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On strengthenings of superstrong cardinals

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Abstract. We consider some natural strengthenings of the well-known notion of superstrong cardinal, looking at their corresponding $C^{(n)}$-versions as well, studying their properties and their connections with other usual large cardinals. In particular, we introduce the notions of $C^{(n)}$-*ultrastrongness* and of $C^{(n)}$-*global superstrongness*. As it turns out, the former is closely related to $C^{(n)}$-extendibility, a rather robust large cardinal assumption that has found applications in other mathematical areas, while for the latter, among other things, we show that appropriate Laver functions exist, making it the second known example of a $C^{(n)}$-hierarchy that has this feature.

Contents

1. Introduction

The hierarchy of large cardinal axioms constitutes a very important set-theoretic theme, not only due to its proper interest and complexity, or to its extensive usage in "measuring" the consistency strength of ZFC-independent statements, but also because it has found many applications in diverse mathematical areas.

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During the last decades, the intensive study of the various large cardinal axioms has produced a remarkably dense picture of properties and connections between them, while the list of large cardinals constantly keeps expanding, with new notions that are yet to be explored.

One family of such notions are the so-called $C^{(n)}$-*cardinals*, which are strengthenings of the usual large cardinal postulates. These were originally introduced by Bagaria (cf. [1]); they were subsequently studied further, among others (see, for instance, [10]), by the second author (cf. [12] and [15]).

Among the various $C^{(n)}$-cardinals, one dominant and important example is that of $C^{(n)}$-extendibility, for which Bagaria established a level-by-level correspondence with *Vopěnka's Principle* (VP), where the latter is a well-known and quite fruitful mathematical assumption of high consistency strength. Both Bagaria's initial work and that of the second author that followed it have underlined the fact that $C^{(n)}$-extendibility exhibits robustness and amenability to standard set-theoretic techniques, while it also has strong reflective properties and desirable related features. Perhaps not surprisingly then, the $C^{(n)}$-extendible cardinals have recently found applications in other mathematical fields, like category theory (see [2]), homotopy theory (see [3]), and model theory (see [4]), thus becoming a large cardinal assumption of wider mathematical interest and significance.

In this present note, we take up the well-known large cardinal notion of superstrongness, which was initially introduced by Gaifman in the 1970's, and we consider natural strengthenings of it. As it turns out, one of these strengthenings, which we call *ultrastrongness*, is equivalent to the usual notion of extendibility, thus providing (yet) another characterization of the latter. In fact, the situation further simplifies a reformulation of extendible cardinals that was proved in [12]. In parallel, we also consider the corresponding $C^{(n)}$-versions of our newly introduced notions, studying them in terms of their properties and of their consistency strength, showing that they enjoy resemblances and close connections with the central example of $C^{(n)}$-extendibility.

1.1. Notation. The notation that we use is standard; we refer the reader to [7] or [8] for further details, as well as for a comprehensive presentation of the theory of large cardinals.

Following [1], for every (meta-theoretic) natural number n, we let $C^{(n)}$ denote the closed proper class of ordinals that are Σ_n-*correct* in the universe V, that is, the class of ordinals α such that V_α is a Σ_n-elementary substructure of V. Note that $C^{(0)}$ is just the class of all ordinals while $C^{(1)}$ is precisely the class of uncountable cardinals α for which $V_\alpha = H_\alpha$. We recall that, for every $n \geqslant 1$, the statement "$\alpha \in C^{(n)}$" is Π_n-expressible (see Section 1 in [1] for details).

If j is a non-trivial elementary embedding, typically being of the form $j : V \longrightarrow M$ with $M \subseteq V$ a transitive class model of ZFC, we write $\mathrm{cp}(j)$

for its critical point, i.e., the least ordinal moved by j. Given an embedding j with $\mathrm{cp}(j) = \kappa$, we write $j^{(n)}(\kappa)$ for the n-th iterate of j at κ, i.e., $j^{(0)}(\kappa) = \kappa$ and, for every $n > 0$, $j^{(n)}(\kappa) = j(j^{(n-1)}(\kappa))$. We now briefly recall the definitions of some relevant large cardinal notions.

A cardinal κ is called λ-*strong*, for some $\lambda \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$; moreover, κ is called *strong* if it is λ-strong for all $\lambda \geqslant \kappa$. A cardinal κ is called *superstrong* if there is some elementary embedding $j : V \longrightarrow M$ such that $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

A cardinal κ is called λ-*supercompact*, for some $\lambda \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$; moreover, κ is called *supercompact* if it is λ-supercompact for all $\lambda \geqslant \kappa$. A cardinal κ is called λ-*extendible*, for some $\lambda > \kappa$, if there is some θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ such that $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$; moreover, κ is called *extendible* if it is λ-extendible for all $\lambda > \kappa$.[1] A cardinal κ is called *huge* if there is some elementary embedding $j : V \longrightarrow M$ such that $\mathrm{cp}(j) = \kappa$ and ${}^{j(\kappa)}M \subseteq M$. We remind the reader that every strong as well as every supercompact cardinal belongs to $C^{(2)}$ (i.e., it is Σ_2-correct in V), while every extendible cardinal belongs to $C^{(3)}$ (i.e., it is Σ_3-correct in V).

Finally, for the sake of completeness, let us also include the definitions of $C^{(n)}$-superstrongness, of $C^{(n)}$-extendibility and of $C^{(n)}$-hugeness, as these were introduced by Bagaria. For every $n \geqslant 1$:

Definition 1.1 ([1])**.** We say that a cardinal κ is $C^{(n)}$**-superstrong** if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\kappa)} \subseteq M$ and $j(\kappa) \in C^{(n)}$.

As already observed by Bagaria (see the comment after Definition 2.1 in [1]), for every $n \geqslant 1$, if the cardinal κ is $C^{(n)}$-superstrong, then $\kappa \in C^{(n)}$: to see this, let j be any $C^{(n)}$-superstrongness embedding for κ and just notice that $V_\kappa \prec V_{j(\kappa)}$. Additionally, a cardinal κ is superstrong if and only if it is $C^{(1)}$-superstrong: this follows from the fact that, for any j that is a superstrongness embedding for κ, we have that $V_{j(\kappa)} \subseteq M$ and, thus, $j(\kappa) \in C^{(1)}$ (see Proposition 2.2 in [1]). Furthermore, the $C^{(n)}$-superstrong cardinals form a proper hierarchy (see Proposition 2.3 in [1]). Finally, for every $n \geqslant 1$, the statement "κ is $C^{(n)}$-superstrong" is Σ_{n+1}-expressible (see the comments after Proposition 2.2 in [1]).

Definition 1.2 ([1])**.** We say that a cardinal κ is λ-$C^{(n)}$**-extendible**, for some $\lambda > \kappa$, if there is some θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Moreover, we say that κ is $C^{(n)}$**-extendible**, if it is λ-$C^{(n)}$-extendible, for all $\lambda > \kappa$.

[1] We remark that it is not necessary to require that "$j(\kappa) > \lambda$" in the definition of (full) extendibility, as this clause follows automatically (see Proposition 23.15 in [8]).

As above, we have that a cardinal is extendible if and only if it is $C^{(1)}$-extendible (see Proposition 3.3 in [1]). Moreover, for $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-extendible" is Σ_{n+1}-expressible; thus, for $n \geqslant 1$, the statement "κ is $C^{(n)}$-extendible" is Π_{n+2}-expressible (these are explained in Section 3 of [1]). Among other things, Bagaria showed that, for every $n \geqslant 1$, if κ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$ (see Proposition 3.4 in [1]), from which it follows that the hierarchy is proper, i.e., the consistency strength (strictly) grows with n (see Proposition 3.5 in [1]).[2]

Definition 1.3 ([1])**.** We say that a cardinal κ is **$C^{(n)}$-huge** if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\text{cp}(j) = \kappa$, ${}^{j(\kappa)}M \subseteq M$ and $j(\kappa) \in C^{(n)}$.

