Rigidity of a family of spherical conical metrics

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ABSTRACT. We study the deformation of spherical conical metrics with at least some of the cone angles larger than 2π. We show in this note via synthetic geometry that for one family of such metrics, there is local rigidity in the choice of cone positions if angles are fixed. This gives evidence of the analytic obstruction considered in recent works of Mazzeo and the author [20, 21].

CONTENTS

1. Introduction

The study of constant curvature metrics with singularities has seen a long and rich history, where a lot of interesting questions are still not completely answered. Among them is the following singular uniformization question: given a compact Riemann surface M, a collection of distinct points $\mathfrak{p} = \{p_1, \ldots, p_k\} \subset M$ and a collection of positive real numbers $\beta_1, \ldots, \beta_k$, is it possible to find a metric g on M with constant curvature and with conic singularities with prescribed cone angles $2\pi\beta_j$ at the points p_j? Here, the sign of its curvature is determined by the 'conic' Gauss-Bonnet formula

$$\frac{1}{2\pi} \int_M K \, dA = \chi(M, \vec{\beta}) := \chi(M) + \sum_{j=1}^{k} (\beta_j - 1). \tag{1}$$

When $\chi(M, \vec{\beta}) \leq 0$, the existence and uniqueness of such solutions are proved by McOwen [22]. In the spherical $K = 1$ case with all cone angles less than 2π, Troyanov [28] gave a set of linear inequalities on the β_j's which

are necessary and sufficient for existence; Luo and Tian [18] later proved uniqueness of the solution in this angle regime.

For all the above cases, there is no restriction on the position of cone points. Deformation theory for these cases has been studied by Mazzeo and Weiss [19] and it is shown that the metrics have smooth dependence on cone angles and positions.

When $K = 1$ with at least some of the cone angles bigger than 2π, the story is much more complicated. Recently, Mondello and Panov [23] discovered that when $M = \mathbb{S}^2$ the cone angles are constrained by a set of linear inequalities

$$d_1(\vec{\beta} - \vec{1}, \mathbb{Z}^k_{\text{odd}}) \geq 1, \tag{2}$$

and showed the existence when the strict inequality holds; the boundary cases have been considered in [10, 17, 12]. The same two authors [24] also showed that when $M \neq \mathbb{S}^2$, the condition $\chi(M, \vec{\beta}) > 0$ is sufficient for existence. In either case, one is unable to specify the marked conformal class, i.e., the location of the points $\mathfrak{p}$.

In this paper, we consider the deformation of the following metrics with four conical points on $\mathbb{S}^2$:

$$2\pi\vec{\beta} = (\alpha, \beta, \alpha + \beta, 4\pi), \ \alpha, \beta, \alpha + \beta \notin \{2\pi\mathbb{Z}, \pi + 2\pi\mathbb{Z}\}, \tag{3}$$

and show that there is local rigidity in the location of cone points. Here, "local rigidity" means that for any small deformation of this metric inside the class of spherical conical metrics with the same cone angle combination, the underlying pointed conformal structure does not change. Note here that equation (3) satisfies the equality in equation (2), and such angle combinations lie on the codimension-two boundary of the admissible region.

For any fixed $\vec{\beta}$ satisfying (3), there exists a real one-parameter family of cone point positions $\{\mathfrak{p}_t, 0 < t < \pi\}$ on $\mathbb{S}^2$ such that there exists a spherical conical metric g_t with angles $2\pi\vec{\beta}$ on $(\mathbb{S}^2, \mathfrak{p}_t)$. The geometric realization of such metrics is obtained by gluing two spherical footballs of angles α and β along part of a meridian with one end at the south pole (see Figure 1) where t parametrizes the length of the cut.

In this note, we approach the rigidity of such metrics via synthetic geometry. We decompose the surface along geodesics to get four spherical triangles (see Figure 2).

