New York Journal of Mathematics

New York J. Math. **25** (2019) 1178–1213.

Solutions of diophantine equations as periodic points of p-adic algebraic functions, II: The Rogers-Ramanujan continued fraction

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Abstract. In this part we show that the diophantine equation $X^5 + Y^5 = \varepsilon^5(1 - X^5Y^5)$, where $\varepsilon = \frac{-1+\sqrt{5}}{2}$, has solutions in specific abelian extensions of quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ in which $-d \equiv \pm 1 \pmod 5$. The coordinates of these solutions are values of the Rogers-Ramanujan continued fraction $r(\tau)$, and are shown to be periodic points of an algebraic function.

Contents

1. Introduction.

In a previous paper [17] integral solutions of the diophantine equation

$$Fer_4: \ X^4 + Y^4 = 1,$$

were constructed in ring class fields Ω_f of odd conductor f over imaginary quadratic fields of the form $K = \mathbb{Q}(\sqrt{-d})$, with $d_K f^2 = -d \equiv 1 \pmod 8$, where d_K is the discriminant of K. The coordinates of these solutions were studied in Part I of this paper [20], and shown to be the periodic points

Received January 23, 2019.

2010 *Mathematics Subject Classification.* 11D41,11G07,11G15,14H05.

Key words and phrases. Periodic points, algebraic functions, 5-adic field, ring class fields, Rogers-Ramanujan continued fraction.

ISSN 1076-9803/2019

of a fixed 2-adic algebraic function on the maximal unramified algebraic extension K_2 of the 2-adic field $\mathbb{Q}_2$. In particular, every ring class field of odd conductor over $K = \mathbb{Q}(\sqrt{-d})$ with $-d \equiv 1 \pmod 8$ is generated over $\mathbb{Q}$ by some periodic point of this algebraic function. This was simplified and extended in [21] to show that all ring class fields over any field K in this family of quadratic fields are generated by individual periodic or pre-periodic points of the 2-adic multivalued algebraic function

$$\hat{F}(z) = \frac{-1 \pm \sqrt{1-z^4}}{z^2}.$$

A similar situation holds for the solutions of

$$Fer_3: \ 27X^3 + 27Y^3 = X^3Y^3,$$

studied in [19], in that they are, up to a finite set, the exact set of periodic points of a fixed 3-adic algebraic function, and all ring class fields of quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ in the family for which $-d \equiv 1 \pmod 3$ are generated by periodic or pre-periodic points of this same 3-adic algebraic function. (See [19] and [21] for a more precise description.)

In this paper I will study the analogous quintic equation

$$\mathcal{C}_5: \ v^5X^5 + v^5Y^5 = 1 - X^5Y^5, \quad v = \frac{1+\sqrt{5}}{2},$$

which can be written in the equivalent form

$$\mathcal{C}_5: \ X^5 + Y^5 = \varepsilon^5(1 - X^5Y^5), \quad \varepsilon = \frac{-1+\sqrt{5}}{2}, \tag{1}$$

in certain abelian extensions of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with $d_K f^2 = -d \equiv \pm 1 \pmod 5$. In Part I [20] these were called *admissible quadratic fields* for the prime $p = 5$: these are the imaginary quadratic fields in which the ideal $(5) = \wp_5\wp_5'$ of the ring of integers R_K of K splits into two distinct prime ideals. In this part I will show that (1) has unit solutions in the abelian extensions $\Sigma_5\Omega_f$ or $\Sigma_5\Omega_{5f}$ of K (according as $d \neq 4f^2$ or $d = 4f^2 > 4$), where Σ_5 is the *ray class field* of conductor $\mathfrak{f} = (5)$ over K and Ω_f, Ω_{5f} are the *ring class fields* of conductors f and $5f$, respectively, over K, for any positive integer f which is relatively prime to $p = 5$. (See [6].)

As is the case for the families of quadratic fields mentioned above, the coordinates of these solutions will be shown in Part III to be the exact set of periodic points (minus a finite set) of a specific 5-adic algebraic function in a suitable extension of the 5-adic field $\mathbb{Q}_5$. This will be used to verify the conjectures of Part I for the prime $p = 5$. In Theorem 5.4 of this paper we establish a preliminary result in this direction, by showing that any ring class field Ω_f over $K = \mathbb{Q}(\sqrt{-d})$ with $(-d/5) = +1$ and $(5, f) = 1$ is generated by a periodic point of a fixed algebraic function, which is independent of d. The 5-adic representation of this function will be explored in Part III.

Let $H_{-d}(x)$ be the class equation for a discriminant $-d \equiv \pm 1 \pmod 5$, and let

$$F_d(x) = x^{5h(-d)}(1 - 11x - x^2)^{h(-d)} H_{-d}(j_5(x)), \tag{2}$$

where

$$j_5(b) = \frac{(1 - 12b + 14b^2 + 12b^3 + b^4)^3}{b^5(1 - 11b - b^2)}. \tag{3}$$

This rational function represents the j-invariant of the Tate normal form

$$E_5(b):\; Y^2 + (1 + b)XY + bY = X^3 + bX^2, \tag{4}$$

on which the point $P = (0, 0)$ has order 5. Note that

$$j_5(b) = -\frac{(z^2 + 12z + 16)^3}{z + 11}, \quad z = b - \frac{1}{b}. \tag{5}$$

The roots of $F_d(x)$ are the values of b for which the curve $E_5(b)$ has complex multiplication by the order R_{-d} of discriminant $-d = d_K f^2$ in K. If $h(-d)$ is the class number of R_{-d}, it turns out that $F_d(x^5)$ has an irreducible factor $p_d(x)$ of degree $4h(-d)$ whose roots give solutions of $\mathcal{C}_5$ in abelian extensions of $K = \mathbb{Q}(\sqrt{-d})$. Furthermore, the roots of $p_d(x)$ are conjugate values over $\mathbb{Q}$ of the Rogers-Ramanujan continued fraction $r(\tau)$ defined by

$$\begin{aligned} r(\tau) &= \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} = \frac{q^{1/5}}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\cdots, \\ &= q^{1/5} \prod_{n \geq 1} (1 - q^n)^{(n/5)}, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}. \end{aligned}$$

See [1], [2], [4], [10]. (We follow the notation in [10].) In the latter formula $(n/5)$ is the Legendre symbol and $\mathbb{H}$ denotes the upper half-plane. The function $r(\tau)$ is a modular function for the congruence group $\Gamma(5)$ [10, p. 149], and $(X, Y) = (r(\tau/5), r(-1/\tau))$ is a modular parametrization of the curve $\mathcal{C}_5$ (see [10, eq. (7.3)]). In Section 4 we prove the following result.

Theorem 1.1. *Let $d \equiv \pm 1$ (mod 5), $K = \mathbb{Q}(\sqrt{-d})$, and*

$$w = \frac{v + \sqrt{-d}}{2} \in R_K, \;\text{ with } \wp_5^2 \mid w \text{ and } (N(w), f) = 1.$$

Then the values $X = r(w/5), Y = r(-1/w)$ of the Rogers-Ramanujan continued fraction give a solution of $\mathcal{C}_5$ in $\Sigma_5\Omega_f$ or $\Sigma_5\Omega_{5f}$, according as $d \neq 4f^2$ or $d = 4f^2$. For a unique primitive 5-th root of unity $\zeta^j = e^{2\pi i j/5}$, depending on w, we have

$$\mathbb{Q}(r(w/5)) = \Sigma_{\wp_5'}\Omega_f, \quad \mathbb{Q}(\zeta^j r(-1/w)) = \Sigma_{\wp_5}\Omega_f, \quad \text{if } d \neq 4f^2;$$

and

$$\mathbb{Q}(r(w/5)) = \Sigma_{2\wp_5'}\Omega_f, \quad \mathbb{Q}(\zeta^j r(-1/w)) = \Sigma_{2\wp_5}\Omega_f, \quad \text{if } d = 4f^2, \; 2 \mid f;$$

where $\wp_5$ is the prime ideal $\wp_5 = (5, w)$, $\wp_5'$ is its conjugate ideal in K, and $\Sigma_{\mathfrak{f}}$ denotes the ray class field of conductor $\mathfrak{f}$ over K. Furthermore,

$$\mathbb{Q}(r(-1/w)) = \mathbb{Q}(r(w)) = \Sigma_5\Omega_f \quad \textit{or} \quad \Sigma_5\Omega_{5f},$$

according as $d \neq 4f^2$ or $d = 4f^2$.

The numbers $\eta = r(w/5), \xi = \zeta^j r(-1/w)$ in this theorem are both roots of the irreducible polynomial $p_d(x)$, and so are conjugate algebraic integers (and units) over $\mathbb{Q}$. Furthermore, they satisfy the relation

$$\xi = \zeta^j r(-1/w) = \frac{-(1+\sqrt{5})\eta^{\tau_5} + 2}{2\eta^{\tau_5} + 1 + \sqrt{5}},$$

(for all $-d = d_K f^2 < -4$) where $\tau_5 = \left(\frac{\mathbb{Q}(\eta)/K}{\wp_5}\right)$ is the Frobenius automorphism (Artin symbol) for $\wp_5$ (which is defined since $\mathbb{Q}(r(w/5))$ is abelian over K and unramified at $\wp_5$). See Tables 1 and 2 for a list of the polynomials $p_d(x)$ for small values of d. As is clear from the tables, these polynomials have relatively small coefficients and discriminants. Moreover, as we show in Section 5, these values of $r(\tau)$ are periodic points of an algebraic function, and can be computed for small values of d and small periods using nested resultants. (See [20, Section 3] and [21].) We prove the following.

Theorem 1.2. *If*

$$g(X, Y) = (Y^4 + 2Y^3 + 4Y^2 + 3Y + 1)X^5 - Y(Y^4 - 3Y^3 + 4Y^2 - 2Y + 1),$$

the roots of $p_d(x)$ are periodic points of the multi-valued algebraic function $\mathfrak{g}(z)$ defined by $g(z, \mathfrak{g}(z)) = 0$. With w chosen as in Theorem 1.1, the period of $\eta = r(w/5)$ with respect to the action of $\mathfrak{g}$ is the order of the Frobenius automorphism $\tau_5 = \left(\dfrac{\mathbb{Q}(\eta)/K}{\wp_5}\right)$ in Gal$(\mathbb{Q}(\eta)/K)$.

As part of our discussion we also prove the following. To state the result, let

$$\mathfrak{s}(z) = \frac{(\zeta + \zeta^2)z + 1}{z + 1 + \zeta + \zeta^2}, \quad \zeta = \zeta_5 = e^{2\pi i/5},$$

a linear fractional map of order 5. The group $\langle \mathfrak{s}(z) \rangle$ generated by $\mathfrak{s}(z)$ under composition is the Galois group of the extension of function fields $\mathbb{Q}(\zeta, z)/\mathbb{Q}(\zeta, \mathfrak{r}(z))$, where

$$\mathfrak{r}(z) = \frac{z(z^4 - 3z^3 + 4z^2 - 2z + 1)}{z^4 + 2z^3 + 4z^2 + 3z + 1}.$$

Theorem 1.3. *With w as in Theorem 1.1 and τ_5 as above, we have the formula*

$$r(w/5)^{\tau_5} = \mathfrak{s}^j(r(w)) = r\left(\frac{w}{1 - jw}\right),$$

where $j \not\equiv 0 \pmod 5$ has the same value as in Theorem 1.1 and j is the unique integer $(\bmod\ 5)$ for which $\mathfrak{s}^j(r(w))$ is an algebraic conjugate of $\eta = r(w/5)$.

This fact is significant, because in the ideal-theoretic formulations of Shimura's Reciprocity Law, such as in [23, p. 123], one has to restrict to ideals that are relatively prime to the level of the modular function being considered. Here $r(\tau) \in \Gamma(5)$, so the level is $N = 5$, but Theorem 1.3 gives information about the automorphism τ_5 corresponding to the prime ideal $\wp_5$ of K.

Theorem 1.3 has the following application. A formula for the real continued fraction

$$r(3i) = \frac{e^{-6\pi/5}}{1+}\ \frac{e^{-6\pi}}{1+}\ \frac{e^{-12\pi}}{1+}\ \frac{e^{-18\pi}}{1+} \cdots$$

was stated by Ramanujan in his notebooks and proved in [3] and [4]. In Section 5 we prove the alternative formula

$$r(3i) = \frac{(1+\zeta^3)\eta^{\tau_5} + \zeta}{\eta^{\tau_5} - \zeta - \zeta^3}, \quad \zeta = e^{2\pi i/5}, \tag{6}$$

where

$$\eta^{\tau_5} = r\left(\frac{4+3i}{5}\right)^{\tau_5} = \frac{-i\omega}{2} - \frac{i\sqrt{3}}{2} + i\frac{\omega^2}{4}\sqrt[4]{3}\left(\sqrt{4+2\sqrt{5}} + i\sqrt{-4+2\sqrt{5}}\right)$$

and $\omega = (-1 + i\sqrt{3})/2$. This formula expresses Ramanujan's value in terms of roots of unity and simpler square-roots than appear in his original formula. (See Example 1 in Section 5.) Similar expressions can be worked out for certain other values of the Rogers-Ramanujan function $r(\tau)$ using Theorem 1.3.

2. Defining the Heegner points.

Throughout the paper we will have occasion to make use of the linear fractional map

$$\tau(b) = \frac{-b + \varepsilon^5}{\varepsilon^5 b + 1} = \frac{-b + \varepsilon_1}{\varepsilon_1 b + 1}, \quad \varepsilon_1 = \varepsilon^5 = \frac{-11 + 5\sqrt{5}}{2}. \tag{7}$$

Whenever the symbol τ appears as a function of b, it denotes the function in (7). We will also have occasion to use τ to denote a complex number in the upper half-plane $\mathbb{H}$ or an automorphism in a suitable Galois group, and which use of τ we mean will be clear from the context. We note that

$$\begin{aligned} j_5(\tau(b)) = j_{5,5}(b) &= \frac{(1 + 228b + 494b^2 - 228b^3 + b^4)^3}{b(1 - 11b - b^2)^5}, \\ &= -\frac{(z^2 - 228z + 496)^3}{(z+11)^5}, \quad z = b - \frac{1}{b}, \end{aligned} \tag{8}$$

where $j_{5,5}(b)$ is the j-invariant of the elliptic curve

$$\begin{aligned} E_{5,5}(b):\ Y^2 + (1+b)XY + 5bY &= X^3 + 7bX^2 + (6b^3 + 6b^2 - 6b)X \\ &\quad + b^5 + b^4 - 10b^3 - 29b^2 - b. \end{aligned}$$

The curve $E_{5,5}(b)$ is isogenous to $E_5(b)$ [18, p. 259], and because of (8), $E_5(\tau(b))$ represents the Tate normal form for $E_{5,5}(b)$.

