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Heegner cycles and congruences between anticyclotomic p-adic L-functions over CM-extensions

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Abstract. Let E be a CM-field, and suppose that $\mathbf{f}, \mathbf{g}$ are two primitive Hilbert cusp forms over E^+ of weight $\underline{2}$ satisfying a congruence modulo λ^r. Under appropriate hypotheses, we show that the complex L-values of $\mathbf{f}$ and $\mathbf{g}$ twisted by a ring class character over E, and divided by the motivic periods, also satisfy a congruence relation mod λ^r (after removing some Euler factors). We treat both the even and odd cases for the sign in the functional equation – this generalizes classical work of Vatsal [23] on congruences between elliptic modular forms twisted by Dirichlet characters. In the odd case, we also show that the p-adic logarithms of Heegner points attached to $\mathbf{f}$ and $\mathbf{g}$ satisfy a congruence relation modulo λ^r, thus extending recent work of Kriz and Li [17] concerning elliptic modular forms.

Contents

1. Introduction and results for elliptic curves

Fix an odd prime p, and suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two elliptic curves defined over $\mathbb{Q}$. Provided that $\mathrm{Re}(s) > 3/2$, their Hasse-Weil L-functions can be expressed in the form of Dirichlet series

$$L(\mathcal{A}_1, s) \;=\; \sum_{m=1}^{\infty} a_m(\mathcal{A}_1)\cdot m^{-s} \quad \text{and} \quad L(\mathcal{A}_2, s) \;=\; \sum_{m=1}^{\infty} a_m(\mathcal{A}_2)\cdot m^{-s}.$$

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Furthermore, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are known to be modular by the deep work in [4] hence these L-functions have an analytic continuation to the whole complex plane.

Definition 1.1. We say *the elliptic curves $\mathcal{A}_1$ and $\mathcal{A}_2$ are congruent mod p^r* if one has a system of p-adic congruences

$$a_m(\mathcal{A}_1) \equiv a_m(\mathcal{A}_2) \pmod{p^r} \quad \text{for each } m \in \mathbb{N} \text{ with } \gcd(m, N_1N_2) = 1,$$

where N_1 denotes the conductor of $\mathcal{A}_1/\mathbb{Q}$, and N_2 denotes the conductor of $\mathcal{A}_2/\mathbb{Q}$.

In the p-ordinary case, Vatsal proved that the Mazur-Tate-Teitelbaum [19] p-adic L-functions $\mathbf{L}_p^{\mathrm{MTT}}(\mathcal{A}_1)$ and $\mathbf{L}_p^{\mathrm{MTT}}(\mathcal{A}_2)$ are congruent modulo $p^r \cdot \mathbb{Z}_p[[\Gamma^{\mathrm{cyc}}]]$, where $\Gamma^{\mathrm{cyc}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$ denotes the Galois group of the cyclotomic $\mathbb{Z}_p$-extension. Since these p-adic L-functions $\mathbf{L}_p^{\mathrm{MTT}}(\mathcal{A}_i)$ interpolate Dirichlet twists of the Hasse-Weil L-function $L(\mathcal{A}_i, \psi, s)$ at $s = 1$, one can view Vatsal's result [23] as a statement about congruences between critical L-values divided by the real Néron periods $\Omega^+_{\mathcal{A}_i}$. It is therefore natural to ask if this result extends to number fields other than $\mathbb{Q}$?

To be more specific, let E be a CM-field that is also a solvable extension of $\mathbb{Q}$, and consider the base-change of $\mathcal{A}_1$ and $\mathcal{A}_2$ to E. Throughout this article, we assume that the Leopoldt defect for E is zero. For a character $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ of finite order, it is reasonable to expect a congruence between the twisted L-values

$$\mathcal{E}_p(\mathcal{A}_1/E, \chi) \cdot \frac{L(\mathcal{A}_1/E, \chi, 1)}{(\Omega^+_{\mathcal{A}_1}\Omega^-_{\mathcal{A}_1})^{[E:\mathbb{Q}]/2}} \quad \text{and} \quad \mathcal{E}_p(\mathcal{A}_2/E, \chi) \cdot \frac{L(\mathcal{A}_2/E, \chi, 1)}{(\Omega^+_{\mathcal{A}_2}\Omega^-_{\mathcal{A}_2})^{[E:\mathbb{Q}]/2}} \tag{1.1}$$

modulo p^r, for a suitable choice of factor $\mathcal{E}_p(\mathcal{A}_i/E, \chi)$ and Néron periods $\Omega^\pm_{\mathcal{A}_i} \in \mathbb{C}^\times$.

For example, if E is an imaginary quadratic field over which the prime p splits then Choi and Kim [6] have established a congruence for the two-variable p-adic L-function over E at cusp forms of different weight. Alternatively, if $E = \mathbb{Q}(\mu_{p^n})$ and $r = 1$, then various types of congruence have been proved in [3, 9, 10, 22]. With the exception of [6], all these aforementioned congruences above are purely cyclotomic in their nature, **so in this paper we shall deal exclusively with the anticyclotomic case**.

Throughout we assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ have good ordinary reduction at p, which means $p \nmid a_p(\mathcal{A}_1) \cdot a_p(\mathcal{A}_2) \cdot N_1 \cdot N_2$ (although we expect that a version of our results should exist if one allows p to divide $N_1 \cdot N_2$, whilst still ensuring that $p \nmid a_p(\mathcal{A}_1) \cdot a_p(\mathcal{A}_2)$). We shall further suppose that the prime p splits inside E. Let $\Gamma_E = \mathrm{Gal}(E_\infty/E)$ be the Galois group of the compositum, E_∞ say, of all the $\mathbb{Z}_p$-extensions of E, which can then be decomposed into $\Gamma_E = \Gamma_E^{\mathrm{cyc}} \times \Gamma_E^{\mathrm{anti}}$ where Γ_E^{cyc} (resp. Γ_E^{anti}) is the Galois group of the cyclotomic (resp. full anti-cyclotomic) extension in E_∞.

Building on earlier results in [15, 20], for each base Hecke character χ_0 the work of Disegni [11, Thm 4.3.4] allows the construction of a p-adic L-function $\mathbf{L}_p(\mathcal{A}_i,\chi_0)\in\mathbb{Z}_p[[\Gamma_E]][1/p]$ interpolating the special values given in Equation (1.1) at specialisations $\chi=\chi_0\cdot\chi^\dagger$, as the character $\chi^\dagger$ ranges over $\mathrm{Hom}(\Gamma_E,\overline{\mathbb{Q}}_p^\times)_{\mathrm{tors}}$. For a fixed topological generator γ_0 of $\Gamma_E^{\mathrm{cyc}}\cong 1+p\mathbb{Z}_p$, one can therefore expand each multi-variable p-adic L-function $\mathbf{L}_p(\mathcal{A}_i,\chi_0)$ into a Taylor series of the form

$$\mathbf{L}_p^{(0)}(\mathcal{A}_i,\chi_0)+\mathbf{L}_p^{(1)}(\mathcal{A}_i,\chi_0)\cdot(\gamma_0-1)+\mathbf{L}_p^{(2)}(\mathcal{A}_i,\chi_0)\cdot\frac{(\gamma_0-1)^2}{2}+\cdots$$

for either choice of $i\in\{1,2\}$. It is therefore natural to ask whether:

Question. *For every non-negative integer j, are the individual coefficients $\mathbf{L}_p^{(j)}(\mathcal{A}_1,\chi_0)$ and $\mathbf{L}_p^{(j)}(\mathcal{A}_2,\chi_0)$ congruent to each other modulo $p^r\cdot\mathbb{Z}_p[[\Gamma_E^{\mathrm{anti}}]]$?*

To make a precise statement, one divides the problem into three disjoint cases. For the rest of the Introduction, we assume that the base Hecke character χ_0 is trivial on $F^\times\backslash\mathbb{A}_F^\times$, where $F=E^+$ denotes the maximal totally real subfield of E. We also assume that the primes of F above p are unramified in the extension E/F. Let $\eta_{E/F}$ be the quadratic character of E/F, and write S_i for the set of F-places

$$S_i \;=\; \big\{\nu \;:\; \nu\big|\infty \text{ or } \eta_{E/F,\nu}\big(\mathrm{cond}(\mathcal{A}_i/F)\big) \;=-1\big\}.$$

Definition 1.2. (a) If the global root numbers satisfy $\epsilon(1/2,\mathcal{A}_i/E,\chi_0)=+1$ for each $i\in\{1,2\}$ and if $\#S_1\equiv\#S_2\equiv 0 \pmod 2$, then we call this the **even case**.

(b) If the global root numbers satisfy $\epsilon(1/2,\mathcal{A}_i/E,\chi_0)=-1$ for each $i\in\{1,2\}$ and if $\#S_1\equiv\#S_2\equiv 1\pmod 2$, then we naturally refer to this as the **odd case**.

(c) If $\epsilon(1/2,\mathcal{A}_1/E,\chi_0)=-\epsilon(1/2,\mathcal{A}_2/E,\chi_0)$ or if $\#S_1\equiv\#S_2+1\pmod 2$, then we shall call this the **mixed parity case**.

In the first two cases (a) and (b), we extend Vatsal's main result [23] as follows.

Theorem 1.3. *In the **even case**, if the conductor of the Hecke character χ_0 is coprime to the $\mathcal{O}_E$-ideal $\prod_{i=1}^2\mathrm{cond}(\mathcal{A}_i/E)$, then*

$$\mathbf{L}_{p,\Sigma'}^{(0)}(\mathcal{A}_1,\chi_0)\;\equiv\;\mathbf{L}_{p,\Sigma'}^{(0)}(\mathcal{A}_2,\chi_0)\quad\bmod p^{r+\mu_0}\cdot\mathbb{Z}_p[[\Gamma_E^{\mathrm{anti}}]]$$

where $\mu_0\in\mathbb{Z}$ is the largest value for which each $\mathbf{L}_p^{(0)}(\mathcal{A}_i,\chi_0)\in p^{\mu_0}\cdot\mathcal{O}_{\mathbb{C}_p}[[\Gamma_E^{\mathrm{anti}}]]$.

Note that in the above result, the subscript '$_{\Sigma'}$' indicates that these L-functions have been stripped of their Euler factors at the finite primes contained in the set

$$\Sigma' \;=\; \left\{\nu \in \mathrm{Spec}(\mathcal{O}_F) \text{ such that } \nu \text{ divides } \mathrm{disc}(E/F)\cdot \prod_{i=1}^{2} \mathrm{cond}(\mathcal{A}_i/F)\right\}.$$

Theorem 1.4. *In the* ***odd case****, if the conductor of the Hecke character χ_0 is coprime to $\prod_{i=1}^2 \mathrm{cond}(\mathcal{A}_i/E)$ and all the primes of F above p split in E, then*

(i) $\mathbf{L}^{(0)}_{p,\Sigma'}\big(\mathcal{A}_1,\chi_0\big) = \mathbf{L}^{(0)}_{p,\Sigma'}\big(\mathcal{A}_2,\chi_0\big) = 0$, *and*

(ii) $\dfrac{\mathcal{E}_{0,\Sigma'}(\mathcal{A}_1)}{\mathcal{E}_{1,\Sigma'}(\mathcal{A}_1)}\cdot \mathbf{L}^{(1)}_{p,\Sigma'}\big(\mathcal{A}_1,\chi_0\big)$ *is congruent to* $\dfrac{\mathcal{E}_{0,\Sigma'}(\mathcal{A}_2)}{\mathcal{E}_{1,\Sigma'}(\mathcal{A}_2)}\cdot \mathbf{L}^{(1)}_{p,\Sigma'}\big(\mathcal{A}_2,\chi_0\big)$ *modulo* $p^{r+\mu_1}\cdot \mathbb{Z}_p\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big]$,

where $\mu_1 \in \mathbb{Z}$ is the largest value for which each $\mathbf{L}^{(1)}_p\big(\mathcal{A}_i,\chi_0\big) \in p^{\mu_1}\cdot \mathcal{O}_{\mathbb{C}_p}\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big]$, and $\mathcal{E}_{k,\Sigma'}(\mathcal{A}_i)$ is an Iwasawa function interpolating the product of Euler factors $\prod_{\nu\in\Sigma'} L_\nu(\mathcal{A}_i/E,\chi,k)$ at each $k\in\mathbb{Z}$.