Definition 1.1. *A **triangulated metric** is defined to be a spherical conical metric with the same geodesic decomposition as in Figure 2 (not necessarily with the same cone angles or geodesic lengths). For two triangulated metrics h and g, h is called **close to** g if lengths of all boundaries of the four geodesic triangles of h are close to the corresponding ones of g.*

In most cases, a small perturbation of metrics preserve the geodesic decomposition. Hence we expect all nearby spherical conical metrics to be triangulated and close to each other in the sense defined above.
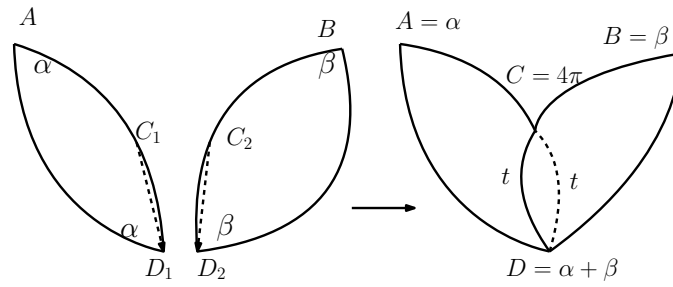
FIGURE 1. A spherical metric with cone angles $\alpha, \beta, \alpha+\beta, 4\pi$ can be obtained by gluing together two spherical footballs along a slit of length t. We cut open each football along a geodesic of length t, and glue points C_1 with C_2 to get a cone point C with angle 4π, D_1 with D_2 to get the point D with angle $\alpha+\beta$, and glue the two pairs of geodesics between $C_1 D_1$ and $C_2 D_2$.
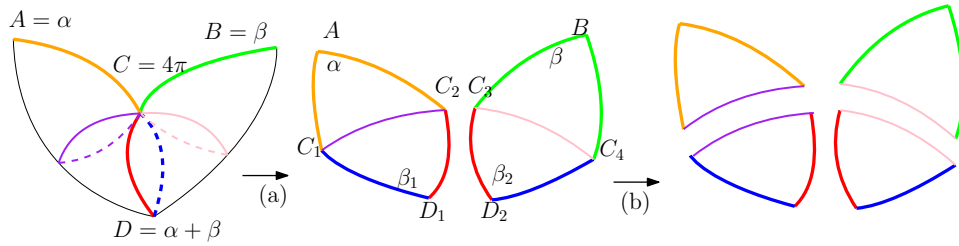

FIGURE 2. The process of cutting the surface into four spherical geodesic triangles (here geodesics with the same color glue together): (a) cut along geodesics AC, BC and two geodesics connecting CD to get two surfaces, each with four sides, where $C_1 - C_4$ glue back to cone point C and $D_1 - D_2$ glue to cone point D; (b) further cut along two geodesics $C_1 C_2$ and $C_3 C_4$ to obtain four triangles.

The space of triangulated metrics are parametrized by six (independent) lengths as long as they satisfy spherical triangle inequalities, see Figure 2 for the color-coded geodesic pairs. If we restrict to such metrics with four fixed cone angles, which imposes four equations on six parameters, then the space of all such metrics is two dimension for a generic angle set $\vec{\beta}$. We remark here that the conformal class of four marked points on a sphere is determined by the (complex) cross ratio which also gives a (real) two-dimension space. Therefore, for a generic angle set $\vec{\beta}$, we expect that the neighborhood of triangulated metrics gives all nearby spherical conical metrics with the same cone angle data.

We will show that for $\vec{\beta}$ as in (3), the glued footballs are the only possible triangulated metrics under perturbation, hence all such metrics form a one-dimensional space. This gives the local rigidity in the geometric sense.

Theorem 1.2. *For any fixed $\vec{\beta}$ as in* (3) *and $t \in (0, \pi)$, if $(\mathbb{S}^2, h)$ is a triangulated spherical metric with the same cone angles and h is close to g_t, then h is isometric to g_s for some s.*

Such geometric rigidity has appeared in the case of a spherical football with noninteger cone angles. By the classical proof of Troyanov [29], the only possible configuration in this case is when all the geodesics connecting two cone points are of length π. By the perturbation argument in [19], the spherical football is the only metric with rigidity when all cone angles are less than 2π. However, when there are more than three cone points with some of angles bigger than 2π, such rigidity is far from clear. This note intends to give a family of explicit metrics in this regime.

