$  on  $\mathbb{Q}_p$  in  $K_1(\mathbf{LCA})$ ; this corresponds to the multiplicativity of the modulus, Equation 2.2. We deduce from this equation that  $\beta = p^{-1} = |p|_p$ , i.e. the standard normalization of the  $p$ -adic absolute value of  $p$ . Next, for  $x \in \mathbb{Z}_p^\times$  by the argument in Example 2.3, this time applied for the prime  $p$  itself, such an  $x$  induces also the trivial class in  $K$ -theory. Since

$$\mathbb{Q}_p^\times \simeq \langle p^\mathbb{Z} \rangle \oplus \mathbb{Z}_p^\times,$$

by multiplicativity, these computations describe the entire map

$$K_1(\mathbb{Q}_p) \longrightarrow K_1(\mathbf{LCA}),$$

and we see that it indeed agrees with the  $p$ -adic valuation. (3) In Equation 12.5 we had already seen that  $K_0(\mathbf{LCA}) = 0$ , so it can only be the zero map.  $\square$

*Remark 13.4.* The reader will observe that the statement  $a = 0$  in the previous proof is precisely the product formula

$$\prod_v |x|_v = 1 \quad \text{for} \quad x \in \mathbb{Q}^\times,$$

(where  $v$  runs through all places), but here we do *not* get it from using the standard normalized definitions of absolute values, but rather by the topological properties of the adèles alone. These considerations work also for adèles of arbitrary number fields and can also be exploited in other degrees of  $K$ -theory to get the vanishing statements of reciprocity laws. See [Cla17], [ArB19].

*Remark 13.5.* Note that  $K_0(\mathbf{LCA}_{\mathfrak{A}})$  is usually not zero (this already happens for  $\mathfrak{A}$  being the ring of integers in a number field), so for the equivariant setting the vanishing statement  $K_0(\mathbf{LCA}) = 0$  is misleading.

## 14. Lower bounds for the cardinality

Unrelated to whether they might find any application in the ETNC or not, we provide some general evidence showing that the higher  $K$ -groups of  $\mathbf{LCA}_{\mathfrak{A}}$  – and that is: the higher relative  $K$ -groups  $K_n(\mathfrak{A}, \mathbb{R})$  – are non-trivial.

**Theorem 14.1.** *Suppose  $A$  is a non-zero finite-dimensional semisimple  $\mathbb{Q}$ -algebra and  $\mathfrak{A} \subset A$  a regular order. Let  $\mathfrak{c}$  be the cardinal of the continuum,  $\mathfrak{c} := |\mathbb{R}|$ . Then*

$$\dim K_n(\text{LCA}_{\mathfrak{A}}) \otimes \mathbb{Q}$$

*satisfies the following lower bounds:*

- (1) *for  $n \equiv 0 \pmod{4}$  and  $n \geq 4$  and if  $A_{\mathbb{R}}$  has a complex or quaternionic factor, it is at least  $\mathfrak{c}$ ;*
- (2) *for  $n \equiv 1 \pmod{4}$  and  $n \geq 1$ , it is at least countable;*
- (3) *for  $n \equiv 2 \pmod{4}$  and  $n \geq 2$ , it is at least  $\mathfrak{c}$ ;*
- (4) *for  $n \equiv 3 \pmod{4}$  and  $n \geq 3$  and if  $A_{\mathbb{R}}$  has a complex or quaternionic factor, it is at least countable.*

*Indeed, for  $n = 1$ ,  $\dim K_1(\text{LCA}_{\mathfrak{A}}) \otimes \mathbb{Q}$  is also at least  $\mathfrak{c}$ . Meanwhile, the dimension  $\dim K_0(\text{LCA}_{\mathfrak{A}}) \otimes \mathbb{Q}$  is finite.*

For  $\mathfrak{A} = \mathbb{Z}$ , we have  $K_0(\text{LCA}_{\mathbb{Z}}) = 0$ , so this value need not be non-zero.

