New York Journal of Mathematics

New York J. Math. **26** (2020) 218–229.

Subgroup growth of all Baumslag-Solitar groups

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Abstract. This paper gives asymptotic formulas for the subgroup growth and maximal subgroup growth of all Baumslag-Solitar groups.

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1. Introduction

For a finitely generated group G, let $a_n(G)$ denote the number of subgroups of G of index n, and let $m_n(G)$ denote the number of maximal subgroups of G of index n. Also, for a, b nonzero integers, let $\mathrm{BS}(a,b)$ denote the Baumslag-Solitar group $\langle x, y \mid y^{-1}x^a y = x^b\rangle$, which was introduced in [1].

In [4], Gelman counts $a_n(\mathrm{BS}(a,b))$ exactly for the case when $\gcd(a,b) = 1$. Exact formulas in the area of subgroup growth are rare, and so his formula (Theorem 3.1 below) is indeed very nice. Can a simple formula also be given for $m_n(\mathrm{BS}(a,b))$ when $\gcd(a,b) = 1$? Yes, see Corollary 3.6. More importantly to this paper, what about the case when $\gcd(a,b) \neq 1$?

From the work of Moldavanskii [8], it is apparent that the largest residually finite quotient of $\mathrm{BS}(a,b)$ is a group, which we will denote $\overline{G}$, which has a normal subgroup of the form $A \cong \mathbb{Z}[1/k]$ (for appropriate k) with $\overline{G}/A \cong \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$, where $m = \gcd(a,b)$. When $m = 1$, this explains why the formula for $a_n(\mathrm{BS}(a,b))$ is so simple; $\overline{G}$ turns out to be of the form

Received September 26, 2018.

2010 *Mathematics Subject Classification.* 20E07.

Key words and phrases. Subgroup growth, Baumslag-Solitar groups.

ISSN 1076-9803/2020

$\mathbb{Z}[1/k] \rtimes \mathbb{Z}$, and so Section 3 gives a more enlightening proof of Gelman's formula.

When $m = \gcd(a,b) > 1$, one has to deal with the free product $\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. In [9], Müller studies such groups (and in fact many more: any free product of groups that are either finite or free). With this, one can give an asymptotic formula for $m_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z})$. Note that Müller's main results are even better than asymptotic formulas.

Next, a small argument shows that the vast majority of maximal subgroups (of any fixed, large index) of $\mathrm{BS}(a,b)$ contain the normal subgroup A (mentioned above), and hence, we obtain an asymptotic formula for $m_n(\mathrm{BS}(a,b))$. As it turns out, the vast majority of *all* subgroups of $\mathrm{BS}(a,b)$ (of any fixed, large index) contain A. As a result, we can combine the two main results of this paper, Theorems 4.10 and 5.3, to obtain the following.

Theorem. *Let* $m = \gcd(a,b)$*, and assume that* $m > 1$*. Then*

$$a_n(\mathrm{BS}(a,b)) \sim m_n(\mathrm{BS}(a,b)) \sim n f(n),$$

where

$$f(n) := K n^{(1-1/m)n} \exp\left(-(1-1/m)n + \sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} \right)$$

with

$$K := \begin{cases} m^{-1/2} & \text{if } m \text{ is odd} \\ m^{-1/2} e^{-1/(2m)} & \text{otherwise.} \end{cases}$$

For related work on the Baumslag-Solitar groups, note that in [2], Button gives an exact formula for counting the normal subgroups of any index in $\mathrm{BS}(a,b)$, when $\gcd(a,b) = 1$. For a survey of subgroup growth up until 2003, see [7], the book by Lubotzky and Segal.

The goal of Section 2 is to describe $\overline{G}$, the largest residually finite quotient of $\mathrm{BS}(a,b)$. In Section 3, a new proof is given for Gelman's formula, and it is shown there what it simplifies to for $m_n(\mathrm{BS}(a,b))$. In Section 4, an asymptotic formula is given for $m_n(\mathrm{BS}(a,b))$ when $\gcd(a,b) > 1$. Finally, in Section 5, it is shown that the asymptotic formula for $m_n(\mathrm{BS}(a,b))$ is also asymptotic to $a_n(\mathrm{BS}(a,b))$ (where still $\gcd(a,b) > 1$).