The angle combination (3) was discussed by Chen, Wang, Wu and Xu [8, Example 4.7]. It was shown that if such a metric has reducible monodromy, then the positions of cone points with the first three angles $\alpha, \beta, \alpha + \beta$ determine the position of the 4th pointand there is a real 1-parameter family of such metrics where the parameter comes from the scaling of the character 1-form. We remark here that this parameter is exactly the length of the cut (denoted by t above). Since such angle combination lies on the boundary of the admissible region of (2), from [23, Corollary 2.25(I)], this implies that the monodromy of such conical metrics is necessarily reducible. In particular, this implies that there is rigidity in the conformal class of the quadruply punctured sphere for this angle combination.

This note intends to give a geometric realization of such rigidity, and to our knowledge, this is the first proof using elementary spherical geometry.

We also hope to give more insight in understanding the rigidity of cone metrics as solutions to the curvature equation with prescribed singularities. We observe that if we write the metrics on a football with angle $2\pi\alpha$ in geodesic coordinates as

$$dr^2 + \alpha^2 \sin^2 r d\theta^2,$$

then $\cos r$ is an eigenfunction of its Laplacian with eigenvalue 2. Moreover, when two footballs (not necessarily with same angles) glue together, this eigenfunction also glue to give a global one. Therefore, the metrics we consider here all satisfy the condition that number 2 lies in the Friedrichs extension of the Laplacian, hence from [21], such partial rigidity in cone positions is expected as a result of the obstruction in solving the curvature equation.

The study of constant curvature conical metrics has seen a lot of recent development. One approach is through complex analysis (see [13, 14, 15, 30]). The angle combination discussed here was also studied in detail as the solutions to Fuchsian equations in [16, 25]. For metrics with special

monodromy which is of particular interest of this note, see the works of Xu and collaborators [8, 26, 27] and Eremenko [12]. We also mention here the variational approach by Malchiodi and collaborators [1, 2, 3, 4] and the Leray-Schauder degree counting method by Chen and Lin [5, 6]. We refer the readers to [21] for a more comprehensive overview. We also mention here another type of closely related objects called HCMU metrics. This exhibit a similar obstruction in existence [9], and geodesic decomposition was used to analyze such metrics [7].

This paper is organized as follows. In §2 we consider the case when $\alpha = \beta$, for which the computation is simpler but keeps the essential feature of the proof. In §3 we give the proof of the general case.

2. The case $\alpha = \beta$

We start with the example when $\alpha = \beta$ to simplify the computation. We first show that by assuming some symmetry, the only possibility would be the glued footballs.

Lemma 2.1. *If $\beta_1 = \beta_2(= \beta)$, then h is equal to g_s for some s.*

Proof. In the proof we focus on the intermediate step of the triangulation given in Figure 2, where we have two pieces, each of which is a spherical domain with four sides. First connect $C_1 C_2$ and $C_3 C_4$ by geodesics and consider their lengths, see Figure 3.
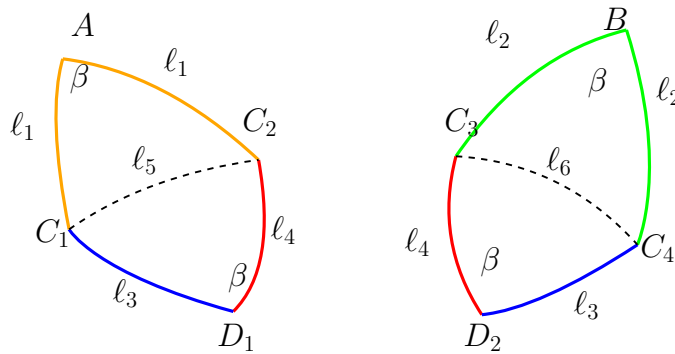

FIGURE 3. When $\beta_1 = \beta_2$, we show that the two pieces should be two identical spherical bigons.

If $\beta_1 = \beta_2 = \beta$, then $\ell_5 = \ell_6$ by spherical cosine law

$$\cos \ell_5 = \cos \ell_3 \cos \ell_4 + \sin \ell_3 \sin \ell_4 \cos \beta = \cos \ell_6. \tag{4}$$

Hence for the two spherical triangles $A_1 C_1 C_2$ and $BC_3 C_4$, there are two possibilities: (a) they are identical; (b) they are not identical but piece together to a spherical bigon.