**Proof.** Firstly, by a result of Kuku, the groups  $K_n(\mathfrak{A})$  are finitely generated for all  $n \geq 0$ , [Kuk86]. Consider the long exact sequence of Theorem 11.3,

$$\cdots \longrightarrow K_4(\text{LCA}_{\mathfrak{A}}) \longrightarrow K_3(\mathfrak{A}) \longrightarrow K_3(A_{\mathbb{R}}) \longrightarrow K_3(\text{LCA}_{\mathfrak{A}}) \longrightarrow K_2(\mathfrak{A}) \longrightarrow \cdots$$

We immediately get our claim for  $n = 0$ , so from now on  $n \geq 1$ . The above sequence is exact in the abelian category of abelian groups  $\text{Ab}$ , but we may regard it in the quotient category  $\text{Ab}/\text{Ab}_{fg}$  modulo finitely generated abelian groups (this is a Serre subcategory). Then, using Kuku’s result, we obtain isomorphisms

$$K_n(A_{\mathbb{R}}) \xrightarrow{\sim} K_n(\text{LCA}_{\mathfrak{A}}) \quad \text{in} \quad \text{Ab}/\text{Ab}_{fg} \quad (14.1)$$

for all  $n$ . Note that since for  $X \in \text{Ab}_{fg}$  we have  $\dim X_{\mathbb{Q}} < \infty$ , the dimension of  $\mathbb{Q}$ -vector spaces in  $\text{Ab}/\text{Ab}_{fg}$  is well-defined as soon as it is at least countable. This will be the key argument throughout. Above,  $A_{\mathbb{R}}$  factors into copies of matrix algebras over  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  (by the classification of division algebras over the reals, [Lam91, Ch. 5, (13.12) Theorem], and the Artin–Wedderburn theorem, [Lam91, Ch. I, (3.5) Theorem]), so by Morita invariance

$$K_n(A_{\mathbb{R}}) \cong K_n(\mathbb{R})^{a_1} \oplus K_n(\mathbb{C})^{a_2} \oplus K_n(\mathbb{H})^{a_3}$$

for some  $a_1, a_2, a_3$ , not all zero. Next, recall that there is a chain of inclusions

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}. \quad (14.2)$$

(Step 0) We first handle the case  $n = 1$ . It is quite easy. For all entries  $X$  in Equation 14.2 we have morphisms  $\mathbb{R}^{\times} \rightarrow K_1(X) \rightarrow \mathbb{R}^{\times}$  such that the composition is multiplication with a number: for the reals, take the identity, for the complex numbers the inclusion and the norm, for the quaternions the inclusion and the reduced norm. Thus,  $\dim K_1(X)_{\mathbb{Q}} \geq \dim(\mathbb{R}^{\times} \otimes \mathbb{Q})$ . However,  $\dim_{\mathbb{Q}} \mathbb{R} = \mathfrak{c}$ . The remaining proof deals with  $n \geq 2$ .

(Step a1) Suppose  $L/F$  is a finite field extension. By the existence of a

transfer (the norm), it follows that the kernel of  $K_n(F) \rightarrow K_n(L)$  is killed by multiplication with the degree  $[L : F]$ . Thus,  $K_n(F)_{\mathbb{Q}} \subseteq K_n(L)_{\mathbb{Q}}$  for all finite field extensions  $L/F$ . Writing an algebraic extension as the union of its finite subextensions, the same follows for every algebraic extension  $L/F$ . This argument also generalizes to the central simple algebra  $\mathbb{H}/\mathbb{R}$  by splitting it over the complex numbers and using the composition

$$K_n(\mathbb{R}) \rightarrow K_n(\mathbb{H}) \rightarrow K_n(M_2(\mathbb{C})) \xrightarrow{\sim} K_n(\mathbb{C}) \rightarrow K_n(\mathbb{R}),$$

where  $M_2$  denotes the  $(2 \times 2)$ -matrix algebra.

(Step a2) Let  $F/\mathbb{Q}$  be a field extension. Then there is a tower  $\mathbb{Q} \subset \mathbb{Q}(t_I)_I \subset F$ , where  $I$  is some set,  $\mathbb{Q}(t_I)_I/\mathbb{Q}$  is purely transcendental and  $F/\mathbb{Q}(t_I)_I$  is algebraic.

(Step b) Now, for every field  $F$  we have Milnor’s exact sequence for algebraic  $K$ -theory,

$$0 \longrightarrow K_n(F) \longrightarrow K_n(F(t)) \longrightarrow \bigoplus_{x \in (\mathbb{A}_F^1)_0} K_{n-1}(\kappa(x)) \longrightarrow 0, \tag{14.3}$$

see [Wei13, Ch. V, Corollary 6.7.1]. Here  $(\mathbb{A}_F^1)_0$  denotes the set of closed points of the affine line  $\mathbb{A}_F^1$  over  $F$ ; or equivalently all maximal ideals of the ring  $F[t]$ . Tensoring with  $\mathbb{Q}$ , taking the dimension and exploiting (a1), applied to each  $\kappa(x)/F$  being algebraic, we obtain

$$\begin{aligned} \dim K_n(F(t))_{\mathbb{Q}} &= \dim K_n(F)_{\mathbb{Q}} + \sum_{x \in (\mathbb{A}_F^1)_0} \dim K_{n-1}(\kappa(x))_{\mathbb{Q}} \\ &\geq \dim K_n(F)_{\mathbb{Q}} + \sum_{x \in (\mathbb{A}_F^1)_0} \dim K_{n-1}(F)_{\mathbb{Q}}. \end{aligned}$$