2. The largest residually finite quotient

The goal of this section is Corollary 2.7. Let $G = \mathrm{BS}(a,b)$.

We will denote the intersection of all finite index subgroups of G by $\mathrm{Res}(G)$. In [8], Moldavanskii determines what $\mathrm{Res}(G)$ is. Let $m := \gcd(a,b)$.

Theorem 2.1 (Moldavanskii, 2010)**.** *The group* $\mathrm{Res}(G)$ *is the normal closure in* G *of the set of commutators* $\{[y^k x^m y^{-k}, x] : k \in \mathbb{Z}\}$.

Let $\overline{G} = G/\operatorname{Res}(G)$ the largest residually finite quotient of $G = \operatorname{BS}(a,b)$. ($\overline{G}$ does depend on a and b.) We then have the following presentation of $\overline{G}$:

$$\overline{G} = \langle x, y \mid y^{-1}x^a y = x^b, [y^k x^m y^{-k}, x] \text{ for all } k \in \mathbb{Z}\rangle.$$

We next define a subgroup of $\overline{G}$ (denoted $\overline{C}$ in [8]):

$$A := \langle y^k x^m y^{-k} : k \in \mathbb{Z}\rangle \leq \overline{G}.$$

Lemma 2.2 (Moldavanskii)**.** *The group A is an abelian normal subgroup of $\overline{G}$.*

Note: This is a small part of Propositions 3 and 4 in [8].

Proof. We have $A \trianglelefteq \overline{G}$ because conjugating the generators of A by y just shifts them and because x commutes with all the generators (because of the commutators in $\operatorname{Res}(G)$).

We have that $[y^k x^m y^{-k}, x] \in \operatorname{Res}(G)$ implies that x^m commutes with $y^k x^m y^{-k}$, and hence for all $j, k \in \mathbb{Z}$ we get $[y^k x^m y^{-k}, y^j x^m y^{-j}] \in \operatorname{Res}(G)$. □

It turns out that $\overline{G}/A$ is the free product of a finite cyclic group with the infinite cyclic group: (Recall that $m := \gcd(a,b)$.)

Corollary 2.3 (Moldavanskii)**.** *The group $\overline{G}/A$ has presentation $\langle x, y \mid x^m\rangle$, and therefore, $\overline{G}/A \cong \mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$.*

Note that the x here does indeed correspond to the x in the presentation of $\overline{G}$.

Proof. Any group with the relation $x^m = 1$ also has the relations $y^k x^m y^{-k} = 1$ as well as $[y^k x^m y^{-k}, x] = 1$, and since m divides a and b, that group will also have the relation $y^{-1}x^a y = x^b$. Therefore, $\overline{G}/A$ has presentation $\langle x, y \mid x^m\rangle$. □

Since we know that A is abelian, we will write A additively instead of multiplicatively. Let $a = um$, $b = vm$. (So $\gcd(u,v) = 1$.)

Proposition 2.4 (Moldavanskii)**.** *The group A has the following presentation as an abelian group (using additive notation)*

$$A = \langle c_i, i \in \mathbb{Z} \mid uc_i = vc_{i+1} \text{ for all } i \in \mathbb{Z}\rangle.$$

Moldavanskii also shows in Proposition 4 of [8] that A is a residually finite abelian group of rank 1. (For A to have rank 1 means that all of its finitely generated subgroups are cyclic.) We will show in Lemma 2.6 something similar, that A is isomorphic to $\mathbb{Z}[u/v, v/u] = \{a_1(u/v)^{t_1} + \cdots + a_k(u/v)^{t_k} : a_i, t_i \in \mathbb{Z}\ \forall i\}$. We remind the reader that the ring $\mathbb{Z}[u/v, v/u]$ is $\mathbb{Z}$ together with the two rational numbers u/v and v/u adjoined. See Lemma 2.5 below for a well-known alternative perspective.

We let $\pi(uv)$ denote the product of the distinct primes that divide uv.