We first show that case (b) is not possible. If $\ell_1 \neq \ell_2$, then we have $\sin \ell_1 \neq 1$ since $\ell_1 + \ell_2 = \pi$. Applying spherical cosine law again, we have

$$\cos \ell_5 = \cos \ell_6 = (\cos \ell_i)^2 + (\sin \ell_i)^2 \cos \beta = 1 + (\cos \beta - 1)(\sin \ell_i)^2, i = 1, 2.$$

Therefore, we know that

$$\frac{\cos \ell_5 - 1}{\cos \beta - 1} < 1. \tag{5}$$

In this case we also have $AC_1 C_2 = AC_2 C_1 = \pi - BC_3 C_4 = \pi - BC_4 C_3 \neq \pi/2$. Since $AC_2 D_1 + AC_1 D_1 + BC_3 D_2 + BC_4 D_2 = 4\pi$ and the two spherical triangles $C_1 C_2 D_1$ and $C_3 C_4 D_2$ are identical, we have

$$C_1 C_2 D_1 + C_2 C_1 D_1 = \pi (= C_3 C_4 D_2 + C_4 C_3 D_2).$$

Then denoting angle $C_2 C_1 D_1 = \alpha$, by cosine law for the triangle $C_1 C_2 D_1$ we have

$$\cos \beta = -\cos \alpha \cos(\pi - \alpha) + \sin \alpha \sin(\pi - \alpha) \cos \ell_5 = 1 + (\cos \ell_5 - 1)(\sin \alpha)^2$$

hence

$$\frac{\cos \beta - 1}{\cos \ell_5 - 1} \leq 1$$

which contradicts (5). Therefore, this shows that case (b) is not possible.

We now consider case (a) and show this gives $h = g_s$ for some s. Since $\ell_1 = \ell_2$, the two 4-sided surfaces in Figure 3 are identical. So we have the following relations of angles:

$$AC_2 D_1 = BC_3 D_2, \ AC_1 D_1 = BC_4 D_2.$$

Since these four angles add up to 4π, this means that

$$AC_2 D_1 + AC_1 D_1 = 2\pi, \ BC_3 D_2 + BC_4 D_2 = 2\pi. \tag{6}$$

Note that when $AC_2 D_1 = AC_1 D_1 = BC_3 D_2 = BC_4 D_2 = \pi$, and $\ell_3 = \ell_4 = \pi - \ell_1$, we have two bigons which satisfy (6). We now prove that this is the only possibility. If $\ell_3 \neq \ell_4$, then the sum of the two angles $C_1 C_2 D_1 + C_2 C_1 D_1$ will be either less than or bigger than the symmetric case by using Lemma 2.3, which will make the total sum $AC_2 D_1 + AC_1 D_1 < 2\pi$ or $> 2\pi$. Therefore, we must have $\ell_3 = \ell_4$. This way we get two bigons, which piece together to give g_s where $s = \ell_3 = \ell_4$. $\square$

We next show that if $\beta_1 \neq \beta_2$, then the angle combination $\vec{\beta}$ cannot be realized.

Lemma 2.2. *If $\beta_1 \neq \beta_2$, then $AC_2 D_1 + AC_1 D_1 + BC_3 D_2 + BC_4 D_2 \neq 4\pi$ in Figure 2.*

Proof. Without loss of generality, we assume $\beta_1 = \beta - 2\epsilon, \beta_2 = \beta + 2\epsilon, \epsilon > 0$.

With these assumptions, for any fixed ℓ_1 and ℓ_2, we will show that no choice of ℓ_3 and ℓ_4 would give the combination of 4π. In fact, we will show that the total sum of the four angles is strictly less than (or bigger than) 4π depending on the length of ℓ_1 and ℓ_2.

Since we are considering perturbation of a metric g_t, ℓ_1 and ℓ_2 are both close to $\pi - t$. Therefore, unless $t = \pi/2$, we can assume $(\ell_1 - \pi/2)(\ell_2 - \pi/2) > 0$. We will discuss the case of $t = \pi/2$ near the end of this proof.