Recall that a quaternion algebra $\mathbb{B}$ is called *coherent* if its ramification set $\Sigma_{\mathbb{B}}$ has even cardinality, and $\mathbb{B}$ is called *incoherent* if the set $\Sigma_{\mathbb{B}}$ has odd cardinality. In the case (c) of mixed parity, we can say nothing about mod p^r congruences as the curves $\mathcal{A}_1, \mathcal{A}_2$ cannot be parameterised by the same quaternion algebra $\mathbb{B}_{/F}$, otherwise $\mathbb{B}$ would have to be simultaneously coherent and incoherent!

There is also a third situation in which one can derive p-adic congruences. Recall that if E is an imaginary quadratic field, the work of Bertolini, Darmon and Prasanna [1] produces a p-adic L-function $\mathfrak{L}(\mathcal{A}_i) \in \mathbb{Z}_p\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big][1/p]$ interpolating critical values of $L\big(\mathcal{A}_i/E,\chi_w,s\big)$ at character twists χ_w of arithmetic weight $w\in\mathbb{N}$. Liu, Zhang and Zhang have extended this to general CM-fields E, constructing a p-adic L-function on Γ_E^{anti} interpolating the complex Rankin-Selberg L-function of each $\mathcal{A}_i$, twisted by characters χ_w of positive weight (see [18, Theorem 3.2.10]). The corresponding p-adic L-functions $\mathfrak{L}(\mathcal{A}_1)$ and $\mathfrak{L}(\mathcal{A}_2)$ exist as elements of

$$\big(\mathrm{Lie}\,\mathcal{A}_i^{+}\otimes_{F^M}\mathrm{Lie}\,\mathcal{A}_i^{-}\big)\otimes_{F^M}\mathcal{D}\big(\mathcal{A}_i, MF_{\mathfrak{p}}^{\mathrm{lt}}\big)$$

in the specific notation of *op. cit*, where $\mathcal{D}\big(\mathcal{A}_i, MF_{\mathfrak{p}}^{\mathrm{lt}}\big)$ is a certain (unbounded) distribution algebra, and $F^M = \mathrm{End}(\mathcal{A}_1)\otimes_{\mathbb{Q}} F = \mathrm{End}(\mathcal{A}_2)\otimes_{\mathbb{Q}} F$.

Aside from the case where E is an imaginary quadratic field, it is not known precisely when $\mathfrak{L}(\mathcal{A}_i)$ arise from p-bounded measures on Γ_E^{anti}. However, if $\mathcal{A}_i$ has good ordinary reduction at p, one might reasonably expect $\mathfrak{L}(\mathcal{A}_i)$ to be an Iwasawa function for each $i\in\{1,2\}$.

In [17], Kriz and Li studied values of the Bertolini-Darmon-Prasanna p-adic L-function via the p-adic logarithms of Heegner points attached to each $\mathcal{A}_i$. In particular, they showed that up to appropriate Euler factors, these logarithms satisfy a congruence relation via Coleman integration. We generalize their method to show that the p-adic logarithms of Heegner points (over ring class fields for a general CM-field E) attached to $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy a similar congruence relation. This allows us to compare special values of $\mathfrak{L}(\mathcal{A}_i)$, and deduce the following result.

Theorem 1.5. *Suppose we are in the odd case, that the primes of F above p split in E, and assuming that **both** $\mathfrak{L}(\mathcal{A}_1), \mathfrak{L}(\mathcal{A}_2)$ **are Iwasawa functions**, then*

$$\mathfrak{L}_{\Sigma'}(\mathcal{A}_1) \ \equiv \ \mathfrak{L}_{\Sigma'}(\mathcal{A}_2) \quad \bmod p^r \cdot \mathcal{L}^{\natural}_{\mathcal{A}_1,\mathcal{A}_2}\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big]$$

where $\mathcal{L}^{\natural}_{\mathcal{A}_1,\mathcal{A}_2}$ is the $\mathcal{O}_{\mathbb{C}_p}$-submodule generated by the values $\chi\big(\mathfrak{L}(\mathcal{A}_1)\big)$ and $\chi\big(\mathfrak{L}(\mathcal{A}_2)\big)$ for $\chi = \chi_0 \cdot \chi^{\dagger}$, as the character $\chi^{\dagger}$ ranges over the elements of $\mathrm{Hom}\big(\Gamma_E^{\mathrm{anti}}, \overline{\mathbb{Q}}_p^{\times}\big)$.

For the remainder of the article, we will work in a more general setting than elliptic curves and solvable CM-fields E. We consider modular abelian varieties $A_\star$ of GL_2-type defined over a totally real field F, parameterised by a common definite quaternion algebra $\mathbb{B}_{/F}$.

Written below is a brief but non-exhaustive summary of our terminology.

- F is a totally real field, E will be a CM-extension of F, and $D_{E/F}$ (resp. D_E) is the relative (resp. absolute) discriminant of E;
- $\eta_{E/F}$ is the quadratic character over F associated to the extension E/F;
- the symbol $\mathfrak{p}$ will indicate a distinguished prime ideal of $\mathcal{O}_F$ lying over p, and we write $\mathfrak{P}$ for any prime $\mathcal{O}_E$-ideal above it ($\mathfrak{p}$ needs not split in E);
- we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and an isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ under which the $\mathcal{O}_E$-ideal $\mathfrak{P}$ is sent into the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$;
- χ always denotes a unitary Hecke character over E (usually a finite order character), which we identify with a Galois character $\mathrm{Gal}\big(E^{\mathrm{ab}}/E\big) \xrightarrow{\chi} \mathbb{C}^\times$;
- for an integral domain R, we shall write R_χ for the ring extension of R which is obtained by adjoining all the values of the character χ above;
- if M is a module equipped with a $\mathrm{Gal}(\overline{E}/E)$-action, $M_{(\chi)} = M \otimes \chi$ denotes the same underlying module M but with its Galois action twisted by χ;
- E^{cyc} indicates the cyclotomic $\mathbb{Z}_p$-extension of E, so that the cyclotomic character κ_{cy} maps $\Gamma_E^{\mathrm{cyc}} := \mathrm{Gal}\big(E^{\mathrm{cyc}}/E\big)$ onto an open subgroup of $1 + p\mathbb{Z}_p$;
- $\mathbf{f}$ and $\mathbf{g}$ denote primitive Hilbert cusp forms over F of parallel weight two, Nebentypus character ω, and levels $N_{\mathbf{f}} \lhd \mathcal{O}_F$ and $N_{\mathbf{g}} \lhd \mathcal{O}_F$ respectively;

- associated to both $\mathbf{f}$ and $\mathbf{g}$ are their modular abelian varieties of GL_2-type, $A_{\mathbf{f}}$ and $A_{\mathbf{g}}$, which are defined over the same totally real number field F;
- $\mathcal{K} = \mathbb{Q}_p\big(C(\mathfrak{n},\mathbf{f}), C(\mathfrak{n},\mathbf{g}) \bigm| \mathfrak{n} \lhd \mathcal{O}_F\big)$ is the finite extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of $\mathbf{f},\mathbf{g}$, and λ denotes a local parameter in $\mathcal{O}_{\mathcal{K}}$;
- for an abelian group M, its (finite) adelisation is given by $\widehat{M} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ where $\widehat{\mathbb{Z}} := \varprojlim_m \mathbb{Z}/m\mathbb{Z} \;\cong\; \prod_{\text{primes } l} \mathbb{Z}_l$ is the profinite completion of $\mathbb{Z}$.

For example, if E is a solvable extension of $\mathbb{Q}$ and $\mathcal{A}_1, \mathcal{A}_2$ are two elliptic curves that are congruent modulo $\lambda^r = p^r$, one can take $\mathbf{f} = \mathbf{BC}_{\mathbb{Q}}^F(f_1)$ and $\mathbf{g} = \mathbf{BC}_{\mathbb{Q}}^F(f_2)$ as their base-changes with each $f_i \in \mathcal{S}_2(\Gamma_0(N_i))$, so that $A_{\mathbf{f}} \cong \mathcal{A}_{1/F}$ and $A_{\mathbf{g}} \cong \mathcal{A}_{2/F}$.

We shall now describe a generalisation of Definition 1.1 to modular abelian varieties over F. Let $\widetilde{N}$ denote the $\mathcal{O}_F$-ideal $\mathrm{lcm}\big(N_{\mathbf{f}}, N_{\mathbf{g}}, \mathcal{Q}^2\big)$ where $\mathcal{Q} = \prod_{\nu | N_{\mathbf{f}} N_{\mathbf{g}}} \nu$. For a prime $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$, $\mathcal{T}(\mathfrak{q})$ denotes the $\mathfrak{q}$-th Hecke operator if $\mathfrak{q}$ is coprime to the level of the HMF, whilst $\mathcal{U}(\mathfrak{q})$ is the $\mathfrak{q}$-th Hecke operator if $\mathfrak{q}$ divides the level of the HMF (see for example [20, Chapter 4, §1.3]). We will also require the diamond operators $\langle \mathfrak{m} \rangle$, as well as the degeneracy maps $\mathcal{V}(\mathfrak{m})$ which act on the Fourier expansions by sending $C(\mathfrak{n},\mathbf{h}) \mapsto C(\mathfrak{n}\mathfrak{m}^{-1},\mathbf{h})$ for either choice of form $\mathbf{h} \in \{\mathbf{f},\mathbf{g}\}$.

Definition 1.6. The $\widetilde{N}$-depletion of $\mathbf{f}$ is the Hilbert cusp form $\widetilde{\mathbf{f}}$ given by

$$\mathbf{f} \,\Bigg|\, \prod_{\mathfrak{q} | N_{\mathbf{g}},\ \mathfrak{q} \nmid N_{\mathbf{f}}} \big(1 - \mathcal{T}(\mathfrak{q}) \circ \mathcal{V}(\mathfrak{q}) + \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q}) \langle \mathfrak{q} \rangle \circ \mathcal{V}(\mathfrak{q}^2)\big) \prod_{\mathfrak{q} \| N_{\mathbf{f}}} \big(1 - \mathcal{U}(\mathfrak{q}) \circ \mathcal{V}(\mathfrak{q})\big).$$

Similarly, the $\widetilde{N}$-depletion $\widetilde{\mathbf{g}}$ of $\mathbf{g}$ is defined by the formula

$$\mathbf{g} \,\Bigg|\, \prod_{\mathfrak{q} | N_{\mathbf{f}},\ \mathfrak{q} \nmid N_{\mathbf{g}}} \big(1 - \mathcal{T}(\mathfrak{q}) \circ \mathcal{V}(\mathfrak{q}) + \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q}) \langle \mathfrak{q} \rangle \circ \mathcal{V}(\mathfrak{q}^2)\big) \prod_{\mathfrak{q} \| N_{\mathbf{g}}} \big(1 - \mathcal{U}(\mathfrak{q}) \circ \mathcal{V}(\mathfrak{q})\big).$$

In particular, $\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}} \in \mathcal{S}_{\underline{2}}\big(\widetilde{N}, \omega\big)$ with $L(\widetilde{\mathbf{f}}, s) = L_{N_{\mathbf{f}} N_{\mathbf{g}}}(\mathbf{f}, s)$ and $L(\widetilde{\mathbf{g}}, s) = L_{N_{\mathbf{f}} N_{\mathbf{g}}}(\mathbf{g}, s)$.