1.2. Reformulating extendible cardinals. Towards providing some motivation for our present study, we briefly review some related earlier work, mainly from [12]. Recall that, traditionally, extendibility is defined locally by set embeddings between rank initial segments of the universe. However, an alternative characterization in terms of class embeddings has also been established. For this, the second author introduced the following notion:

Definition 1.4 ([12])**.** We say that a cardinal κ is **jointly λ-supercompact and θ-superstrong**, for some $\lambda, \theta \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $V_{j(\theta)} \subseteq M$. In this case, we say that j *is jointly λ-supercompact and θ-superstrong for κ.*

For a fixed $\theta \geqslant \kappa$, we say that κ is **jointly supercompact and θ-superstrong**, if it is jointly λ-supercompact and θ-superstrong, for every $\lambda \geqslant \kappa$; moreover, we say that κ is **jointly supercompact and superstrong**, if κ is jointly λ-supercompact and λ-superstrong, for every $\lambda \geqslant \kappa$.

In other words, this definition "blends", simultaneously, two separate requirements for the elementary embedding j: on the one hand, the term "λ-supercompact" refers to the clause ${}^{\lambda}M \subseteq M$ while, on the other, the term "θ-superstrong" refers to the clause $V_{j(\theta)} \subseteq M$. Observe that, in this terminology, "κ-superstrong" means just ordinary superstrongness, i.e., $V_{j(\kappa)} \subseteq M$.

We note that if κ is the least supercompact, then it is not jointly λ-supercompact and κ-superstrong, for any λ (see Fact 2.4 in [15]). Actually, the following holds:

Theorem 1.5 ([12])**.** *A cardinal κ is extendible if and only if it is jointly supercompact and κ-superstrong if and only if it is jointly supercompact and superstrong.*

[2] We refer the reader directly to [1] for the initial (general) study of the various $C^{(n)}$-cardinals. For further explorations in this context, see the subsequent [12] and [15].

This follows from Corollary 2.31 in [12] (and its subsequent remarks). In fact, it is shown there that such a characterization is valid for $C^{(n)}$-extendibility as well.

The above reformulation of extendibility has been used, for instance, in [13], in order to motivate the introduction and to derive the consistency of the so-called *unbounded resurrection axioms*. Moreover, it has been employed in order to further develop the general theory of $C^{(n)}$-extendible cardinals in the more recent [15]: as an example, it is shown there that every $C^{(n)}$-extendible cardinal carries an appropriate Laver function and that such cardinals are compatible with forcing globally the GCH in the universe, both of which are desirable features, in general.

In the context of our present work, one important aspect of the aforementioned characterization is the following: if one strengthens the notion of supercompactness by requiring that the witnessing embeddings are, in addition, sufficiently superstrong above their target $j(\kappa)$, then one arrives at the notion of extendibility.[3]

We now wish to isolate this requirement of "sufficient superstrongness above the target" and study it in its own right, in the next section.

2. Ultrastrong cardinals

Let us start by giving the official definition.

Definition 2.1. We say that a cardinal κ is **λ-superstrong**, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\operatorname{cp}(j) = \kappa$ and $V_{j(\lambda)} \subseteq M$. In such a case, we say that j is *λ-superstrong for κ*. Moreover, we say that κ is **ultrastrong**, if it is λ-superstrong, for all $\lambda \geqslant \kappa$.

It is clear that every ultrastrong cardinal is superstrong. In addition, it is easily seen, as in the case of the usual superstrong cardinals, that for any λ-superstrong embedding j for κ we have that $j(\kappa) \in C^{(1)}$, because $V_{j(\kappa)} \subseteq M$. Similarly, if $\lambda \in C^{(1)}$ as well, then $j(\lambda) \in C^{(1)}$ too. Moreover, one can readily check that the statement "κ is λ-superstrong" is Σ_2-expressible, via the existence of an appropriate extender; thus, the statement "κ is ultrastrong" is Π_3-expressible.

The attentive reader may be wondering why the (usual) clause "$j(\kappa) > \lambda$" is missing from the above definition. The reason is that it follows automatically, exactly as in the case of extendible cardinals (see Proposition 23.15 in [8]). In fact, as we now show, extendibility is equivalent to ultrastrongness.

Proposition 2.2. *A cardinal is extendible if and only if it is ultrastrong.*

[3] Let us also mention that, in a separate work, this additional requirement of "sufficient superstrongness above the target $j(\kappa)$" has been "blended" with the notion of huge cardinals, producing *ultrahugeness*; see [14].

Proof. The forward direction follows directly from Corollary 2.29 in [12].

For the converse, assume that the cardinal κ is ultrastrong and fix some $\lambda > \kappa$. Let $j : V \longrightarrow M$ be an embedding that is λ-superstrong for κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$ and $V_{j(\lambda)} \subseteq M$. Then, the restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ witnesses the λ-extendibility of κ, as desired. □

The previous proposition further enligthens us, in the following sense. As stated in Theorem 1.5 above, by the work in [12], we know that the notion of extendibility can be captured by embeddings that are, simultaneously, "λ-supercompact" and "λ-superstrong". However, by the last result, it turns out after all that the clause regarding "λ-supercompactness" is superfluous in this characterization: one only needs to check the superstrongness requirement in order to verify extendibility. Put differently, the clause regarding "λ-supercompactness" already follows from the assumption of sufficient superstrongness.

Consequently, ultrastrong cardinals enjoy the same properties as the extendibles; for instance, every ultrastrong cardinal is Σ_3-correct and carries an appropriate Laver function (for the latter, see Theorem 1.7 in [13]). By the way, and as a curiosity, let us also observe the following.

Fact 2.3. If κ is ultrastrong, then, for any $\lambda \geqslant \kappa$, there exists an embedding $j : V \longrightarrow M$ that is λ-superstrong for κ, such that $j(\kappa)$ is a superstrong cardinal.

Proof. Fix some $\lambda \geqslant \kappa$ and let $\delta > \lambda$ be any Σ_2-correct ordinal; then, notice that $V_\delta \models$ "κ is superstrong", since the property of being superstrong is Σ_2-expressible (via the existence of one single extender). Let $j : V \longrightarrow M$ be an embedding that is δ-superstrong for κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$ and $V_{j(\delta)} \subseteq M$. By elementarity, we have that $V_{j(\delta)} \models$ "$j(\kappa)$ is superstrong". But note that $j(\delta) \in C^{(1)}$ (since $V_{j(\delta)} \subseteq M$) and, thus, $j(\kappa)$ is indeed a superstrong cardinal in V. □

In particular, we have (re)confirmed the (known) fact that if κ is extendible, then there are unboundedly many superstrong cardinals in the universe.

With the equivalence of Proposition 2.2 in mind, we now turn to the more general setting of $C^{(n)}$-ultrastrong cardinals, where some interesting subtleties emerge.

2.1. $C^{(n)}$-ultrastrong cardinals.

As expected, we begin by giving the relevant definition, in the spirit of [1]. For every $n \geqslant 1$:

Definition 2.4. We say that a cardinal κ is **λ-$C^{(n)}$-superstrong**, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) \in C^{(n)}$. In such a case, we say that the embedding j is *λ-$C^{(n)}$-superstrong for κ*. Moreover, we say that κ is **$C^{(n)}$-ultrastrong**, if it is λ-$C^{(n)}$-superstrong, for all $\lambda \geqslant \kappa$.

As already remarked, every ultrastrong cardinal is $C^{(1)}$-ultrastrong. Additionally, it is easy to check that, for each $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-superstrong" is Σ_{n+1}-expressible, via the existence of an appropriate extender; thus, the statement "κ is $C^{(n)}$-ultrastrong" is Π_{n+2}-expressible.