We first look at the case when $\ell_1, \ell_2 < \pi/2$. In this case $\ell_3, \ell_4 > \pi/2$. We will show that even in the case when $\ell = \ell_3 = \ell_4 > \pi/2$ (which will give the maximal possible angle sum by Lemma 2.3), the total sum of the four angles in question is still less than 4π.

When $\ell_3 = \ell_4$, the two pieces each have a $\mathbb{Z}_2$ symmetry. Therefore, we can consider a half of each piece, which give us two spherical triangles. See picture 4.
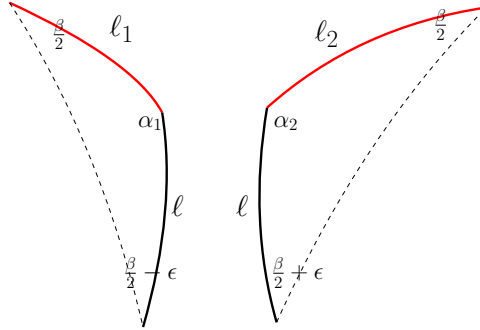

FIGURE 4. The two half pieces

By spherical sine rule we have

$$\frac{\sin \ell_1}{\sin \frac{\beta_1}{2}} = \frac{\sin \ell}{\sin \frac{\beta}{2}}, \ \frac{\sin \ell_2}{\sin \frac{\beta_2}{2}} = \frac{\sin \ell}{\sin \frac{\beta}{2}} \tag{7}$$

In particular, since $\sin \frac{\beta_2}{2} > \sin \frac{\beta}{2} > \sin \frac{\beta_1}{2}$ this implies that $\ell_1 < \pi - \ell < \ell_2$. By Napier's analogies,

$$\cot(\frac{1}{2}\alpha_1) = \frac{\tan(\frac{\beta}{2} - \frac{\beta_1}{2}) \sin[\frac{1}{2}(\ell_1 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_1)]},$$

and

$$\cot(\frac{1}{2}\alpha_2) = \frac{\tan(\frac{\beta}{2} - \frac{\beta_2}{2}) \sin[\frac{1}{2}(\ell_2 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_2)]}.$$

Since $\frac{\beta}{2} - \frac{\beta_1}{2} = -(\frac{\beta}{2} - \frac{\beta_2}{2}) = \epsilon$, if we can show

$$\frac{\sin[\frac{1}{2}(\ell_1 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_1)]} > \frac{\sin[\frac{1}{2}(\ell_2 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_2)]}$$

then this would imply $\alpha_1 + \alpha_2 < 2\pi$, or equivalently $AC_2D_1 + AC_1D_1 + BC_3D_2 + BC_4D_2 < 4\pi$. Now to show the above inequality holds, we just need to expand it and see that after cancellation it is equivalent to

$$\sin \ell \cos \ell \sin[\frac{1}{2}(\ell_1 - \ell_2)] > 0 \qquad (8)$$

which is true by our assumption $\ell > \pi/2$ and $\ell_1 < \ell_2$.

Now, consider the other case when $\ell_1, \ell_2 > \pi/2$. In this case, we show that even in the minimum $\ell = \ell_3 = \ell_4 < \pi/2$ that the total sum is still greater than 4π. Note here that we have $\ell_1 > \pi - \ell > \ell_2$. To show $\alpha_1 + \alpha_2 > 2\pi$, we need to have

$$-\frac{\sin[\frac{1}{2}(\ell_1 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_1)]} > -\frac{\sin[\frac{1}{2}(\ell_2 + \ell)]}{\sin[\frac{1}{2}(\ell - \ell_2)]}$$

and we arrive at the same inequality (note here $\sin[\frac{1}{2}(\ell - \ell_i)] < 0$)

$$\sin \ell \cos \ell \sin[\frac{1}{2}(\ell_1 - \ell_2)] > 0$$

which holds this time since $\ell < \pi/2$ and $\ell_1 - \ell_2 > 0$. This shows that even in the minimum case we still have $\alpha_1 + \alpha_2 > 2\pi$. Therefore, the sum of the four angles is always bigger than 4π in this case cannot be realized.