Hypothesis. **($\mathbf{f} \equiv \mathbf{g}\ (\lambda^r)$)** There is an identity of depleted Hilbert cusp forms

$$\widetilde{\mathbf{f}} \;=\; \widetilde{\mathbf{g}} \;+\; \lambda^r \cdot \sum_j c_j \cdot \mathbf{h}_j$$

with each scalar term $c_j \in \mathcal{O}_{\mathcal{K}}$, and where the $\mathbf{h}_j$'s denote normalised eigenforms of parallel weight two, level dividing into $\widetilde{N}$, and with Nebentypus character ω.

To reassure the reader, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two elliptic curves as before that are congruent modulo p^r, then their base-changes $\mathbf{f} = \mathbf{BC}_{\mathbb{Q}}^F(f_1)$ and

$\mathbf{g} = \mathbf{BC}_{\mathbb{Q}}^{F}(f_2)$ automatically satisfy Hypothesis $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ upon choosing the uniformizer $\lambda = p$. Indeed to verify this claim, we first observe that

$$\tilde{f}_1 \;=\; \sum_{\gcd(m,N_1N_2)=1} a_m(\mathcal{A}_1)\cdot q^m \quad\text{and}\quad \tilde{f}_2 \;=\; \sum_{\gcd(m,N_1N_2)=1} a_m(\mathcal{A}_2)\cdot q^m$$

satisfy $\tilde{f}_1 - \tilde{f}_2 = p^r \cdot f^{\natural}$ for some $f^{\natural} \in \mathcal{S}_2\Big(\Gamma_0\big(\mathrm{lcm}\big(N_1, N_2, \prod_{l|N_1N_2} l^2\big)\big)\Big) \cap \mathbb{Z}[\![q]\!]$. However, this latter module has an integral basis consisting of elements of the type $h_j\big|\mathcal{V}(d)$ where h_j is a newform of level C_j, and $d \geq 1$ ranges over integers such that dC_j divides the common level $\mathrm{lcm}\big(N_1, N_2, \prod_{l|N_1N_2} l^2\big)$; one can therefore express

$$\tilde{f}_1 \;=\; \tilde{f}_2 \;+\; p^r \cdot \sum_{j,d} c_j^{(d)} \cdot h_j\big|\mathcal{V}(d) \quad \text{where the scalars } c_j^{(d)} \in \mathbb{Z}.$$

After base-changing each of the cusp forms $\tilde{f}_1$, $\tilde{f}_2$ and the $h_j|\mathcal{V}(d)$'s from $\mathbb{Q}$ to F, we respectively obtain the HMFs $\widetilde{\mathbf{f}}$, $\widetilde{\mathbf{g}}$ and the $\mathbf{h}_j$'s in Hypothesis $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$.

The proof of our main results (Theorems 1.3, 1.4 and 1.5) makes heavy use of three recent spectacular but rather technical formulae, due to various authors. To treat the even case, we use a version of the Waldspurger formula from [5, 26].To treat the odd case, we apply the p-adic Gross-Zagier formula in [11, 12]. Lastly, to prove congruences for the Liu-Zhang-Zhang p-adic L-functions, we use the connection between its special values and the logarithms of Heegner cycles [1, 18]. The demonstrations themselves are written up in Sections 2, 3, and 4, respectively.

2. The even case: Waldspurger's formula

Let $\mathbb{B}$ be a totally definite quaternion algebra defined over the totally real field F. We suppose that $\pi_{\mathbf{f}}$ and $\pi_{\mathbf{g}}$ are two cuspidal automorphic representations of $\mathbb{B}_{\mathbb{A}_F}^{\times}$, associated to the Hilbert modular forms $\mathbf{f}$ and $\mathbf{g}$ respectively under the Jacquet-Langlands correspondence on GL_2/F, with a common central character ω on $\mathbb{A}_{F,\mathrm{fin}}^{\times}$. Let us also consider a fixed finite order Hecke character χ defined on $E^{\times}\backslash\mathbb{A}_E^{\times}$, corresponding to a weight one theta-series automorphic representation π_χ of $\mathbb{B}_{\mathbb{A}_F}^{\times}$.

Hypothesis. (Even) The product $\omega \cdot \chi\big|_{\mathbb{A}_F^{\times}}$ is trivial, the three finite sets

$$\begin{aligned} S_{N_{\mathbf{f}}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(N_{\mathbf{f}}) = -1\big\} \\ S_{N_{\mathbf{g}}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(N_{\mathbf{g}}) = -1\big\} \\ S_{\widetilde{N}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(\widetilde{N}) = -1\big\} \end{aligned}$$

each have *even* cardinality, and for all places ν of F

$$\epsilon\big(1/2, \pi_{\mathbf{f},\nu}, \pi_{\chi,\nu}\big) \;=\; \epsilon\big(1/2, \pi_{\mathbf{g},\nu}, \pi_{\chi,\nu}\big) \;=\; \chi_\nu(-1)\cdot \eta_{E/F,\nu}(-1)\cdot \xi\big(\mathbb{B}_\nu\big)$$

where the sign $\xi\big(\mathbb{B}_\nu\big) = -1$ if $\mathbb{B}_\nu$ is a division algebra, and $\xi\big(\mathbb{B}_\nu\big) = +1$ otherwise.

Here we have written $\epsilon\big(1/2, \pi_{\star,\nu}, \pi_{\chi,\nu}\big)$ for the local root number associated to the complex tensor product L-series $L\big(s, \pi_\star \times \pi_\chi\big)$, for each choice of $\star \in \{\mathbf{f}, \mathbf{g}\}$. **The above hypothesis then implies that both the global root numbers $\epsilon\big(1/2, \pi_{\mathbf{f}}, \pi_\chi\big)$ and $\epsilon\big(1/2, \pi_{\mathbf{g}}, \pi_\chi\big)$ in the Rankin L-functions are equal to** $+1$, and there is an F-embedding of E into $\mathbb{B}$ that identifies $E^\times$ with a sub-torus in $\mathbb{B}^\times$.

Proposition 2.1. *If the Hypotheses* $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ *and* **(Even)** *both hold, and if* $\mathfrak{f}_\chi := \mathrm{cond}(\chi)$ *is coprime to* $N_{\mathbf{f}} N_{\mathbf{g}} \cdot \mathcal{O}_E$*, there is a congruence of* p*-integral elements*

$$\sqrt{|D_E| \cdot \|\mathfrak{c}_\chi\|^2} \cdot \frac{L_\Sigma(1/2, \pi_{\mathbf{f}} \times \pi_\chi)}{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\mathbf{f})} \equiv$$
$$\sqrt{|D_E| \cdot \|\mathfrak{c}_\chi\|^2} \cdot \frac{L_\Sigma(1/2, \pi_{\mathbf{g}} \times \pi_\chi)}{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\mathbf{g})} \mod \lambda^r \mathcal{O}_{\mathcal{K},\chi}$$

where $\mathfrak{c}_\chi$ *is the largest* $\mathcal{O}_F$*-ideal so that* χ *is trivial on* $\prod_{\nu \nmid \mathfrak{c}_\chi} \mathcal{O}^\times_{E,\nu} \times \prod_{\nu | \mathfrak{c}} \big(1 + \mathfrak{c}\mathcal{O}_{E,\nu}\big)$*,* $\|\mathfrak{c}_\chi\|$ *denotes the norm* $\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{c}_\chi)$*, the finite set* Σ *consists of the places of* F *dividing* $N_{\mathbf{f}} \cdot N_{\mathbf{g}} \cdot D_{E/F} \cdot \mathfrak{c}_\chi \cdot \infty$*, and* $\Omega^{\mathrm{aut},(0)}_{\infty,K}(\star)$ *is the automorphic period (see Equation (2.1)) associated to each* $\star \in \{\mathbf{f}, \mathbf{g}\}$*.*

Proof. The key ingredient is the generalised Waldspurger formula in [5, 26]. In particular, we shall take as our common level structure $\widetilde{N} := \mathrm{lcm}\big(N_{\mathbf{f}}, N_{\mathbf{g}}, \mathcal{Q}^2\big)$ where $\mathcal{Q} = \prod_{\nu | N_{\mathbf{f}} N_{\mathbf{g}}} \nu \ \lhd \mathcal{O}_F$. Firstly, one defines a finite subset of $\mathrm{Spec}(\mathcal{O}_F)$ by

$$\Sigma_1 \ := \ \Big\{\ \nu \big| \widetilde{N} \ \text{ where } \eta_{E/F}(\nu) = -1 \text{ and } \mathrm{ord}_\nu(\mathfrak{c}_\chi) < \mathrm{ord}_\nu\big(\widetilde{N}\big) \Big\}$$

and next constructs a pair of $\mathcal{O}_F$-ideals via

$$c_1 := \prod_{\substack{\nu | \mathfrak{c}_\chi, \\ \nu \notin \Sigma_1}} \nu^{\mathrm{ord}_\nu(\mathfrak{c}_\chi)} \quad \text{and} \quad N_1 := \prod_{\substack{\nu | \widetilde{N}, \\ \nu \notin \Sigma_1}} \nu^{\mathrm{ord}_\nu(\widetilde{N})}.$$

Now let $\mathcal{R}$ be an admissible $\mathcal{O}_F$-order for the pairs $(\pi_{\mathbf{f}}, \chi)$ and $(\pi_{\mathbf{g}}, \chi)$ in the sense of [5, Sect 1], so that in addition $\mathcal{R}$ has discriminant $\widetilde{N}$ and $\mathcal{R} \cap E = \mathcal{O}_F + c_1 \mathcal{O}_E$. We shall also fix a compact open subgroup $U = \prod_\nu U_\nu \subset \mathbb{B}^\times_{\mathbb{A}_F}$ such that $U_\nu = \mathcal{R}^\times_\nu$ at all finite places ν of F, and moreover if the place $\nu | N_1$ then $\mathbb{B}_\nu$ must be split. The (zero-dimensional) Shimura variety $\mathbf{X} = \mathbf{X}_U(\mathbb{B})$ is then defined by

$$\mathbf{X}_U(\mathbb{B}) \ := \ \mathbb{B}^\times \backslash \widehat{\mathbb{B}}^\times / \widehat{U}$$

and let $g_1, \ldots, g_n \in \widehat{\mathbb{B}}^\times$ be a complete set of representatives for $\mathbf{X}$, so that $[g_i] \in \mathbf{X}$.

If $\mathbb{Z}[\mathbf{X}]$ denotes the free $\mathbb{Z}$-module consisting of formal sums $\sum_i a_i[g_i]$, then there is a height pairing $\big[-,-\big]_{\mathbf{X}} : \mathbb{Z}[\mathbf{X}] \times \mathbb{Z}[\mathbf{X}] \to \mathbb{C}[\mathbf{X}]$ from [14], sending each pair $\big(\sum_i a_i[g_i], \sum_i b_i[g_i]\big)$ to the element $\sum_i a_i b_i w_i$ with $w_i = \#\big(\mathbb{B}^\times \cap g_i \widehat{\mathcal{R}}^\times g_i^{-1}\big)/\mathcal{O}_F^\times$. There exists a canonical direct sum decomposition

$$\mathbb{Z}[\mathbf{X}] \;=\; \bigoplus_{c \in C_+} \mathbb{Z}\big[\mathbf{X}_c\big]$$

where $\mathbf{X}_c$ is the preimage of $c \in C_+ := F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times$ under the natural surjection $\mathbf{X}_U(\mathbb{B}) = \mathbb{B}^\times \backslash \widehat{\mathbb{B}}^\times / \widehat{U} \twoheadrightarrow F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times$. One may also consider the submodules $\mathbb{Z}\big[\mathbf{X}_c\big]^0 \subset \mathbb{Z}\big[\mathbf{X}_c\big]$ containing degree zero classes, and set $\mathbb{Z}[\mathbf{X}]^0 := \bigoplus_{c \in C_+} \mathbb{Z}\big[\mathbf{X}_c\big]^0$.