Let $K = \mathbb{Q}(\sqrt{-d})$, where $-d = d_K f^2 \equiv \pm 1 \pmod 5$ and d_K is the discriminant of K. As usual, let $\eta(\tau)$ be the Dedekind η-function. From Weber [26, p.256] the function

$$x_1 = x_1(w) = \left(\frac{\eta(w/5)}{\eta(w)}\right)^2$$

satisfies the equation

$$x_1^6 + 10x_1^3 - \gamma_2(w)x_1 + 5 = 0, \quad \gamma_2(w) = j(w)^{1/3}.$$

Thus

$$j(w) = \frac{(x_1^6 + 10x_1^3 + 5)^3}{x_1^3}. \tag{9}$$

On the other hand,

$$x_1^3 = y^5 + 5y^4 + 15y^3 + 25y^2 + 25y = (y+1)^5 + 5(y+1)^3 + 5(y+1) - 11,$$

with $y = y(w) = \frac{\eta(w/25)}{\eta(w)}$. By Theorem 6.6.4 of Schertz [23, p. 159], both x_1^3 and y are elements of the ring class field $\Omega_f = K(j(w))$ if

$$w = \begin{cases} \frac{v+\sqrt{-d}}{2}, & 2 \nmid d, \ v^2 \equiv -d \pmod{5^2}, \ (v, 2f) = 1, \\ v + \frac{\sqrt{-d}}{2}, & 2 \mid d, \ 2 \nmid f, \ v^2 \equiv -d/4 \pmod{5^2}, \quad (v, f) = 1, \\ v + \frac{\sqrt{-d}}{2}, & 2 \mid d, \ 2 \mid f, \ v^2 \equiv -d/4 \pmod{5^2}, \quad (v, f_{odd}) = 1; \end{cases} \tag{10}$$

in the last case f_{odd} is the largest odd divisor of f and $v \not\equiv d/4 \pmod 2$ is chosen to guarantee that $(N(w), f) = 1$. (The latter condition is needed to insure that (w) is a proper ideal of R_{-d} in Section 4.) These conditions on w are equivalent to the conditions imposed on w in Theorem 1.1.

Now we set

$$z = z(w) = b - \frac{1}{b} = -11 - x_1^3 = -11 - \left(\frac{\eta(w/5)}{\eta(w)}\right)^6, \tag{11}$$

so that b is one of the two roots of the equation

$$b^2 - zb - 1 = 0, \quad z = -11 - x_1^3.$$

From the identity

$$\frac{1}{r^5(\tau)} - 11 - r^5(\tau) = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6, \quad \tau \in \mathbb{H},$$

for the Rogers-Ramanujan function $r(\tau)$ (see [10]), we see that

$$\frac{1}{b} - b - 11 = \frac{1}{r^5(w/5)} - r^5(w/5) - 11,$$

from which it follows that

$$b = r^5(w/5) \quad \text{or} \quad \frac{-1}{r^5(w/5)} \tag{12}$$

and

$$z = r^5(w/5) - \frac{1}{r^5(w/5)}. \tag{13}$$

We find from (5), (11), and (9) that

$$\begin{aligned} j_5(b) =& \frac{((-11 - x_1^3)^2 + 12(-11 - x_1^3) + 16)^3}{x_1^3} \\ =& \frac{(x_1^6 + 10x_1^3 + 5)^3}{x_1^3} = j(w). \end{aligned} \tag{14}$$

When z is given by (11), $j(w)$ is the j-invariant of $E_5(b)$. Weber [26, p.256] also gives the equation

$$j(w/5) = \frac{(x_1^6 + 250x_1^3 + 3125)^3}{x_1^{15}} = j_{5,5}(b), \tag{15}$$

for the same substitution (11), by (8). Thus, $j(w/5)$ is the j-invariant of the isogenous curve $E_{5,5}(b)$.

The functions $z(w)$ and $y(w)$ are modular functions for the group $\Gamma_0(5)$, by Schertz [23, p. 51]. Moreover, w and $w/5$ are basis quotients for proper ideals in the order R_{-d} of discriminant $-d$ in K. Hence, we have the following.

Theorem 2.1. *If $z = b - 1/b$ satisfies (11), where w is given by (10), then $j_5(b) = j(w)$ and $j_{5,5}(b) = j(w/5)$ are roots of the class equation $H_{-d}(x) = 0$, and the isogeny $E_5(b) \rightarrow E_{5,5}(b)$ represents a Heegner point on $\Gamma_0(5)$. Furthermore, z lies in the ring class field of conductor f over $K = \mathbb{Q}(\sqrt{-d})$, where $-d = f^2 d_K$ and d_K is the discriminant of K.*

Exactly the same arguments apply if w is replaced in (9)-(15) by w/a, where $(a, f) = 1$ and $5a \mid N(w)$. (To guarantee $y(w/a) \in \Omega_f$ we would also need $5^2 a \mid N(w)$.) Then w/a and $w/(5a)$ are basis quotients for proper ideals in R_{-d} and $j(w/a)$ and $j(w/(5a))$ are roots of $H_{-d}(x)$. Thus, $j(w), j(w/a) \in \Omega_f$ are conjugate to each other over K. Theorem 6.6.4 of Schertz [23] implies that the corresponding values $z(w), z(w/a)$ in (11) are also conjugate to each other over K if $5 \nmid a$, but in Section 4 we will need to relax this restriction on a. To do this, we prove the following lemma. Let $J(z)$ denote the rational function

$$J(z) = -\frac{(z^2 + 12z + 16)^3}{z + 11}.$$

Recall that an ideal $\mathfrak{a}$ of the order R_{-d} corresponds to the ideal $\mathfrak{a}R_K$ of the maximal order $R_K = \mathsf{R}_{d_K}$ of K, and conversely, an ideal $\mathfrak{b}$ in R_K corresponds to the ideal $\mathfrak{b}_d = \mathfrak{b} \cap \mathsf{R}_{-d}$ in R_{-d} (see [6, p. 130]).

Lemma 2.2. *For a given ideal $\mathfrak{a} = (a, w) \subseteq \mathsf{R}_{-d}$ with ideal basis quotient w/a, where $(a, f) = 1$ and $5a \mid N(w)$, there is a unique value of $z_1 \in \Omega_f$ for which $J(z_1) = j(w/a)$ and $z_1 + 11 \cong \wp_5'^3$, and this value is $z_1 = z^{\sigma^{-1}}$, where $\sigma = \left(\frac{\Omega_f/K}{\mathfrak{a}R_K}\right)$. ($\alpha \cong \beta$ denotes equality of the divisors (α) and (β).)*

Proof. If σ is the Frobenius automorphism given in the statement of the lemma, $j(w/a)^\sigma = j(\mathfrak{a})^\sigma = j(\mathsf{R}_{-d}) = j(w) = J(z)$, it follows that $J(z^{\sigma^{-1}}) = j(w/a)$. Suppose there is a $z_2 \in \Omega_f$, different from $z_1 = z^{\sigma^{-1}}$, for which $J(z_2) = J(z_1)$ and $z_2 + 11 \cong z_1 + 11$. Then (z_1, z_2) is a point on the curve $F(u, v) = 0$, where

$$\begin{aligned} F(u,v) &= -(u+11)(v+11)\frac{J(u)-J(v)}{u-v} \\ &= (v+11)u^5 + (v^2+47v+396)u^4 + (v^3+47v^2+876v+5280)u^3 \\ &\quad + (v^4+47v^3+876v^2+8160v+31680)u^2 \\ &\quad + (v^5+47v^4+876v^3+8160v^2+39360v+84480)u \\ &\quad + 11v^5+396v^4+5280v^3+31680v^2+84480v+97280. \end{aligned}$$

A calculation on Maple shows that this is a curve of genus 0, parametrized by the rational functions

$$\begin{aligned} u &= -\frac{11t^5+55t^4+165t^3+275t^2+275t+125}{t(t^4+5t^3+15t^2+25t+25)} \\ v &= -\frac{t^5+11t^4+55t^3+165t^2+275t+275}{t^4+5t^3+15t^2+25t+25}. \end{aligned}$$

Hence, $F(z_1, z_2) = 0$ gives that

$$z_1 + 11 = \frac{-125}{t(t^4+5t^3+15t^2+25t+25)},$$

or

$$t^5+5t^4+15t^3+25t^2+25t+\frac{125}{z_1+11} = 0,$$

for some algebraic number t. Since $z_1 + 11 \cong z + 11 \cong \wp_5'^3$ (see eq. (28) below), we have $(z_1 + 11) \mid 5^3$ and t is an algebraic integer which is not divisible by any prime divisor of $\wp_5'$ in $\Omega_f(t)$. Then

$$z_2 + 11 = \frac{-t^5}{t^4+5t^3+15t^2+25t+25} = \frac{t^5}{\frac{125}{t(z_1+11)}} = t^6\frac{(z_1+11)}{125}.$$

But the equality of the ideals $(z_2 + 11) = (z_1 + 11)$ implies that $t^6 \cong 5^3$, so t *is divisible* by some prime divisor of $\wp_5'$ in $\Omega_f(t)$. This contradiction establishes the claim. $\square$

3. Points of order 5 on $E_5(b)$.

From [22] we take the following. The X-coordinates of points of order 5 on $E_5(b)$ which are not in the group

$$\langle(0,0)\rangle = \{O, (0,0), (0,-b), (-b,0), (-b,b^2)\}$$

can be given in the form

$$\begin{aligned} X =& \frac{(5-\alpha)}{100}\{(-18-12b+6b\alpha+8\alpha-2b^2)u^4 \\ &+(-4b\alpha+2b^2+3\alpha-7+12b)u^3 \\ &+(7b\alpha+\alpha-3-2b^2-7b)u^2 \\ &+(22b-2+2b^2)u-3-7b+3b\alpha-2b^2-\alpha\} \\ =& \frac{(5-\alpha)}{100}(A_4u^4+A_3u^3+A_2u^2+A_1u+A_0), \end{aligned}$$

where $\alpha = \pm\sqrt{5}$,

$$u^5 = \phi_1(b) = \frac{2b+11+5\alpha}{-2b-11+5\alpha} = \frac{b-\bar{\varepsilon}^5}{-b+\varepsilon^5} \tag{16}$$

and

$$\varepsilon = \frac{-1+\alpha}{2}, \quad \bar{\varepsilon} = \frac{-1-\alpha}{2}.$$

Equation (16) shows that $u^5 = 1/(\varepsilon^5\tau(b))$, i.e., $\tau(b) = (\varepsilon u)^{-5}$. Solving for b in (16) gives

$$b = \frac{\varepsilon^5 u^5 + \bar{\varepsilon}^5}{u^5+1}. \tag{17}$$

Now the Weierstrass normal form of $E_5(b)$ is given by

$$Y^2 = 4X^3 - g_2X - g_3, \quad g_2 = \frac{1}{12}(b^4+12b^3+14b^2-12b+1),$$
$$g_3 = \frac{-1}{216}(b^2+1)(b^4+18b^3+74b^2-18b+1),$$

with

$$\Delta = g_2^3 - 27g_3^2 = b^5(1-11b-b^2).$$

By Theorem 2.1, $E_5(b)$ has complex multiplication by the order R_{-d}, so the theory of complex multiplication implies that if $K \neq \mathbb{Q}(i)$, i.e. $d \neq 4f^2$, the X-coordinates $X(P)$ of points of order 5 on $E_5(b)$ have the property that the quantities

$$\frac{g_2g_3}{\Delta}\left(X(P) + \frac{1}{12}(b^2+6b+1)\right)$$

generate the field $\Sigma_5\Omega_f$ over Ω_f, where Σ_5 is the ray class field of conductor 5 over $K = \mathbb{Q}(\sqrt{-d})$. (See [11]; or [25] for $f = 1$.)

In the case that $d = 4f^2 > 4$, the argument leading to Theorem 2 of [11] shows that these quantities generate a class field Σ'_{5f} over $K = \mathbb{Q}(i)$ whose corresponding ideal group H consists of the principal ideals generated by elements of K, prime to $5f$, which are congruent to rational numbers (mod f) and congruent to ± 1 (mod 5). H is an ideal group because it contains the ray mod $5f$. Thus $\mathsf{H} \subset \mathsf{S}_5 \cap \mathsf{P}_f$ is contained in the intersection of the principal ring class mod f, P_f, and the ray mod 5, S_5. If $(\alpha) \in \mathsf{S}_5 \cap \mathsf{P}_f$, then we may take $\alpha \equiv r$ (mod f) and $r \in \mathbb{Q}$; and then $i^a\alpha \equiv 1$ (mod 5) for some power of i. If $2 \mid a$, then $(\alpha) \in \mathsf{H}$; while if $2 \nmid a$, then $\alpha^2 \equiv -1 \pmod 5$, so $(\alpha)^2 \in \mathsf{H}$, and the product of any two such ideals lies in H. This implies that $[\mathsf{S}_5 \cap \mathsf{P}_f : \mathsf{H}] = 2$ and Σ'_{5f} is a quadratic extension of $\Sigma_5\Omega_f$ (when $K = \mathbb{Q}(i)$). Moreover, H is a subgroup of the principal ring class P_{5f} and $[\mathsf{P}_{5f} : \mathsf{H}] = 2$, so that $[\Sigma'_{5f} : \Omega_{5f}] = 2$. Since $\mathsf{P}_{5f} \neq \mathsf{S}_5 \cap \mathsf{P}_f$, it follows that $\Sigma'_{5f} = \Omega_{5f}(\Sigma_5\Omega_f) = \Sigma_5\Omega_{5f}$. Noting that $\mathsf{P}_f/\mathsf{P}_{5f}$ is cyclic of order 4, generated by $(\alpha)\mathsf{P}_{5f}$ with $\alpha \equiv 2 \pmod{\wp_5}$ and $\equiv 1 \pmod{\wp'_5}$, it follows from Artin Reciprocity that Ω_{5f}/Ω_f is a cyclic quartic extension.

Let F denote the field $\Sigma_5\Omega_f$, for $d \neq 4f^2$; and $\Sigma'_{5f} = \Sigma_5\Omega_{5f}$, for $d = 4f^2 > 4$. Also, let $\phi(\mathfrak{a})$ denote the Euler ϕ-function for ideals $\mathfrak{a}$ of R_K. Since $p = 5 = \wp_5\wp'_5$ splits in K, the degree of Σ_5/Σ_1 is given by

$$[\Sigma_5 : \Sigma_1] = \frac{1}{2}\phi(\wp_5)\phi(\wp'_5) = 8, \quad \text{if } d \neq 4f^2;$$

and since every intermediate field of Σ_5/Σ_1 is ramified over $p = 5$ we have that

$$[F : \Omega_f] = [\Sigma_5\Omega_f : \Omega_f] = 8, \quad d \neq 4f^2.$$

On the other hand,

$$[F : \Omega_f] = [\Sigma'_{5f} : \Omega_f] = 2 \cdot [\Sigma_5\Omega_f : \Omega_f] = 8, \quad d = 4f^2 > 4,$$

since in this case

$$[\Sigma_5 : K] = \frac{1}{4}\phi(\wp_5)\phi(\wp'_5) = 4, \quad d = 4f^2;$$

so that $\Sigma_5 = K(\zeta_5)$ when $K = \mathbb{Q}(i)$. Thus, $[F : \Omega_f] = 8$ in all cases (with $d \neq 4$).

In Cho's notation [5], the ideal group H coincides with the ideal group declared modulo $5f$ given by

$$P_{(5),\mathcal{O}} = \{(\alpha) | \alpha \in \mathcal{O}_K, \alpha \equiv a \pmod{5f}, a \in \mathbb{Z}, (a, f) = 1, a \equiv 1 \pmod 5\};$$

and F equals the corresponding field $K_{(5),\mathcal{O}}$, with $\mathcal{O} = \mathsf{R}_{-d}$. Since $(5, f) = 1$, this holds whether $d \neq 4f^2$ or $d = 4f^2$. Cox [6, p. 313] denotes this field as $F = L_{\mathcal{O},5}$ and calls it an *extended ring class field.*

We henceforth take $\alpha = \sqrt{5}$ in the above formulas, and we prove the following.