Lemma 2.5. *Assume still that* $\gcd(u,v)=1$. *As subrings of* $\mathbb{Q}$, *we have*

$$\mathbb{Z}[v/u,u/v]=\mathbb{Z}[1/u,1/v]=\mathbb{Z}[1/(uv)]=\mathbb{Z}[1/\pi(uv)]$$

Lemma 2.5 is well-known.

Lemma 2.6. *We have that* $A\cong\mathbb{Z}[u/v,v/u]$ *as groups.*

Proof. Let $\varphi:\{c_i : i\in\mathbb{Z}\}\to\mathbb{Z}[u/v,v/u]$ be defined by $\varphi(c_i):=(u/v)^i$.

Step 1. φ gives a homomorphism: To get a homomorphism from A to $\mathbb{Z}[u/v,v/u]$, all we need to check is that $u\varphi(c_i)=v\varphi(c_{i+1})$. And indeed, it is true that $u(u/v)^i=v(u/v)^{i+1}$.

Step 2. φ is surjective: This is evident because for all i, $(u/v)^i$ is in the image of φ.

Step 3. φ is injective: Let $g\in\ker(\varphi)$. Assume by contradiction that $g\neq 0$. Then there exist $n_i\in\mathbb{Z}$ such that $g=\sum_{i=s}^t n_ic_i$ with $n_s,n_t\neq 0$. We will show that we can assume that the sum has only one term in it (i.e. that $s=t$) and then easily get a contradiction.

We have $\varphi(g)=\sum_{i=s}^t n_i(u/v)^i=0$. Assume $t>s$. Multiplying by v^t and dividing by u^s yields

$$n_sv^{t-s}+n_{s+1}v^{t-s-1}u+n_{s+2}v^{t-s-2}u^2+\cdots+n_tu^{t-s}=0.$$

Therefore $u\mid n_sv^{t-s}$, and since $\gcd(u,v^{t-s})=1$, we get $u\mid n_s$. Thus we can rewrite g and then apply the relation $uc_i=vc_{i+1}$ to get

$$g=\frac{n_s}{u}uc_i+\sum_{i=s+1}^{t}n_ic_i=\frac{n_s}{u}vc_{i+1}+\sum_{i=s+1}^{t}n_ic_i.$$

Since we assumed $t>s$, we showed that we can rewrite g as $\sum_{i=s+1}^t\tilde{n}_ic_i$, decreasing the number of terms in the summation (by at least 1). Continuing in this way, we see that $g=nc_t$ for some $n\in\mathbb{Z}$. Because we assumed $g\neq 0$, we know that $n\neq 0$. Therefore $0=\varphi(g)=\varphi(nc_t)=n(u/v)^t$, and this is a contradiction since $n\neq 0$. □

Recall that $m=\gcd(a,b)$, and $a=um$, $b=vm$.

Corollary 2.7. *The group* $\overline{G}$ *(defined after Theorem* 2.1*) satisfies a short exact sequence of the form*

$$1\to\mathbb{Z}[1/(uv)]\to\overline{G}\to\mathbb{Z}*(\mathbb{Z}/m\mathbb{Z})\to 1.$$

Writing $\mathbb{Z}*\mathbb{Z}/m\mathbb{Z}=\langle x,y\mid x^m\rangle$, *the action of* x *on* $\mathbb{Z}[1/(uv)]$ *is trivial, and the action of* y *on* $\mathbb{Z}[1/(uv)]$ *is multiplication by* u/v.

Proof. Indeed, this is just a summary of the previous results: By Lemma 2.2, $A\trianglelefteq G$. By Lemma 2.6, $A\cong\mathbb{Z}[u/v,v/u]$, which is isomorphic to $\mathbb{Z}[1/(uv)]$ by Lemma 2.5. Finally, Corollary 2.3 gives us the rest of the short exact sequence.

We know that x acts trivially on $\mathbb{Z}[1/(uv)]$ (by conjugation) because in $\overline{G}$, the element x commutes with x^m, and x^m normally generates $A=\mathbb{Z}[1/(uv)]$.