For each choice of $\star \in \{\mathbf{f}, \mathbf{g}\}$, let $V(\pi_\star, \chi)$ indicate the space of 'test vectors' in the sense of [5, Defn 3.6]. Because we are working at level $\widetilde{N}$ rather than level $N_\star$, it is no longer true in general that $V(\pi_\star, \chi)$ is one-dimensional over $\mathbb{C}$; in fact

$$\dim_{\mathbb{C}}\big(V(\pi_\star, \chi)\big) \;=\; \prod_{\nu | \widetilde{N}} \Big(1 + \mathrm{ord}_\nu\big(\widetilde{N}\big) - \mathrm{ord}_\nu\big(N_\star\big)\Big)$$

(of course, if $\widetilde{N} = N_{\mathbf{f}} = N_{\mathbf{g}}$ then both $V(\pi_{\mathbf{f}}, \chi)$ and $V(\pi_{\mathbf{g}}, \chi)$ correspond to $\mathbb{C}$-lines). There are injections $V(\pi_\star, \chi) \hookrightarrow \mathbb{Z}[\mathbf{X}]^0 \otimes \mathbb{C}$ obtained from $\Phi \mapsto \sum_i \Phi\big([g_i]\big) w_i^{-1} [g_i]$, which respect the natural action of the Hecke algebra on both $\mathbb{C}$-vector spaces.

Remarks. (a) Considering the $\widetilde{N}$-depletions $\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}} \in \mathcal{S}_{\underline{2}}\big(\widetilde{N}, \omega\big)$ in Definition 1.6, the images of $\mathbb{C} \cdot \widetilde{\mathbf{f}}$ and $\mathbb{C} \cdot \widetilde{\mathbf{g}}$ inside $\mathbb{Z}[\mathbf{X}]^0 \otimes \mathbb{C}$ define unique dimension one subspaces.

(b) The action of the Hecke operators $T(\mathfrak{n})$ on $\mathbb{C} \cdot \widetilde{\mathbf{f}}$ (resp. $\mathbb{C} \cdot \widetilde{\mathbf{g}}$) coincide with their action on $\mathbb{C} \cdot \mathbf{f}$ (resp. $\mathbb{C} \cdot \mathbf{g}$) if $\mathfrak{n}$ is coprime to $\widetilde{N}$, whilst the $U(\mathfrak{q})$-operators annihilate both of the depleted lines $\mathbb{C} \cdot \widetilde{\mathbf{f}}$ and $\mathbb{C} \cdot \widetilde{\mathbf{g}}$ whenever $\gcd\big(\mathfrak{q}, \widetilde{N}\big) \neq \mathcal{O}_F$.

(c) In the notation of [5, Thm 1.9], we can take as test vectors any $f_1', f_2' \in \mathbb{C} \cdot \widetilde{\mathbf{f}}$ (resp. $f_1', f_2' \in \mathbb{C} \cdot \widetilde{\mathbf{g}}$) viewed inside $\mathbb{Z}[\mathbf{X}]^0 \otimes \mathbb{C}$, and then apply the variation of Waldspurger's formula to $f_1' \in V(\pi_\star, \chi)$, $f_2' \in V(\pi_\star^\vee, \chi^{-1})$ for $\star = \mathbf{f}$ (resp. $\star = \mathbf{g}$).

We now relate the Rankin L-function to twisted CM-cycles living in $\mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$. Recall the fixed embedding $E \hookrightarrow \mathbb{B}$ induces a group homomorphism $\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi}) \to \mathbf{X}$ sending $t \mapsto x_t$ where $\mathcal{O}_{\mathfrak{c}_\chi}$ denotes the order $\mathcal{O}_F + \mathfrak{c}_\chi \mathcal{O}_E$, with $\mathfrak{c}_\chi \lhd \mathcal{O}_F$ indicating the largest $\mathcal{O}_F$-ideal such that χ becomes trivial on $\prod_{\nu \nmid \mathfrak{c}_\chi} \mathcal{O}_{E,\nu}^\times \times \prod_{\nu | \mathfrak{c}_\chi} \big(1 + \mathfrak{c}_\chi \mathcal{O}_{E,\nu}\big)$. One defines a pair of

($\widetilde{N}$-depleted) CM-cycles by

$$\widetilde{P}_\chi(\mathbf{f}) \;:=\; \sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi^{-1}(t)\cdot\widetilde{\mathbf{f}}(x_t) \quad\text{and}\quad \widetilde{P}_\chi(\mathbf{g}) \;:=\; \sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi^{-1}(t)\cdot\widetilde{\mathbf{g}}(x_t)$$

which à priori lie inside $\mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]$. However, if χ is a non-trivial character then $\sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})}\chi^{-1}(t)=0$, so clearly $\widetilde{P}_\chi(\mathbf{f}),\widetilde{P}_\chi(\mathbf{g})\in\mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$ both have degree zero[1].

We initially focus on the HMF $\mathbf{f}$, and its depleted CM-cycle $\widetilde{P}_\chi(\mathbf{f})\in\mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$. Viewing $\mathbf{f}$ as a holomorphic function $\phi_{\mathbf{f}}:\mathcal{H}^{[F:\mathbb{Q}]}\to\mathbb{C}$, let us denote by $\langle\phi_{\mathbf{f}},\phi_{\mathbf{f}}\rangle_{\mathrm{Pet}}$ the Petersson self-product of $\phi_{\mathbf{f}}$, computed using the invariant measure induced on

$$\mathrm{PGL}_2(F)\backslash\mathcal{H}^{[F:\mathbb{Q}]}\times\mathrm{PGL}_2(\mathbb{A}_{F,\mathrm{fin}})/U_0\big(\widetilde{N}\big)$$

from the standard hyperbolic volume $\mathrm{d}x\mathrm{d}y/y^2$ on the extended upper half-plane. Applying Waldspurger's formula in the format of [5, Thm 1.9] and [26, Thm 7.1],

$$L_{\Sigma'}\big(1/2,\pi_{\mathbf{f}}\times\pi_\chi\big) = 2^{-\#\Sigma_D}\cdot\frac{(8\pi^2)^{[F:\mathbb{Q}]}\cdot\frac{1}{2}\mathrm{Vol}\big(\mathbf{X}_{U_0(\widetilde{N})}\big)\cdot\langle\phi_{\mathbf{f}},\phi_{\mathbf{f}}\rangle_{\mathrm{Pet}}}{u^2\sqrt{|D_E|\cdot\|\mathfrak{c}_\chi\|^2}}\cdot\big[\widetilde{P}_\chi(\mathbf{f}),\widetilde{P}_\chi(\mathbf{f})\big]_{\mathbf{X}},$$

where Σ' consists of those primes dividing $\gcd\big(N_{\mathbf{f}}\cdot N_{\mathbf{g}},\mathfrak{c}_\chi\cdot D_{E/F}\big)\cdot\infty$ such that if $\nu\|\widetilde{N}$ then $\nu\nmid D_{E/F}$, whilst Σ_D denotes the set of primes of F dividing $\gcd\big(\widetilde{N},D_{E/F}\big)$.

Furthermore, we claim that $u:=\#\mathrm{Ker}\big(\mathrm{Pic}(\mathcal{O}_F)\to\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})\big)\times[\mathcal{O}_{\mathfrak{c}_\chi}^\times:\mathcal{O}_F^\times]$ is always a p-adic unit. To see why this is so, observe that $\mathrm{Ker}\big(\mathrm{Pic}(\mathcal{O}_F)\to\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})\big)$ is either 1 or 2 by [24, Theorem 10.3]. Writing W_E for the roots of unity of E, then $[W_E\mathcal{O}_F^\times:\mathcal{O}_F^\times]$ is coprime to p as the primes of F above p are unramified in E. Moreover $[\mathcal{O}_E^\times:W_E\mathcal{O}_F^\times]$ is either 1 or 2 by [24, Theorem 4.12], consequently both $[\mathcal{O}_E^\times:\mathcal{O}_F^\times]$ and hence $[\mathcal{O}_{\mathfrak{c}_\chi}^\times:\mathcal{O}_F^\times]$ are coprime to p.

It is also easy to check that there is an inclusion of sets of finite places $\Sigma'\hookrightarrow\Sigma$. If we now attach a (complex) automorphic period to $\mathbf{f}$ over K by setting

$$\Omega_{\infty,K}^{\mathrm{aut},(0)}(\mathbf{f}) \;:=\; (8\pi^2)^{[F:\mathbb{Q}]}\cdot\mathrm{Vol}\big(\mathbf{X}_{U_0(\widetilde{N})}\big)\cdot\langle\phi_{\mathbf{f}},\phi_{\mathbf{f}}\rangle_{\mathrm{Pet}} \tag{2.1}$$

then rearranging Waldspurger's formula yields the equality

$$\big[\widetilde{P}_\chi(\mathbf{f}),\widetilde{P}_\chi(\mathbf{f})\big]_{\mathbf{X}} \;=\; u^2\cdot 2^{\#\Sigma_D+1}\times\sqrt{|D_E|\cdot\|\mathfrak{c}_\chi\|^2}\cdot\frac{L_{\Sigma'}(1/2,\pi_{\mathbf{f}}\times\pi_\chi)}{\Omega_{\infty,K}^{\mathrm{aut},(0)}(\mathbf{f})}\,.$$

[1]If χ is trivial then one takes instead $\big[\widetilde{P}_\chi(\mathbf{f})-\deg(\widetilde{P}_\chi)\cdot\xi\big],\big[\widetilde{P}_\chi(\mathbf{g})-\deg(\widetilde{P}_\chi)\cdot\xi\big]\in\mathrm{Pic}(\mathbf{X})\otimes\mathbb{C}$, where ξ denotes the absolute Hodge class [26, Eqn (6.8)] which has degree one on each component.

An entirely similar argument, applied to $\mathbf{g}$ and $\widetilde{P}_\chi(\mathbf{g}) \in \mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$, establishes that

$$\left[\widetilde{P}_\chi(\mathbf{g}), \widetilde{P}_\chi(\mathbf{g})\right]_{\mathbf{X}} \;=\; u^2 \cdot 2^{\#\Sigma_D+1} \times \sqrt{|D_E| \cdot \|\mathfrak{c}_\chi\|^2} \cdot \frac{L_{\Sigma'}(1/2, \pi_{\mathbf{g}} \times \pi_\chi)}{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\mathbf{g})} .$$

Crucially, for each eigenform $\mathbf{h}$ lying in the $(\mathbf{f},\mathbf{g})$-isotypic component, the depleted cycles $\widetilde{P}_\chi(\mathbf{h})$ belongs to the dual lattice $\left(\mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0\right)^\vee$ under the pairing $[-,-]_{\mathbf{X}}$. Using the $\mathcal{O}_{\mathcal{K},\chi}$-bilinearity of this pairing, it therefore suffices to show that

$$\widetilde{P}_\chi(\mathbf{f}) \;=\; \widetilde{P}_\chi(\mathbf{g}) + \lambda^r \cdot Q \quad \text{for some } Q \in \mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$$

because if this is indeed the case, then as a direct corollary,

$$\left[\widetilde{P}_\chi(\mathbf{f}), \widetilde{P}_\chi(\mathbf{f})\right]_{\mathbf{X}} \;=\; \left[\widetilde{P}_\chi(\mathbf{g}), \widetilde{P}_\chi(\mathbf{g})\right]_{\mathbf{X}} + \lambda^r \times \Big(2 \cdot \left[\widetilde{P}_\chi(\mathbf{g}), Q\right]_{\mathbf{X}} + \lambda^r \cdot \left[Q, Q\right]_{\mathbf{X}}\Big),$$

so that $\left[\widetilde{P}_\chi(\mathbf{f}), \widetilde{P}_\chi(\mathbf{f})\right]_{\mathbf{X}} \equiv \left[\widetilde{P}_\chi(\mathbf{g}), \widetilde{P}_\chi(\mathbf{g})\right]_{\mathbf{X}} \mod \lambda^r$.