Theorem 3.1. *If $z = b - 1/b$ is given by (13), where w is given by (10), with $d \neq 4$, then the roots u of the equation (16) lie in the field $F = \Sigma_5\Omega_f$, if $d \neq 4f^2$, and in $F = \Sigma_5\Omega_{5f}$, if $d = 4f^2 > 4$. Thus, the value b is given by*

$$b = \frac{\varepsilon^5 u^5 + \bar{\varepsilon}^5}{u^5 + 1}, \quad \varepsilon = \frac{-1 + \sqrt{5}}{2}, \quad \bar{\varepsilon} = \frac{-1 - \sqrt{5}}{2},$$

where

$$u = -\frac{r(w) - \bar{\varepsilon}}{r(w) - \varepsilon} \quad or \quad -\frac{\bar{\varepsilon} r(w) + 1}{\varepsilon r(w) + 1},$$

according as $b = r^5(w/5)$ *or* $b = \frac{-1}{r^5(w/5)}$. *Moreover,* $r(w), r(w/5)$ *and* $r(-1/w)$ *lie in the field* F.

Proof. Note first that

$$\begin{aligned}\frac{g_2 g_3}{\Delta} &= \frac{-1}{2^5 3^4}\frac{(b^4+12b^3+14b^2-12b+1)(b^2+1)(b^4+18b^3+74b^2-18b+1)}{b^5(1-11b-b^2)}\\ &= \frac{1}{2^5 3^4}\frac{(z^2+12z+16)(z^2+18z+76)}{z+11}\frac{b^2+1}{b^2},\end{aligned}$$

where $z = b - \frac{1}{b} = -11 - x_1^3$ lies in Ω_f. It follows that

$$\frac{b^2+1}{b^2}\left(X(P) + \frac{1}{12}(b^2+6b+1)\right) \in F$$

for any point $P \in E_5[5]$. In particular, with $P = (-b, 0)$ we have that

$$\frac{b^2+1}{12b^2}(b^2-6b+1) = \frac{1}{12}\left(b+\frac{1}{b}\right)\left(b+\frac{1}{b}-6\right) \in F.$$

Since $b - \frac{1}{b}$ lies in Ω_f, the field F contains the quantity

$$\left(b-\frac{1}{b}\right)^2 + 4 = b^2 + \frac{1}{b^2} + 2 = \left(b+\frac{1}{b}\right)^2,$$

and therefore also $\left(b+\frac{1}{b}\right)$ and $\left(b+\frac{1}{b}\right) + \left(b-\frac{1}{b}\right) = 2b$. Therefore, $b \in F$ and we have that

$$X(P) \in F, \quad \text{for } P \in E_5[5].$$

Since $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\zeta_5) \subseteq \Sigma_5$, we deduce from the formula for X above that

$$A_4u^4 + A_3u^3 + A_2u^2 + A_1u + A_0 \in F$$

for any root of (16). Hence, for any fixed root u of (16) we have that

$$A_4\zeta^{4i}u^4 + A_3\zeta^{3i}u^3 + A_2\zeta^{2i}u^2 + A_1\zeta^i u + A_0 = B_i \in F, \quad 0 \le i \le 4. \tag{18}$$

This gives a system of 5 equations in the 5 "unknowns" u^i, with coefficients in F. The determinant of this system is

$$\begin{aligned}D = -\frac{5^2}{8}&(\zeta-\zeta^2-\zeta^3+\zeta^4)(-3-7b+3b\alpha-2b^2-\alpha)(-2b-1+\alpha)\\ &\times(2b+\alpha+1)(2b+11+5\alpha)(-b+2+\alpha)(-2b-11+5\alpha)^4,\end{aligned} \tag{19}$$

which I claim is not zero.

Ignoring the constant term $\frac{\pm 5^2\sqrt{5}}{8}$ in front, multiply the rest by the polynomial in (19) obtained by replacing α with $-\alpha$. This gives the polynomial

$$2^{16}(b^2-4b-1)(b^4+7b^3+4b^2+18b+1)(b^2+11b-1)^5(b^2+b-1)^2.$$

If b is a root of any of the quadratic factors, then $z = b - \frac{1}{b}$ is rational: $z = 4, -11$, or -1, respectively. In these cases $j(w) = -102400/3, \infty$, or $-25/2$, all of which are impossible, since $j(w)$ is an algebraic integer.

Now $E_5(b)$ has complex multiplication by an order in the field $K = \mathbb{Q}(\sqrt{-d})$ whose discriminant is not divisible by 5. Therefore, $j(w) = j(E_5(b))$

generates an extension of $\mathbb{Q}$ which is not ramified at $p = 5$. If b is a root of $h(x) = x^4 + 7x^3 + 4x^2 + 18x + 1$, then $\mathrm{disc}(h(x)) = -5^8 19$ and $\mathrm{Gal}(h(x)/\mathbb{Q}) \cong D_4$ imply that $K(b)$ can only be abelian over the quadratic field $K = \mathbb{Q}(\sqrt{-19})$ and $f = 1$. Then $j_5(b)$ is a root of the irreducible polynomial

$$H(x) = x^4 + 5584305x^3 - 32305549025x^2 + 63531273863125x - 5^6 31^3 449^3,$$

which is impossible, since $K = \mathbb{Q}(\sqrt{-19})$ has class number 1. This shows that the determinant D in (19) is nonzero, and therefore, since the coefficients A_i and D lie in the field F, we get that the solution $(u^4, u^3, u^2, u, 1)$ of the system (18) lies in F also. This proves that $u \in F$. In particular, $\tau(b) = (\varepsilon u)^{-5}$ is a 5-th power in F.

We can find formulas for u from the identities

$$r^5\left(\frac{-1}{5\tau}\right) = \frac{-r^5(\tau) + \varepsilon^5}{\varepsilon^5 r^5(\tau) + 1} \quad \text{and} \quad r\left(\frac{-1}{w}\right) = \frac{\bar{\varepsilon} r(w) + 1}{r(w) - \bar{\varepsilon}}. \tag{20}$$

See [10, pp. 150, 142]. If $\tau = w/5$ and $b = r^5(w/5)$, we have

$$r^5\left(\frac{-1}{w}\right) = \frac{-b + \varepsilon^5}{\varepsilon^5(b - \bar{\varepsilon}^5)} = \frac{1}{\varepsilon^5 u^5},$$

and we can take

$$u = \frac{1}{\varepsilon r\left(\frac{-1}{w}\right)} = \frac{r(w) - \bar{\varepsilon}}{\varepsilon(\bar{\varepsilon} r(w) + 1)} = -\frac{r(w) - \bar{\varepsilon}}{r(w) - \varepsilon}, \quad b = r^5(w/5). \tag{21}$$

On the other hand, if $b = \frac{-1}{r^5(w/5)}$, then we can choose

$$u = -\frac{\bar{\varepsilon} r(w) + 1}{\varepsilon r(w) + 1}.$$

In either case it is clear that $r(w), r(-1/w) \in F$.

We can apply the same analysis with b replaced by $\tau(b)$, since $E_{5,5}(b) \cong E_5(\tau(b))$, so that the latter curve also has complex multiplication by R_{-d}. Furthermore,

$$b = r^5(w/5) \implies \tau(b) = r^5\left(\frac{-1}{w}\right),$$

while

$$b = \frac{-1}{r^5(w/5)} \implies \tau(b) = \frac{-1}{r^5(-1/w)}.$$

Note also that when b is replaced by $\tau(b)$ in the determinant D, its factors in b are

$$\frac{(2b+1)(b-2)(b+3)(-3-7b+3b\alpha-2b^2-\alpha)b^4}{(2b+11+5\alpha)^{10}},$$

and so are nonzero by the same reason as before. Using (16) again, we get a solution $u_1 \in F$ of the equation

$$u_1^5 = \phi_1(\tau(b)) = -\frac{\bar{\varepsilon}^5}{b} = \frac{1}{\varepsilon^5 b}.$$

Therefore, $b = 1/(\varepsilon u_1)^5$ is also a 5-th power in F, i.e. $r(w/5) \in F$. $\square$

Remarks. (1) The fact that $r(w), r(w/5) \in F$ also follows from [6, Theorem 15.16], since $F = L_{\mathcal{O},5}$. The above proof does not make use of Shimura's reciprocity law.

(2) The result $r(w), r(w/5) \in F$ is sharper than what is obtained from [23, Thm. 5.1.2, p. 123]. That theorem only yields that $r(w), r(w/5)$ lie in Σ_{5f}, the ray class field of conductor $5f$. Also, the coefficients of the q-expansion of $r(-1/\tau)$ are in $\mathbb{Q}(\sqrt{5})$ but not all in $\mathbb{Q}$, so [23, Theorem 5.2.1] does not apply.

(3) The results of [22] show that the coordinates of all the points in $E_5(b)[5] - \langle (0,0) \rangle$ are rational functions of the quantity u, and therefore of the quantity $r(w)$, with coefficients in $\mathbb{Q}(\zeta_5)$, by (21). It follows from the theory of complex multiplication that $L_{\mathcal{O},5} = F = K(\zeta_5, r(w))$. In Corollary 4.7 and Theorem 4.8 below we will prove that $L_{\mathcal{O},5} = F = \mathbb{Q}(r(w))$ when $d > 4$. See the discussion in [6, pp. 315-316] for the case $d = 4$.

Now b satisfies the equation $b - \frac{1}{b} = z = -11 - x_1^3 \in \Omega_f$, so b is at most quadratic over Ω_f. Hence, its degree over $\mathbb{Q}$ is at most $4h(-d)$. This degree is also at least $h(-d)$ since $j(w) \in \mathbb{Q}(b)$.

Proposition 3.2. *If $d > 4$, the degree of $z = b - 1/b$ over $\mathbb{Q}$ is $2h(-d)$. Thus, $\Omega_f = \mathbb{Q}(z)$, and the minimal polynomial $\mathcal{R}_d(X)$ of z over $\mathbb{Q}$ is normal.*

Remark. Our use of $\mathcal{R}_d(X)$ in this paper is unrelated to the polynomial $R_n(x)$ discussed in Part I.

Proof. Recall from above that

$$j(w) = j_5(b) = -\frac{(z^2 + 12z + 16)^3}{z + 11},$$

and

$$j(w/5) = j_{5,5}(b) = -\frac{(z^2 - 228z + 496)^3}{(z + 11)^5}.$$

Since $z = -11 - x_1^3 \in \Omega_f$ and the real number $j(w)$ has degree $h(-d)$ over $\mathbb{Q}$, it is clear that the degree of z is either $h(-d)$ or $2h(-d)$. Suppose the degree is $h(-d)$. Then $\mathbb{Q}(z) = \mathbb{Q}(j(w))$, which implies that z is real, and therefore $j(w/5)$ is also real. We also know $j(w/5) = j(\wp_{5,d})$, where $\wp_{5,d} = \wp_5 \cap \mathsf{R}_{-d}$, so that $j(\wp_{5,d}) = \overline{j(\wp_{5,d})} = j(\wp_{5,d}^{-1})$ implies that $\wp_5$ must have order 1 or 2 in the ring class group of K (mod f).

If $\wp_5 \sim 1$ (mod f), then $4 \cdot 5 = x_2^2 + dy_2^2$ for some integers x_2, y_2, which implies that $d = 4, 11, 16, 19$, the first of which is excluded. In the last three cases we have, respectively

$$H_{-11}(x) = x + 32^3, \quad H_{-16}(x) = x - 66^3, \quad H_{-19}(x) = x + 96^3.$$

(See [6].) In these cases there is only one irreducible polynomial $Q_d(x)$ of degree $4h(-d) = 4$ or less which divides $F_d(x)$ in (2), which must therefore be the minimal polynomial of b. We have

$$Q_{11}(x) = x^4 + 4x^3 + 46x^2 - 4x + 1, \quad Q_{16}(x) = x^4 + 18x^3 + 200x^2 - 18x + 1,$$
$$Q_{19}(x) = x^4 + 36x^3 + 398x^2 - 36x + 1.$$

To each of these polynomials with root b corresponds the minimal polynomial $\mathcal{R}_d(x)$ with root $z = b - \frac{1}{b}$. These are:

$$\mathcal{R}_{11}(x) = x^2 + 4x + 48, \quad \mathcal{R}_{16}(x) = x^2 + 18x + 202, \quad \mathcal{R}_{19}(x) = x^2 + 36x + 400,$$

each of which has the correct degree $2h(-d) = 2$.

Now suppose that the order of $\wp_5$ is 2. Then $\wp_5^2 \sim 1 \pmod{f}$ implies that $4 \cdot 5^2 = x_2^2 + dy_2^2$ for $x_2, y_2 \in \mathbb{Z}$ with $x_2 \equiv y_2 \pmod{2}$, if d is odd, giving the possibilities:

$$d = 51, 91, 99, \quad \text{with} \quad h(-51) = h(-91) = h(-99) = 2;$$

and $5^2 = x_2^2 + \frac{d}{4}y_2^2$, if d is even, in which case we have the following possibilities: $d = 24, 36, 64, 84, 96$, with

$$h(-24) = h(-36) = h(-64) = 2, \quad h(-84) = h(-96) = 4.$$

We use the following class equations (see Fricke [12, III, pp. 401, 405, 420] for $D = -24, -36, -64, -91$; and Fricke [13, III, p. 201] for $D = -51$):

$$H_{-24}(x) = x^2 - 4834944x + 14670139392,$$
$$H_{-36}(x) = x^2 - 153542016x - 1790957481984,$$
$$H_{-51}(x) = x^2 + 5541101568x + 6262062317568,$$
$$H_{-64}(x) = x^2 - 82226316240x - 7367066619912,$$
$$H_{-91}(x) = x^2 + 10359073013760x - 3845689020776448,$$
$$H_{-99}(x) = x^2 + 37616060956672x - 56171326053810176.$$

These polynomials yield the following minimal polynomials for z:

$$\mathcal{R}_{24}(x) = x^4 - 12x^3 + 20x^2 + 3120x + 16912,$$
$$\mathcal{R}_{36}(x) = x^4 + 60x^3 + 3020x^2 + 51984x + 287248,$$
$$\mathcal{R}_{51}(x) = x^4 - 24x^3 + 6800x^2 + 155136x + 852736,$$
$$\mathcal{R}_{64}(x) = x^4 - 216x^3 + 17234x^2 + 430380x + 2362354,$$
$$\mathcal{R}_{91}(x) = x^4 - 216x^3 + 154448x^2 + 3449088x + 18965248,$$
$$\mathcal{R}_{99}(x) = x^4 + 872x^3 + 292624x^2 + 6230016x + 34284288.$$

We computed $H_{-99}(x)$ and $\mathcal{R}_{99}(x)$ directly from (11). In the same way we find

$$\mathcal{R}_{84}(x) = x^8 - 468x^7 + 81124x^6 + 3053232x^5 + 65642496x^4 + 1156633920x^3$$
$$+13586087488x^2 + 88268813568x + 244368064768,$$

$$\mathcal{R}_{96}(x) = x^8 + 324x^7 + 230848x^6 + 5080248x^5 + 32351604x^4 + 88662672x^3 \\ +675333328x^2 + 2681910144x + 7697193232.$$

Each of these polynomials is irreducible, so the quantity z always has degree $2h(-d)$ over $\mathbb{Q}$. Since $z \in \Omega_f$, it follows that $\Omega_f = \mathbb{Q}(z)$. This proves the claim. $\square$

Remark. The class equations appearing in the above proof are all the irreducible factors of the discriminant $\mathrm{disc}_y(\Phi_5(x, y))$ of the classical modular equation $\Phi_5(x, y)$ for $N = 5$.

Theorem 3.3. *With z as in (13) and $d > 4$, the quantities b and $\tau(b) = \frac{-b+\varepsilon^5}{\varepsilon^5 b + 1}$ are 5-th powers in the field F, and if*

$$\xi^5 = \tau(b) \quad \textit{and} \quad \eta^5 = b, \tag{22}$$

then $(X, Y) = (\xi, \eta)$ is a solution in F of the equation

$$X^5 + Y^5 = \varepsilon^5(1 - X^5Y^5). \tag{23}$$

Such numbers ξ and η exist for which $\xi \in \mathbb{Q}(\tau(b))$ and $\eta \in \mathbb{Q}(b)$.