Finally, consider the relation $y^{-1}x^a y = x^b$ in $\overline{G}$. Recall that $a = um$ and $b = vm$. So solving the relation for x^a, we can rewrite it as $(x^m)^u = y(x^m)^v y^{-1}$. Written additively, this says that y acts on x^m (a generator of A) by multiplication by u/v. □

3. When $\gcd(a,b) = 1$: redoing Gelman's formula

In this section, we give a new proof of a beautiful result of Gelman (Theorem 3.1 below). In my opinion, this proof better explains the result. Gelman's formula makes sense in light of the free product $\mathbb{Z} * (\mathbb{Z}/\gcd(a,b)\mathbb{Z})$ simplifying to $\mathbb{Z}$ and so giving the semidirect product in Lemma 3.2.

As before, we let $\mathrm{BS}(a,b) := \langle x, y \mid y^{-1}x^a y = x^b \rangle$. Assume $\gcd(a,b) = 1$. In [4], Gelman gives the following exact formula for $a_n(\mathrm{BS}(a,b))$, the number of *all* subgroups of index n in $\mathrm{BS}(a,b)$:

Theorem 3.1 (Gelman, 2005)**.** *Recall that* $\gcd(a,b) = 1$*. We have*

$$a_n(\mathrm{BS}(a,b)) = \sum_{\substack{d \mid n \\ \gcd(d,ab)=1}} d$$

In order to (re)prove this, we state a few lemmas. First, we state the isomorphism type of $\overline{G}$, the largest residually finite quotient of $\mathrm{BS}(a,b)$.

Lemma 3.2. *Let* $\overline{G}$ *be the group defined just after Theorem* 2.1*. Then*

$$\overline{G} \cong \mathbb{Z}[1/(ab)] \rtimes \mathbb{Z},$$

where the action of $1 \in \mathbb{Z}$ *on* $\mathbb{Z}[1/(ab)]$ *is multiplication by* a/b*.*

Proof. By Corollary 2.7, (and since $d = \gcd(a,b) = 1$), this is exact:

$$1 \to \mathbb{Z}[1/(ab)] \to \overline{G} \to \mathbb{Z} \to 1.$$

Because $\mathbb{Z}$ is a free group, every such short exact sequence splits.

The statement about the action also follows from Corollary 2.7: Indeed, recall that since $m = \gcd(a,b) = 1$, we have in the notation of that corollary, $u = a$ and $v = b$. □

Once we have Lemma 3.2, proving Theorem 3.1 is standard. Notice that the group $\overline{G}$ is an example of a group included in Lemma 3.4, part (i) in [11], and Shalev has the formula (i.e. the one in Theorem 3.1) there in his remark (on page 3804) following his proof of his Lemma 3.4. Nevertheless, we will give a few more details anyways.

Lemma 3.3 is well-known. (We will use it in the following section as well.)

Lemma 3.3. *Let* $0 \neq k \in \mathbb{Z}$*. We have*

$$a_n(\mathbb{Z}[1/k]) = \begin{cases} 1 & \text{if } \gcd(n,k) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, the nonzero ideals of $\mathbb{Z}[1/k]$ *are exactly the subgroups of finite index.*

Definition 3.4. Let G be a group acting on an abelian group N. A derivation is a function $\delta\colon G \to N$ such that $\delta(gh) = \delta(g) + g \cdot \delta(h)$ for all $g, h \in G$. The set of all derivations from G to N is denoted $\mathrm{Der}(G, N)$.

Lemma 3.5 (quoted from Shalev)**.** *Suppose A is an abelian group, and let $G = A \rtimes B$. Then*

$$a_n(G) = \sum_{A_0, B_0} |\mathrm{Der}(B_0, A/A_0)|,$$

where the sum is taken over all subgroups $A_0 \le A$, $B_0 \le B$ such that A_0 is B_0-invariant, and $[A : A_0][B : B_0] = n$.

This is Lemma 2.1 part (iii) in [11].

Proof of Theorem 3.1. In the notation of Lemma 3.5, let $A = \mathbb{Z}[1/(ab)]$ and $B = \mathbb{Z}$, so that as in Lemma 3.2, we have $\overline{G} \cong A \rtimes B$.