We now exploit the relation between $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{g}}$ given in Hypothesis **($\mathbf{f} \equiv \mathbf{g}$ (λ^r))**, observing that this relation is preserved when we apply the Jacquet-Langlands correspondence and shift to the quaternion algebra $\mathbb{B}$. One thereby deduces that

$$\begin{aligned}
\widetilde{P}_\chi(\mathbf{f}) \;&=\; \sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi^{-1}(t) \cdot \widetilde{\mathbf{f}}(x_t) \\
&=\; \sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi^{-1}(t) \cdot \left(\widetilde{\mathbf{g}}(x_t) + \lambda^r \cdot \sum_j c_j \cdot \mathbf{h}_j(x_t)\right) \\
&=\; \widetilde{P}_\chi(\mathbf{g}) \;+\; \lambda^r \cdot \sum_j c_j \;\cdot \sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi^{-1}(t) \cdot \mathbf{h}_j(x_t),
\end{aligned}$$

and setting $Q = \sum_j c_j \cdot \sum_t \chi^{-1}(t) \cdot \mathbf{h}_j(x_t) \in \mathcal{O}_{\mathcal{K},\chi}[\mathbf{X}]^0$, the result follows at once. $\square$

Let E_∞ denote the maximal $\mathbb{Z}_p$-power extension of E unramified outside p, so $\Gamma_E := \mathrm{Gal}(E_\infty/E) \cong \mathbb{Z}_p^{1+[F:\mathbb{Q}]+\delta}$ where $\delta \geq 0$ is the defect in Leopoldt's conjecture. If we choose a base character χ_0 such that $\omega \cdot \chi_0\big|_{\mathbb{A}_F^\times}$ is trivial, it follows that the family of characters $\left\{\chi_0 \cdot \chi^\dagger \;\middle|\; \chi^\dagger : \Gamma_E^{\mathrm{anti}} \to \mu_{p^\infty}\right\}$ also satisfies Hypothesis **(Even)**. Henceforth we define $\rho_0 := \mathrm{Ind}_E^F(\chi_0) : G_F \to \mathrm{GL}_2(\mathcal{O}_\chi)$ which is a two-dimensional Artin representation, as are $\rho_\chi := \mathrm{Ind}_E^F(\chi)$ for every character $\chi = \chi_0 \cdot \chi^\dagger$ as above. **For the rest of this section, we shall assume that all the primes of F lying above p split in the CM-extension E.**

Remarks. (a) Building on earlier work of Hida and Panchishkin [15, 20] for the cyclotomic deformation, Disegni [12] has attached p-adic L-functions to $\mathrm{GL}_2 \times \mathrm{GL}_2$ interpolating the Rankin product L-functions $L(s, \pi_{\mathbf{f}} \times \pi_\chi)$ and

$L(s,\pi_{\mathbf{g}}\times\pi_\chi)$ at the critical point $s=1/2$: since we are taking the unitarizations, let us identify these L-values (respectively) with $L\big(\mathbf{f}\otimes\mathrm{Ind}_E^F(\chi),s\big)$ and $L\big(\mathbf{g}\otimes\mathrm{Ind}_E^F(\chi),s\big)$ at $s=1$.

(b) Let Γ_E^{anti} denote the Galois group of the anticyclotomic extension of E inside E_∞, which by definition is the (-1)-eigenspace of the complex conjugation $c\in\mathrm{Gal}(E/F)$ inside Γ_E. For a topological generator γ_0 of Γ_E^{cyc} and the particular choice $\star=\mathbf{f}$ say, one expands the $\big(1+[F:\mathbb{Q}]\big)$-variable Disegni-Hida-Panchishkin p-adic L-function $\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\in\mathcal{O}_{\mathcal{K}}\big[\big[\Gamma_E\big]\big][1/\lambda]$ into a Taylor series of the form

$$\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\;=\;\mathbf{L}^{(0)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)+\mathbf{L}^{(1)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\cdot(\gamma_0-1)+\frac{1}{2}\,\mathbf{L}^{(2)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\cdot(\gamma_0-1)^2+\cdots$$

where $\mathbf{L}^{(i)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\in\mathcal{O}_{\mathcal{K}}\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big][1/\lambda]$ under the decomposition $\Gamma_E=\Gamma_E^{\mathrm{cyc}}\times\Gamma_E^{\mathrm{anti}}$. Here the subscript '$_\Sigma$' above indicates that the p-adic L-function $\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)$ has been completely stripped[2] of its Euler factors at those finite places $\nu\in\Sigma$, $\nu\nmid p$.

(c) Note also the condition **(Even)** implies either $\mathbf{L}^{(0)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\neq 0$, or instead that $\mathbf{L}^{(0)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)=\mathbf{L}^{(1)}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)=0$, because the global root number $\epsilon\big(1/2,\pi_{\mathbf{f}},\pi_\chi\big)$ is equal to $+1$ under our assumptions.

If $\chi=\chi_0\cdot\chi^\dagger$ where $\chi^\dagger$ is anticyclotomic, then

$$\chi^\dagger\left(\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\right)=\chi^\dagger\big(\mathbf{L}^{(0)}_{p,\Sigma}(\mathbf{f},\rho_0)\big)$$

as $\chi^\dagger(\gamma_0-1)=0$. The exact interpolation rule from [11, Thm 4.3.4] states that

$$\chi^\dagger\Big(\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)\Big)\;=\;\frac{\chi\big(d_F^{(p)}\big)\cdot G\big(\overline{\chi}\big)\cdot\sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\cdot\mathcal{N}_{E/F}(\mathfrak{f}_\chi)\big)}\cdot\overline{\chi}(D_{E/F})}{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}$$
$$\times\;\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right)\times\;\frac{L_{\Sigma\backslash\{\mathfrak{p}|p\}}\big(\mathbf{f}\otimes\mathrm{Ind}_E^F(\overline{\chi}),1\big)}{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\mathbf{f})}\,. \tag{2.2}$$

An analogous formula holds for the value of $\mathbf{L}_{p,\Sigma}\big(\mathbf{g},\rho_0\big)$ at each twist $\chi=\chi_0\cdot\chi^\dagger$.

Theorem 2.2. *Assuming Hypothesis* $(\mathbf{f}\equiv\mathbf{g}\ (\lambda^r))$, *and that Hypothesis* **(Even)** *for the base character* χ_0 *holds true with the conductor of* χ_0 *coprime*

[2] We have deliberately removed the Euler factors from $\mathbf{L}_{p,\Sigma}\big(\mathbf{f},\rho_0\big)$ at the finite places in Σ, so that we can obtain a congruence modulo λ^r; it follows that the p-adic L-functions we are considering correspond to Σ-imprimitive versions of the Disegni-Hida-Panchishkin construction.

to $N_{\mathbf{f}}N_{\mathbf{g}} \cdot \mathcal{O}_E$, there is a congruence of p-adic L-functions

$$\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{f},\rho_0) \;\equiv\; \mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{g},\rho_0) \mod \lambda^r \cdot \mathcal{O}_{\mathcal{K}}[[\Gamma_E^{\mathrm{anti}}]].$$

If either $\lambda^r \nmid \frac{\epsilon(\rho_0,0)\cdot L_\Sigma(\mathbf{f},\rho_0)}{\Omega_{\infty,K}^{\mathrm{aut},(0)}(\mathbf{f})}$ or $\lambda^r \nmid \frac{\epsilon(\rho_0,0)\cdot L_\Sigma(\mathbf{g},\rho_0)}{\Omega_{\infty,K}^{\mathrm{aut},(0)}(\mathbf{g})}$ with $\epsilon(\rho_0,s)$ the ϵ-factor for ρ_0, then both sides of this anticyclotomic congruence must be non-trivial modulo λ^r.

Proof. To establish this p-adic congruence, clearly it is sufficient to prove that $\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{f},\rho_0)\right)$ and $\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{g},\rho_0)\right)$ are congruent modulo λ^r, at $\chi = \chi_0 \cdot \chi^\dagger$ where $\chi^\dagger$ ranges over finite order characters on the anticyclotomic component Γ_E^{anti}. Because $\left|G(\overline{\chi})\right|_p^{-1} = \left|\mathcal{N}_{E/\mathbb{Q}}(\mathfrak{f}_\chi)\right|_p^{-1/2} = \left|\|\mathfrak{c}_\chi\|\right|_p^{-1/2}$, the ratio of algebraic numbers

$$r_\chi \;:=\; \frac{\chi\big(d_F^{(p)}\big)\cdot G(\overline{\chi})\cdot\sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\cdot\mathcal{N}_{E/F}(\mathfrak{f}_\chi)\big)}\cdot\overline{\chi}(D_{E/F})}{\sqrt{|D_E|\cdot\|\mathfrak{c}_\chi\|^2}}$$

is a p-adic unit, independent of choosing $\star \in \{\mathbf{f},\mathbf{g}\}$ but dependent on χ obviously. From the interpolation in Equation (2.2), and after replacing the Hecke character χ by its dual $\overline{\chi}$, one can reinterpret the congruence in Proposition 2.1 as the statement:

$$\begin{aligned}
&\frac{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}{\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right)} \times r_\chi^{-1}\cdot\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{f},\rho_0)\right)\\
\equiv\;&\frac{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\mathbf{g})^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}{\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\mathbf{g})}\right)} \times r_\chi^{-1}\cdot\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{g},\rho_0)\right) \mod \lambda^r\cdot\mathcal{O}_{\mathcal{K},\chi}.
\end{aligned}$$

However, for $\star \in \{\mathbf{f},\mathbf{g}\}$, we can identify $\alpha_{\mathfrak{p}}(\star)$ with the eigenvalue of $\mathrm{Frob}_{\mathfrak{p}}$ acting on the maximal unramified quotient of $\mathrm{Ta}_p(A_\star)$ as a $G_{F_{\mathfrak{p}}}$-module, in which case $\alpha_{\mathfrak{p}}(\mathbf{f}) \equiv \alpha_{\mathfrak{p}}(\mathbf{g}) \mod \lambda^r\cdot\mathcal{O}_{\mathcal{K},\chi}$ since $\mathrm{Ta}_p(A_{\mathbf{f}})/\lambda^r \cong \mathrm{Ta}_p(A_{\mathbf{g}})/\lambda^r$ as $G_{F_{\mathfrak{p}}}$-modules. Consequently, the reciprocals of these extra terms satisfy

$$\begin{aligned}
&\left(\frac{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}{\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right)}\right)^{-1}\\
\equiv\;&\left(\frac{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\mathbf{g})^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}{\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\mathbf{g})}\right)}\right)^{-1} \mod \lambda^r\cdot\mathcal{O}_{\mathcal{K},\chi},
\end{aligned}$$

which completes the proof of the main congruence.