Proof. From (22) and the last part of the proof of Theorem 3.1, we have

$$b = \frac{1}{\varepsilon^5 u_1^5} = \eta^5, \quad \tau(b) = \frac{1}{\varepsilon^5 u^5} = \xi^5;$$

with

$$\eta = \delta\zeta^i r^\delta \left(\frac{w}{5}\right), \quad \xi = \delta\zeta^{\delta j} r^\delta \left(\frac{-1}{w}\right), \quad \delta = \pm 1. \tag{24}$$

The relation $\xi^5 = \tau(\eta^5)$ implies that $(X, Y) = (\xi, \eta)$ lies on (23). It only remains to prove that $\eta = \frac{1}{\varepsilon u_1} = b^{1/5}$ can be chosen to lie in $\mathbb{Q}(b)$. The polynomial $q(X) = X^5 - b$ has the root η and splits completely in F. Since the degree $[F : \Omega_f] = 8$ is not divisible by 5 or by 3, and the degree $[\mathbb{Q}(b) : \Omega_f] = [\mathbb{Q}(b) : \mathbb{Q}(z)]$ divides 2, $q(X)$ has to factor into a product of a linear and a quartic polynomial, or a linear times a product of two quadratics over $\mathbb{Q}(b)$. Hence, at least one root of $q(X)$ has to lie in $\mathbb{Q}(b)$, and we can assume this root is η. In the same way, we can assume $\xi \in \mathbb{Q}(\tau(b))$. $\square$

Remark. When $d = 4$, $(X, Y) = (\xi, \eta) = (-i, i)$ is a solution of the equation (23), corresonding to the values $b = i, z = 2i$.

Using (22), we see that

$$j(w/5) = j(E_5(\tau(b))) = j(E_5(\xi^5)) = \frac{(1 - 12\xi^5 + 14\xi^{10} + 12\xi^{15} + \xi^{20})^3}{\xi^{25}(1 - 11\xi^5 - \xi^{10})},$$

while $\xi^5 = \tau(\eta^5)$ and (8) imply that

$$j(w/5) = \frac{(1 + 228\eta^5 + 494\eta^{10} - 228\eta^{15} + \eta^{20})^3}{\eta^5(1 - 11\eta^5 - \eta^{10})^5}. \tag{25}$$

In the same way we have

$$j(w) = \frac{(1 - 12\eta^5 + 14\eta^{10} + 12\eta^{15} + \eta^{20})^3}{\eta^{25}(1 - 11\eta^5 - \eta^{10})}$$
$$= \frac{(1 + 228\xi^5 + 494\xi^{10} - 228\xi^{15} + \xi^{20})^3}{\xi^5(1 - 11\xi^5 - \xi^{10})^5}.$$

It follows that the minimal polynomials of ξ and η divide the polynomial $F_d(x^5)$, where $F_d(x)$ is given by (2), as well as the polynomial $G_d(x^5)$, where

$$G_d(x^5) = x^{5h(-d)}(1 - 11x^5 - x^{10})^{5h(-d)} H_{-d}(j_{5,5}(x^5)). \tag{26}$$

4. Fields generated by values of $r(\tau)$.

If $\mathcal{R}_d(X)$ is the minimal polynomial of $z = b - 1/b$ over $\mathbb{Q}$, as in Proposition 3.2, define the polynomial $Q_d(X)$ by

$$Q_d(X) = X^{2h(-d)} \mathcal{R}_d\left(X - \frac{1}{X}\right). \tag{27}$$

The case $d = 4$ is unusual, in that

$$F_4(x) = (x^2 + 1)^2(x^4 + 18x^3 + 74x^2 - 18x + 1)^2$$

is divisible by a square factor, so that $Q_4(x) = x^2 + 1$. In all other cases we have the following result. We will need the well-known fact that

$$-z - 11 = x_1(w)^3 \cong \wp_5'^3. \tag{28}$$

(See [9, p.32].)

Proposition 4.1. *If $d > 4$, the polynomial $Q_d(x)$ defined by (27) is an irreducible factor of $F_d(x)$ of degree $4h(-d)$. Both b and $\tau(b)$ are roots of $Q_d(x)$. Furthermore, $Q_d(x^5)$ is divisible by an irreducible factor $p_d(x)$ of degree $4h(-d)$ having η as a root.*

Proof. Certainly, b is a root of $Q_d(x)$. If $Q_d(x)$ were reducible, it would have to factor into a product of two polynomials of degree $2h(-d)$ over $\mathbb{Q}$. Neither of these polynomials would be invariant under $z \to U(z) = \frac{-1}{z}$, since this would imply that $\mathcal{R}_d(x)$ factors. Hence, b would have to lie in Ω_f, and

$$Q_d(x) = f(x) \cdot x^{2h(-d)} f(-1/x)$$

for some irreducible $f(x)$ having b as a root. Next, note that

$$\tau(b) - \frac{1}{\tau(b)} = \bar{\varepsilon}^5 \frac{b - \varepsilon^5}{b - \bar{\varepsilon}^5} + \varepsilon^5 \frac{b - \bar{\varepsilon}^5}{b - \varepsilon^5} = \frac{-11b^2 + 4b + 11}{b^2 + 11b - 1} = \frac{-11z + 4}{z + 11}.$$

Putting $z_1 = \tau(b) - \frac{1}{\tau(b)}$, the last equation gives

$$-z_1 - 11 = \frac{125}{-z - 11} = \frac{125}{x_1(w)^3} = x_1(-5/w)^3,$$

by the transformation formula $\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\eta(\tau)$ for the Dedekind η-function. Furthermore,

$$\frac{-5}{w} = \frac{-5w'}{N(w)} = \frac{-w'}{a} = \frac{-v+\sqrt{-d}}{2a}$$

is an ideal basis quotient for the ideal $\mathfrak{a}' = (a, -w')$, where $\wp_5\mathfrak{a} = (w)$ and therefore $\wp_5'\mathfrak{a}' = (-w')$. It follows that

$$x_1(-5/w)^3 = \left(\frac{\eta\left(\frac{-w'}{5a}\right)}{\eta\left(\frac{-w'}{a}\right)}\right)^6 = \overline{x_1(w/a)^3}.$$

From [9, p.32] we have with $z_2 = \bar{z}_1$ that

$$-z_2 - 11 = x_1(w/a)^3 \cong \wp_5'^3 \cong -z - 11$$

and $J(z_2) = j(w/a)$, in the notation of Lemma 2.2. That lemma implies that $z_2 = z^{\sigma^{-1}}$ is a conjugate of z over K. Hence z_1 is a conjugate of z over $\mathbb{Q}$, and therefore also a root of $\mathcal{R}_d(X) = 0$. This shows that $\tau(b)$ is also a root of $Q_d(x) = 0$. But then either $\tau(b)$ or $\frac{-1}{\tau(b)}$ is a conjugate of b over $\mathbb{Q}$. From the formula (7) for $\tau(b)$, which is linear fractional in ε^5 with determinant $b^2+1 \neq 0$ (for $d > 4$), this would imply that $\sqrt{5} \in \Omega_f$, which is not the case, since $p = 5$ is not ramified in Ω_f. Therefore $Q_d(x)$ is irreducible over $\mathbb{Q}$.

The last assertion of this proposition follows from the equation $\eta^5 = b$ and the above arguments. We have chosen η so that $\eta \in \mathbb{Q}(b)$, so the minimal polynomial of η, namely $p_d(x)$, has degree $4h(-d)$. $\square$

As a corollary of this argument we have:

Corollary 4.2. *The roots of $\mathcal{R}_d(x) = 0$ are invariant under the map $x \to \frac{-11x+4}{x+11}$:*

$$(x+11)^{2h(-d)}\mathcal{R}_d\left(\frac{-11x+4}{x+11}\right) = 5^{3h(-d)}\mathcal{R}_d(x).$$

Note that the substitution $z \to V(z) = \frac{-11z+4}{z+11}$ has the effect of interchanging $j(w)$ and $j(w/5)$, as functions of $z = b - \frac{1}{b}$.

Proposition 4.3. *If $d > 4$, the minimal polynomial $p_d(x)$ of $\eta = b^{1/5}$ over $\mathbb{Q}$ is irreducible and normal over $L = \mathbb{Q}(\zeta_5)$. Furthermore,*

$$F = (\Sigma_5\Omega_f \text{ or } \Sigma_5\Omega_{5f}) = \mathbb{Q}(b, \zeta_5) = \mathbb{Q}(\eta, \zeta_5)$$

is the disjoint compositum of $\mathbb{Q}(b) = \mathbb{Q}(\eta)$ and $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$. The same facts hold with b replaced by $\tau(b)$ and η replaced by ξ.

Proof. We know that a root of $p_d(x)$ generates a quadratic extension of Ω_f over $\mathbb{Q}$. Hence, the field $L(\eta)$ contains $L\Omega_f$. On the other hand, the roots u of (16) are contained in $L(\eta)$, since $\xi = (\varepsilon u)^{-1}$ lies in $\mathbb{Q}(\tau(b)) \subseteq \mathbb{Q}(b, \sqrt{5}) \subseteq$

$L(\eta)$, by Theorem 3.3. Since the X-coordinates of points in $E_5[5]$ generate F over Ω_f, and these X-coordinates are rational functions in u with coefficients in L, by the formulas in [22], it follows that $F = L(\eta) = \mathbb{Q}(b, \zeta_5)$, and therefore $[L(\eta) : L] = \frac{16h(-d)}{4} = 4h(-d)$. This shows that $p_d(x)$ is irreducible over $L = \mathbb{Q}(\zeta_5)$ and implies that $\mathbb{Q}(b) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. □

This proposition also shows that the polynomial $Q_d(x)$ is not normal over $\mathbb{Q}$, since it has both b and $\tau(b)$ as roots, and $\sqrt{5} \notin \mathbb{Q}(b)$. Hence, $p_d(x)$ is also not normal over $\mathbb{Q}$. But $\mathbb{Q}(b) \subset F$ is abelian over K and $\mathbb{Q}(b)$ and $\Omega_f(\zeta_5)$ are linearly disjoint over Ω_f.

Corollary 4.4. *If $Q_d(x^5) = p_d(x)q_d(x)$, then $q_d(x)$ is irreducible over $\mathbb{Q}$, of degree $16h(-d)$, and $p_d(\xi) = 0$. Moreover, $x^{4h(-d)}p_d(-1/x) = p_d(x)$ and $x^{16h(-d)}q_d(-1/x) = q_d(x)$.*

Proof. To show that the polynomial $q_d(x)$ in $Q_d(x^5) = p_d(x)q_d(x)$ is irreducible, note that $b \in \mathbb{Q}(\zeta\eta)$ implies η and therefore also ζ lies in this field. Thus, $\mathbb{Q}(\zeta\eta) = \mathbb{Q}(\zeta, \eta) = F$ has degree 8 over Ω_f and degree $16h(-d)$ over $\mathbb{Q}$. This implies that $\zeta\eta$, which is a root of $Q_d(x^5)$, must be a root of $q_d(x)$, hence $q_d(x)$ is irreducible. Since the set of roots of $Q_d(x^5)$ is stable under the mapping $x \to -1/x$ and $p_d(x)$ and $q_d(x)$ have different degrees, the respective sets of roots of the latter polynomials must also be stable under this map. The fact that $x^{4h(-d)}p_d(-1/x) = p_d(x)$ now follows from the norm formula

$$N_{\mathbb{Q}(\eta)/\mathbb{Q}}(\eta) = N_{\Omega_f/\mathbb{Q}}(N_{\mathbb{Q}(\eta)/\Omega_f}(\eta)) = 1.$$

This holds because (11) implies η is a unit (z is an algebraic integer) and Ω_f is complex. Finally, ξ must also be a root of $p_d(x)$, by Proposition 4.1, since ξ and $\tau(b)$ have degree $4h(-d)$ over $\mathbb{Q}$. □

This corollary allows us to prove the following.

Theorem 4.5. *The quantities η and ξ satisfy*

$$\eta = \delta r^\delta \left(\frac{w}{5}\right), \quad \xi = \delta\zeta^{\delta j} r^\delta \left(\frac{-1}{w}\right), \quad \delta = \pm 1, \ \zeta^j \neq 1, \tag{29}$$

and are roots of $p_d(x)$. Thus, the roots of $p_d(x)$ are conjugates over $\mathbb{Q}$ of the values $r(w/5)$ and $\zeta^j r(-1/w)$ of the Rogers-Ramanujan function $r(\tau)$.

Remark. This and Theorem 3.3 prove the first assertion of Theorem 1.1.

Proof. First note that the map $\sigma : b \to -1/b$ is an automorphism of $\mathbb{Q}(b)$ which fixes $\Omega_f = \mathbb{Q}(z)$. Since η is the only fifth root of b contained in $\mathbb{Q}(b)$, this automorphism takes η to $\eta^\sigma = -1/\eta$ and therefore $\eta - 1/\eta \in \Omega_f$. Furthermore, $\eta' = \zeta\eta$ is a root of the polynomial $q_d(x)$ in Corollary 4.4, and $\eta' \to -1/\eta'$ is likewise an automorphism of order 2 of the field F. But then $\eta' - 1/\eta'$ has degree $8h(-d)$ over $\mathbb{Q}$, since η' is a primitive element for F over

$\mathbb{Q}$, so that $\eta' - 1/\eta' \notin \Omega_f$. On the other hand, the function $r(\tau)$ satisfies the identity

$$r^{-1}(\tau) - 1 - r(\tau) = \frac{\eta(\tau/5)}{\eta(5\tau)},$$

by [10, p. 149]. Putting $\tau = w/5$ therefore gives that

$$r(w/5) - r^{-1}(w/5) = -1 - \frac{\eta(w/25)}{\eta(w)} = -1 - y(w) \in \Omega_f.$$

Now the first formula in (24) implies that $i = 0$, i.e., that the first formula in (29) holds. On the other hand, putting $\tau = -1/w$ gives

$$\begin{aligned} r(-1/w) - r^{-1}(-1/w) =& \frac{\bar{\varepsilon} r(w) + 1}{r(w) - \bar{\varepsilon}} - \frac{r(w) - \bar{\varepsilon}}{\bar{\varepsilon} r(w) + 1} \\ =& -\frac{r^2(w) - 4r(w) - 1}{r^2(w) + r(w) - 1}, \end{aligned} \tag{30}$$

and the last expression is linear fractional (with determinant -5) in the expression

$$r(w) - r^{-1}(w) = -1 - \frac{\eta(w/5)}{\eta(5w)} = -1 - y(5w). \tag{31}$$

In this case, $y(5w) \in \Omega_{5f}$ [23, p. 159], but $y(5w) \notin \Omega_f$, since

$$y(5w)^{24} = \left(\frac{\eta(w/5)}{\eta(w)}\right)^{24} \left(\frac{\eta(w)}{\eta(5w)}\right)^{24} = x_1(w)^{12} \frac{\Delta(w,1)}{\Delta(5w,1)} = x_1(w)^{12} \frac{5^{12}}{\varphi_P(w)},$$

where P is the 2×2 diagonal matrix with entries 5 and 1, in the notation of Hasse [14] and Deuring [9]. By [9, p.43], $\varphi_P(w)$ is a unit, so this gives that $y(5w)^{24} \cong \wp_5'^{12} 5^{12} = \wp_5'^{24} \wp_5^{12}$, i.e. $y(5w)^2 \cong \wp_5'^2 \wp_5$. This equation implies that $\wp_5$ is the square of an ideal in $\Omega_f(y(5w))$, which shows that $y(5w) \notin \Omega_f$. Since $\xi - \xi^{-1} \in \Omega_f$, this proves that $\zeta^j \neq 1$ in (24), i.e. that (29) holds. $\square$