Furthermore, for each $n \geqslant 1$, exactly as for $C^{(n)}$-superstrongness, we have that if κ is $C^{(n)}$-ultrastrong, then $\kappa \in C^{(n)}$. Evidently, for $n = 1$, if κ is $C^{(1)}$-ultrastrong (that is, extendible), then $\kappa \in C^{(3)}$. It remains open whether this property generalizes for $n > 1$, i.e., we may ask:

Question 2.5. Suppose that $n > 1$ and that κ is a $C^{(n)}$-ultrastrong cardinal. Does it follow that $\kappa \in C^{(n+2)}$?

Regarding this question, let us point out that the (obvious) argument, using a meta-theoretic induction (appropriately adapting the proof of Proposition 3.2 of the next section), does not seem to go through. The reason is that it remains unclear whether the clause "$j(\kappa) > \lambda$" can be added, automatically, in the general $C^{(n)}$-version of ultrastrongness (for $n > 1$), as is the case when $n = 1$. See also Question 2.9 and the relevant discussion in (sub)section 2.1.1 below.

An immediate application of Corollary 2.29 in [12] gives the following (upper) bound on $C^{(n)}$-ultrastrongness (but see also Theorem 2.10 below). For every $n \geqslant 1$:

Proposition 2.6. *If κ is $C^{(n)}$-extendible, then κ is $C^{(n)}$-ultrastrong.*

We now show that not only the consistency strength of the hierarchy of $C^{(n)}$-ultrastrong cardinals grows with the number n but, also, that this hierarchy is closely tied to that of $C^{(n)}$-extendible cardinals. For every $n \geqslant 1$:

Proposition 2.7. *If κ is $C^{(n+1)}$-ultrastrong, then there is a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is } C^{(n)}\text{-extendible”}\} \in \mathcal{U}$.*

Proof. Suppose that κ is $C^{(n+1)}$-ultrastrong and let $j : V \longrightarrow M$ be a λ-$C^{(n+1)}$-superstrongness embedding for κ, for some (any) $\lambda > \kappa$ with $\lambda \in C^{(n+2)}$; that is, M is transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) \in C^{(n+1)}$. We consider two cases.

First, let us assume that $j(\kappa) \leqslant \lambda$. In this case, for any $\gamma < j(\kappa)$, note that the restricted embedding $j \restriction V_\gamma : V_\gamma \longrightarrow V_{j(\gamma)}$ is a γ-$C^{(n)}$-extendibility embedding for κ. Since being γ-$C^{(n)}$-extendible is a Σ_{n+1}-expressible statement and $j(\kappa) \in C^{(n+1)}$, it follows that, for every $\gamma < j(\kappa)$, the γ-$C^{(n)}$-extendibility of κ is reflected in $V_{j(\kappa)}$, i.e., we actually have that $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-extendible".

Alternatively, suppose that $j(\kappa) > \lambda$. In this case, let us observe that $V_{j(\kappa)} \models \lambda \in C^{(n+2)}$, because Π_{n+2}-statements reflect downwards to Σ_{n+1}-correct ordinals. Moreover, via a similar argument as in the previous case, one gets that κ is $C^{(n)}$-extendible in V_λ now: notice that, for every $\gamma < \lambda$, the γ-$C^{(n)}$-extendibility of κ, which is again witnessed by the map $j \restriction V_\gamma$, reflects inside V_λ. In turn, the (full) $C^{(n)}$-extendibility of κ reflects from V_λ up to $V_{j(\kappa)}$, due to the correctness of the ordinal λ in the latter model (given that being $C^{(n)}$-extendible is Π_{n+2}-expressible); in other words, we again have that κ is $C^{(n)}$-extendible in $V_{j(\kappa)}$.

In either case, we have that $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-extendible". The conclusion now follows from a standard reflection argument, via the usual normal measure $\mathcal{U}$ on κ that is derived from j. $\square$

In particular, from the existence of a $C^{(n+1)}$-ultrastrong cardinal we get a (ZFC) model with a proper class of $C^{(n)}$-extendible cardinals.

One important fact that we should underline here is that, as it follows from Propositions 2.6 and 2.7, it turns out that the hierarchy of $C^{(n)}$-ultrastrong cardinals is (consistency-wise) intertwined with the hierarchy of the $C^{(n)}$-extendibles, highlighting even more the close connection between the two notions.

Moreover, if we assume the existence of a $C^{(n+2)}$-ultrastrong cardinal κ, then we get the existence of many (actual) $C^{(n)}$-extendibles below κ. For every $n \geqslant 1$:

Corollary 2.8. *If κ is $C^{(n+2)}$-ultrastrong, then there is a normal measure $\mathcal{U}$ on κ such that* $\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-extendible}\} \in \mathcal{U}$.

Proof. Since the property of being $C^{(n)}$-extendible is Π_{n+2}-expressible, the desired conclusion follows from the previous proposition and the fact that every $C^{(n+2)}$-ultrastrong cardinal is itself a member of $C^{(n+2)}$, as noted after Definition 2.4. $\square$

The following question (annoyingly) remains open, for $n > 1$:

Question 2.9. Is it consistent to have a $C^{(n)}$-ultrastrong cardinal that is not $C^{(n)}$-extendible?

2.1.1. Variants of $C^{(n)}$-ultrastrongness. There are two plausible (and natural) ways to further strengthen the notion of $C^{(n)}$-ultrastrongness: one is to require that the witnessing embeddings satisfy the usual clause "$j(\kappa) > \lambda$" as well, while another one is to consider the corresponding "$C^{(n)+}$" version of ultrastrongness (similarly to that of $C^{(n)+}$-extendibility, which was defined in Section 4 of [1]; we prompt the reader to recall, at this point, the relevant definition from [1]).

In this final part of Section 2, we show that both of these variants, along with a further modification of the second one, are in fact all equivalent to $C^{(n)}$-extendibility. For every $n \geqslant 1$:

Theorem 2.10. *Given any cardinal κ, the following are equivalent:*

(*i*) *κ is $C^{(n)}$-extendible.*
(*ii*) *For all $\lambda \geqslant \kappa$, there exists an elementary embedding $j : V \longrightarrow M$ with M transitive,* $\text{cp}(j) = \kappa$, *$j(\kappa) > \lambda$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) \in C^{(n)}$.*

(*iii*) *For all* $\lambda \geqslant \kappa$ *with* $\lambda \in C^{(n)}$*, there exists an elementary embedding* $j : V \longrightarrow M$ *with* M *transitive,* $\mathrm{cp}(j) = \kappa$*,* $V_{j(\lambda)} \subseteq M$ *and* $\{j(\kappa), j(\lambda)\} \subseteq C^{(n)}$*.*

(*iv*) *For all* $\lambda \geqslant \kappa$ *with* $\lambda \in C^{(n)}$*, there are* $\gamma \in C^{(n+1)}$ *and* $\theta \in C^{(n)}$*, with* $\kappa \leqslant \gamma$ *and* $\{\gamma, \lambda\} \subseteq \theta$*, and there is an elementary embedding* $j : V \longrightarrow M$ *with* M *transitive,* $\mathrm{cp}(j) = \kappa$*,* $V_{j(\theta)} \subseteq M$*,* $\lambda < j(\gamma)$ *and* $\{j(\kappa), j(\gamma)\} \subseteq C^{(n)}$*.*

(*v*) κ *is* $C^{(n)+}$*-extendible.*

Proof. Let us first mention that the equivalence between (i) and (v) is not a novelty, since it has already been established by the second author in [15] (see Corollary 3.5 there), as well as, independently, by Gitman and Hamkins in [6]: their Theorem 15 in fact gives more equivalent formulations of $C^{(n)}$-extendibility.[4] At any rate, we include the case of $C^{(n)+}$-extendibility in the enunciation of our theorem for the sake of completeness. We now proceed with the rest of the equivalences.