Finally, identifying $G(\overline{\chi}_0)\cdot\sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\cdot\mathcal{N}_{E/F}(\mathfrak{f}_{\chi_0})\big)}$ with the factor $\epsilon(\rho_0,0)$, the non-triviality of either $\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{f},\rho_0)\right)$ or $\chi^\dagger\left(\mathbf{L}_{p,\Sigma}^{(0)}(\mathbf{g},\rho_0)\right)$ mod

λ^r at $\chi^\dagger = \mathbf{1}$, directly implies $\mathbf{L}^{(0)}_{p,\Sigma}(\mathbf{f},\rho_0) \equiv \mathbf{L}^{(0)}_{p,\Sigma}(\mathbf{g},\rho_0) \not\equiv 0 \bmod \lambda^r \cdot \mathcal{O}_{\mathcal{K}}[[\Gamma_E^{\mathrm{anti}}]]$. □

3. The odd case: p-adic Gross-Zagier formula

We now treat the opposite situation, where

$$\epsilon(1/2,\pi_{\mathbf{f}},\pi_\chi) = \epsilon(1/2,\pi_{\mathbf{g}},\pi_\chi) = -1.$$

In particular $\mathbf{L}^{(0)}_{p,\Sigma}(\star,\rho_0)$ is identically zero, whence

$$\frac{\mathbf{L}_{p,\Sigma}(\star,\rho_0)}{\gamma_0-1} \;=\; \mathbf{L}^{(1)}_{p,\Sigma}(\star,\rho_0) \;+\; \frac{1}{2}\,\mathbf{L}^{(2)}_{p,\Sigma}(\star,\rho_0)\cdot(\gamma_0-1) \;+\; O\big((\gamma_0-1)^2\big)$$

so that $\chi^\dagger\Big(\mathbf{L}^{(1)}_{p,\Sigma}(\star,\rho_0)\Big) = \big(\log_p \kappa_{\mathrm{cy}}(\gamma_0)\big)^{-1}\cdot\chi^\dagger\Big(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\star,\rho_0)}{\mathrm{d}s}\Big)\Big|_{s=1}$ for $\star\in\{\mathbf{f},\mathbf{g}\}$. Therefore, our goal is to establish a congruence modulo $\lambda^r\cdot\log_p\kappa_{\mathrm{cy}}(\gamma_0)$ between $\chi^\dagger\Big(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\mathbf{f},\rho_0)}{\mathrm{d}s}\Big)\Big|_{s=1}$ and $\chi^\dagger\Big(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\mathbf{g},\rho_0)}{\mathrm{d}s}\Big)\Big|_{s=1}$ under Hypothesis **($\mathbf{f}\equiv\mathbf{g}$ (λ^r))**. Again $\mathbb{B}$ denotes a totally definite quaternion algebra over F, with the property that the automorphic representations $\pi_{\mathbf{f}}$ and $\pi_{\mathbf{g}}$ are both parameterised by $\mathbb{B}^\times_{\mathbb{A}_F}$. Likewise $\chi: E^\times\backslash\mathbb{A}_E^\times\to\mathbb{C}^\times$ will be a fixed Hecke character of finite order, as before.

Hypothesis. (Odd) The product $\omega\cdot\chi\big|_{\mathbb{A}_F^\times}$ is trivial, the three finite sets

$$\begin{aligned} S_{N_{\mathbf{f}}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(N_{\mathbf{f}}) = -1\big\}\\ S_{N_{\mathbf{g}}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(N_{\mathbf{g}}) = -1\big\}\\ S_{\widetilde{N}} &= \big\{\nu \;:\; \nu|\infty \text{ or } \eta_{E/F,\nu}(\widetilde{N}) = -1\big\}\end{aligned}$$

each have *odd* cardinality, and for all places ν of F

$$\epsilon(1/2,\pi_{\mathbf{f},\nu},\pi_{\chi,\nu}) \;=\; \epsilon(1/2,\pi_{\mathbf{g},\nu},\pi_{\chi,\nu}) \;=\; \chi_\nu(-1)\cdot\eta_{E/F,\nu}(-1)\cdot\xi(\mathbb{B}_\nu).$$

The above hypothesis implies that both the global root numbers $\epsilon(1/2,\pi_{\mathbf{f}},\pi_\chi)$ and $\epsilon(1/2,\pi_{\mathbf{g}},\pi_\chi)$ in the Rankin L-functions are equal to -1, and that the quaternion algebra $\mathbb{B}$ is incoherent. Henceforth, we shall further assume that all primes of F above p split inside the extension E.

As explained in [12, Sect 1.1], one can interpret the modular parameterizations of the abelian varieties $A_{\mathbf{f}}$ and $A_{\mathbf{g}}$ in terms of Shimura curves. For a compact open subgroup U of $\mathbb{B}^\times_{\mathbb{A}_F}$, the complex points of the algebraic curve X_U are given by

$$X_U(\mathbb{C}) \;=\; \mathbb{B}\backslash\mathcal{H}^\pm\times\widehat{\mathbb{B}}^\times/U.$$

In fact, there exists an infinite tower of Shimura curves $\{X_U\}_U$ indexed by the compact open subgroups $U\subset\mathbb{B}^\times_{\mathbb{A}_F}$, and we shall set $X(\mathbb{B}) := \varprojlim_U X_U$.

The canonical Hodge class $\xi_U \in \mathrm{Pic}\big(X_U\big) \otimes \mathbb{Q}$ which has degree one on each component induces an embedding $X_U \overset{\iota_{\xi_U}}{\hookrightarrow} \mathrm{Jac}X_U$. Because the HMFs $\mathbf{f}$ and $\mathbf{g}$ are parameterised by $\mathbb{B}^\times_{\mathbb{A}_F}$, the $\mathrm{End}^0(A_\star)$-vector spaces

$$\varinjlim_U \mathrm{Hom}^0_{\xi_U}\big(\mathrm{Jac}X_U, A_{\mathbf{f}}\big) \quad \text{and} \quad \varinjlim_U \mathrm{Hom}^0_{\xi_U}\big(\mathrm{Jac}X_U, A_{\mathbf{g}}\big)$$

are both non-empty; let $\pi_{A_\star} \in \varinjlim_U \mathrm{Hom}^0\big(\mathrm{Jac}X_U, A_\star\big)$ be the smooth irreducible representation of $\mathbb{B}^\times_{\mathbb{A}_F}$ corresponding to $\pi_\star$, for each choice of cusp form $\star \in \{\mathbf{f}, \mathbf{g}\}$. Taking $U_{\mathbf{f}} = U_0(N_{\mathbf{f}})$, $U_{\mathbf{g}} = U_0(N_{\mathbf{g}})$ and $\widetilde{U} = U_0\big(\widetilde{N}\big)$, there exists a factorisation

$$X(\mathbb{B}) \overset{\sim}{\to} \varprojlim_U X_U \xrightarrow{\iota_{\underline{\xi}}} \varprojlim_U \mathrm{Jac}X_U \twoheadrightarrow \mathrm{Jac}X_{\widetilde{U}} \begin{array}{l} \twoheadrightarrow \mathrm{Jac}X_{U_{\mathbf{f}}} \twoheadrightarrow A_{\mathbf{f}} \\ \twoheadrightarrow \mathrm{Jac}X_{U_{\mathbf{g}}} \twoheadrightarrow A_{\mathbf{g}} \end{array} \tag{3.1}$$

and the top sequence of maps yields $\pi_{A_{\mathbf{f}}} \circ \iota_{\underline{\xi}}$, whilst the bottom maps yield $\pi_{A_{\mathbf{g}}} \circ \iota_{\underline{\xi}}$.

Before we state our main result below, for each choice of HMF $\star \in \{\mathbf{f}, \mathbf{g}\}$ let us introduce the ratio of Euler factors

$$\mathcal{E}_{\widetilde{N}}(\star, \chi) := \prod_{\mathfrak{q}|\widetilde{N}} \frac{L_{\mathfrak{q}}(\star \otimes \mathrm{Ind}_E^F(\chi), s-1)}{L_{\mathfrak{q}}(\star \otimes \mathrm{Ind}_E^F(\chi), s)}\Bigg|_{s=1}.$$

Whilst the denominator can never vanish, the numerator can sometimes vanish (for example, if $q \| N_\star$ and $C(\mathfrak{q}, \star) = \chi(\mathfrak{Q})$ for some place $\mathfrak{Q}$ of E lying above $\mathfrak{q}$). Furthermore, these algebraic values can be interpolated by the ratio of two elements of $\mathcal{O}_{\mathcal{K},\chi}\big[\big[\Gamma_E^{\mathrm{anti}}\big]\big]$, denoted by $\mathcal{E}_{0,\widetilde{N}}(\star)$ and $\mathcal{E}_{1,\widetilde{N}}(\star)$, so that

$$\frac{\chi^\dagger\big(\mathcal{E}_{0,\widetilde{N}}(\star)\big)}{\chi^\dagger\big(\mathcal{E}_{1,\widetilde{N}}(\star)\big)} = \prod_{\mathfrak{q}|\widetilde{N}} \frac{L_{\mathfrak{q}}(\star \otimes \mathrm{Ind}_E^F(\chi), 0)}{L_{\mathfrak{q}}(\star \otimes \mathrm{Ind}_E^F(\chi), 1)}$$

for all characters $\chi = \chi_0 \cdot \chi^\dagger$ in the standard formulation above.

Theorem 3.1. *Assume Hypothesis* $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$, *and that Hypothesis* **(Odd)** *for the base character χ_0 holds true with the conductor of χ_0 coprime to $N_{\mathbf{f}}N_{\mathbf{g}} \cdot \mathcal{O}_E$. Then one has the twin relations*

(i) $\mathbf{L}^{(0)}_{p,\Sigma}\big(\mathbf{f}, \rho_0\big) = \mathbf{L}^{(0)}_{p,\Sigma}\big(\mathbf{g}, \rho_0\big) = 0$, *and*

(ii) $\dfrac{\mathcal{E}_{0,\widetilde{N}}(\mathbf{f})}{\mathcal{E}_{1,\widetilde{N}}(\mathbf{f})}\mathbf{L}^{(1)}_{p,\Sigma}\big(\mathbf{f}, \rho_0\big) \equiv \dfrac{\mathcal{E}_{0,\widetilde{N}}(\mathbf{g})}{\mathcal{E}_{1,\widetilde{N}}(\mathbf{g})}\mathbf{L}^{(1)}_{p,\Sigma}\big(\mathbf{g}, \rho_0\big)$ *mod* $\lambda^{r-r_0+\delta_E} \cdot \log_p \kappa_{\mathrm{cy}}(\gamma_0)$.

Here $r_0 := 2 \cdot \sum_{\mathfrak{P}|p} \mathrm{ord}_\lambda\Big(\#\widetilde{A}_\star\big(\mathcal{O}_E/\mathfrak{P}\big)\Big)$ *with* $\star \in \{\mathbf{f}, \mathbf{g}\}$[3], *while* $\delta_E \in \mathbb{Q}^\times$ *depends on the CM-extension E/F but does not depend on either* $\mathbf{f}, \mathbf{g}$, *nor on the prime p.*

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[3]Note that $\widetilde{A}_{\mathbf{f}}\big(\mathcal{O}_E/\mathfrak{P}\big)[\lambda^r] \cong \widetilde{A}_{\mathbf{g}}\big(\mathcal{O}_E/\mathfrak{P}\big)[\lambda^r]$ since we are assuming $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ holds here.