Theorem 4.6. *If $d \neq 4f^2$ and $z = b - \frac{1}{b}$ is given by (11), then $\mathbb{Q}(b) = \Sigma_{\wp_5'} \Omega_f$ is the compositum of Ω_f with the ray class field of conductor $\wp_5'$ over K; and $\mathbb{Q}(\tau(b)) = \Sigma_{\wp_5} \Omega_f$. Furthermore, the normal closure of $\mathbb{Q}(b)$ over $\mathbb{Q}$ is $\mathbb{Q}(b, \sqrt{5}) = \Sigma_{\wp_5} \Sigma_{\wp_5'} \Omega_f$.*

Proof. First note that $[\Sigma_{\wp_5'} : \Sigma] = \phi(\wp_5')/2 = 2$, so that $[\Sigma_{\wp_5'} \Omega_f : \Omega_f] = 2$. Moreover, the quadratic extensions $\Sigma_{\wp_5'} \Omega_f$ and $\Sigma_{\wp_5} \Omega_f$ are contained in $F = \Sigma_5 \Omega_f$, because $\Sigma_{\wp_5'}, \Sigma_{\wp_5} \subset \Sigma_5$. On the other hand, $\mathrm{Gal}(F/\Omega_f) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, so that F has three quadratic subfields over Ω_f. These subfields are $F_1 = \Omega_f(b), F_2 = \Omega_f(\tau(b)), F_3 = \Omega_f(\sqrt{5})$. The field F_3 is normal over $\mathbb{Q}$, while F_1 and F_2 must coincide with the fields $\Sigma_{\wp_5'} \Omega_f$ and $\Sigma_{\wp_5} \Omega_f$. The quantity b satisfies the equation $b^2 - bz - 1 = 0$, whose discriminant $z^2 + 4 =$

$(z+1)(z-1)+5$ is divisible by $\wp_5'$ (by (28)). Now note the congruence (from (5))

$$j(w) \equiv -\frac{(z^2+2z+1)^3}{z+1} \equiv -(z+1)^5 \pmod{\wp_5}.$$

This implies that $j(w)$ is conjugate to $-(z+1) \pmod{\mathfrak{p}}$ for every prime divisor $\mathfrak{p}$ of $\wp_5$ in Ω_f. Further, the discriminant of $H_{-d}(x)$ is not divisible by $p=5$, since the Legendre symbol $\left(\frac{-d}{5}\right)=+1$ (see [8]). Hence, the minimal polynomial $m_d(x)$ of z over K satisfies

$$m_d(x) \equiv (-1)^{h(-d)} H_{-d}(-x-1) \pmod{\wp_5},$$

and factors into irreducibles of degree $f_1 = \mathrm{ord}(\wp_5)$, where f_1 is the order of $\wp_5$ in the ring class group (mod f) of K. If $f_1 \geq 2$, then certainly $x=1$ is not a root of $m_d(z) \pmod{\wp_5}$, so no prime divisor of $\wp_5$ divides $z-1$. If $f_1=1$, then by the calculations of Proposition 3.2, d is 11 or 19 (since $d \neq 16$ by assumption); and it can be checked that

$$\mathcal{R}_{11}(x) \equiv (x+1)(x+3), \quad \mathcal{R}_{19}(x) \equiv x(x+1) \pmod{5}.$$

It follows that no prime divisor of $\wp_5$ divides $z-1$, for any d. Hence, only the prime divisors of $\wp_5'$ in Ω_f can be ramified in $\Omega_f(b)/\Omega_f$. It follows that $\wp_5'$ must divide the conductor of F_1, which proves the first assertion. Then the field $\Sigma_{\wp_5}\Sigma_{\wp_5'}\Omega_f = F_1F_2$ is obviously the smallest normal extension of $\mathbb{Q}$ containing $\mathbb{Q}(b)$. $\square$

Corollary 4.7. *If $d \neq 4f^2$, w is defined by (10), and ζ^j is as in (29), then*

$$\mathbb{Q}(r(w/5)) = \mathbb{Q}(b) = \Sigma_{\wp_5'}\Omega_f, \quad \mathbb{Q}(\zeta^j r(-1/w)) = \mathbb{Q}(\tau(b)) = \Sigma_{\wp_5}\Omega_f,$$

and $\mathbb{Q}(r(-1/w)) = \mathbb{Q}(r(w)) = F = \Sigma_5\Omega_f$. The field $F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(r(w/5))$ is the inertia field for $\wp_5$ in the abelian extension F/K.

Remark. This and Theorem 4.8 prove the remaining assertions in Theorem 1.1. In Cho's notation [5], the field $\Sigma_{\wp_5'}\Omega_f = K_{\wp_5',\mathcal{O}}$, where $\mathcal{O} = \mathsf{R}_{-d}$.

Proof. The first assertion follows directly from Theorems 4.5 and 4.6, since $\mathbb{Q}(r(w/5)) = \mathbb{Q}(\eta) = \mathbb{Q}(b)$. The fact that $\mathbb{Q}(r(-1/w)) = F$ follows from $r^\delta(-1/w) = \delta\zeta^{-\delta j}\xi$ and the proof of Corollary 4.4, which shows that $\zeta^{-\delta j}\xi$ is a root of the irreducible polynomial $q_d(x)$. By (30), $r(w)$ generates a field over $\mathbb{Q}$ containing Ω_f whose degree is at least $8h(-d)$, since $r(-1/w) - r^{-1}(-1/w)$ generates the fixed field of the automorphism

$$r(-1/w) \rightarrow -r^{-1}(-1/w),$$

which also contains $\xi^5 - 1/\xi^5 = \tau(b) - 1/\tau(b)$, i.e., a root of $\mathcal{R}_d(X) = 0$. Hence, $r(w)$ must have degree at least 4 over Ω_f. If this degree equals 4, so that $[\mathbb{Q}(r(w)) : \mathbb{Q}] = 8h(-d)$, then $\mathbb{Q}(r(w))/\Omega_f \subseteq F/\Omega_f$ is a quartic extension which contains $\sqrt{5}$. (This is easiest to see using the correspondence between abelian extensions of Ω_f and characters of $\mathrm{Gal}(F/\Omega_f) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, as in [16, p. 5].) Therefore $r(-1/w) \in \mathbb{Q}(r(w))$ by (20) and would not generate F. This contradiction proves that $r(w)$ has degree $16h(-d)$ over $\mathbb{Q}$ and $\mathbb{Q}(r(w)) = F$. The last assertion follows from the fact that the ramification index of the prime divisors of $\wp_5$ in F/K is $e = 4 = [F : F_1]$, so that F_1 is the maximal subextension of F which is unramified at $\wp_5$. $\square$

In the case $K = \mathbb{Q}(i)$, we have $\Sigma_{\wp_5} = \Sigma_{\wp_5'} = K$, so the conclusion of Theorem 4.6 cannot hold. However, the fact that $\wp_5'$ ramifies and $\wp_5$ does not ramify in the quadratic extension $\Omega_f(b)/\Omega_f$ follows in exactly the same way, since $\mathcal{R}_{16}(x) \equiv (x+1)(x+2) \pmod 5$. This gives the following result.

Theorem 4.8. *If $K = \mathbb{Q}(i)$, $d = 4f^2 > 4$ and $2 \mid f$, then with the value of j in (29),*

$$\mathbb{Q}(r(w/5)) = \mathbb{Q}(b) = \Sigma_{2\wp_5'}\Omega_f \quad \textit{and} \quad \mathbb{Q}(\zeta^j r(-1/w)) = \mathbb{Q}(\tau(b)) = \Sigma_{2\wp_5}\Omega_f.$$

In general, if $d = 4f^2 > 4$, then $\mathbb{Q}(r(-1/w)) = \mathbb{Q}(r(w)) = F = \Sigma_5\Omega_{5f}$; and $F_1 = \mathbb{Q}(\eta)$ is the inertia field for $\wp_5$ in the abelian extension F/K.

Remark. The result $F = \mathbb{Q}(r(w)) = L_{\mathcal{O},5}$ in Corollary 4.7 and Theorem 4.8 generalizes the example in [6, p. 316], which deals with the case $d = 4$.

Proof. In this case we have $f = 2f'$ and $\Omega_{5f} = \Omega_{10}\Omega_f$, by Hasse's Zusatz in [15, p. 326]. Therefore $F = \Sigma_5\Omega_{10}\Omega_f$. On the other hand, $\mathsf{S}_5 \cap \mathsf{P}_{10} \subset \mathsf{S}_{2\wp_5'}$ in $K = \mathbb{Q}(i)$, when these ideal groups are declared modulo 10, so we have that $\Sigma_{2\wp_5'} \subset \Sigma_5\Omega_{10}$ and $\Sigma_{2\wp_5'}\Omega_f \subset F$. Since $[\Sigma_{2\wp_5'} : K] = 2$ and $\wp_5'$ ramifies in $\Sigma_{2\wp_5'}$, it is clear that $[\Sigma_{2\wp_5'}\Omega_f : \Omega_f] = 2$. Now the proof of Theorem 4.6 shows that $\mathbb{Q}(b) = \Sigma_{2\wp_5'}\Omega_f$ and $\mathbb{Q}(\tau(b)) = \Sigma_{2\wp_5}\Omega_f$ and the rest is a consequence of Theorem 4.5 and the same arguments as in the last corollary. $\square$

Remark. When $K = \mathbb{Q}(i)$ and f is odd, the conductor $\mathfrak{f}(F_1/K)$ of F_1/K divides $\wp_5'(f)$, and is divisible by the conductor $\mathfrak{f}(\Omega_f/K)$. Since f is odd, $\mathfrak{f}(\Omega_f/K) = (f)$, so that $\mathfrak{f}(F_1/K) = \wp_5'(f)$. (See [6, Ex. 9.20, pp. 195-196].) In the general case $d > 4$ it is not hard to see that the equality $\mathfrak{f}(F_1/K) = \wp_5'(f)$ still holds, unless $-d = d_K f^2 \neq -4f^2$, $d_K \equiv 1 \pmod 8$, and $f = 2f'$ with odd f'; in which case $\mathfrak{f}(F_1/K) = \wp_5'(f')$. As an example of the latter phenomenon, see the polynomial $p_{124}(x)$ in Table 2 below, for which $f = 2$, but whose discriminant is not divisible by 2.

In Tables 1 and 2 are listed the minimal polynomials $p_d(x)$ of the values $r(w/5)$ for all $d < 150$. For most values of d, $p_d(x)$ was computed from $H_{-d}(x)$ using the fact that $p_d(x) \mid F_d(x^5)$ with $F_d(x)$ in (2). For $d \neq 4f^2$ for which $H_{-d}(x)$ was not available, $p_d(x)$ was computed by approximating to high accuracy the values of $r(\tau) = r(w/(5a))$ at ideal basis quotients of representatives $\wp_5\mathfrak{a} = (5a, w)$ of the classes in the *ray class group modulo* $\mathfrak{f} = \wp_5'$ of R_{-d}, for which $\wp_5^2 \mid (w)$, in line with (10). (See [23, p.88].) This gives $2h(-d)$ values $r(w/(5a))$, which are class invariants for the ideal class group $\mathsf{A}/\mathsf{H}_{\wp_5' f}$, where A is the group of fractional ideals of K prime to $\wp_5'(f)$ and $\mathsf{H} = \mathsf{H}_{\wp_5' f}$ is the ideal group of conductor $\wp_5'(f)$ (or $\wp_5'(f')$) corresponding to the class field $\mathbb{Q}(r(w/5))/K$. Then

$$p_d(x) = \prod_{\mathfrak{a} \ mod \ \mathsf{H}} (x - r\left(\frac{w}{5a}\right))(x - \bar{r}\left(\frac{w}{5a}\right)).$$

A similar computation was carried out for $d = 4f^2$. In Section 5 below we will give an algebraic method for verifying these calculations. The discriminants of these polynomials seem to satisfy the following.

Conjecture. (1) *If $q > 5$ is a prime which divides d_K but does not divide f, then $q^{2h(-d)}$ exactly divides disc$(p_d(x))$.*
(2) *If $h = h(-d)$, $5^{h(2h-1)}$ exactly divides disc$(p_d(x))$.*
(3) *disc$(p_d(x))$ is only divisible by primes $q \le d$.*
(4) *If $q \ne 5$ is a prime dividing disc$(p_d(x))$, then the Kronecker symbol $\left(\frac{-d}{q}\right) \ne 1$.*

5. Periodic points of an algebraic function.

5.1. Preliminary facts on the group G_{60}. In this section we shall make use of the fact that the rational function

$$f_5(z) = \frac{(1 + 228z^5 + 494z^{10} - 228z^{15} + z^{20})^3}{z^5(1 - 11z^5 - z^{10})^5}$$

is invariant under a group G_{60} of linear fractional substitutions:

$$G_{60} = \langle S, T\rangle, \quad S(z) = \zeta z, \quad T(z) = \frac{-(1+\sqrt{5})z + 2}{2z + 1 + \sqrt{5}},$$

which is isomorphic to the icosahedral group A_5. (In this subsection, z is taken to be an indeterminate.) The coefficients of the maps in G_{60} are in the field $\mathbb{Q}(\zeta_5)$. The transformations S and T have orders 5 and 2, respectively, while the transformation

$$U(z) = \frac{-1}{z}$$

is given in terms of S and T by $U = T \cdot S^2 \cdot T \cdot S^3 \cdot T \cdot S^2$. (See [12, II, pp. 42-43].) Furthermore,

$$H = \{1, T, U, TU\}$$

is a Klein-4 subgroup of G_{60}, where $TU(z) = UT(z) = -1/T(z) = T_2(z)$, and

$$T_2(z) = \frac{-(1-\sqrt{5})z + 2}{2z + 1 - \sqrt{5}}.$$

Thus, $U = TT_2 = T_2T$. The normalizer of H in G_{60} is $N = \langle A, H\rangle \cong A_4$, where $A = STS^{-2}$ is the map

$$A(z) = \zeta^3 \frac{(1+\zeta)z + 1}{z - 1 - \zeta^4}$$

TABLE 1. The minimal polynomial $p_d(x)$ of $r(w/5)$, $w = \frac{v+\sqrt{-d}}{2}$, $5^2 \mid N(w)$, $11 \leq d \leq 99$.