Fix some (meta-theoretic) $n \geqslant 1$. We initially deal with the equivalence between (i) and (ii), which is similar to that between ultrastrong and (ordinary) extendible cardinals. First of all, note that the implication (i) $\Longrightarrow$ (ii) follows directly from Corollary 2.29 in [12].

For the converse, assume that κ satisfies (ii), fix some $\lambda > \kappa$ and let $j : V \longrightarrow M$ be an elementary embedding with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) \in C^{(n)}$. Then, the restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ witnesses the λ-$C^{(n)}$-extendibility of κ, as desired.

We now turn to the equivalence between (i) and (iii). We start by observing that the implication (i) $\Longrightarrow$ (iii) follows from the equivalence between (i) and (v) and the fact that every $C^{(n)+}$-extendible cardinal satisfies (iii); the latter implication is verified using (the proof of) Theorem 2.28 in [12].

Next, for the implication (iii) $\Longrightarrow$ (i), we argue as follows. Suppose that κ satisfies (iii) and fix some $\lambda > \kappa$. We further fix some $\gamma > \lambda$ such that $\gamma \in C^{(n)}$, $\mathrm{cf}(\gamma) > \omega$ and the following condition holds: whenever $\beta < \gamma$ and there exists a δ and an elementary embedding $e : V_\lambda \longrightarrow V_\delta$ with $\mathrm{cp}(e) = \kappa$ and $e(\kappa) = \beta$, then there exists such an embedding e with $\delta < \gamma$. Recalling that $C^{(n)}$ is a closed proper class of ordinals, we note that we may find such a γ by a straightforward closure argument, e.g., iterating (at most) ω_1-many times above λ.

By our assumption, we fix some $\eta \in C^{(n)}$ and an embedding $j : V_\gamma \longrightarrow V_\eta$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) \in C^{(n)}$.[5] Of course, we may assume that $j(\kappa) < \gamma$,

[4] Among them, of particular interest are those involving their concept of "A-extendibility", which indeed constitutes a more general setting that is certainly worth exploring further.

[5] In fact, given that κ satisfies (iii) and that $\kappa < \gamma$ with $\gamma \in C^{(n)}$, we first fix some $j' : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j') = \kappa$, $V_{j'(\gamma)} \subseteq M$ and $\{j'(\kappa), j'(\gamma)\} \subseteq C^{(n)}$. Then, setting $j = j' \restriction V_\gamma$ and $\eta = j'(\gamma)$ gives the stated (set) embedding $j : V_\gamma \longrightarrow V_\eta$.

since otherwise we are done. In light of Kunen's celebrated "inconsistency theorem" (see, for instance, Theorem 23.12 and Corollary 23.14 in [8]), given that $\mathrm{cf}(\gamma) > \omega$, there must exist some $m \in \omega$ such that $j^{(m)}(\kappa) < \gamma \leqslant j^{(m+1)}(\kappa)$.[6] At this point, we note that, for each $i \leqslant m+1$, we have that $j^{(i)}(\kappa) \in C^{(n)}$. To see this, proceed inductively: for $i = 0$ (or $i = 1$) it is clear, since $\{\kappa, j(\kappa)\} \subseteq C^{(n)}$. Now, assume that $j^{(i)}(\kappa) \in C^{(n)}$, for some $i \leqslant m$. Then, since $\gamma \in C^{(n)}$ and $j^{(i)}(\kappa) < \gamma$, we have that $V_\gamma \models j^{(i)}(\kappa) \in C^{(n)}$ and, hence, by elementarity, $V_\eta \models j^{(i+1)}(\kappa) \in C^{(n)}$, which is correctly computed because $\eta \in C^{(n)}$ as well.

We now let $P(i)$ be the statement: there exists a δ and an elementary embedding $e : V_\lambda \longrightarrow V_\delta$ with $\mathrm{cp}(e) = \kappa$ and $e(\kappa) = j^{(i+1)}(\kappa)$. We observe that it suffices to establish that $P(m)$ holds. For this, once again we proceed inductively: for $i = 0$, it is clear that $P(0)$ holds, since this is witnessed by the embedding $e = j \restriction V_\lambda$ (with $\delta = j(\lambda)$). Now, assume that $P(i)$ holds, for some $i < m$.

By $P(i)$ and the choice of γ, since $j^{(i+1)}(\kappa) < \gamma$, there must exist some $\delta < \gamma$ and an elementary embedding $e : V_\lambda \longrightarrow V_\delta$ with $\mathrm{cp}(e) = \kappa$ and $e(\kappa) = j^{(i+1)}(\kappa) \in C^{(n)}$. This embedding is witnessed inside V_γ (because $e \in V_\gamma$ and $\gamma \in C^{(n)}$ as well) and, therefore, by elementarity, we have that, in V_η, there exists an embedding $\bar{e} : V_{j(\lambda)} \longrightarrow V_{j(\delta)}$ with $\mathrm{cp}(\bar{e}) = j(\kappa)$ and $\bar{e}(j(\kappa)) = j^{(i+2)}(\kappa) \in C^{(n)}$. Thus, we may now consider the composed embedding $h = \bar{e} \circ (j \restriction V_\lambda) : V_\lambda \longrightarrow V_{j(\delta)}$, with $\mathrm{cp}(h) = \kappa$ and $h(\kappa) = j^{(i+2)}(\kappa) \in C^{(n)}$. This shows that $P(i+1)$ holds, as desired.

Finally, we establish the equivalence between (*iii*) and (*iv*), which is enough in order to conclude the theorem. For the implication (*iii*) $\Longrightarrow$ (*iv*), given some $\lambda \geqslant \kappa$ with $\lambda \in C^{(n)}$, let $\gamma \in C^{(n+1)}$ and $\theta \in C^{(n)}$ be some (any) Σ_{n+1}-correct and Σ_n-correct cardinals, respectively, with $\lambda < \gamma < \theta$. Now, by the assumption that κ satisfies (*iii*), there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\theta)} \subseteq M$ and $\{j(\kappa), j(\theta)\} \subseteq C^{(n)}$. Note that $\lambda < j(\gamma)$ and that, by elementarity and the correctness of θ, we have that $V_{j(\theta)} \models j(\gamma) \in C^{(n)}$. But the latter must hold in V, since $j(\theta) \in C^{(n)}$. In other words, this same embedding j witnesses the fact that κ satisfies (*iv*), for this choice of λ.

Conversely, for the implication (*iv*) $\Longrightarrow$ (*iii*), fix some $\lambda \geqslant \kappa$ such that $\lambda \in C^{(n)}$. By the assumption that κ satisfies (*iv*), let $\gamma \in C^{(n+1)}$ and $\theta \in C^{(n)}$ with $\kappa \leqslant \gamma$ and $\{\gamma, \lambda\} \subseteq \theta$, and let $j : V \longrightarrow M$ be an elementary embedding with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\theta)} \subseteq M$, $\lambda < j(\gamma)$ and so that $\{j(\kappa), j(\gamma)\} \subseteq C^{(n)}$. Recalling that Π_{n+1}-expressible statements reflect downwards to Σ_n-correct cardinals, by elementarity, it actually follows that $V_{j(\theta)} \models j(\kappa), j(\lambda) \in C^{(n)} \wedge j(\gamma) \in C^{(n+1)}$.

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[6] Observe that $m \neq 0$, because $\{\kappa, j(\kappa)\} \subseteq \gamma$.