Before supplying the proof, we first need to establish some preliminary results. For a CM-point $\mathbf{x} \in X^{E^\times}$, we begin by considering the Heegner points

$$\mathcal{P}(\mathbf{f},\chi) := \sum_{t\in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)\cdot \pi_{A_{\mathbf{f}}}\big(\iota_{\underline{\xi}}(t\cdot \mathbf{x})\big) \quad \text{and}$$
$$\mathcal{P}(\mathbf{g},\chi) := \sum_{t\in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)\cdot \pi_{A_{\mathbf{g}}}\big(\iota_{\underline{\xi}}(t\cdot \mathbf{x})\big),$$

which lie inside $\big(A_{\mathbf{f}}(E^{\mathrm{ab}})\otimes\chi\big)^{\mathrm{Gal}(E^{\mathrm{ab}}/E)}$ and $\big(A_{\mathbf{g}}(E^{\mathrm{ab}})\otimes\chi\big)^{\mathrm{Gal}(E^{\mathrm{ab}}/E)}$ respectively. In general, we do not expect their pre-images in $\mathrm{Jac}X_{\widetilde{U}}$ to be congruent modulo λ^r so instead work with their $\widetilde{N}$-depletions, for which we do expect congruences.

Fix a choice of cusp form $\star \in \{\mathbf{f},\mathbf{g}\}$. At each $\mathcal{O}_F$-ideal $\mathfrak{a}$ such that $\widetilde{N} \subset \mathfrak{a}\cdot N_\star$, we write $\mathcal{V}(\mathfrak{a}) : \mathrm{Jac}X_{U_\star} \to \mathrm{Jac}X_{\widetilde{U}}$ for the degeneration map induced on jacobians. Clearly, $\mathcal{V}(\mathfrak{a})$ induces a p-integral map on the ordinary components

$$\text{‘}\mathcal{V}(\mathfrak{a})\text{’} : \mathrm{Ta}_p\left(\mathrm{Jac}X_{U_\star}\right)^{\mathrm{ord}} \to \mathrm{Ta}_p\left(\mathrm{Jac}X_{\widetilde{U}}\right)^{\mathrm{ord}}$$

where $\mathrm{Ta}_p(J) := \varprojlim_m J[p^m]$ and $\mathrm{Ta}_p(J)^{\mathrm{ord}} := \mathrm{Ta}_p(J)\Big|\lim_{n\to\infty} U(p\mathcal{O}_F)^{n!}$.

For every finite place $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$, there are associated Hecke correspondences $\mathcal{T}(\mathfrak{q})$ and $\langle\mathfrak{q}\rangle$ (resp. $\mathcal{U}(\mathfrak{q})$) if $\mathfrak{q}+N_\star = \mathcal{O}_F$ (resp. if $\mathfrak{q}+N_\star \neq \mathcal{O}_F$) [25, Section 1.4]. Using these correspondences, one constructs a depletion map on Jacobian varieties

$$\mathrm{dep}_{U_\star}^{\widetilde{U}} : \mathrm{Jac}X_{U_\star} \to \mathrm{Jac}X_{\widetilde{U}}$$

sending a point $P_{U_\star} \in \mathrm{Jac}X_{U_\star}$ to its $\widetilde{N}$-depleted version (cf. Definition 1.6)

$$P_{U_\star}\Bigg| \prod_{\mathfrak{q}|\widetilde{N},\ \mathfrak{q}\nmid N_\star} \big(1-\mathcal{T}(\mathfrak{q})\circ\mathcal{V}(\mathfrak{q})+\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\cdot\langle\mathfrak{q}\rangle\circ\mathcal{V}(\mathfrak{q}^2)\big)\cdot\prod_{\mathfrak{q}|N_\star}\big(1-\mathcal{U}(\mathfrak{q})\circ\mathcal{V}(\mathfrak{q})\big).$$

In particular, under the composition $\pi_{\mathrm{Jac}X_{U_\star}}\circ\iota_{\underline{\xi}} : X(\mathbb{B}) \to \mathrm{Jac}X_{U_\star}$ one may define $\widetilde{\mathcal{P}}(\star,\chi)$ to be

$$\mathrm{dep}_{U_\star}^{\widetilde{U}}\Bigg(\sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})}\chi(t)\cdot\pi_{\mathrm{Jac}X_{U_\star}}\big(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\big)\Bigg) \in \big(\mathrm{Jac}X_{\widetilde{U}}(E^{\mathrm{ab}})\otimes\chi\big)^{\mathrm{Gal}(E^{\mathrm{ab}}/E)}.$$

Our strategy in proving Theorem 3.1 is to initially establish that:

(I) the pair of Heegner points $\widetilde{\mathcal{P}}(\mathbf{f},\chi)$ and $\widetilde{\mathcal{P}}(\mathbf{g},\chi)$ are congruent modulo λ^r;

(II) their projections to $\big(A_\star\otimes\chi(E)\big)_{\mathbb{Q}}$ equal $\mathcal{P}(\star,\chi)$, up to some Euler factors;

(III) their p-adic heights equal $\chi^\dagger\Big(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\star,\rho_0)}{\mathrm{d}s}\Big)\Big|_{s=1}$.

Let us begin with the middle task (II), and then deal with (I) and (III) afterwards.

Lemma 3.2. *For each $\star \in \{\mathbf{f},\mathbf{g}\}$, if we factorise $\pi_{A_\star}$ into $\mathrm{pr}^{\widetilde{U}}_{A_\star} \circ \pi_{\mathrm{Jac}X_{\widetilde{U}}}$ where $\mathrm{pr}^{\widetilde{U}}_{A_\star} : \mathrm{Jac}X_{\widetilde{U}} \twoheadrightarrow A_\star$, then inside $A_\star \otimes \chi(E)$ we have the identities:*

$$\mathrm{pr}^{\widetilde{U}}_{A_{\mathbf{f}}}\left(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\right) = \prod_{\mathfrak{q}|\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{f})\chi(\mathfrak{Q}) + \chi^2(\mathfrak{Q})\omega(\mathfrak{q}) \cdot \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\right) \cdot \mathcal{P}(\mathbf{f},\chi),$$

$$\mathrm{pr}^{\widetilde{U}}_{A_{\mathbf{g}}}\left(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\right) = \prod_{\mathfrak{q}|\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{g})\chi(\mathfrak{Q}) + \chi^2(\mathfrak{Q})\omega(\mathfrak{q}) \cdot \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\right) \cdot \mathcal{P}(\mathbf{g},\chi).$$

N.B. Here for each $\mathfrak{q}|\widetilde{N}$, we have fixed a choice of prime $\mathcal{O}_E$-ideal $\mathfrak{Q}$ lying above $\mathfrak{q}$.

Proof. To simplify the exposition, we will focus exclusively on the HMF $\star = \mathbf{f}$. Throughout we write $\mathfrak{c}$ for $\mathfrak{c}_\chi$, and fix a lift of the level structure $\widetilde{\mathfrak{N}} \lhd \mathcal{O}_E$ such that $\mathcal{O}_E/\widetilde{\mathfrak{N}} \cong \mathcal{O}_F/\widetilde{N}$; without loss of generality, we may represent $t \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})$ with ideals coprime to $\widetilde{\mathfrak{N}}$.

Following Katz [16, Section 1], for a ring $R \subset \mathbb{C}$ one can view the R-points of $X_{\widetilde{U}}$ as a triple (A, C, ϖ) where A is a $\mathfrak{C}$-polarized Hilbert-Blumenthal abelian variety over R, the finite group C denotes a cyclic R-subscheme of $A\big[\widetilde{\mathfrak{N}}\big]$, ϖ is a nowhere vanishing differential form on A, and $\mathfrak{C}$ runs through a set of coset representatives for the narrow class group of F. We denote the natural action of $t \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})$ on the R-points of $X_{\widetilde{U}}$ by $(A, C, \varpi) \mapsto t * (A, C, \varpi)$.

At a prime $\mathcal{O}_F$-ideal $\mathfrak{q}$ such that $\mathfrak{Q}\big|\widetilde{\mathfrak{N}}$ lies over it and for a class $t \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})$, the map $\mathcal{V}(\mathfrak{q}^r)$ sends a point $t * (A, C, \varpi)$ to the point $\left(\overline{\mathfrak{Q}}^{\,-r} t\right) * \left(A, C \cap A[\widetilde{\mathfrak{N}}\mathfrak{Q}^{-r}], \varpi\right)$. Consequently, for either choice of exponent $r \in \{1, 2\}$, the image $\mathcal{V}(\mathfrak{q}^r)\Big(\sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})}\chi(t) \cdot \big(t * (A, C, \varpi)\big)\Big)$ is equal to

$$\begin{aligned} &\sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \chi(t) \cdot (\overline{\mathfrak{Q}}^{\,-r} t) * \left(A, C[\widetilde{\mathfrak{N}}\mathfrak{Q}^{-r}], \varpi\right) \\ &= \sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \chi\big(t\overline{\mathfrak{Q}}^r\big) \cdot \left(t * \left(A, C[\widetilde{\mathfrak{N}}\mathfrak{Q}^{-r}], \varpi\right)\right) \\ &= \chi\big(\overline{\mathfrak{Q}}\big)^r \cdot \sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \chi(t) \cdot \big(t * (A, C, \varpi)\big) \end{aligned}$$

since $\big\{\overline{\mathfrak{Q}}^r t\big\}_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})}$ also yields a complete set of representative classes for $\mathrm{Pic}(\mathcal{O}_\mathfrak{c})$. It follows that

$$\mathcal{P}_{U_\mathbf{f}}(\mathbf{f},\chi)\Big|\mathcal{V}(\mathfrak{q}^r) \;=\; \chi\big(\overline{\mathfrak{Q}}\big)^r \cdot \mathcal{P}_{\widetilde{U}}(\mathbf{f},\chi) \quad \text{at each } r \in \{1,2\}. \tag{3.2}$$

On the other hand, the projection $\mathrm{pr}^{U_\mathbf{f}}_{A_\mathbf{f}} : \mathrm{Jac}X_{U_\mathbf{f}} \twoheadrightarrow A_\mathbf{f}$ is obtained via quotienting by the elements $\mathcal{T}(\mathfrak{q}) - C(\mathfrak{q},\mathbf{f})$ and $\langle\mathfrak{q}\rangle - \omega(\mathfrak{q})$ if $\mathfrak{q} + N_\mathbf{f} = \mathcal{O}_F$, and by $\mathcal{U}(\mathfrak{q})-C(\mathfrak{q},\mathbf{f})$ if $\mathfrak{q}+N_{\mathbf{f}}\neq\mathcal{O}_F$. As an immediate corollary, one obtains the corresponding relations

$$\mathcal{P}(\mathbf{f},\chi)\Big|\mathcal{T}(\mathfrak{q}) \text{ or } \mathcal{U}(\mathfrak{q}) = C(\mathfrak{q},\mathbf{f})\cdot\mathcal{P}(\mathbf{f},\chi) \quad \text{and} \quad \mathcal{P}(\mathbf{f},\chi)\Big|\langle\mathfrak{q}\rangle = \omega(\mathfrak{q})\cdot\mathcal{P}(\mathbf{f},\chi). \tag{3.3}$$

Combining the various identities in (3.2) and (3.3) together, one thereby deduces

$$\begin{aligned}
&\mathrm{pr}_{A_{\mathbf{f}}}^{\widetilde{U}}\left(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\right)\\
=\;& \mathrm{pr}_{A_{\mathbf{f}}}^{\widetilde{U}}\circ\mathrm{dep}_{U_{\mathbf{f}}}^{\widetilde{U}}\left(\sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}\chi(t)\cdot\pi_{\mathrm{Jac}X_{U_{\mathbf{f}}}}\left(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\right)\right)\\
=\;& \mathrm{pr}_{A_{\mathbf{f}}}^{\widetilde{U}}\left(\sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}\chi(t)\cdot\pi_{\mathrm{Jac}X_{U_{\mathbf{f}}}}\left(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\right)\Bigg|\prod_{\mathfrak{q}|N_\star}\left(1-\mathcal{U}(\mathfrak{q})\circ\mathcal{V}(\mathfrak{q})\right)\right.\\
&\left.\cdot\prod_{\mathfrak{q}|\widetilde{N},\ \mathfrak{q}\nmid N_\star}\left(1-\mathcal{T}(\mathfrak{q})\circ\mathcal{V}(\mathfrak{q})+\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\cdot\langle\mathfrak{q}\rangle\circ\mathcal{V}(\mathfrak{q}^2)\right)\right)\\
=\;& \sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}\chi(t)\cdot\pi_{A_{\mathbf{f}}}\left(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\right)\\
&\times\prod_{\mathfrak{q}|\widetilde{N}}\left(1-C(\mathfrak{q},\mathbf{f})\chi(\mathfrak{Q})+\chi^2(\mathfrak{Q})\omega(\mathfrak{q})\cdot\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\right).
\end{aligned}$$