d	$p_d(x)$	$\mathrm{disc}(p_d(x))$
11	$x^4 - x^3 + x^2 + x + 1$	$5 \cdot 11^2$
16	$x^4 - 2x^3 + 2x + 1$	$2^6 5$
19	$x^4 + x^3 + 3x^2 - x + 1$	$5 \cdot 19^2$
24	$x^8 - 2x^7 + x^6 - 4x^5 + 3x^4 + 4x^3 + x^2 + 2x + 1$	$2^{12}3^4 5^6$
31	$x^{12} - x^{11} + 5x^{10} - 4x^9 + 8x^8 - 2x^7 + 19x^6 + 2x^5$ $+8x^4 + 4x^3 + 5x^2 + x + 1$	$3^8 5^{15} 31^6$
36	$x^8 + x^6 - 6x^5 + 9x^4 + 6x^3 + x^2 + 1$	$2^8 3^6 5^6 11^4$
39	$x^{16} - 3x^{15} + 7x^{14} - 9x^{13} + 21x^{12} - 15x^{11} + 17x^{10}$ $+3x^9 + 11x^8 - 3x^7 + 17x^6 + 15x^5 + 21x^4$ $+9x^3 + 7x^2 + 3x + 1$	$3^8 5^{28} 7^8 13^8$
44	$x^{12} - x^{11} + 6x^{10} + 15x^8 + 9x^6 + 15x^4 + 6x^2 + x + 1$	$2^8 5^{15} 11^6 19^4$
51	$x^8 + x^7 + x^6 - 7x^5 + 12x^4 + 7x^3 + x^2 - x + 1$	$2^{12} 3^4 5^6 17^4$
56	$x^{16} + 8x^{14} - 4x^{13} + 15x^{12} - 12x^{11} + 50x^{10} + 4x^9$ $+91x^8 - 4x^7 + 50x^6 + 12x^5 + 15x^4 + 4x^3 + 8x^2 + 1$	$2^{40} 5^{28} 7^8 31^4$
59	$x^{12} - 4x^{11} + 5x^{10} - 2x^9 + 14x^8 - 2x^7 - 24x^6 + 2x^5$ $+14x^4 + 2x^3 + 5x^2 + 4x + 1$	$2^{20} 5^{15} 59^6$
64	$x^8 + 4x^7 + 10x^6 + 8x^5 + 12x^4 - 8x^3 + 10x^2 - 4x + 1$	$2^{18} 3^8 5^6$
71	$x^{28} - 6x^{27} + 17x^{26} - 45x^{25} + 104x^{24} - 164x^{23}$ $+277x^{22} - 357x^{21} + 388x^{20} - 319x^{19} + 316x^{18}$ $+135x^{17} - 144x^{16} + 83x^{15} - 551x^{14} - 83x^{13}$ $-144x^{12} - 135x^{11} + 316x^{10} + 319x^9 + 388x^8 + 357x^7$ $+277x^6 + 164x^5 + 104x^4 + 45x^3 + 17x^2 + 6x + 1$	$5^{91} 7^{16} 23^8 71^{14}$
76	$x^{12} - 5x^{11} + 12x^{10} - 2x^9 - 21x^8 + 12x^7 + 35x^6 - 12x^5$ $-21x^4 + 2x^3 + 12x^2 + 5x + 1$	$2^8 3^{12} 5^{15} 19^6$
79	$x^{20} + 9x^{18} - 12x^{17} + 18x^{16} - 9x^{15} + 117x^{14} - 33x^{13}$ $+99x^{12} - 207x^{11} + 353x^{10} + 207x^9 + 99x^8 + 33x^7$ $+117x^6 + 9x^5 + 18x^4 + 12x^3 + 9x^2 + 1$	$3^{28} 5^{45} 29^8 79^{10}$
84	$x^{16} + 2x^{15} - 4x^{14} - 12x^{13} + 25x^{12} - 18x^{11} + 68x^{10}$ $-112x^9 + 13x^8 + 112x^7 + 68x^6 + 18x^5 + 25x^4 + 12x^3$ $-4x^2 - 2x + 1$	$2^{32} 3^{20} 5^{28} 7^8 59^4$
91	$x^8 + 4x^7 - x^6 - 14x^5 + 23x^4 + 14x^3 - x^2 - 4x + 1$	$2^8 3^4 5^6 7^4 13^4$
96	$x^{16} + 4x^{15} + 29x^{12} - 24x^{11} + 86x^{10} - 32x^9 + 105x^8$ $+32x^7 + 86x^6 + 24x^5 + 29x^4 - 4x + 1$	$2^{32} 3^{24} 5^{28} 71^4$
99	$x^8 + 7x^7 + 15x^6 + 15x^5 + 16x^4 - 15x^3 + 15x^2 - 7x + 1$	$2^{12} 3^4 5^6 11^4$

TABLE 2. The minimal polynomial $p_d(x)$ of $r(w/5)$, $w = \frac{v+\sqrt{-d}}{2}$, $5^2 \mid N(w)$, $104 \le d \le 144$.

d	$p_d(x)$	$\mathrm{disc}(p_d(x))$
104	$x^{24} - 4x^{23} + 20x^{22} - 40x^{21} + 53x^{20} - 28x^{19} + 94x^{18}$	$2^{84}5^{66}13^{12}$
	$-92x^{17} + 42x^{6} - 76x^{15} + 782x^{14} - 328x^{13} - 272x^{12}$	$\times 29^{8}79^{4}$
	$+328x^{11} + 782x^{10} + 76x^{9} + 42x^{8} + 92x^{7} + 94x^{6}$	
	$+28x^{5} + 53x^{4} + 40x^{3} + 20x^{2} + 4x + 1$	
111	$x^{32} - 4x^{31} + 21x^{30} - 31x^{29} + 144x^{28} - 180x^{27}$	$3^{52}5^{120}11^{12}$
	$+563x^{26} - 435x^{25} + 1398x^{24} - 653x^{23} + 2108x^{22}$	$\times 37^{16}43^{8}61^{8}$
	$+380x^{21} + 4093x^{20} + 1273x^{19} + 4560x^{18} - 990x^{17}$	
	$+7975x^{16} + 990x^{15} + 4560x^{14} - 1273x^{13} + 4093x^{12}$	
	$-380x^{11} + 2108x^{10} + 653x^{9} + 1398x^{8} + 435x^{7}$	
	$+563x^{6} + 180x^{5} + 144x^{4} + 31x^{3} + 21x^{2} + 4x + 1$	
116	$x^{24} - 6x^{23} + 12x^{22} - 24x^{21} + 99x^{20} - 58x^{19} + 136x^{18}$	$2^{80}5^{66}7^{8}$
	$-256x^{17} + 144x^{16} + 410x^{15} + 436x^{14} + 274x^{13}$	$\times 29^{12}41^{8}$
	$-1192x^{12} - 274x^{11} + 436x^{10} - 410x^{9} + 144x^{8} + 256x^{7}$	
	$+136x^{6} + 58x^{5} + 99x^{4} + 24x^{3} + 12x^{2} + 6x + 1$	
119	$x^{40} - x^{39} + 12x^{38} - 51x^{37} + 146x^{36} - 248x^{35} + 569x^{34}$	$5^{190}7^{20}11^{24}$
	$-951x^{33} + 2005x^{32} - 3810x^{31} + 8702x^{30} - 14440x^{29}$	$\times 17^{20}19^{12}$
	$+26580x^{28} - 35295x^{27} + 47491x^{26} - 45351x^{25}$	$\times 23^{16}47^{8}$
	$+53426x^{24} - 29809x^{23} + 41387x^{22} - 6812x^{21}$	
	$+31769x^{20} + 6812x^{19} + 41387x^{18} + 29809x^{17}$	
	$+53426x^{16} + 45351x^{15} + 47491x^{14} + 35295x^{13}$	
	$+26580x^{12} + 14440x^{11} + 8702x^{10} + 3810x^{9} + 2005x^{8}$	
	$+951x^{7} + 569x^{6} + 248x^{5} + 146x^{4} + 51x^{3}$	
	$+12x^{2} + x + 1$	
124	$x^{12} - 7x^{11} + 9x^{10} + 8x^{9} + 24x^{8} + 6x^{7} - 67x^{6} - 6x^{5}$	$3^{12}5^{15}11^{4}31^{6}$
	$+24x^{4} - 8x^{3} + 9x^{2} + 7x + 1$	
131	$x^{20} + 20x^{18} + 8x^{17} + 48x^{16} + 4x^{15} + 72x^{14} + 88x^{13}$	$2^{76}5^{45}31^{4}$
	$+348x^{12} + 168x^{11} + 446x^{10} - 168x^{9} + 348x^{8} - 88x^{7}$	$\times 131^{10}$
	$+72x^{6} - 4x^{5} + 48x^{4} - 8x^{3} + 20x^{2} + 1$	
136	$x^{16} + 6x^{15} + 25x^{14} + 24x^{13} - 3x^{12} + 119x^{10} + 174x^{9}$	$2^{56}3^{16}5^{28}11^{8}$
	$+404x^{8} - 174x^{7} + 119x^{6} - 3x^{4} - 24x^{3} + 25x^{2} - 6x + 1$	$\times 17^{8}$
139	$x^{12} - 5x^{11} + 12x^{10} + 16x^{9} + 33x^{8} + 12x^{7} - 55x^{6}$	$2^{24}3^{12}5^{15}139^{6}$
	$-12x^{5} + 33x^{4} - 16x^{3} + 12x^{2} + 5x + 1$	
144	$x^{16} - 2x^{15} + 18x^{14} + 24x^{13} + 83x^{12} + 78x^{11} + 74x^{10}$	$2^{24}3^{12}5^{28}7^{8}$
	$+40x^{9} + 9x^{8} - 40x^{7} + 74x^{6} - 78x^{5} + 83x^{4} - 24x^{3}$	$\times 11^{4}19^{8}$
	$+18x^{2} + 2x + 1$	

of order 3, and $ATA^{-1} = U, AUA^{-1} = T_2$. Also, $A^\sigma = A^{-1}U$ is the conjugate map

$$A^\sigma(z) = \zeta \frac{(1+\zeta^2)z + 1}{z - 1 - \zeta^3},$$

obtained by applying the automorphism $\sigma : \zeta \to \zeta^2$ to the coefficients. In particular, $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is a subgroup of the automorphism group $\mathrm{Aut}(N)$.

It is clear from (8) and (26) that $\deg(G_d(x^5)) = 60h(-d)$. The group G_{60} acts on the irreducible factors $p(x)$ of $G_d(x^5)$ over $L = \mathbb{Q}(\zeta_5)$, one of which is $p_d(x)$ (Proposition 4.3), by

$$p^\sigma(x) = (cx+d)^{deg(p)} p(\sigma(x)) = (cx+d)^{deg(p)} p\left(\frac{ax+b}{cx+d}\right), \quad \sigma \in G_{60},$$

ignoring constant factors. Moreover, G_{60} acts transitively on these irreducible factors over the field L (see the analogous argument in [17, p. 1982]), so $G_d(x^5)$ splits into 15 irreducible factors of degree $4h(-d)$ over L, by Proposition 4.3. In particular, these considerations show that every root of $G_d(x^5)$ has the form $\sigma(\alpha)$ for some root α of $p_d(x)$ and some $\sigma \in G_{60}$.

The group $G_{60} \cong A_5$ has no elements of order 4, so the stabilizer of $p_d(x)$ is one of the five conjugate subgroups in G_{60} of the subgroup H. We have that

$$S^{-1}US(z) = \frac{-\zeta^3}{z}, \quad S^{-1}TS(z) = \frac{-(1+\sqrt{5})z + 2\zeta^4}{2\zeta z + (1+\sqrt{5})}.$$

Hence, only one these conjugate subgroups, namely H, contains the map U, and since U fixes $p_d(x)$ by Corollary 4.4, we have

$$\mathrm{Stab}_{G_{60}}(p_d(x)) = H = \{1, T, U, TU\}.$$

As a consequence, we have that

$$\left(z + \frac{1+\sqrt{5}}{2}\right)^{4h(-d)} p_d(T(z)) = \left(\frac{5+\sqrt{5}}{2}\right)^{2h(-d)} p_d(z).$$

It can be checked that the factor on the right side of this equation is correct by putting z equal to

$$z_1 = \frac{-1 - \sqrt{5} + \sqrt{10 + 2\sqrt{5}}}{2},$$

which is a fixed point of $T(z)$, and noting that $p_d(z_1) \neq 0$, since $\mathbb{Q}(z_1)$ is a cyclic quartic extension of $\mathbb{Q}$ in which $p = 5$ is totally ramified.

We also note that all of the roots of $p_d(x)$ are values of the Rogers-Ramanujan function $r(\tau)$. This follows from the identity (see [10, p. 138]):

$$j(\tau) = \frac{(r^{20} - 228r^{15} + 494r^{10} + 228r^5 + 1)^3}{r^5(1 - 11r^5 - r^{10})^5} = f_5(r), \quad r = r(\tau).$$

Any root α of $p_d(x)$ satisfies $f_5(\alpha) = j(w/a)$ for some w of the form (10) and some positive integer a, by (26). However, the above identity implies that $f_5(r(w/a)) = j(w/a)$. It follows that α and $r(w/a)$ are related by an element M of the group G_{60}. Now we use Proposition 2 of [10], according to which

$$r(\tau+1) = S(r(\tau)), \quad r\left(\frac{-1}{\tau}\right) = T(r(\tau)) \quad \tau \in \mathbb{H}.$$

It follows that the action of any mapping $M \in G_{60}$ on a value $r(\tau)$ can be represented by a suitable element $\mu \in \Gamma = SL_2(\mathbb{Z})$, such that $M(r(\tau)) = r(\mu(\tau))$; hence,

$$\alpha = M(r(w/a)) = r(\mu(w/a))$$

is a value of the function $r(\tau)$ with $\tau \in K$. This argument applies to all the roots of $G_d(x^5)$. (Since $r(\tau)$ is a Hauptmodul for $\Gamma(5)$, the above formulas imply that $G_{60} \cong \bar{\Gamma}(5)$; see [24, p. 76].)

5.2. Automorphisms of F_1/K. Now let ψ be an automorphism of the extension $F = \Omega_f(\xi, \zeta_5)$ which fixes $\Omega_f(\xi) = \Omega_f(\tau(b))$ and sends ζ to ζ^2. Then ψ takes $\sqrt{5}$ to $-\sqrt{5}$, so that

$$(\eta^5)^\psi = b^\psi = \tau(\xi^5)^\psi = \frac{-\xi^5 + \bar{\varepsilon}^5}{\bar{\varepsilon}^5\xi^5 + 1} = -\frac{\varepsilon^5\xi^5 + 1}{-\xi^5 + \varepsilon^5} = \frac{-1}{\eta^5}.$$

It follows that $\eta^\psi = \frac{-\zeta^i}{\eta}$, for some i. Thus, $\zeta^i \in \Omega_f(\eta)$ and $i \equiv 0 \pmod 5$, giving $\eta^\psi = \frac{-1}{\eta}$.

Next, let ϕ be an automorphism of F which takes η to ξ and fixes ζ (this exists by Proposition 4.3 and Corollary 4.4). Then

$$\tau(b)^\phi = (\xi^5)^\phi = \tau(\eta^5)^\phi = \tau(\xi^5) = \eta^5 = b,$$

so that $\xi^\phi = \eta$ by Theorem 3.3, since $\zeta \notin \mathbb{Q}(b)$. Hence ϕ has order 2 in $\mathrm{Gal}(F/\mathbb{Q})$. Furthermore, since

$$-z^\phi - 11 = -\left(b - \frac{1}{b}\right)^\phi - 11 = -\left(\tau(b) - \frac{1}{\tau(b)}\right) - 11 = -z_1 - 11,$$

we see from (28) and $-z_1 - 11 \cong \wp_5^3$ (see the proof of Proposition 4.1) that ϕ interchanges the ideals $\wp_5'$ and $\wp_5$. Thus, ϕ does not fix the field K.