Now, if we let E be the $(\kappa, j(\lambda))$-extender that is derived from j, then we have that $E \in V_{j(\theta)}$. Hence, as witnessed by this extender, we get that the statement "there exists an extender E whose associated embedding $j_E : V \longrightarrow M_E$ is such that $\mathrm{cp}(j_E) = \kappa$, $V_{j_E(\lambda)} \subseteq M_E$ and $\{j_E(\kappa), j_E(\lambda)\} \subseteq C^{(n)}$" holds in $V_{j(\theta)}$. But notice that this is a Σ_{n+1}-expressible statement, in the parameters κ and λ. Hence, it reflects from $V_{j(\theta)}$ down to $V_{j(\gamma)}$, given that $V_{j(\theta)} \models j(\gamma) \in C^{(n+1)}$. Finally, and since $j(\gamma) \in C^{(n)}$, this statement reflects from $V_{j(\gamma)}$ up to V, establishing this last implication and, effectively, concluding the proof. □

Note that, regarding the variant appearing in item (iii) of the previous theorem, the (additional) idea behind it — in accordance with the initial introduction of such a variant by Bagaria in [1] — is to ensure that the (Σ_n-) correctness of the chosen λ is carried over to its target $j(\lambda)$; this would be, in Bagaria's terminology, the corresponding "$C^{(n)+}$" version of ultrastrongness. On the other hand, regarding the variant appearing in item (iv), the idea behind it is to (try to) "weaken" this condition: we do not require, a priori, that the correctness of the chosen λ is carried over to its own target but, rather, that there are some "sufficiently correct ordinals around the chosen λ" whose correctness is, at least to some extent, carried over to their targets. As it turns out, this "some extent" is already enough.

In our view, the moral of the previous theorem is that there are two main paths that we can follow in order to arrive at $C^{(n)}$-extendibility from $C^{(n)}$-ultrastrongness: we can either require the additional clause "$j(\kappa) > \lambda$", or we can consider (what would be) the corresponding "$C^{(n)+}$" version, as mentioned above.[7] Of course, as underlined in Question 2.9, it is open whether these extra assumptions are actually proper strengthenings of $C^{(n)}$-ultrastrongness.[8] In this context, it is worthwhile noting the following, which may be considered as an indication that the hierarchies of $C^{(n)}$- ultrastrongness and of $C^{(n)}$-extendibility perhaps coincide. For $n > 1$:

Proposition 2.11. *Suppose that the cardinal κ is $C^{(n)}$-ultrastrong but not $C^{(n)}$-extendible. Then, κ is $C^{(n)}$-huge.*

Proof. Our assumption means that the cardinal κ does not satisfy property (ii) of Theorem 2.10. That is, there exists some $\lambda_0 > \kappa$ such that there is no elementary embedding $e : V \longrightarrow N$ with N is transitive, $\mathrm{cp}(e) = \kappa$, $e(\kappa) > \lambda_0$, $V_{e(\lambda_0)} \subseteq N$ and $e(\kappa) \in C^{(n)}$. However, given that κ is $C^{(n)}$-ultrastrong, there must exist some $\mu \leqslant \lambda_0$ with $\mu \in C^{(n)}$ and a proper class $\mathcal{C}$ of ordinals such that, for all $\lambda \in \mathcal{C}$, there exists an elementary embedding $j : V \longrightarrow M$ with M is transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) = \mu$.

[7] Regarding requiring the additional clause "$j(\kappa) > \lambda$", see also some relevant remarks made by Gitman and Hamkins, in their discussion right after Theorem 9 in [6].

[8] If the notions of $C^{(n)}$-extendibility and $C^{(n)}$-ultrastrongness are indeed different, then one might try some forcing construction in order to "separate" them.

In other words, by a pigeonhole argument, for proper class many λ's we must have a corresponding embedding j that is λ-$C^{(n)}$-superstrong for κ and whose target $j(\kappa)$ is a fixed $C^{(n)}$-cardinal $\mu \leqslant \lambda_0$.

Now pick any $\lambda \in \mathcal{C}$ with $\lambda > \mu$ and let $j : V \longrightarrow M$ be an elementary embedding with M transitive, $\mathrm{cp}(j) = \kappa$, $V_{j(\lambda)} \subseteq M$ and $j(\kappa) = \mu$. Note that $V_{j^{(2)}(\kappa)} \subseteq V_{j(\lambda)} \subseteq M$. Hence, by the usual ultrafilter characterization (see Theorem 24.8 and the discussion before Proposition 26.12, in [8]), it follows that κ is huge. Indeed, since $j(\kappa) = \mu \in C^{(n)}$, we actually have that κ is $C^{(n)}$-huge, as claimed. □

At any rate, Theorem 2.10 gives us (yet) more equivalent characterizations of $C^{(n)}$-extendibility (adding to the ones obtained by Gitman and Hamkins in [6]), indicating that it constitutes a rather robust large cardinal hierarchy. Moreover, given that the $C^{(n)}$-extendible cardinals have recently found applications in other mathematical contexts, this is something certainly worth remembering, which moreover concludes our current treatment of ($C^{(n)}$-)ultrastrong cardinals. Let us now turn to another strengthening of superstrongness, in the next section.

3. Globally superstrong cardinals

From our discussion so far, it turns out that the (global) requirement of "λ-superstrongness" (i.e., $V_{j(\lambda)} \subseteq M$) is "too much to ask", when trying to strengthen (usual) superstrongness: it already implies extendibility, which is a much stronger large cardinal assumption.

We now relax this requirement by asking only for "κ-superstrongness" (i.e., $V_{j(\kappa)} \subseteq M$); however, we ask that it occurs unboundedly often in the ordinals. This can be viewed as a "globalization" of ordinary superstrongness: recall that superstrong cardinals are witnessed "locally" by the existence of one single elementary embedding (or, equivalently, of one single appropriate extender). It is only natural to consider the corresponding "global" notion, as made precise in the following definition.

Definition 3.1. We say that a cardinal κ is **superstrong above** λ, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \subseteq M$. In such a case, we say that the embedding j is *superstrong above* λ *for* κ. Moreover, we say that κ is **globally superstrong**, if it is superstrong above λ, for all $\lambda \geqslant \kappa$.

Clearly, every globally superstrong cardinal is superstrong (and, also, strong). On the other hand, global superstrongness is a weakening of the previously defined notion of ultrastrongness: intuitively, here we require only the usual κ-superstrongness (i.e., $V_{j(\kappa)} \subseteq M$) for the embeddings in question. More accurately, since ultrastrongness is equivalent to extendibility, it is straightforward to verify that ultrastrongness directly entails global superstrongness (appealing to Theorem 1.5); in fact, the former is a strictly stronger assumption than the latter (see Proposition 3.9 below).

It is easy to see that the statement "κ is superstrong above λ" is Σ_2-expressible, via the existence of an appropriate extender. Hence, the statement "κ is globally superstrong" is Π_3-expressible which (modulo the consistency of this notion) is an optimal complexity bound since, as we now show, every globally superstrong cardinal is Σ_3-correct.

Proposition 3.2. *If κ is globally superstrong, then $\kappa \in C^{(3)}$.*

Proof. It is clear that every globally superstrong cardinal is strong and, thus, a member of $C^{(2)}$. So, fix a Σ_3-formula $\varphi(v) \equiv (\exists\, x)\, \psi(x, v)$, where ψ is Π_2, and fix some $a \in V_\kappa$. It is enough to check that if $\varphi(a)$ holds, then $V_\kappa \models \varphi(a)$.

For this, fix some $x_0 \in V$ such that $\psi(x_0, a)$ holds. We may assume that $\text{rank}(x_0) \geqslant \kappa$. Let $\lambda > \text{rank}(x_0)$ and fix some elementary embedding $j : V \longrightarrow M$ that is superstrong above λ for κ; that is, $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \subseteq M$. By choice of λ, we have that $x_0 \in V_{j(\kappa)} \subseteq M$. Now, since $\psi(x_0, a)$ is Π_2, with $\{a, x_0\} \subseteq V_{j(\kappa)}$ and $j(\kappa) \in C^{(1)}$, it follows that $V_{j(\kappa)} \models \psi(x_0, a)$, i.e., $V_{j(\kappa)} \models (\exists\, x)\, \psi(x, a)$. Thus, by elementarity and the fact that $j(a) = a$, we get that $V_\kappa \models \varphi(a)$, as desired. ☐

Regarding consistency lower bounds, we now show that globally superstrong cardinals transcend ordinary (super)strongs, in the following sense.