Finally, we show that in the case when $t = \pi/2$, there is still no possible choice of perturbation. It is easy to show that ℓ_1 cannot be equal to $\pi/2$. If $\ell_1 = \pi/2$ which implies $\cos \ell_5 = \cos \beta$, then by applying cosine law to the triangle $C_1C_2D_1$ one gets $\cos \beta = (\cos \ell_1)^2 + (\sin \ell_1)^2 \cos(\beta - 2\epsilon)$ and this will force $(\cos \ell_1)^2 < 0$, which is impossible. Therefore, either $\ell_1 < \pi/2$ or $\ell_1 > \pi/2$. Using the same sine rule (7), we will have either $\ell_1 < \pi - \ell < \ell_2 \leq \pi/2$ or $\ell_1 > \pi - \ell > \ell_2 \geq \pi/2$. In particular, we also have $\ell \neq \pi/2$. The same computation for (8) would follow on both cases and we see that either $\alpha_1 + \alpha_2 < 2\pi$ or $\alpha_1 + \alpha_2 > 2\pi$.

This completes the proof.

$\square$

We now prove the fact we have used in the proofs above, that for a spherical triangle, if the length of one side and its opposite angle are fixed, then the extremal of the sum of its three angles is achieved by an isosceles spherical triangle. Note that for a fixed ℓ and β with $\cos \ell \neq \cos \beta$, there are two different isosceles triangles, one with angle $\alpha < \pi/2$ and the other with $\alpha' > \pi/2$. See picture 5.

Lemma 2.3. *For the spherical triangle ABC in Figure 6, if angle $ACB = \beta$ and length $AB = \ell \notin 2\pi\mathbb{Z}$ are fixed and they satisfy $\cos \ell \neq \cos \beta$, then the sum of two angles $CAB + CBA$ reaches its minimum and maximum when $AC = BC$. Denoting angle $CAB = \alpha$, the minimum is achieved when $\alpha < \pi/2$, and the maximum is achieved when $\alpha > \pi/2$.*
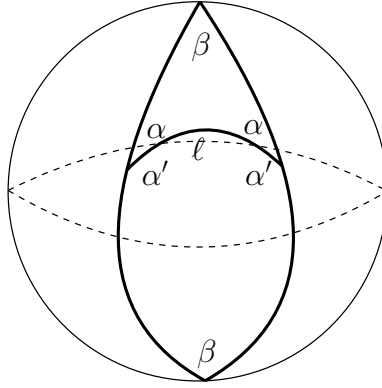
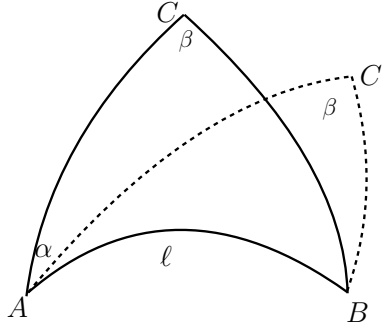FIGURE 5. Two spherical isosceles triangles sharing one side and having the same opposite angle

FIGURE 6. The extremal value of the total angle sum is achieved by an isosceles triangle

Proof. Let s be the sum of the two angles $CAB + CBA$, and let α be the angle CAB. Using the spherical cosine law, we have

$$\cos \beta = -\cos \alpha \cos(s - \alpha) + \sin \alpha \sin(s - \alpha) \cos \ell$$

Here we treat s as an implicit function of α. Differentiating the two sides with respect to α, we get

$$0 = \sin \alpha \cos(s - \alpha) + \cos \alpha \sin(s - \alpha) \cos \ell$$
$$+ (s' - 1)[\cos \alpha \sin(s - \alpha) + \sin \alpha \cos(s - \alpha) \cos \ell]$$

The critical point of s is obtained when $s' = 0$, which corresponds to

$$(1 - \cos \ell) \sin(2\alpha - s) = 0.$$

With the conditions $\alpha > 0$ and $s - \alpha > 0$, we obtain that the critical point satisfies $\alpha = s/2$, i.e. it is an isosceles triangle.