Let $B_0 \le_f B$ (i.e. let B_0 be a subgroup of finite index in B). Then a subgroup $A_0 \le_f A$ is B_0-invariant iff it is B-invariant iff A_0 is an ideal of A. Recall that since $\mathbb{Z}$ is a free group, regardless of its action on $\mathbb{Z}/d\mathbb{Z}$, we get that $|\mathrm{Der}(\mathbb{Z}, \mathbb{Z}/d\mathbb{Z})| = d$. Combining the previous two sentences with Lemmas 3.5 and 3.3, we conclude that

$$a_n(\overline{G}) = \sum_{d|n} a_{n/d}(\mathbb{Z}) a_d(\mathbb{Z}[1/(ab)]) d. \tag{1}$$

But $a_{n/d}(\mathbb{Z}) = 1$, and then using Lemma 3.3 again, (1) becomes

$$a_n(\overline{G}) = \sum_{\substack{d|n \\ \gcd(d,ab)=1}} d.$$

We are done because $\overline{G}$ is the largest residually finite quotient of $\mathrm{BS}(a, b)$. ☐

Gelman's formula simplifies to the following when counting maximal subgroups:

Corollary 3.6. *Recall that here,* $\gcd(a, b) = 1$*. Every maximal subgroup of* $\mathrm{BS}(a, b)$ *has prime index, and*

$$m_p(\mathrm{BS}(a, b)) = \begin{cases} p + 1 & \text{if } p \nmid ab \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The reason why $\mathrm{BS}(a, b)$ has no maximal subgroups of non-prime index is if $M \le G$ with M maximal of index n then $M \cap \mathbb{Z}[1/(ab)]$ is a maximal ideal of $\mathbb{Z}[1/(ab)]$ of index n, and such an n can only be prime. The present corollary then follows from Theorem 3.1. ☐

4. When $\gcd(a,b) \neq 1$: an asymptotic formula for $m_n(\mathrm{BS}(a,b))$

Let $m := \gcd(a,b)$ and assume $m > 1$. The goal of this section is to prove Theorem 4.10. We first state formula (8) on page 115 of [9].

Theorem 4.1 (Müller, 1996)**.** *Let G be a finite group of order m. (Recall $m > 1$.) Then $|\mathrm{Hom}(G, \mathrm{Sym}(n))|$ is asymptotic to*

$$K_G n^{(1-1/m)n} \exp\left(-(1-1/m)n + \sum_{\substack{d<m \\ d|m}} \frac{a_d(G)}{d} n^{d/m} \right),$$

where

$$K_G := \begin{cases} m^{-1/2} & \text{if } m \text{ is odd} \\ m^{-1/2} e^{-a_{m/2}(G)^2/(2m)} & \text{otherwise.} \end{cases}$$

We will use the following easy consequence of Theorem 4.1:

Corollary 4.2. *Recall $m > 1$. We have $|\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Sym}(n))|$ is asymptotic to $f(n)$, where*

$$f(n) := K n^{(1-1/m)n} \exp\left(-(1-1/m)n + \sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} \right),$$

and

$$K := \begin{cases} m^{-1/2} & \text{if } m \text{ is odd} \\ m^{-1/2} e^{-1/(2m)} & \text{otherwise.} \end{cases}$$

We will use this function f throughout the rest of this paper.

Notation 4.3. We let

$$\begin{aligned} h_n(G) &:= |\mathrm{Hom}(G, \mathrm{Sym}(n))|, \\ i_n(G) &:= t_n(G) - p_n(G), \end{aligned}$$

where $t_n(G)$ is the number of transitive permutation representations of G of degree n and $p_n(G)$ is the number of primitive permutation representations of G of degree n.

The i is because we say that an imprimitive permutation representation is a transitive permutation representation that is not primitive.

Lemma 4.4. *With the above notation, we have*

$$a_n(G) = t_n(G)/(n-1)!$$

and

$$m_n(G) = p_n(G)/(n-1)!$$

for all n.

For a proof, see Proposition 1.1.1 on page 12 of [7].