Proposition 3.3. *Suppose that κ is globally superstrong. Then, there exists a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : \alpha \text{ is (super)strong}\} \in \mathcal{U}$.*

Proof. Let κ be globally superstrong, fix some $\lambda > \kappa$ and let $j_0 : V \longrightarrow M_0$ be an elementary embedding that is superstrong above λ for κ, i.e., $\text{cp}(j_0) = \kappa$, $j_0(\kappa) > \lambda$ and $V_{j_0(\kappa)} \subseteq M_0$. Now, let $\gamma = j_0(\kappa)$ and let $j : V \longrightarrow M$ be some (other) elementary embedding that is superstrong above γ for κ, i.e., $\text{cp}(j) = \kappa$, $j(\kappa) > \gamma$ and $V_{j(\kappa)} \subseteq M$. We shall use this particular j and the (usual) normal measure $\mathcal{U}$ on κ derived from it, in order to treat both cases, i.e., "superstrong" and "strong", simultaneously. By the standard reflection arguments, it is enough to verify that $M \models$ "κ is superstrong" and that $M \models$ "κ is strong", correspondingly.

For "superstrongness", let E_γ be the (κ, γ)-extender derived from j_0 and note that $E_\gamma \in V_{j(\kappa)} \subseteq M$. Then, it is easy to see that E_γ witnesses the superstrongness of κ inside M, as desired.

For "strongness", fix some $\alpha < j(\kappa)$ and let E_α be the $(\kappa, |V_\alpha|^+)$-extender derived from j. Since $j(\kappa) \in C^{(1)}$, we have that all such derived extenders, for all $\alpha < j(\kappa)$, belong to $V_{j(\kappa)}$ and, thus, to M. Therefore, for each ordinal $\alpha < j(\kappa)$, we get that $M \models$ "κ is α-strong", as witnessed by the corresponding E_α. Now, by Proposition 3.2, it follows that $M \models j(\kappa) \in C^{(3)}$. Hence, all these Σ_2-expressible statements of the form "κ is α-strong" (for $\alpha < j(\kappa)$) are reflected from M down to $V_{j(\kappa)}$, where they collectively become the universal statement "κ is strong". The latter statement then goes up from $V_{j(\kappa)}$ to M, again because $j(\kappa)$ is Σ_3-correct in M. □

For a consistency upper bound, we have already mentioned that every extendible (i.e., every ultrastrong) cardinal is globally superstrong; in fact, more is true: see Proposition 3.9 below. A (much) better consistency upper bound will be given later on in this section (see Theorem 3.8), in the context of the corresponding $C^{(n)}$-version to which we now turn.

3.1. $C^{(n)}$-globally superstrong cardinals. We begin by giving the relevant definition. For every $n \geqslant 1$:

Definition 3.4. We say that a cardinal κ is **$C^{(n)}$-superstrong above** λ, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $V_{j(\kappa)} \subseteq M$ and $j(\kappa) \in C^{(n)}$. In such a case, we say that the embedding j is *$C^{(n)}$-superstrong above λ for κ*. Moreover, we say that κ is **$C^{(n)}$-globally superstrong**, if it is $C^{(n)}$-superstrong above λ, for all $\lambda \geqslant \kappa$.

Once again, as in the case of the usual superstrong cardinals, every globally superstrong cardinal is $C^{(1)}$-globally superstrong. It is easy to see that, for each $n \geqslant 1$, the statement "κ is $C^{(n)}$-superstrong above λ" is Σ_{n+1}-expressible, via the existence of an appropriate extender. Hence, the statement "κ is $C^{(n)}$-globally superstrong" is Π_{n+2}-expressible. As before, modulo the consistency of such notions, this is an optimal complexity bound: a straightforward adaptation of Proposition 3.2 (performing a meta-theoretic induction on n) shows that, for every $n \geqslant 1$:

Proposition 3.5. *If κ is $C^{(n)}$-globally superstrong, then $\kappa \in C^{(n+2)}$.*

It follows from the last proposition that the $C^{(n)}$-globally superstrong cardinals, if consistent, form a proper hierarchy: for each $n \geqslant 1$, the least $C^{(n)}$-globally superstrong cardinal is below the least $C^{(n+1)}$-globally superstrong cardinal, assuming that both of them exist.

En passant, let us now give one useful observation in the general context of $C^{(n)}$-superstrong embeddings. As a matter of notation, we write $\mathrm{Lim}(C^{(n)})$ for the collection of limit points of the (closed and unbounded) class $C^{(n)}$. For every $n \geqslant 1$:

Fact 3.6. Suppose that the embedding $j : V \longrightarrow M$ is $C^{(n)}$-superstrong for κ. Then, $j(\kappa) \in \mathrm{Lim}(C^{(n)})$.

Proof. As we have already remarked, every $C^{(n)}$-superstrong cardinal belongs to $C^{(n)}$ itself. Given an embedding $j : V \longrightarrow M$ that is $C^{(n)}$-superstrong for κ, it is then clear that $V_{j(\kappa)} \models \kappa \in C^{(n)}$ and, therefore, $M \models \kappa \in C^{(n)}$ since, by elementarity, $j(\kappa)$ is Σ_n-correct in M. By reflection, it follows that there are unboundedly many $\alpha < \kappa$ such that $\alpha \in C^{(n)}$.

Therefore, by elementarity, there are unboundedly many $\alpha < j(\kappa)$ such that $M \models \alpha \in C^{(n)}$. But note that, for each $\alpha < j(\kappa)$, we have that $\alpha \in C^{(n)}$ if and only if $V_{j(\kappa)} \models \alpha \in C^{(n)}$ if and only if $M \models \alpha \in C^{(n)}$. Hence, we conclude that $j(\kappa) \in \mathrm{Lim}(C^{(n)})$, as desired. □

As a consequence of (the proof of) Proposition 3.3, easily adapting the case of "superstrongness", we get the following consistency lower bound. For every $n \geqslant 1$:

Proposition 3.7. *Suppose that κ is $C^{(n)}$-globally superstrong. Then, there is a normal measure $\mathcal{U}$ on κ such that* $\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-superstrong}\} \in \mathcal{U}$.

Next, we establish a consistency upper bound for the case of $C^{(n)}$-globally superstrong cardinals which, in particular, takes care of the globally superstrong ones as well.

Theorem 3.8. *If κ is $\kappa+1$-extendible, then there is a normal measure $\mathcal{U}$ on κ such that* $\{\alpha < \kappa : (\forall n \in \omega)\, V_\kappa \models \text{"}\alpha \text{ is } C^{(n)}\text{-globally superstrong"}\} \in \mathcal{U}$.

Proof. Let $j_0 : V_{\kappa+1} \longrightarrow V_{j_0(\kappa)+1}$ be an embedding witnessing the $\kappa+1$-extendibility of κ and let E be the $(\kappa, j_0(\kappa))$-extender derived from j_0. Then, if $j : V \longrightarrow M$ is the corresponding extender embedding, we have that $\mathrm{cp}(j) = \kappa$, $j(\kappa) = j_0(\kappa)$ (which is an inaccessible cardinal) and $V_{j(\kappa)} = V_{j_0(\kappa)} \subseteq M$.

We now perform an elementary chain construction below the inaccessible $j(\kappa)$, as follows. Let us first fix an initial limit ordinal $\beta_0 \in (\kappa, j(\kappa))$. Then, we let:

$$X_0 = \{j(f)(x) : f \in V,\ f : V_\kappa \longrightarrow V,\ x \in V_{\beta_0}\} \prec M.$$

Given any $\xi+1 < j(\kappa)$ and given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$ and

$$X_{\xi+1} = \{j(f)(x) : f \in V,\ f : V_\kappa \longrightarrow V,\ x \in V_{\beta_{\xi+1}}\} \prec M.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for each $\alpha < \xi$, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha \prec M$. This concludes the definition of the elementary chain.