On the other hand, by analyzing the sign of

$$s' = 1 - \frac{\sin \alpha \cos(s - \alpha) + \cos \alpha \sin(s - \alpha) \cos \ell}{\cos \alpha \sin(s - \alpha) + \sin \alpha \cos(s - \alpha) \cos \ell}$$

we obtain the following two cases:

- When $\alpha_0 = s/2 > \pi/2$, we have $s' > 0$ when $\alpha < \alpha_0$ and $s' < 0$ when $\alpha > \alpha_0$, hence it is a maximum point.
- When $\alpha_0 = s/2 < \pi/2$, the signs are reversed and hence it is a minimum point for s.

$\square$

Remark. When $\cos\beta = \cos\ell$, the critical point corresponds to the triangle with angles $\beta, \pi/2, \pi/2$ (so the two extremal triangles are identical). However, in this case this point is neither minimum nor maximum, as the family of triangles are given by any combination of $\alpha = \pi/2$ and $s - \alpha \in (0, \pi)$. The proofs of lemma 1 and lemma 2 do not need to use this fact.

3. Proof of Theorem 1.2

Now we give the proof of our main result.

Proof of Theorem 1.2. When $\alpha = \beta$, we combine Lemmas 2.1 and 2.2, and see that the only possibility is the gluing of two footballs.

When $\alpha \neq \beta$, by replacing the statement $\beta_1 = \beta_2$ to

$$\beta_1 = \alpha, \beta_2 = \beta,$$

the proof is similar. We list out the steps as follows.

Step 1: The only possible case is $\alpha = \beta_1, \beta = \beta_2$.

Assuming $\beta_1 = \alpha - 2\epsilon$ and $\beta_2 = \beta + 2\epsilon$ for some small ϵ. Similar to Lemma 2.2, we look at the extremal case which is achieved when $\ell_3 = \ell_4$ by Lemma 2.3. In this case we get a similar picture as Figure 4 except the one of the two angles $\beta/2$ is replaced by $\alpha/2$. Using the same computation of Napier's analogies, we get that the sum of $\alpha_1 + \alpha_2$ is either strictly bigger than 4π in the minimal case or strictly smaller than 4π in the maximal case. Hence this is not possible when $\epsilon \neq 0$.

Step 2: When $\alpha = \beta_1, \beta = \beta_2$ we get g_s for some s.

We then show that when $\alpha = \beta_1, \beta = \beta_2$ in Figure 7, the only possible configuration is indeed the gluing of two footballs.

When $\alpha \neq \beta$, one no longer has $\ell_5 = \ell_6$ as in the proof of Lemma 2.1. However, by Lemma 2.3, one can see that only the case of $\ell_3 = \ell_4$ would give the possible sum of 4π (otherwise it would either be bigger or smaller than 4π). Then again by symmetry, we can see that the only possible case is when two spherical triangles AC_1C_2 and $C_1C_2D_1$ (similarly, BC_3C_4 and $C_3C_4D_2$) piece together to a bigon, which corresponds to the gluing of two footballs. $\square$
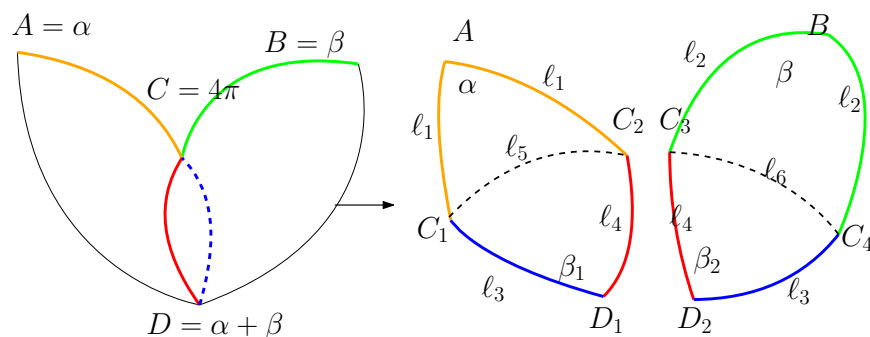
FIGURE 7. Cutting the manifold into two pieces, each is a surface with 4 sides. A priori they do not need to be spherical bigons as in the glued football construction.

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