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Nodal curves on K3 surfaces

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Abstract. In this paper, we study the Severi variety $V_{L,g}$ of genus g curves in $|L|$ on a general polarized K3 surface (X, L). We show that the closure of every component of $V_{L,g}$ contains a component of $V_{L,g-1}$. As a consequence, we see that the general members of every component of $V_{L,g}$ are nodal.

Contents

1. Introduction

It was proved that every complete linear system on a very general polarized K3 surface (X, L) contains a nodal rational curve [C1] and furthermore every rational curve in $|L|$ is nodal, i.e., has only nodes $xy = 0$ as singularities [C2]. The purpose of this note is to prove an analogous result on singular curves in $|L|$ of geometric genus $g > 0$.

For a line bundle A on a projective surface X, we use the notation $V_{A,g}$ to denote the Severi varieties of integral curves of geometric genus g in the complete linear series $|A| = \mathbb{P}H^0(A)$. For a K3 surface X, it is well known that every component of $V_{A,g}$ has the expected dimension g. Furthermore, using theory of deformation of maps, one can show that $\nu : \widehat{C} \to X$ is an immersion for ν the normalization of a general member $[C] \in V_{A,g}$ if $g > 0$ [HM, Chap. 3, Sec. B].

It was claimed that a general member of $V_{A,g}$ is nodal on every projective K3 surface X and every $A \in \mathrm{Pic}(X)$ as long as $g > 0$ in [C1, Lemma 3.1]. However, as kindly pointed out to the author by Edoardo Sernesi [DS, Sec. 3.3], the proof there is wrong. So this note provides a partial fix for this

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problem, albeit only for singular curves in the primitive class $|L|$ on a general polarized K3 surface (X, L). Our main theorem is

Theorem 1.1. *For a general polarized K3 surface (X, L), every (irreducible) component of $\overline{V}_{L,g}$ contains a component of $V_{L,g-1}$ for all $1 \le g \le p_a(L)$, where $\overline{V}_{L,g}$ is the closure of $V_{L,g}$ in $|L|$ and $p_a(L) = L^2/2+1$ is the arithmetic genus of L.*

Clearly, the above theorem, combining with the fact that every rational curve in $|L|$ is nodal [C2], implies the following corollary by induction:

Corollary 1.2. *For a general polarized K3 surface (X, L), the general members of every component of $V_{L,g}$ are nodal for all $0 \le g \le p_a(L)$.*

It was proved in [KLM, Theorem 1.3, 5.3 and Remark 5.6] that the general members of every component of $V_{L,g}$ are not trigonal for $g \ge 5$. Combining with [DS, Theorem B.4], it shows that the corollary holds for $5 \le g \le p_a(L)$. Of course, we have settled it for all genus g here. As an application, it shows that the genus g Gromov-Witten invariant computed in [BL] is the same as the number of genus g curves in $|L|$ passing through g general points.

A comprehensive treatment for $V_{mL,g}$ is planned in a future paper.

As another potential application of Theorem 1.1, we want to mention the conjecture of the irreducibility of universal Severi variety $\mathcal{V}_{L,g}$ on K3 surfaces:

Conjecture 1.3. *Let $\mathcal{K}_p$ be the moduli space of polarized K3 surfaces (X, L) of genus $p = p_a(L)$ and let*

$$\mathcal{V}_{L,g} = \{(X, L, C) : (X, L) \in \mathcal{K}_p, C \in V_{L,g}\} \tag{1.1}$$

be the universal Severi variety of genus g curves in $|L|$ over $\mathcal{K}_p$. Then $\mathcal{V}_{L,g}$ is irreducible.

If we approach the conjecture along the line of argument of J. Harris for the irreducibility of Severi variety of plane curves [H], we need to establish two facts:

- Every component of $\overline{\mathcal{V}}_{L,g}$ contains a component of $\mathcal{V}_{L,0}$.
- $\mathcal{V}_{L,0}$ is irreducible and the monodromy action on the p nodes of a rational curve $C \in V_{L,0}$ is the full symmetric group Σ_p as (X, L, C) moves in $\mathcal{V}_{L,0}$.