Lemma 4.5 (Müller, 1996)**.** *Let f be as in Corollary 4.2. Then*

$$a_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \sim nf(n).$$

Proof. This is one small case of the Corollary on page 123 of [9]. □

Lemma 4.6. *We have that $t_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \sim h_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \sim n!f(n)$.*

Proof. By Corollary 4.2 we have that $h_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \sim n!f(n)$. Also, by Lemmas 4.4 and 4.5, we get $t_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \sim n!f(n)$. □

Theorem 4.7. *Let $G = \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. Let f be as in Corollary 4.2. Then*

$$m_n(G) \sim nf(n).$$

Proof. Recall that $i_n(G) := t_n(G) - p_n(G)$. So $t_n(G) = p_n(G) + i_n(G)$. Thus, $1 = p_n(G)/t_n(G) + i_n(G)/t_n(G)$. By Lemma 4.6, we have $t_n(G) \sim h_n(G)$. So in order to show $p_n(G) \sim t_n(G)$, we need only show that $i_n(G)/h_n(G) \to 0$. The present theorem would then follow by Lemmas 4.4 and 4.6.

This paragraph is based on Dixon's Lemma 2 in [3]. First, notice that the number of imprimitive permutation representations of G with r blocks each of size $d = n/r$ is at most $t_r(G)(h_d(G))^r < h_r(G)(h_d(G))^r$. Also, the number of ways an n element set can be partitioned into r blocks, each with d elements is $n!/((d!)^r r!)$. Therefore,

$$i_n(G) \le \sum_{\substack{d,r>1 \\ dr=n}} \frac{h_r(G)(h_d(G))^r n!}{(d!)^r r!}.$$

Let $a = 1 - 1/m$. Let c be such that $\sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} < c\sqrt{n}$; (obviously, $c = m$ works). We have by Lemma 4.6 that (for large j and n) that

$$h_j(G) \le K j^{aj} e^{-aj + c\sqrt{j}} j! \quad \text{and} \quad h_n(G) > K n^{an} e^{-an} n!$$

Recall $n = dr$. For large n, and assuming c is large enough,

$$\begin{aligned}
\frac{i_n(G)}{h_n(G)} &\le \sum_{\substack{d,r>1 \\ dr=n}} \frac{K r^{ar} e^{-ar + c\sqrt{r}} r! (K d^{ad} e^{-ad + c\sqrt{d}} d!)^r n!}{K n^{an} e^{-an} n! (d!)^r r!} \\
&= \sum_{\substack{d,r>1 \\ dr=n}} \frac{r^{ar} e^{-ar + c\sqrt{r}} K^r d^{an} e^{-an + c\sqrt{d} r}}{d^{an} r^{an} e^{-an}} \\
&= \sum_{\substack{d,r>1 \\ dr=n}} \frac{K^r e^{-ar + c\sqrt{r} + c\sqrt{d} r} r^{ar}}{r^{an}} \\
&< \sum_{\substack{d,r>1 \\ dr=n}} \frac{e^{c\sqrt{r} + c\sqrt{d} r}}{e^{ar + a(n-r)\ln(r)}} = \sum_{\substack{d,r>1 \\ dr=n}} \frac{e^{c\sqrt{r} + c\sqrt{nr}}}{e^{ar + a(n-r)\ln(r)}},
\end{aligned}$$

where the last inequality is because $0 < K < 1$. Let

$$g(n,r) := e^{c\sqrt{r}+c\sqrt{nr}-ar-a(n-r)\ln(r)}.$$

Also let

$$\begin{aligned} G(n) &:= \sum_{r=2}^{\lfloor n/2 \rfloor} g(n,r), \\ A(n) &:= \sum_{r=2}^{\lfloor \sqrt{n} \rfloor} g(n,r), \\ B(n) &:= \sum_{r=\lceil \sqrt{n} \rceil}^{\lfloor n/2 \rfloor} g(n,r). \end{aligned}$$

We have then that $G(n) \le A(n) + B(n)$ with

$$A(n) < \sqrt{n} e^{cn^{1/4}+cn^{3/4}-2a-a(n-\sqrt{n})\ln(2)} \to 0$$

and

$$B(n) < ne^{c\sqrt{n}+cn-a\sqrt{n}-\frac{a}{2}n\ln(\sqrt{n})} \to 0$$

as $n \to \infty$. Thus $G(n) \to 0$, and so $i_n(G)/h_n(G) \to 0$ as $n \to \infty$. □

We almost have Theorem 4.10. We only need to show that the groups $\mathrm{BS}(a,b)$ have very few maximal subgroups that are not contained in the quotient $\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. So our goal is to show that Theorem 4.7 is sufficient to count almost all of the maximal subgroups.