We shall use both the equality and inequality variant of this (we only really need the inequality, but we can see clearer what happens if we keep the equality in mind as well).

(Step c) Suppose  $n \equiv 1 \pmod{4}$  and  $n \geq 5$ ; the case  $n = 1$  has already been dealt with in Step (0). Then for the real cyclotomic field  $F_m := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$  we have  $\dim K_n(F_m)_{\mathbb{Q}} = [F_m : \mathbb{Q}]$  by Borel’s rank computation, [Wei13, Ch. IV, Theorem 1.18]. Let  $m$  go to infinity (partially ordered by divisibility), combined with (a1), we deduce  $\dim K_n(\mathbb{Q}^{\text{sep}} \cap \mathbb{R})_{\mathbb{Q}} \geq \aleph_0$  as all  $F_m$  lie in  $\mathbb{Q}^{\text{sep}} \cap \mathbb{R}$ . It follows by (a2) that there is a tower  $\mathbb{Q}^{\text{sep}} \cap \mathbb{R} \subset \mathbb{Q}(t_I)_I \subset \mathbb{R}$ , where the first extension is purely transcendental and  $\mathbb{R}/\mathbb{Q}(t_I)_I$  is algebraic (and possibly non-trivial). We have  $\dim K_{n-1}(\mathbb{Q}^{\text{sep}} \cap \mathbb{R})_{\mathbb{Q}} = 0$  by Borel’s rank computation, [Wei13, Ch. IV, Theorem 1.18] (so, by this computation this follows for any finite subextension of  $\mathbb{Q}^{\text{sep}} \cap \mathbb{R}$  over  $\mathbb{Q}$ , but since any possible non-trivial basis vector lies in some finite subextension, none such can exist). Hence, by the equality from (b), we get

$$\dim K_n(F(t))_{\mathbb{Q}} = \dim K_n(F)_{\mathbb{Q}} + 0 \tag{14.4}$$

where  $F$  is  $\mathbb{Q}^{\text{sep}} \cap \mathbb{R}$ , or (by transfinite induction) any purely transcendental extension thereof. Finally, by (a1) we obtain

$$\dim K_n(\mathbb{R})_{\mathbb{Q}} \geq \dim K_n(\mathbb{Q}(t_I)_I)_{\mathbb{Q}} = \dim K_n(\mathbb{Q}^{\text{sep}} \cap \mathbb{R})_{\mathbb{Q}} \geq \aleph_0.$$

Since we have the chain in Equation 14.2, we obtain claim (2).

(Step d) Suppose  $n \equiv 3 \pmod{4}$  and  $n \geq 3$ . Then for the complex cyclotomic field  $G_m := \mathbb{Q}(\zeta_m)$ ,  $m$  big enough, we have  $\dim K_n(G_m)_{\mathbb{Q}} = [G_m : \mathbb{Q}]/2$  by Borel’s rank computation. Combined with (a1) we deduce

$$\dim K_n(\mathbb{Q}^{\text{sep}})_{\mathbb{Q}} \geq \aleph_0.$$

We proceed as in (c) and obtain  $\dim K_n(\mathbb{C})_{\mathbb{Q}} \geq \aleph_0$ . This settles claim (4).