The second fact comes easily for plane curves, while the establishment of the first fact is the focus of Harris' proof (see also [HM, Chap. 6, Sec. E]). The situation for $\mathcal{V}_{L,g}$ is somewhat reversed at the moment: the first fact follows from our main theorem, while the difficulty lies in the second fact:

Conjecture 1.4. *Let $\mathcal{V}_{L,0}$ be the universal Severi variety of rational curves in $|L|$ over the moduli space $\mathcal{K}_p$ of polarized K3 surfaces (X, L) of genus p*

and let

$$\mathcal{W}_{L,0} = \big\{(X, L, C, s_1, s_2, ..., s_p) : (X, L, C) \in \mathcal{V}_{L,0}, \\ C_{sing} = \{s_1, s_2, ..., s_p\}\big\}. \tag{1.2}$$

Then $\mathcal{W}_{L,0}$ is irreducible.

Our above discussion shows that Conjecture 1.4 implies 1.3.

Conventions. We work exclusively over $\mathbb{C}$. A K3 surface in this paper is always projective. A polarized K3 surface is a pair (X, L), where X is a K3 surface and L is an indivisible ample line bundle on X.

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2. Proof of Theorem 1.1

We start with the following observation:

Proposition 2.1. *Let W be a component of $V_{L,g}$ for a polarized K3 surface (X, L) with $\operatorname{Pic}(X) = \mathbb{Z}$. The following are equivalent:*

(1) *The closure $\overline{W}$ of W in $|L|$ contains a component of $V_{L,g-1}$.*
(2) $\dim(\overline{W}\backslash W) = g - 1$.
(3) *For a set σ of $g - 1$ general points on X, $W \cap \Lambda_\sigma$ is not projective (i.e. complete), where $\Lambda_\sigma \subset |L|$ is the locus of curves $C \in |L|$ passing through σ.*

Proof. (1) $\Rightarrow$ (2) is obvious. Since every curve in $|L|$ is integral, we have

$$\overline{W}\backslash W \subset \bigcup_{i<g} V_{L,i}. \tag{2.1}$$

And since $\dim V_{L,i} \le i$, we have (2) $\Rightarrow$ (1).

Let $\partial W = \overline{W}\backslash W$. Obviously, $\dim(\partial W \cap \Lambda_\sigma) = \dim \partial W - (g-1)$. Therefore, (2) $\Rightarrow$ (3). On the other hand, if $W \cap \Lambda_\sigma$ is not complete, then there exists $C_\sigma \in \partial W$ passing through σ. Then $\dim \partial W \ge g-1$. So (3) $\Rightarrow$ (2). □

So it suffices to show that $W \cap \Lambda_\sigma$ is not complete for every component W of $V_{L,g}$. We prove this using a degeneration argument similar to the one in [C2]. A general K3 surface can be specialized to a *Bryan-Leung* (BL) K3 surface X_0, which is a K3 surface with Picard lattice

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

It can be polarized by the line bundle $C + mF$, where C and F are the generators of $\operatorname{Pic}(X_0)$ satisfying $C^2 = -2$, $CF = 1$ and $F^2 = 0$. A general polarized K3 surface of genus m can be degenerated to $(X_0, C+mF)$. Such X_0 has an elliptic fibration $X_0 \to \mathbb{P}^1$ with fibers in $|F|$. For a general BL

K3 surface X_0, there are exactly 24 nodal fibers in $|F|$. A key fact here is that every member of $|C + mF|$ is "completely" reducible in the sense that it is a union of C and m fibers in $|F|$ (counted with multiplicities).

Let X be a family of K3 surfaces of genus m over a smooth quasi-projective curve T such that X_0 is a general BL K3 surface for a point $0 \in T$, X_t are K3 surfaces of $\text{Pic}(X_t) = \mathbb{Z}$ for $t \neq 0$ and L is a line bundle on X with $L_0 = C + mF$. After a base change, there exists $W \subset \mathcal{V}_{L,g}$ flat over T such that W_t is a component of $V_{L_t,g}$ for all $t \neq 0$. Let σ be a set of $g - 1$ general sections of X/T. It suffices to prove that $W_t \cap \Lambda_\sigma$ is not projective for t general.