Lemma 4.8. *Let G be a f.g. group with $A \trianglelefteq G$ and A abelian. Then*

$$m_n(G) \le m_n(G/A) + \sum_{A_0} |\mathrm{Der}(G/A, A/A_0)| \tag{2}$$

where the sum is taken over all A_0 such that $A_0 \trianglelefteq G$, $A_0 \le A$ and such that A/A_0 is a simple $\mathbb{Z}[G/A]$-module with $|A/A_0| = n$. When we have $G \cong A \rtimes G/A$, then the inequality in (2) is an equality.

Lemma 4.8 is Lemma 5 in [6].

Lemma 4.9. *Let $\overline{G}$ and A be as in Section 2. Also, we will let*

$$m_n^c(\overline{G}) := m_n(\overline{G}) - m_n(\overline{G}/A).$$

Then $m_n^c(\overline{G}) = 0$ if n is not prime and $m_p^c(\overline{G}) \le p^2$ if p is prime.

Proof. Because $A \cong \mathbb{Z}[1/(uv)]$ has no maximal submodules that are not of prime index, Lemma 4.8 implies that $m_n^c(\overline{G}) = 0$ for such n.

Let $n = p$ be prime. We know that $\mathbb{Z}[1/(uv)]$ has at most 1 maximal ideal of index p (by, say Lemma 3.3). Therefore, to show that $m_p^c(\overline{G}) \le p^2$, by Lemma 4.8, we just need to show that

$$|\mathrm{Der}(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})| \le p^2.$$

This is immediate because the number of functions from a two element generating set of $\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}$ is at most p^2.

□

Theorem 4.10. *Let $m = \gcd(a,b)$, and assume that $m > 1$. Then*

$$m_n(\mathrm{BS}(a,b)) \sim K n^{(1-1/m)n+1} \exp\left(-(1-1/m)n + \sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} \right),$$

where

$$K := \begin{cases} m^{-1/2} & \text{if } m \text{ is odd} \\ m^{-1/2} e^{-1/(2m)} & \text{otherwise.} \end{cases}$$

Proof. Let f be as in Corollary 4.2, $\overline{G}$ from immediately after Theorem 2.1, A from Lemma 2.2, and $m_n^c(\overline{G})$ as in Lemma 4.9.

We know that $m_n(\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}) \leq m_n(\mathrm{BS}(a,b))$, because Corollary 2.7 tells us that $\overline{G}/A \cong \mathbb{Z}*\mathbb{Z}/m\mathbb{Z}$. So by Theorem 4.7, we observe that $m_n(\mathrm{BS}(a,b)) = m_n(\overline{G})$ grows at least as fast as $nf(n)$.

By definition of $m_n^c(\overline{G})$ (in Lemma 4.9) we can write

$$m_n(\overline{G}) = m_n(\overline{G}/A) + m_n^c(\overline{G}).$$

We are done because Lemma 4.9 gives us that $m_n^c(\overline{G})$ is bounded above by a polynomial of degree 2. □

5. When $\gcd(a,b) \neq 1$: an asymptotic formula for $a_n(\mathrm{BS}(a,b))$

The goal of this section is Theorem 5.3. We will again denote $\gcd(a,b)$ by m, and we assume $m > 1$.

Lemma 5.1. *Let G be a group, and let $A \trianglelefteq G$ with A abelian. Then*

$$a_n(G) \leq \sum_{d|n} a_{n/d}(G/A) a_d(A) D_{n,d},$$

where $D_{n,d} = \max_{A_0,G_0} |\mathrm{Der}(G_0/A, A/A_0)|$, where the max *is over the subgroups $A_0 \leq A \leq G_0 \leq G$ with $[A : A_0] = d$, $[G : G_0] = n/d$, and $A_0 \trianglelefteq G_0$.*

This is part of Lemma 2.1 part (ii) in [11].

In what follows, $a = um$ and $b = vm$.