For any limit ordinal length $\gamma < j(\kappa)$, we consider $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and the corresponding $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is:

$$X_\gamma = \{j(f)(x) : f \in V,\ f : V_\kappa \longrightarrow V,\ x \in V_{\beta_\gamma}\} \prec M.$$

We note that from the inaccessibility of $j(\kappa)$ it follows that $\beta_\gamma < j(\kappa)$, for all $\gamma < j(\kappa)$. We now consider the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$ and we let $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$ be the composed embedding. As expected, this produces a commutative diagram of elementary embeddings (where the third arrow of the diagram is the map $k_\gamma = \pi_\gamma^{-1}$). Now, by results in [12] (cf. Proposition 2.13), we have that j_γ, which is a factor embedding of j, is superstrong for κ with target $j_\gamma(\kappa) = \beta_\gamma$.

In other words, for any initial limit ordinal $\beta_0 \in (\kappa, j(\kappa))$, this construction produces a collection of targets $j_\gamma(\kappa) > \beta_0$ of superstrongness embeddings for κ, for various limit lengths $\gamma < j(\kappa)$. It is easily checked that this collection is actually a full club in $j(\kappa)$ (see, e.g., Corollary 2.14 in [12]). Moreover, all these factor embeddings j_γ are witnessed by extenders inside $V_{j(\kappa)}$, with the latter being a (ZFC) model that faithfully verifies that all such extenders are superstrong for κ.

But now, for each $n \in \omega$, we can intersect the aforementioned club of superstrong targets with $(C^{(n)})^{V_{j(\kappa)}}$, which is the (club) subset of $j(\kappa)$ consisting of the ordinals that are Σ_n-correct in the sense of the model $V_{j(\kappa)}$. That is, we can find unboundedly many ordinals below $j(\kappa)$ that are targets of superstrong embeddings for κ and that, moreover, belong to the class $C^{(n)}$ as this is computed in $V_{j(\kappa)}$. This shows that, for each $n \in \omega$, we have that $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-globally superstrong". The desired result now follows from a standard reflection argument, using the usual normal measure $\mathcal{U}$ on κ derived from j. □

Thus, the assumption of $\kappa + 1$-extendibility is an adequate consistency upper bound for $C^{(n)}$-global superstrongness, in a strong sense: we get the consistency of the latter notion for all natural numbers simultaneously. Of course, the question of whether this (upper) bound is optimal remains open.

Nevertheless, it should certainly be remarked that there is substantial difference, in terms of consistency strength, between the notion of $C^{(n)}$-ultrastrongness and that of $C^{(n)}$-global superstrongness: modulo their consistency, the former notion is much stronger than the latter; this follows from the previous theorem and our discussion and results in Section 2.

We now take care of a small issue that we left pending after Proposition 3.3, regarding the relationship between ultrastrong (i.e., extendible) and globally superstrong cardinals.

Proposition 3.9. *Suppose that κ is extendible. Then, there is a normal measure $\mathcal{U}$ on κ such that $\{\alpha < \kappa : \alpha \text{ is globally superstrong}\} \in \mathcal{U}$. In particular, the least globally superstrong cardinal is below the least extendible, assuming that both of them exist.*

Proof. Having fixed an embedding $j : V \longrightarrow M$ with $\text{cp}(j) = \kappa$ and $V_{j(\kappa)+1} \subseteq M$, which we can obtain from the extendibility of κ, we employ an argument similar to the one given in the proof of Theorem 3.8 (disregarding the "$C^{(n)}$" part). We omit the details. We merely note that, in the current situation, the cardinal κ, being fully extendible, is Σ_3-correct and, hence, for every $\alpha < \kappa$, we have that α is globally superstrong if and only if $V_\kappa \models$ "α is globally superstrong". □

Via a straightforward adaptation of the previous result in the context of the corresponding $C^{(n)}$-versions, recalling that every $C^{(n)}$-extendible cardinal belongs to $C^{(n+2)}$, we also get that, for every $n \geqslant 1$:

Proposition 3.10. *Suppose that κ is a $C^{(n)}$-extendible cardinal. Then, there exists a normal measure $\mathcal{U}$ on κ such that:*

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-globally superstrong}\} \in \mathcal{U}$$

The following question remains open. For every $n > 1$:

Question 3.11. Does the implication stated in the previous proposition remain valid if we replace $C^{(n)}$-extendibility by $C^{(n)}$-ultrastrongness in the assumption?

Evidently, this question makes sense only if the two notions do not coincide (cf. Question 2.9). In such a case, observe that, in light of Theorem 3.8, an affirmative answer to Question 2.5 would indeed answer affirmatively Question 3.11 as well.

3.2. $C^{(n)}$-global superstrongness Laver functions. It is a matter of fact that Laver functions are an extremely flexible and useful tool in the context of large cardinals, with several of the usual large cardinal notions having this desirable feature (that is, they carry their own versions of such functions).

Historically, the initial example was given by Laver himself, who showed that supercompact cardinals have this property (see [9]). In the subsequent years, the list has been expanded substantially; here, we show that globally superstrong cardinals share this feature as well. Before anything else, we have to define the exact sort of "Laver function" that we have in mind. For every $n \geqslant 1$:

Definition 3.12. Let κ be $C^{(n)}$-globally superstrong. A function $\ell \vdots \kappa \longrightarrow V_\kappa$ is called a $C^{(n)}$-**global superstrongness Laver function** for κ if, for every cardinal $\lambda \geqslant \kappa$ and for every $x \in H_{\lambda^+}$, there exists an (extender) elementary embedding $j : V \longrightarrow M$ that is $C^{(n)}$-superstrong above λ for κ, and is such that $j(\ell)(\kappa) = x$.

Let us recall that the "three-dot" notation $\ell \vdots \kappa \longrightarrow V_\kappa$ means that $dom(\ell) \subseteq \kappa$ (i.e., a partial function). We now show that, for every $n \geqslant 1$:

Theorem 3.13. *If the cardinal κ is $C^{(n)}$-globally superstrong, then κ carries a $C^{(n)}$-global superstrongness Laver function.*

Proof. Fix $n \geqslant 1$, suppose that κ is $C^{(n)}$-globally superstrong and fix a well-ordering $\lhd_\kappa$ of V_κ. Aiming for a contradiction, let us assume that there does not exist any $C^{(n)}$-global superstrongness Laver function for κ.

In a recursive manner, we define a partial function $\ell \vdots \kappa \longrightarrow V_\kappa$, as follows. Given $\alpha < \kappa$ and $\ell \restriction \alpha$, we define $\ell(\alpha)$ only if $\ell``\alpha \subseteq V_\alpha$ and the following condition is satisfied: there is some $\lambda \geqslant \alpha$ and some $x \in H_{\lambda^+}$ such that, for every extender embedding $j : V \longrightarrow M$ that is $C^{(n)}$-superstrong above λ for α, we have that $j(\ell \restriction \alpha)(\alpha) \neq x$. We note that this condition is

Σ_{n+2}-expressible, in the parameters α and $\ell \restriction \alpha$;[9] thus, given that κ is $C^{(n)}$-globally superstrong and, hence, a member of $C^{(n+2)}$, if this condition is satisfied, then it reflects down to V_κ. In that case, we let $\lambda_\alpha < \kappa$ be the least such cardinal $\lambda \geqslant \alpha$ and we define $\ell(\alpha)$ to be the $\lhd_\kappa$-minimal witness $x \in H_{\lambda_\alpha^+}$; otherwise, we leave the function ℓ undefined. This concludes the definition of $\ell \colon \kappa \longrightarrow V_\kappa$.