By stable reduction, there exists a family $f : Y \to X$ of genus g stable maps over a smooth surface S with the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\pi} & T \end{array} \tag{2.3}$$

where S is flat and projective over T, $f_* Y_s \in \overline{W}_t \cap \Lambda_\sigma$ on X_t for all $s \in S_t$ and $t \in T$ and S dominates $\overline{W} \cap \Lambda_\sigma$ via the map sending $s \to [f_* Y_s]$. In other words, $f : Y \to X$ is the stable reduction of the universal family over $\overline{W}$ such that $f : Y_s \to X$ is the normalization of a general member $G \in W_t$ passing through the $g - 1$ points $\sigma(t)$ for $s \in S_t$ general and $t \neq 0$.

Let us consider the moduli map $\rho : S \to \overline{\mathcal{M}}_g \times T$ sending $s \to ([Y_s], \pi(s))$, where $\overline{\mathcal{M}}_g$ is the moduli space of stable curves of genus g with $\mathcal{M}_g$ its open subset parameterizing smooth curves. To show that $W_t \cap \Lambda_\sigma$ is not complete, it suffices to show that

$$\rho^{-1}(\Delta \times T) \cap S_t \neq \emptyset \tag{2.4}$$

for $t \neq 0$, where $\Delta = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ is the boundary divisor of $\overline{\mathcal{M}}_g$.

Let $F_1, F_2, ..., F_{g-1} \subset X_0$ be $g - 1$ fibers in $|F|$ passing through the $g - 1$ points $\sigma(0)$, respectively. Since $\sigma(0)$ are in general position, $F_1, F_2, ..., F_{g-1}$ are $g - 1$ general fibers in $|F|$ and $\sigma(0) \cap C = \emptyset$.

For every $s \in S_0$, $f_* Y_s \in |C + mF|$ passes through $\sigma(0)$. Therefore, we must have

$$f_* Y_s = C + m_1 F_1 + m F_2 + ... + m_{g-1} F_{g-1} + M_s \tag{2.5}$$

for some $m_1, m_2, ..., m_{g-1} \in \mathbb{Z}^+$. Since the curves in $W_t \cap \Lambda_\sigma$ cover X_t for $t \neq 0$, f is surjective. Hence $f_* Y_s$ covers X_0 as s moves in S_0. Therefore, M_s contains a moving fiber in $|F|$. More precisely, there exists a component Γ of S_0 such that $\cup_{s \in \Gamma} M_s = X_0$.

For a general point $s \in \Gamma$, M_s contains a general fiber F_s in $|F|$. Therefore, Y_s has components $\widehat{F}_{1,s}, \widehat{F}_{2,s}, ..., \widehat{F}_{g-1,s}, \widehat{F}_s$ dominating $F_1, F_2, ..., F_{g-1}, F_s$, respectively. And since $p_a(Y_s) = g$, $\widehat{F}_{1,s}, \widehat{F}_{2,s}, ..., \widehat{F}_{g-1,s}, \widehat{F}_s$ are all elliptic curves. Indeed, it is very easy to see that its moduli $[Y_s]$ in $\overline{\mathcal{M}}_g$

$$[Y_s] = [\widehat{C}_s \cup \widehat{F}_{1,s} \cup \widehat{F}_{2,s} \cup ... \cup \widehat{F}_{g-1,s} \cup \widehat{F}_s] \tag{2.6}$$

is a smooth rational curve $\widehat{C}_s$ with g elliptic "tails" $\widehat{F}_{1,s}, \widehat{F}_{2,s}, ..., \widehat{F}_{g-1,s}, \widehat{F}_s$ attached to it, where $\widehat{C}_s$ is the component of Y_s dominating C. Of course, when $g \le 2$, $\widehat{C}_s$ is contracted under the moduli map.

Note that $\widehat{F}_{1,s}, \widehat{F}_{2,s}, ..., \widehat{F}_{g-1,s}, \widehat{F}_s$ are isogenous to $F_1, F_2, ..., F_{g-1}, F_s$, respectively. As s moves on Γ, F_s moves in $|F|$. So $\widehat{F}_s$ has varying moduli. This shows that ρ maps S generically finitely onto its image. That is,

$$\dim \rho(S) = 2. \tag{2.7}$$

Furthermore, when F_s becomes one of 24 nodal fibers in $|F|$, $\widehat{F}_s$ becomes a union of rational curves. Therefore, there exists $b \in \Gamma$ such that $\widehat{F}_b$ is a connected union of rational curves with normal crossings and $p_a(\widehat{F}_b) = 1$. The moduli $[Y_b]$ of Y_b is thus a smooth rational curve with $g-1$ elliptic tails and one nodal rational curve attached to it. Consequently,

$$\rho(b) \in \Delta_0 \times T \tag{2.8}$$

where Δ_0 is the component of Δ whose general points parameterize curves of genus $g-1$ with one node. Combining (2.7), (2.8) and the fact that Δ_0 is $\mathbb{Q}$-Cartier, we conclude that

$$\rho(S) \cap (\Delta_0 \times T) \ne \emptyset \text{ has pure dimension } 1. \tag{2.9}$$