(Step e) Suppose  $n \equiv 2 \pmod{4}$  and  $n \geq 2$ . We apply (a2) to the extension  $\mathbb{R}/\mathbb{Q}$ . Since  $\text{trdeg}_{\mathbb{Q}}(\mathbb{R}) = \mathfrak{c}$ , it follows by (a2) that there is a tower  $\mathbb{Q} \subset \mathbb{Q}(t_I)_I \subset \mathbb{R}$ , where the first extension is purely transcendental of degree  $\mathfrak{c}$ , and  $\mathbb{R}/\mathbb{Q}(t_I)_I$  is algebraic (and possibly non-trivial). Let  $F$  be any purely transcendental extension of  $\mathbb{Q}$ . By (b) we have

$$\dim K_n(F(t))_{\mathbb{Q}} \geq \dim K_n(F)_{\mathbb{Q}} + \sum_{x \in (\mathbb{A}_F^1)_0} \dim K_{n-1}(F)_{\mathbb{Q}}.$$

and since  $n - 1 \equiv 1 \pmod{4}$ , we have

$$\dim K_{n-1}(F)_{\mathbb{Q}} \geq \dim K_{n-1}(\mathbb{Q})_{\mathbb{Q}} = 1,$$

where the first inequality holds since  $F$  will certainly contain  $\mathbb{Q}$ , and the latter follows from Borel’s rank computations. Indeed, recall that this inequality comes from the Sequence 14.3. It is known to be split exact. Indeed, one can check that for the closed point  $x := F[t]/(t - 1)$  (or any other  $F$ -rational closed point), the  $K$ -theory product

$$\omega := \{(t - 1)\} \smile \alpha \quad \text{in} \quad K_1(F) \times K_{n-1}(F) \rightarrow K_n(F)$$

with  $\alpha \in K_{n-1}(F)$  and using  $\kappa(x) = F$ , defines a class in  $K_n(F(t))$  which lies in the direct complement of the inclusion of  $K_n(F)$  from the left, see the proof of [Wei13, Ch. V, Corollary 6.7.2 and Lemma 6.7.3]. This follows since  $\partial_x \omega = \alpha$  for the boundary map at  $x$ , while  $\partial_x K_n(F) = 0$ . Let  $\alpha_i$  for  $i \in I$  (of cardinality  $\mathfrak{c}$ ) be such classes for each individual purely transcendental extension (Axiom of Choice). As each  $\alpha_i$  contributes a dimension to  $K_n(\mathbb{Q}(t_I)_I)_{\mathbb{Q}}$ , we obtain  $\dim K_n(\mathbb{Q}(t_I)_I)_{\mathbb{Q}} \geq \mathfrak{c}$ . As  $\mathbb{R}/\mathbb{Q}(t_I)_I$  is algebraic, (a1) implies that  $\dim K_n(\mathbb{R})_{\mathbb{Q}} \geq \mathfrak{c}$ . This settles claim (3).

(Step f) A similar argument works for  $n \equiv 0 \pmod{4}$ ,  $n \geq 4$ , based on  $n - 1 \equiv 3 \pmod{4}$  then, and using (d) for this. Copying the argument from (e) then settles claim (1).  $\square$

*Remark 14.2.* It is tempting to think of an abelian group  $A$  of cardinality  $\mathfrak{c}$  as huge and incomprehensible. However, we might just have  $A \simeq \mathbb{R}$ , a group nobody is afraid of. The question is whether it will be possible to equip these higher  $K$ -groups with such an interpretation or not. Certainly, the above style of proof is of no use in this respect.

### 14.1. Dreams.

**14.1.1. Equivariant measure.** As much as the term “equivariant Haar measure” and the surrounding picture is suggestive, we have only constructed the associated torsor, but not an actual formalism of a measure. For example, our torsor description tells us how to relate the equivariant measures on objects sitting in exact sequences  $A \hookrightarrow B \twoheadrightarrow C$ , where  $A$  is a subobject. By using translation-invariance as a guiding principle, one would like to believe that this can be extended to all cosets  $A + b$  with  $b \in B$ , too. An approach using classical  $K$ -theory is not possible since cosets like  $A + b$  for  $b \neq 0$  are not subobjects. Perhaps Zakharevich’s  $K$ -theory of assemblers [Zak17] could be the basic language for attacking these issues. More optimistically, there should be a form of a  $\sigma$ -algebra style calculus, where not just finite scissor operations are allowed, but also countable ones. It would be wonderful to have some formalism where computations in the style of Tate’s thesis could exist. All of this, of course, is fairly utopic and speculative at the moment.

Nonetheless, in the non-equivariant situation,  $\mathfrak{A} = \mathbb{Z}$  and  $A_{\mathbb{R}} = \mathbb{R}$ , the classical theory of the Haar measure has all these features, so *one* example exists. Clearly, such a conjectural theory would not be real-valued, and the glueing laws when taking the union or intersection of measurable subsets will involve homotopical structures, controlled by the symmetry constraint of the Picard groupoid  $V(\mathrm{LCA}_{\mathfrak{A}})$  of Definition 2.4.

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