Since $T \in H$, the map $\sigma_1 = (\eta \to T(\eta))$ also represents an automorphism of order 2 of F/L. Setting $\upsilon = \eta - \frac{1}{\eta} \in \Omega_f$, and noting that υ is an algebraic integer, we have

$$T(\eta) - \frac{1}{T(\eta)} = -\frac{\eta^2 - 4\eta - 1}{\eta^2 + \eta - 1} = -\frac{\upsilon - 4}{\upsilon + 1} = -1 + \frac{5}{\upsilon + 1},$$

so that

$$(\upsilon + 1)^{\sigma_1} = \frac{5}{\upsilon + 1}. \tag{32}$$

The identity

$$x^5 - \frac{1}{x^5} = \left(x - \frac{1}{x}\right)^5 + 5\left(x - \frac{1}{x}\right)^3 + 5\left(x - \frac{1}{x}\right)$$

gives that

$$z = b - \frac{1}{b} = \upsilon^5 + 5\upsilon^3 + 5\upsilon,$$

and implies

$$z \equiv \upsilon^5 \pmod 5.$$

It follows that

$$z + 11 \equiv z + 1 \equiv (\upsilon + 1)^5 \pmod 5,$$

so $\upsilon + 1$ is divisible by $\wp_5'$ but not by any prime divisors of $\wp_5$. Equation (32) implies that $(\upsilon + 1) = \left(\frac{\eta^2+\eta-1}{\eta}\right) = \wp_5'$, and that σ_1 interchanges the ideals $\wp_5$ and $\wp_5'$. This also shows that

$$\wp_5 = \left(\frac{5\eta}{\eta^2 + \eta - 1}\right) = \left(\frac{\xi^2 + \xi - 1}{\xi}\right) \text{ in } \Omega_f.$$

5.3. Periodic points. Thus, the automorphism $\sigma_1\phi$ fixes the field K, and it follows from (25) and the fact that σ_1 fixes the rational function $f_5(\eta)$ that

$$j(w/5)^{\sigma_1\phi} = \frac{(1 + 228\xi^5 + 494\xi^{10} - 228\xi^{15} + \xi^{20})^3}{\xi^5(1 - 11\xi^5 - \xi^{10})^5} = j(w).$$

Since $\sigma_1\phi$ fixes the quadratic field K and $K(j(w)) = \Omega_f$, we deduce that

$$(\sigma_1\phi)|_{\Omega_f} = \left(\frac{\Omega_f/K}{\wp_5}\right).$$

We would like to extend this automorphism to the abelian extension $F_1 = \mathbb{Q}(\eta) = \Omega_f(\eta)$ of K, in which $\wp_5$ is still unramified. This can be done in two ways. On the one hand, the restriction of

$$\tau_5 = \left(\frac{F_1/K}{\wp_5}\right) = \left(\frac{\mathbb{Q}(b)/K}{\wp_5}\right)$$

to Ω_f is certainly the same as $(\sigma_1\phi)|_{\Omega_f}$. But the automorphism $\rho = \psi|_{F_1} = (\eta \to \frac{-1}{\eta})$ of F_1 fixes Ω_f, so that $\rho\tau_5 = \tau_5\rho \in \mathrm{Gal}(F_1/K)$ also restricts to $(\sigma_1\phi)|_{\Omega_f}$. Hence we have that

$$\tau_5 = \sigma_1\phi \quad \text{or} \quad \tau_5\rho = \sigma_1\phi \quad \text{on } F_1.$$

This gives

$$\eta^{\tau_5} = \eta^{\sigma_1\phi} = T(\eta)^{\phi} = T(\xi), \quad \text{or} \quad \eta^{\tau_5\rho} = \eta^{\sigma_1\phi} = T(\xi).$$

Hence,

$$\xi = T(\eta^{\tau_5}) = \frac{-(1+\sqrt{5})\eta^{\tau_5} + 2}{2\eta^{\tau_5} + 1 + \sqrt{5}} \quad \text{or} \quad \xi = T_2(\eta^{\tau_5}) = \frac{-(1-\sqrt{5})\eta^{\tau_5} + 2}{2\eta^{\tau_5} + 1 - \sqrt{5}}.$$

In the following theorem we eliminate the second of these possibilities.

Theorem 5.1. *If* $\tau_5 = \left(\frac{\Omega_f(\eta)/K}{\wp_5}\right)$, *the coordinates of the solution* (ξ, η) *of* $\mathcal{C}_5$ *satisfy*

$$\xi = T(\eta^{\tau_5}) = \frac{-(1+\sqrt{5})\eta^{\tau_5} + 2}{2\eta^{\tau_5} + 1 + \sqrt{5}}. \tag{33}$$

Proof. Assume that $d > 4$. It suffices to show that $T(\xi) = \eta^{\tau_5}$, and to do this we show that $T(\xi) \equiv \eta^5 \pmod{\wp_5}$ in $F_1 = \mathbb{Q}(\eta)$. We have

$$\begin{aligned} T(\xi) - \eta^5 = T(\xi) - \tau(\xi^5) &= \frac{\bar{\varepsilon}\xi + 1}{\xi - \bar{\varepsilon}} - \frac{-\xi^5 + \varepsilon^5}{\varepsilon^5\xi^5 + 1}. \\ &= \frac{-\xi + \varepsilon}{\varepsilon\xi + 1} + \frac{\xi^5 - \varepsilon^5}{\varepsilon^5\xi^5 + 1} \\ &= \frac{(5+2\sqrt{5})(\xi^2+1)(\xi-\varepsilon)^2}{(\xi^2 + \xi + \frac{3+\sqrt{5}}{2})(\xi^2 - \frac{3+\sqrt{5}}{2}\xi + \frac{3+\sqrt{5}}{2})}, \end{aligned}$$

by factoring this rational function in ξ and $\sqrt{5}$ on Maple. Now multiply this expression by

$$(T(\xi) - \eta^5)^{\psi} = T_2(\xi) + \frac{1}{\eta^5}.$$

This yields the equation

$$(T(\xi) - \eta^5)\left(T_2(\xi) + \frac{1}{\eta^5}\right) = \frac{5(\xi^2+1)^2(\xi^2+\xi-1)^2}{p_1(\xi)p_2(\xi)} \tag{34}$$

in F_1, where

$$p_1(\xi) = \xi^4 + 2\xi^3 + 4\xi^2 + 3\xi + 1, \quad p_2(\xi) = \xi^4 - 3\xi^3 + 4\xi^2 - 2\xi + 1.$$

Expanding the element $\xi^{-4}p_1(\xi)p_2(\xi)$ of Ω_f π-adically in terms of the generating element $\pi = (\xi^2+\xi-1)/\xi$ of $\wp_5$ gives

$$\xi^{-4}p_1(\xi)p_2(\xi) = \pi^4 - 5\pi^3 + 15\pi^2 - 25\pi + 25, \quad \pi = \frac{\xi^2+\xi-1}{\xi},$$

and shows that the squares of prime divisors $\mathfrak{q}$ of $\wp_5$ in F_1 exactly divide $p_1(\xi)p_2(\xi)$ (recall that $\wp_5$ is unramified in F_1 and ξ is a unit). This shows that $\frac{(\xi^2+1)^2(\xi^2+\xi-1)^2}{p_1(\xi)p_2(\xi)}$ is a $\mathfrak{q}$-adic integer of F_1 for each $\mathfrak{q} \mid \wp_5$, and (34) gives that

$$(T(\xi) - \eta^5)\left(T_2(\xi) + \frac{1}{\eta^5}\right) \equiv 0 \bmod \wp_5.$$

It follows that $T(\xi) \equiv \eta^5$ or $T_2(\xi) = \frac{-1}{T(\xi)} \equiv \frac{-1}{\eta^5} \pmod{\mathfrak{q}}$ for each $\mathfrak{q}$. Since $T(\xi)$ and η are units, the latter congruence implies that $T(\xi) \equiv \eta^5$ (mod $\mathfrak{q}$), which therefore holds for all $\mathfrak{q}$ dividing $\wp_5$. Thus we have $T(\xi) \equiv \eta^5$ (mod $\wp_5$). This implies finally that $T(\xi) = \eta^{\tau_5}$, since $T(\xi) = \eta^{\tau_5\rho}$ would give $\eta^{\rho} \equiv \eta \pmod{\mathfrak{q}}$, so $\eta \equiv \pm 2 \pmod{\mathfrak{q}}$ and $z \equiv \pm 1$ (mod $N_{F_1/\Omega_f}(\mathfrak{q})$). As in the proof of Theorem 4.6, this can only happen when $f_1 = \mathrm{ord}(\wp_5) = 1$

in the ring class group (mod f) of K and $d = 11, 16, 19$. In these cases $[\mathbb{Q}(\eta) : K] = 2$, so $\mathrm{Gal}(\mathbb{Q}(\eta)/K) = \{1, \rho\}$. In the first two cases τ_5 has order 2, so $\tau_5 = \rho$, while in the third case $\tau_5 = 1$. In all three cases the formula (33) can be checked directly. $\square$

Note that $\tau_5 = 1$ on $K = \mathbb{Q}(i)$ and $T(i) = T_2(i) = -i$, so the solution $(\xi, \eta) = (-i, i)$ of $\mathcal{C}_5$ is covered by Theorem 5.1.

If we substitute the expression in Theorem 5.1 for ξ into the equation for $\mathcal{C}_5$ and simplify, we obtain:

$$(\eta^{4\tau_5} + 2\eta^{3\tau_5} + 4\eta^{2\tau_5} + 3\eta^{\tau_5} + 1)\eta^5 = \eta^{\tau_5}(\eta^{4\tau_5} - 3\eta^{3\tau_5} + 4\eta^{2\tau_5} - 2\eta^{\tau_5} + 1). \quad (35)$$

Thus, we have:

Theorem 5.2. *If*

$$g(X, Y) = (Y^4 + 2Y^3 + 4Y^2 + 3Y + 1)X^5 - Y(Y^4 - 3Y^3 + 4Y^2 - 2Y + 1),$$

then $(X, Y) = (\eta, \eta^{\tau_5})$ *is a point on the curve* $g(X, Y) = 0$.

From this we deduce the following.

Theorem 5.3. *The roots of* $p_d(x)$ *are periodic points of the multi-valued algebraic function* $\mathfrak{g}(z)$ *defined by* $g(z, \mathfrak{g}(z)) = 0$. *The period of* η *with respect to the action of* $\mathfrak{g}$ *is the order of* $\tau_5 = \left(\frac{\mathbb{Q}(\eta)/K}{\wp_5}\right)$ *in Gal*$(\mathbb{Q}(\eta)/K)$.

Remark. See the Introduction of Part I for the definition of a periodic point of an algebraic function.

Proof. Since $g(X, Y)$ has rational coefficients, applying τ_5^i $(1 \le i \le n-1)$ to the equation $g(\eta, \eta^{\tau_5}) = 0$ gives that

$$g(\eta, \eta^{\tau_5}) = g(\eta^{\tau_5}, \eta^{\tau_5^2}) = \cdots = g(\eta^{\tau_5^{n-1}}, \eta) = 0,$$

where $n = \mathrm{ord}(\tau_5)$. Thus, η is one of the values of the iterate $\mathfrak{g}^{(n)}(\eta)$, i.e., is periodic with period n. Any conjugate over $\mathbb{Q}$ of a periodic point of $\mathfrak{g}(z)$ is also a periodic point, and this proves the theorem. $\square$

Using the same idea as in Part I, Section 3 ([20]; see also [19, p. 875]), it can be shown that the order of τ_5 is the *minimal* period of a root of $p_d(x)$ in Theorem 5.3. Details will be provided in Part III of this paper.

By Artin Reciprocity, the order of τ_5 is equal to the order of $\wp_5$ in the quotient group $\mathsf{A}/(\mathsf{S}_{\wp_5'} \cap \mathsf{P}_f)$ (when $d \ne 4f^2$), where A is the group of fractional ideals in K which are relatively prime to $\wp_5'(f)$. If this order is n, then there is an equation $\wp_5^n = (\frac{x+y\sqrt{-d}}{2})$, and since $y\sqrt{-d} \equiv x \pmod{\wp_5'}$, it follows that $\alpha = \frac{x+y\sqrt{-d}}{2} \equiv 2x/2 = x \equiv \pm 1 \pmod{\wp_5'}$. Therefore, when $d \ne 4f^2$, the period n of the roots of $p_d(x)$ is the smallest positive integer n for which there is an equation $4 \cdot 5^n = x^2 + dy^2$ with $x \equiv \pm 1 \pmod 5$ and $(x, y) \mid 2$.

The substitution $(X,Y) \rightarrow \left(\frac{-1}{X}, \frac{-1}{Y}\right)$ represents an automorphism of the curve $g(X,Y)=0$, since

$$X^5Y^5 g\left(\frac{-1}{X}, \frac{-1}{Y}\right) = g(X,Y).$$

The equation connecting $t = X - \frac{1}{X}$ and $u = Y - \frac{1}{Y}$ in the function field of this curve is

$$\begin{aligned} h(t,u) = u^5 &- (6+5t+5t^3+t^5)u^4 + (21+5t+5t^3+t^5)u^3 \\ &- (56+30t+30t^3+6t^5)u^2 + (71+30t+30t^3+6t^5)u \\ &- 120 - 55t - 55t^3 - 11t^5 = 0; \end{aligned} \tag{36}$$

this follows from the calculation

$$-h(t,u)^2 = Res_y(Res_x(g(x,y), x^2 - tx - 1), y^2 - uy - 1).$$

From $g(\eta, \eta^{\tau_5}) = 0$ and $\upsilon^{\tau_5} = \eta^{\tau_5} - \frac{1}{\eta^{\tau_5}}$ we obtain

$$h(\upsilon, \upsilon^{\tilde{\tau}_5}) = 0, \quad \tilde{\tau}_5 = \tau_5|_{\Omega_f} = \left(\frac{\Omega_f/\mathbb{Q}(\sqrt{-d})}{\wp_5}\right).$$

This yields the following result.

Theorem 5.4. *If Ω_f is the ring class field of conductor f (relatively prime to 5) over the field $K = \mathbb{Q}(\sqrt{-d})$, where $-d = d_K f^2$ and $\left(\frac{-d}{5}\right) = +1$, then $\Omega_f = K(\upsilon)$, where $\upsilon = \eta - \frac{1}{\eta}$ is a periodic point of the algebraic function $\mathfrak{f}(z)$ defined by $h(z, \mathfrak{f}(z)) = 0$, and $h(t,u)$ is given by equation (36). The period of υ is the order of $\tilde{\tau}_5 = \tau_5|_{\Omega_f}$ in $Gal(\Omega_f/K)$.*

Now we compare (35) with Ramanujan's modular equation

$$r^5(\tau) = r(5\tau)\frac{r^4(5\tau) - 3r^3(5\tau) + 4r^2(5\tau) - 2r(5\tau) + 1}{r^4(5\tau) + 2r^3(5\tau) + 4r^2(5\tau) + 3r(5\tau) + 1}$$

for $r(\tau)$. Letting z be an indeterminant and setting

$$\mathfrak{r}(z) = \frac{z(z^4 - 3z^3 + 4z^2 - 2z + 1)}{z^4 + 2z^3 + 4z^2 + 3z + 1},$$

we conclude from (35) and Theorem 4.5 that

$$\mathfrak{r}(\eta^{\tau_5}) = \eta^5 = r^5(w/5) = \mathfrak{r}(r(w)), \quad \text{if } b = r^5(w/5). \tag{37}$$

It is easily checked on Maple that the quintic extension of function fields $\mathbb{Q}(\zeta_5, z)/\mathbb{Q}(\zeta_5, \mathfrak{r}(z))$ is normal and cyclic, with generating automorphism

$$z \rightarrow \mathfrak{s}(z) = \frac{(\zeta + \zeta^2)z + 1}{z + 1 + \zeta + \zeta^2},$$

where $\mathfrak{s}(z) = S^{-2}AS(z) = S^{-1}TS^{-1}(z)$ is an element of G_{60}. It follows from (37) that

$$\eta^{\tau_5} = \mathfrak{s}^i(r(w)), \text{ for some } \ i, \ 0 \le i \le 4.$$

From Corollary 4.7 and Theorem 4.8 we know that $i \neq 0$, since $\eta^{\tau_5} \in F_1$, but $r(w)$ generates F. More specifically, we have the following.

Theorem 5.5. *With notation as above, if* $\xi = \zeta^j r(-1/w)$, $1 \leq j \leq 4$, *we have the formula*

$$r(w/5)^{\tau_5} = \mathfrak{s}^j(r(w)) = T(\xi),$$

and j *is the unique integer* (*mod* 5) *for which* $\mathfrak{s}^j(r(w))$ *is a root of* $p_d(x)$.