Therefore, for every connected component G of $\rho^{-1}(\Delta_0 \times T)$, we have

$$\dim \rho(G) = 1. \tag{2.10}$$

If $\rho^{-1}(\Delta_0 \times T) \cap S_t \ne \emptyset$ for $t \ne 0$, then (2.4) follows and we are done. Otherwise,

$$\rho^{-1}(\Delta_0 \times T) \subset S_0. \tag{2.11}$$

Let G be the connected component of $\rho^{-1}(\Delta_0 \times T)$ containing the point b. Then $G \subset S_0$ and $\dim \rho(G) = 1$.

Let B be an irreducible component of G passing through b. For Y_b, we have

$$f_* Y_b = C + m_1 F_1 + m F_2 + ... + m_{g-1} F_{g-1} + M_b \tag{2.12}$$

with M_b supported on the union F_Σ of 24 nodal rational curves in $|F|$. Therefore, for $s \in B$ general, M_s must also be supported on F_Σ; otherwise, M_s contains a general member F_s of $|F|$, the moduli $[Y_s]$ of Y_s is given by (2.6) and $[Y_s] \notin \Delta_0$. Consequently, $M_s \equiv M_b$ for all $s \in B$ and ρ is constant on B.

For a component Q of G with $q \in B \cap Q \ne \emptyset$, the same argument shows that $M_s \equiv M_q$ is supported on F_Σ for all $s \in Q$ and ρ is constant on Q. And since G is connected, we can use this argument to show that ρ is constant on every component of G, i.e., constant on G. This contradicts (2.10).

References

[BL] Bryan, Jim; Leung, Naichung Conan. The enumerative geometry of $K3$ surfaces and modular forms. *J. Amer. Math. Soc.* **13** (2000), no. 2, 371–410. MR1750955, Zbl 0963.14031, arXiv:alg-geom/9711031, doi: 10.1090/S0894-0347-00-00326-X. 169

[C1] Chen, Xi. Rational curves on $K3$ surfaces. *J. Algebraic Geom.* **8** (1999), no. 2, 245–278. MR1675158, Zbl 0940.14024, arXiv:math/9804075. 168, 170

[C2] Chen, Xi. A simple proof that rational curves on K3 are nodal. *Math. Ann.* **324** (2002), no. 1, 71–104. MR1931759, Zbl 1039.14019, arXiv:math/0011190, doi: 10.1007/s00208-002-0329-1. 168, 169, 170

[DS] Dedieu, Thomas; Sernesi, Edoardo. Equigeneric and equisingular families of curves on surfaces. *Publ. Mat.* **61** (2017), no. 1, 175–212. MR3590119, Zbl 1374.14025, arXiv:1410.4221, doi: 10.5565/PUBLMAT_61117_07. 168, 169

[H] Harris, Joseph. On the Severi problem. *Invent. Math.* **84** (1986), no. 3, 445–461. MR0837522, Zbl 0596.14017, doi: 10.1007/BF01388741. 169

[HM] Harris, Joseph; Morrison, Ian. Moduli of curves. Graduate Texts in Mathematics, 187. *Springer-Verlag, New York*, 1998. xiv+366 pp. ISBN: 0-387-98438-0; 0-387-98429-1. MR1631825, Zbl 0913.14005, doi: 10.1007/b98867. 168, 169

[KLM] Knutsen, Andreas Leopold; Lelli-Chiesa, Margherita; Mongardi, Giovanni. Severi varieties and Brill–Noether theory of curves on abelian surfaces. *J. Reine Angew. Math.* (2016). arXiv:1503.04465v2, doi: 10.1515/crelle-2016-0029. 169, 170

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