By our assumption, there exists a least (cardinal) $\lambda^* \geqslant \kappa$ and some $x^* \in H_{\lambda^{*+}}$ such that for every extender embedding j that is $C^{(n)}$-superstrong above λ^* for κ, we have that $j(\ell)(\kappa) \neq x^*$. We fix a Π_{n+1}-expressible formula, say $\varphi(\lambda^*, x^*)$, that asserts this fact (in the parameters κ and ℓ). Further, we fix some $\theta > \lambda^*$ with $\theta \in C^{(n+1)}$ and an elementary embedding $j : V \longrightarrow M$ that is $C^{(n)}$-superstrong above θ for κ; i.e., M is transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, $V_{j(\kappa)} \subseteq M$ and $j(\kappa) \in C^{(n)}$. Note that, trivially, j is also $C^{(n)}$-superstrong above λ^* for κ.[10] Further, observe that, in the model M, we also have that $\theta \in C^{(n+1)}$.[11] It now follows that, in M, the cardinal λ^* is the least μ for which φ holds for some $x \in H_{\mu^+}$, since this is correctly reflected inside V_θ; in other words, M believes that $\lambda^* = \lambda_\kappa$ in the above notation. Thus, by elementarity, there exists some $y \in H_{\lambda^{*+}}$ such that $j(\ell)(\kappa) = y$, as computed in M. By definition of the function $j(\ell)$, we have that $M \models \varphi(\lambda^*, y)$; this will lead us to a contradiction.

For the final part of the argument, we have to find an appropriate extender E, inside M, that witnesses the $C^{(n)}$-superstrongness above λ^* for κ and whose corresponding embedding j_E is such that $j_E(\ell)(\kappa) = y$. But note that, for every Σ_n-correct cardinal $\alpha \in (\lambda^*, \theta)$,[12] the statement "there is some extender E that is $C^{(n)}$-superstrong above α for κ and whose support contains V_α" is Σ_{n+1}-expressible in the parameters κ and α. This statement is clearly true in V, as witnessed by the $(\kappa, j(\kappa))$-extender derived from j. Hence, it reflects down to V_θ and, then, from there, it reflects upwards to M (again, since θ is Σ_{n+1}-correct in M). Given any such extender E in M, note that all four: κ, λ^*, $H_{\lambda^{*+}}$ and y belong to V_α (i.e., are in the support of E) and are thus fixed by the usual third factor embedding k_E (that commutes with j and j_E). It follows that M correctly computes the value $j_E(\ell)(\kappa) = j(\ell)(\kappa) = y$, which is the contradiction that concludes the proof. □

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[9] See the similar Claim 4.3 in [15], for more details on this complexity computation.

[10] Observe that, here, the fact that we pick the embedding j with target above some Σ_{n+1}-correct ordinal above λ^*, in this case θ, in particular means that $j(\kappa)$ is far from being the least target among the possible embeddings that are $C^{(n)}$-superstrong above λ^* for κ. This is because the statement "there is some extender E that is $C^{(n)}$-superstrong above λ^* for κ" is Σ_{n+1}-expressible. This statement holds in V (as witnessed by j itself) and, thus, it must reflect (correctly) down to V_θ.

[11] For this, just observe that $V_{j(\kappa)} \models \theta \in C^{(n+1)}$ and that, by elementarity, the cardinal $j(\kappa)$ is Σ_{n+2}-correct in M.

[12] Of which there are many, since $\theta \in C^{(n+1)}$.

The $C^{(n)}$-globally superstrong cardinals are, thus, the second known example of a $C^{(n)}$-hierarchy that has its own, appropriate, Laver functions. The first example was shown to be the case of $C^{(n)}$-extendibility (see Theorem 4.2 in [15]).

Towards concluding, let us say a few words regarding the interaction of $C^{(n)}$-globally superstrong (and of $C^{(n)}$-ultrastrong) cardinals with forcing techniques. First of all, as usual in the context of large cardinals, we get the preservation of these notions under small forcing. For every $n \geqslant 1$:

Proposition 3.14. *Suppose that κ is a $C^{(n)}$-globally superstrong (resp. $C^{(n)}$-ultrastrong) cardinal and let $\mathbb{P}$ be a poset with $|\mathbb{P}| < \kappa$. Then, in the forcing extension $V^{\mathbb{P}}$, the cardinal κ remains $C^{(n)}$-globally superstrong (resp. $C^{(n)}$-ultrastrong).*

Proof. The proposition follows from standard arguments, as when dealing with ordinary superstrong cardinals, which we omit. The only detail that we should mention, in the current setting, is that one needs to employ Lemma 4.2 (i) from [12], in order to get the preservation of Σ_n-correct ordinals under small forcing. □

Furthermore, Friedman showed that every superstrong cardinal is preserved by the canonical class poset $\mathbb{P}$ that forces the global GCH in the universe (see [5] for definitions, details, etc.; in particular, the relevant result appears as Theorem 2 there). Indeed, he showed that if κ is superstrong, then every ground model superstrongness embedding j for κ lifts in the forcing extension $V^{\mathbb{P}}$ in order to witness, there, the superstrongness of κ (with the same target $j(\kappa)$). As an immediate consequence of this result, we get:

Proposition 3.15. *Every globally superstrong cardinal is preserved by the canonical forcing for global* GCH.

We should remark that a similar result holds for ultrastrong (i.e., extendible) cardinals as well: see [11].

This concludes our current treatment of globally superstrong cardinals. Just before the end, let us point out that both some earlier work (cf. [12], [13], [14], [15]) and the current note suggest that the elementary chain method (as exemplified in the proof of Theorem 3.8 above) ties nicely with embeddings that are sufficiently superstrong above their target; such a construction is typically performed below some inaccessible cardinal bound (e.g., in the case of extendible cardinals, this bound could be taken to be the target $j(\kappa)$). This is a quite general observation that has some value as to what sort of methods can be used in the context of particular large cardinal notions. We expect further exploitation of it in the future.

With an eye to subsequent research investigations, let us end by mentioning some open themes that, we believe, are worth looking into. As far as the $C^{(n)}$-ultrastrong cardinals are concerned, in our view, the main issue revolves around their exact relationship with the $C^{(n)}$-extendibles, something that has already been highlighted in Questions 2.5 and 2.9 of Section 2, but also in Question 3.11 of the current section.

Let us close by stating a few more questions. For every $n > 1$:

Question 3.16. Do $C^{(n)}$-ultrastrong cardinals carry (appropriate) Laver functions?

Question 3.17. Are $C^{(n)}$-globally superstrong (resp. $C^{(n)}$-ultrastrong) cardinals preserved by the canonical forcing for global GCH?

The next two questions also make sense for $n = 1$. The first one is more of a general inquiry than a precise statement. For any $n \geqslant 1$:

Question 3.18. What forcing posets preserve the $C^{(n)}$-global superstrongness (resp. $C^{(n)}$-ultrastrongness) of a given cardinal κ?

Question 3.19. Let κ be $C^{(n+1)}$-globally superstrong. Is there a forcing that destroys the $C^{(n+1)}$-global superstrongness of κ while preserving its $C^{(n)}$-global superstrongness?

This last question, appropriately modified, is also open in the context of $C^{(n)}$-supercompact, of $C^{(n)}$-ultrastrong and of $C^{(n)}$-extendible cardinals, respectively.[13] Regrettably, in all these cases, we have no clue as to what the corresponding answer might be.

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[13] By the way, let us note that, in the case of $C^{(n)}$-extendible cardinals, there is an easy argument (without employing any forcing constructions) that "separates" $C^{(n+1)}$-extendibility from $C^{(n)}$-extendibility; see Section 6 in [15]. Whether even this "easy argument" is applicable in the other cases remains unclear.

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