Proof. We have that $\xi = \zeta^j r(-1/w) = S^j T(r(w))$, by the transformation formula for $r(-1/w)$, so $T(\xi) = TS^jT(r(w))$. On the other hand, $\mathfrak{s}(z) = S^{-1}TS^{-1}(z) = TST(z)$, since $(ST)^3 = 1$. Therefore, $\mathfrak{s}^j(r(w)) = (TST)^j(r(w)) = TS^jT(r(w)) = T(\xi)$ since T is its own inverse. The above formula now follows from (33). This proves that $\mathfrak{s}^j(r(w))$ is a root of $p_d(x)$, since $p_d(x)$ is stabilized by T. There is only one value of i for which $\mathfrak{s}^i(r(w))$ is a root of $p_d(x)$, since $T(\mathfrak{s}^i(r(w))) = S^iT(r(w)) = \zeta^i r(-1/w)$ must also be a root of $p_d(x)$. ∎

Remark. Since $\mathfrak{s}(z) = TST(z)$, $\mathfrak{s}(r(w)) = TST(r(w)) = TS(r(-1/w)) = T(r(1-1/w)) = r(-w/(w-1))$. Thus, $\mathfrak{s}^j(r(w)) = r(w/(1-jw))$.

Example 1. Consider Ramanujan's remarkable value

$$r(3i) = \sqrt{c^2+1} - c, \quad 2c = \frac{60^{1/4} + 2 - \sqrt{3} + \sqrt{5}}{60^{1/4} - 2 + \sqrt{3} - \sqrt{5}}\sqrt{5} + 1$$

established in [3] and [4, p.142]. A calculation on Maple shows that the minimal polynomial of $r(3i) = \zeta_5 r(4+3i) = \zeta r(w)$ is

$$\begin{aligned} m(x) = x^{16} &+ 38x^{15} - 240x^{14} - 300x^{13} - 235x^{12} - 726x^{11} + 92x^{10} - 1840x^9 \\ &- 675x^8 + 1840x^7 + 92x^6 + 726x^5 - 235x^4 + 300x^3 - 240x^2 - 38x + 1, \end{aligned}$$

which is a factor of $G_{36}(x^5)$ in (26). (Use the polynomial $H_{-36}(x)$ given in the proof of Proposition 3.2.) Thus, $r(3i)$ is a linear fractional expression in some conjugate of $\eta = r\left(\frac{4+3i}{5}\right)$ with coefficients in $L = \mathbb{Q}(\zeta_5)$, and the minimal polynomial of the latter value is

$$p_{36}(x) = x^8 + x^6 - 6x^5 + 9x^4 + 6x^3 + x^2 + 1,$$

from Table 1. Using Maple to compare approximations of $r\left(\frac{4+3i}{5}\right)$ and the roots of $p_{36}(x)$, we find

$$r\left(\frac{4+3i}{5}\right) = \frac{-i\omega^2}{2} + \frac{i\sqrt{3}}{2} - \frac{\omega}{4}\sqrt[4]{3}\left(\sqrt{4+2\sqrt{5}} + i\sqrt{-4+2\sqrt{5}}\right), \quad (38)$$

with $\omega = \frac{-1+i\sqrt{3}}{2}$.

We determine the linear fractional expression in a root of $p_{36}(x)$ which will equal $r(3i)$. Since

$$p_{36}(x) \equiv (x+3)^4(x^4 + 3x^3 + x^2 + 2x + 1) \pmod 5,$$

the Frobenius automorphism τ_5 has order 4. A calculation on Maple shows that

$$\mathfrak{s}^2(r(w)) = \frac{(\zeta+\zeta^3)r(w)+1}{r(w)+1+\zeta+\zeta^3} = 1.375418808... - (.899074105...)i$$

is the unique value $\mathfrak{s}^j(r(w))$ which is a root of $p_{36}(x)=0$. By Theorem 5.5 we have

$$\eta^{\tau_5} = \mathfrak{s}^2(r(w)) = \frac{(\zeta+\zeta^3)r(w)+1}{r(w)+1+\zeta+\zeta^3} = \frac{(1+\zeta^2)r(3i)+1}{\zeta^4 r(3i)+1+\zeta+\zeta^3}. \tag{39}$$

Inverting the linear fractional map in the last equality gives

$$r(3i) = \frac{(1+\zeta^3)\eta^{\tau_5}+\zeta}{\eta^{\tau_5}-\zeta-\zeta^3};$$

this is the desired expression for $r(3i)$. Another calculation on Maple using (38) and (39) shows that

$$\eta^{\tau_5} = r\left(\frac{4+3i}{5}\right)^{\tau_5} = \frac{-i\omega}{2} - \frac{i\sqrt{3}}{2} + i\frac{\omega^2}{4}\sqrt[4]{3}\left(\sqrt{4+2\sqrt{5}} + i\sqrt{-4+2\sqrt{5}}\right).$$

This expresses $r(3i)$ in terms of 3rd, 4th, and 5th roots of unity and shows that τ_5 can be given by

$$\tau_5 = \left(\sqrt[4]{3} \rightarrow -i\sqrt[4]{3}, i \rightarrow i, \sqrt{4+2\sqrt{5}} \rightarrow \sqrt{4+2\sqrt{5}}\right)|_{F_1}.$$

This proves formula (6) of the Introduction.

Remark. In this example, $F = \Sigma_5\Omega_{15}$ has degree $8h(-36) = 16$ over $K = \mathbb{Q}(i)$, so its real subfield F^+ has degree 16 over $\mathbb{Q}$ and the value $r(3i)$ generates F^+. In particular, $K(r(3i)) = \Sigma_5\Omega_{15}$. Since $\sqrt{3} \in \Omega_3 \subset \Omega_{15}$ and $\sqrt{5} \in \Omega_5 \subset \Omega_{15}$, Ramanujan's formula shows that $60^{1/4} \in \Sigma_5\Omega_{15}$. On the other hand, $\Omega_3(60^{1/4})$ is a cyclic quartic extension of Ω_3. As in the proof of Theorem 4.6, there are only two cyclic quartic extensions of Ω_3 contained in $\Sigma_5\Omega_{15}$, namely, $\Sigma_5\Omega_3 = \Omega_3(\zeta_5)$ and Ω_{15} (see Section 3); and the former is abelian over $\mathbb{Q}$. Hence, we have $\Omega_{15} = K(\sqrt{3}, \sqrt[4]{60})$. As a corollary, this shows that the rational primes which split completely in Ω_{15}, which are the primes representable as $p = a^2 + 15^2 b^2$, are characterized by the two conditions $p \equiv 1 \pmod{12}$ and $\left(\frac{60}{p}\right)_4 = +1$.

Given that the period of η in the above example is $n = 4$, $p_{36}(x)$ can be calculated by a threefold iterated resultant, as in Part I, Section 3, pp. 727-730. Namely, $p_{36}(x)$ is a factor of

$$R_4(x) = Res_{x_3}(Res_{x_2}(Res_{x_1}(g(x,x_1), g(x_1,x_2)), g(x_2,x_3)), g(x_3,x)).$$

Unfortunately, this calculation takes an extremely long time to complete, since $\deg(R_4(x)) = 2\cdot 5^4 - 1 = 1249$.

To get around this difficulty, we let g_1 be the polynomial $g_1(X,Y) = Y^5 g(X, \frac{-1}{Y})$, i.e.,

$$g_1(X,Y) = Y(Y^4 - 3Y^3 + 4Y^2 - 2Y + 1)X^5 + (Y^4 + 2Y^3 + 4Y^2 + 3Y + 1).$$

The class number $h(-36) = 2$, so $[F_1 : K] = 4$; hence $\mathrm{Gal}(F_1/K) = \langle \tau_5 \rangle$, implying that $\tau_5^2 = \rho$ on F_1. Putting $\tau = \tau_5$, we have

$$g(\eta, \eta^\tau) = g(\eta^\tau, \eta^{\tau^2}) = 0.$$

However, $g(\eta^\tau, \eta^{\tau^2}) = g(\eta^\tau, \eta^\rho) = g(\eta^\tau, -1/\eta)$, so that

$$g(\eta, \eta^\tau) = g_1(\eta^\tau, \eta) = 0.$$

Therefore, $p_{36}(x)$ should be a factor of the resultant

$$\begin{aligned}
\tilde{R}_2(x) &= Res_{x_1}(g(x,x_1), g_1(x_1,x)) \\
&= -(x^2+1)(x^8 + x^7 + x^6 - 7x^5 + 12x^4 + 7x^3 + x^2 - x + 1) \\
&\times (x^8 + 4x^7 - x^6 - 14x^5 + 23x^4 + 14x^3 - x^2 - 4x + 1) \\
&\times (x^8 - 2x^7 + x^6 - 4x^5 + 3x^4 + 4x^3 + x^2 + 2x + 1) \\
&\times (x^8 + x^6 - 6x^5 + 9x^4 + 6x^3 + x^2 + 1) \\
&\times (x^{16} + 4x^{15} + 29x^{12} - 24x^{11} + 86x^{10} - 32x^9 + 105x^8 \\
&\quad + 32x^7 + 86x^6 + 24x^5 + 29x^4 - 4x + 1) \\
&= -(x^2+1)p_{51}(x)p_{91}(x)p_{24}(x)p_{36}(x)p_{96}(x).
\end{aligned}$$

Hence, the discriminants with $d \in \{24, 36, 51, 91, 96\}$ are *all* the discriminants for which $\tau_5^2 = \rho$. An analysis similar to the above for $d = 36$ can be applied for these integers d to yield formulas for the corresponding values of the Rogers-Ramanujan continued fraction $r(w)$, namely,

$$r(12+\sqrt{-6}),\ r\left(\frac{7+\sqrt{-51}}{2}\right),\ r\left(\frac{3+\sqrt{-91}}{2}\right),\ r(1+2\sqrt{-6}).$$

In addition, for small values of n, the $(n-1)$-fold iterated resultant

$$\tilde{R}_n(x) = R_{x_{n-1}}(...(R_{x_2}(R_{x_1}(g(x,x_1), g(x_1,x_2)), g(x_2,x_3)), ..., g_1(x_{n-1},x)),$$

where R_{x_i} on the right side of this equation denotes the resultant with respect to x_i, can be used to determine minimal polynomials of $r(w/5)$ for the values of $d \equiv \pm 1 \pmod 5$ for which $\rho \in \langle \tau_5 \rangle$ and $\tau_5^n = \rho$.

Example 2. For example, $\tilde{R}_3(x)$ has degree 226 and is the product of (x^2+1) and 2 factors of degree 4, 3 factors of degree 12, 4 factors of degree

24, and one factor each of degree 36 and 48. The degree 36 factor is

$$\begin{aligned} p_{491}(x) = {} & x^{36} + 28x^{35} + 206x^{34} - 324x^{33} + 2163x^{32} + 2080x^{31} + 1600x^{30} \\ & + 19440x^{29} + 9145x^{28} + 60876x^{27} + 21486x^{26} - 5532x^{25} + 220279x^{24} \\ & + 208904x^{23} + 453304x^{22} - 117152x^{21} - 62271x^{20} + 142940x^{19} \\ & + 1116798x^{18} - 142940x^{17} - 62271x^{16} + 117152x^{15} + 453304x^{14} \\ & - 208904x^{13} + 220279x^{12} + 5532x^{11} + 21486x^{10} - 60876x^{9} + 9145x^{8} \\ & - 19440x^{7} + 1600x^{6} - 2080x^{5} + 2163x^{4} + 324x^{3} + 206x^{2} - 28x + 1, \end{aligned}$$

with discriminant $D = 2^{316}5^{153}7^{16}19^{4}23^{8}29^{16}191^{8}491^{18}$. The value $d = 491$ is a guess based on the conjecture at the end of Section 4. This can be verified by factoring $p_{491}(x)$ modulo primes of the form $p = (x^2 + 491y^2)/4$, with $x + 3y \equiv \pm 2 \pmod 5$ (assuming that $w = \frac{3+\sqrt{-491}}{2}$), to check that it splits into linear and quadratic factors. For example, $p_{491}(x)$ factors into a product of linear polynomials modulo the primes $179 = \frac{15^2+491}{4}, 3251 = \frac{27^2+5^2\cdot 491}{4}$, and $3989 = 45^2 + 2^2 \cdot 491$; while it splits into a product of 18 linear factors and 9 quadratics modulo $1237 = \frac{23^2+3^2\cdot 491}{4}$, corresponding to the fact that $(\alpha) = \left(\frac{23+3\sqrt{-491}}{2}\right)$ satisfies $\alpha \equiv 1$, but $\alpha' \equiv 2 \pmod{\wp_5'}$. As an additional check, $\eta = r\left(\frac{3+\sqrt{-491}}{10}\right)$ is a root of $p_{491}(x)$ (to an accuracy of at least 60 decimal places). Note that $\mathrm{ord}(\tau_5) = 6$, since $\tau_5^3 = \rho$ has order 2, so the roots of $p_{491}(x)$ have period 6 with respect to the action of $\mathfrak{g}(z)$. This aligns with the fact that $4 \cdot 5^3 = 3^2 + 491$ and $4 \cdot 5^6 = 241^2 + 3^2 \cdot 491$ and that

$$\alpha_1 = \frac{3+\sqrt{-491}}{2} \notin \mathsf{S}_{\wp_5'} \quad \text{but} \quad \alpha_2 = \frac{241+3\sqrt{-491}}{2} \in \mathsf{S}_{\wp_5'}.$$

In general, it is more convenient to work with a lower degree polynomial derived from $p_d(x)$ using the fact that it is stabilized by the subgroup H. First write $p_d(x) = x^{2h(-d)} t_d(x - 1/x)$, which is possible since $p_d(x)$ is stabilized by $U(z) = -1/z$ (or $\eta^\rho = -1/\eta$ is an automorphism fixing Ω_f). Then $t_d(x)$ is a normal polynomial with root $\upsilon = \eta - 1/\eta$ generating Ω_f. By (32), we can write $t_d(x-1) = x^{h(-d)} u_d\left(x + \frac{5}{x}\right)$. This yields the polynomial $u_d(x)$ having degree $h(-d)$ and smaller discriminant. In the above example we find

$$\begin{aligned} u_{491}(x) = {} & x^9 + 10x^8 - 144x^7 - 840x^6 + 18354x^5 - 110972x^4 + 345800x^3 \\ & - 601496x^2 + 550293x - 205102, \end{aligned}$$

whose discriminant is $D_1 = 2^{76}7^{2}29^{4}191^{2}491^{4}$. It is straightforward to check that $7, 29, 191$ divide the index and 491 does not (using Dedekind's method in [7, pp. 214-218], for example), so we only have to exclude $q = 2$ and $q = 29$ as divisors of d. However, $h(-4\cdot 29) = 6$ and $h(-491) = 9$ yield that $d = 491f^2$, where $f = 2^a$. If $a \geq 2$, then $h(-d)$ is even, while $h(-4\cdot 491) = 27$, so the only possibility is $d = 491$.

A similar analysis was applied to check the polynomials in Tables 1 & 2.

We will continue this discussion in Part III, by showing that the only irreducible factors of iterated resultants of the form $R_n(x)$ or $\tilde{R}_n(x)$ are the polynomials x, x^2+1, and $p_d(x)$, for $d \equiv \pm 1 \pmod 5$. This will prove that the polynomial $p_{491}(x)$ given above actually is the minimal polynomial of $r(w/5)$ for $w = \frac{3+\sqrt{-491}}{2}$.

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This paper is available via http://nyjm.albany.edu/j/2019/25-49.html.