Lemma 5.2. *Let $\overline{G}$ be the group defined just after Theorem 2.1. Let $A \cong \mathbb{Z}[1/(uv)]$ be the subgroup of $\overline{G}$ in Corollary 2.7 so that $\overline{G}/A \cong \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. Fix $n > 1$, and let $d \mid n$. Let $G_0 \trianglelefteq \overline{G}$ with $[\overline{G} : G_0] = n/d$, and let $A_0 \leq_d A$. Then*

$$|\mathrm{Der}(G_0/A, A/A_0)| \leq 3^{2n/3}.$$

This basically follows from the proof of Proposition 1.3.2 part (i) in [7].

Proof. Recall that for a f.g. group H, we let $d(H)$ denote the minimal size of a generating set for H. Hopefully this notation will not be confusing because n/d is the index of G_0 in $\overline{G}$.

We have that $2 = d(\overline{G}/A)$. By Schreier's formula (Result 6.1.1 in [10]), we have that

$$d(G_0/A) \le 1 + [\overline{G} : G_0](2-1) = 1 + \frac{n}{d} \le \frac{2n}{d}.$$

Therefore,

$$|\mathrm{Der}(G_0/A, A/A_0)| \le |A/A_0|^{d(G_0/A)} \le d^{2n/d} \le 3^{2n/3},$$

since $d^{1/d} \le 3^{1/3}$ for every $d \in \mathbb{N}$. □

Theorem 5.3. *Let $m = \gcd(a,b)$, and assume that $m > 1$. Then*

$$a_n(\mathrm{BS}(a,b)) \sim K n^{(1-1/m)n+1} \exp\left(-(1-1/m)n + \sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} \right),$$

where

$$K := \begin{cases} m^{-1/2} & \text{if } m \text{ is odd} \\ m^{-1/2} e^{-1/(2m)} & \text{otherwise.} \end{cases}$$

Proof. Let $G = \mathrm{BS}(a,b)$, and let $\overline{G}$ and A be as in §2. (So $\overline{G}/A \cong \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$.) It follows from Lemma 4.5 that the function $a_n(\overline{G}/A)$ is eventually increasing. Also, we have $a_d(A) \le 1$ for all d. Therefore by Lemmas 5.1 and 5.2, for large (even) n,

$$\begin{aligned} a_n(G) - a_n(\overline{G}/A) &\le \sum_{d|n, d>1} a_{n/d}(\overline{G}/A) \cdot a_d(A) \cdot 3^{2n/3} \\ &\le n \cdot 3^{2n/3} \cdot a_{n/2}(\overline{G}/A). \end{aligned} \tag{3}$$

Let $a = 1 - 1/m$. By Lemma 4.5 (and since $\sum_{\substack{d<m \\ d|m}} \frac{n^{d/m}}{d} = O(n)$), for some $\beta > 1$, we have that $a_{n/2}(\overline{G}/A) \le K n^{an/2+1} \beta^n$ and $a_n(\overline{G}/A) \ge K n^{an+1} e^{-an}$. We have then that for some $C > 1$ that

$$\frac{a_{n/2}(\overline{G}/A)}{a_n(\overline{G}/A)} \le \frac{C^n}{n^{an/2}}.$$

Combining this with (3), we get that

$$\frac{a_n(G)}{a_n(\overline{G}/A)} - 1 \le \frac{n \cdot 3^{2n/3} C^n}{n^{an/2}} \to 0 \quad \text{as } n \to \infty.$$

Therefore $a_n(G) \sim a_n(\overline{G}/A)$. We are done by Lemma 4.5. □

Corollary 5.4. *Assume that* $\gcd(a,b) > 1$. *Then* $m_n(\mathrm{BS}(a,b)) \sim a_n(\mathrm{BS}(a,b))$.

Proof. This follows from Theorems 4.10 and 5.3. □

Acknowledgments

I completed much of the work in this paper while I was a graduate student at Binghamton University. See [5], my dissertation. Also, I would like to thank the referee for pointing out that Lemma 2 of [3] shortens my proof of Theorem 4.7 and for showing me the present proof of Theorem 5.3. (Longer proofs of these theorems, which I found, are in the arXiv version of this paper.)

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This paper is available via http://nyjm.albany.edu/j/2020/26-11.html.