Not surprisingly, the argument for the other HMF $\star=\mathbf{g}$ is almost identical. $\square$

We shall now establish statements (I) and (III) mentioned in our strategy above. For a compact open subgroup U of $\mathbb{B}_{\mathbb{A}_F}^{\times}$, there are group homomorphisms

$$\left(\mathrm{Jac}X_U\otimes\chi\right)(E)\xrightarrow{-\otimes 1}\left(\mathrm{Jac}X_U\otimes\chi\right)(E)\widehat{\otimes}\mathbb{Z}_p\xrightarrow{\partial} H_f^1\left(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\left(\mathrm{Jac}X_U\right)\otimes\chi\right),$$

where

$$H_f^1\left(E\otimes\mathbb{Q}_p,\mathbf{T}\right):=\mathrm{Ker}\left(H^1\left(E\otimes\mathbb{Q}_p,\mathbf{T}\right)\xrightarrow{-\otimes 1}H^1\left(E\otimes\mathbb{Q}_p,\mathbf{T}\otimes_{\mathbb{Z}_p}B_{\mathrm{cris}}\right)\right),$$

and the right-hand arrow ∂ is the Kummer map – see [2, Section 3] for further details. We shall label the composition of this whole sequence as '∂_U'.

Now set $U:=\widetilde{U}=U_0(\widetilde{N})$: the depleted points $\widetilde{\mathcal{P}}(\mathbf{f},\chi)$ and $\widetilde{\mathcal{P}}(\mathbf{g},\chi)$ each belong to $\left(\mathrm{Jac}X_{\widetilde{U}}\otimes\chi\right)(E)$, so we can apply the mapping $\partial_{\widetilde{U}}$ to them. In fact $\partial_{\widetilde{U}}\left(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\right)$ and $\partial_{\widetilde{U}}\left(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\right)$ lie inside $H_f^1\left(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\left(\mathrm{Jac}X_{\widetilde{U}}\right)^{\mathrm{ord}}\otimes\chi\right)$, since $\mathbf{f}$ and $\mathbf{g}$ are both p-ordinary Hilbert cusp forms.

Remark. Disegni's normalisation of the p-adic L-function in [12, Theorem A] is slightly different to that of $\mathbf{L}_{p,\Sigma}(\star,\rho_0)$, for each $\star\in\{\mathbf{f},\mathbf{g}\}$ and set of places Σ. Note that his interpolation formula is almost the same as that in Equation (2.2), except that the automorphic period $\Omega^{\mathrm{aut},(0)}_{\infty,K}(\star)$ is instead replaced by

$$\Omega^{\mathrm{aut},(1)}_{\infty,K}(\star)\ :=\ \frac{2\cdot L(1,\eta_{E/F})\cdot L\big(1,\mathrm{ad}(\star)\big)}{\pi^{2[F:\mathbb{Q}]}\cdot|D_F|^{1/2}}\ .$$

We will write

$$\mathbf{L}^{\mathrm{Dis}}_{p,\Sigma}\big(\star,\rho_0\big)=\frac{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\star)}{\Omega^{\mathrm{aut},(1)}_{\infty,K}(\star)}\times\mathbf{L}_{p,\Sigma}\big(\star,\rho_0\big),$$

while $\mathbf{L}^{\mathrm{Dis}}_{p,\emptyset}\big(\star,\rho_0\big)=\dfrac{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\star)}{\Omega^{\mathrm{aut},(1)}_{\infty,K}(\star)}\times\mathbf{L}_{p,\emptyset}\big(\star,\rho_0\big)$ denotes the primitive p-adic L-function in Theorem A of *op. cit.*, which has not yet had its Euler factors at $\mathfrak{q}\in\Sigma$ removed.

Lemma 3.3. *Recall under Hypothesis* **(Odd)** *that* $\mathbf{L}^{\mathrm{Dis},(0)}_{p,\Sigma}\big(\star,\rho_0\big)$ *is always zero.*

(a) Assuming that Hypothesis $(\mathbf{f}\equiv\mathbf{g}\ (\lambda^r))$ *holds true as well, there exists a crystalline* 1*-cocycle* $Q(\mathbf{f},\mathbf{g},\chi)\in H^1_f\Big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}\otimes\chi\Big)$ *such that*

$$\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big)\ =\ \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big)\ +\ \lambda^r\cdot Q(\mathbf{f},\mathbf{g},\chi).$$

(b) For either choice of HMF $\star\in\{\mathbf{f},\mathbf{g}\}$ *and at the Hecke character* $\chi=\chi_0\cdot\chi^\dagger$*,*

$$\begin{aligned}&\chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}^{\mathrm{Dis}}_{p,\Sigma}(\star,\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1}\\ &=\frac{\chi\big(d_F^{(p)}\big)G\big(\overline{\chi}\big)\sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\mathcal{N}_{E/F}(\mathfrak{f}_\chi)\big)}\ \overline{\chi}(D_{E/F})}{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\star)^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}\times\ \mathcal{E}_{\widetilde{N}}(\star,\chi)^{-1}\\ &\quad\cdot\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\star)}\right)\times\frac{2}{c_E}\ \Big(\!\!\Big(\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big)\ ,\ \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\Big)\!\!\Big)_{\widetilde{U},E}\end{aligned}$$

where the scalar $c_E:=\frac{\zeta_F(2)}{(\pi/2)^{[F:\mathbb{Q}]}|D_E|^{1/2}L(1,\eta_{E/F})}\neq 0$ *is independent of* $\star$ *and* χ*, and*

$$\begin{aligned}&(\!(-,-)\!)_{\widetilde{U},E}:\\ &H^1_f\Big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}_{(\chi)}\Big)\times H^1_f\Big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}_{(\chi^{-1})}\Big)\to\mathbb{Q}_p\end{aligned}$$

denotes the p*-adic height pairing of Perrin-Riou et al (e.g. see* [21, Section 1.2]*).*

Before giving the demonstration of this result, it is important to point out that for a p-ordinary G_F-lattice $\mathbf{T}$, the p-adic height pairing is between $H^1_f\big(E\otimes\mathbb{Q}_p,\mathbf{T}\big)$ and $H^1_f\big(E\otimes\mathbb{Q}_p,\mathbf{T}^*(1)\big)$. In particular, if $\mathbf{T}=\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)\otimes\chi$ then its Kummer dual is isomorphic to $\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)\otimes\chi^{-1}$ because Jacobian varieties are auto-dual; therefore, cutting out the ordinary parts, the height pairing reduces to the above.

Proof. We begin with the first assertion. Let us write

$$\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}=\mathfrak{h}^{\mathrm{ord}}\big(U_0(\widetilde{N});\mathcal{O}_{\mathcal{K}}\big)$$

for the Hecke algebra acting on the ordinary part of the jacobian of $X_{U_0(\widetilde{N})}$, taking coefficients in $\mathcal{O}_{\mathcal{K}}$. In particular for $\star\in\{\mathbf{f},\mathbf{g}\}$, the composition of the projection map from $\mathrm{Jac}X_{\widetilde{U}}$ to $A_\star$ with the homomorphism $\mathrm{dep}^{\widetilde{U}}_{U_\star}$ from $A_\star$ back up to $\mathrm{Jac}X_{\widetilde{U}}$ is obtained by tensoring (over $\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}$) by the integral domain $\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}/\mathbb{I}_\star$, where the ideal

$$\mathbb{I}_\star:=\Big[\mathcal{T}(\mathfrak{q})-C(\mathfrak{q},\star),\langle\mathfrak{q}\rangle-\omega(\mathfrak{q})\text{ if }\mathfrak{q}+\widetilde{N}=\mathcal{O}_F,\text{ and }\mathcal{U}(\mathfrak{q})\text{ if }\mathfrak{q}+\widetilde{N}\neq\mathcal{O}_F\Big].$$

In other words, $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big)\in H^1_f\big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}\otimes\chi\big)$ will coincide exactly with the image of $\sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})}\chi(t)\cdot\pi_{\mathrm{Jac}X_{\widetilde{U}}}\big(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\big)\otimes 1$ under $\partial_{\widetilde{U}}$ in the specialisation $H^1_f\big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}\otimes\chi\big)\otimes_{\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}}\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}/\mathbb{I}_\star$.

To establish the congruence between $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big)$ and $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big)$ modulo λ^r, we introduce the ideals '$\mathbb{I}_{\star,\lambda^r}$' generated over $\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}$ by $\mathbb{I}_\star$ and the element $\lambda^r\in\mathcal{O}_{\mathcal{K}}$. For the HMF $\star=\mathbf{f}$, this alternative description for the image of $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big)$ above means that $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big)\mod\lambda^r$ is equal to

$$\sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})}\chi(t)\cdot\partial_{\widetilde{U}}\circ\pi_{\mathrm{Jac}X_{\widetilde{U}}}\big(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\big)\otimes 1$$
$$\in H^1_f\Big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}_{(\chi)}\Big)\otimes_{\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}}\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}/\mathbb{I}_{\mathbf{f},\lambda^r}.$$

Likewise for the HMF $\star=\mathbf{g}$, the 1-cocycle $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big)\mod\lambda^r$ equals

$$\sum_{t\in\mathrm{Pic}(\mathcal{O}_\mathfrak{c})}\chi(t)\cdot\partial_{\widetilde{U}}\circ\pi_{\mathrm{Jac}X_{\widetilde{U}}}\big(\iota_{\underline{\xi}}(t\cdot\mathbf{x})\big)\otimes 1$$
$$\in H^1_f\Big(E\otimes\mathbb{Q}_p,\mathrm{Ta}_p\big(\mathrm{Jac}X_{\widetilde{U}}\big)^{\mathrm{ord}}_{(\chi)}\Big)\otimes_{\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}}\mathfrak{h}^{\mathrm{ord}}_{\widetilde{U}}/\mathbb{I}_{\mathbf{g},\lambda^r}.$$

But, Hypothesis $(\mathbf{f}\equiv\mathbf{g}\ (\lambda^r))$ implies that $C(\mathfrak{q},\mathbf{f})\equiv C(\mathfrak{q},\mathbf{g})\mod\lambda^r$ at all primes $\mathfrak{q}\lhd\mathcal{O}_F$ satisfying $\mathfrak{q}+\widetilde{N}=\mathcal{O}_F$, in which case $\mathbb{I}_{\mathbf{f},\lambda^r}$ and $\mathbb{I}_{\mathbf{g},\lambda^r}$ are the same. Thus, $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big)$ and $\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big)$ must be congruent mod λ^r, which proves (a).

To show that assertion (b) is true, a simple direct calculation reveals that

$$\begin{aligned}
&\left\langle\!\left\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\right\rangle\!\right\rangle_{\widetilde{U}} \\
&= \left\langle\!\left\langle \mathrm{pr}^{\widetilde{U}}_{A_\star}\circ\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big)\ ,\ \mathrm{pr}^{\widetilde{U}}_{A_\star}\circ\partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\right\rangle\!\right\rangle_{U_\star} \\
&= \left\langle\!\left\langle \partial_{U_\star}\circ\mathrm{pr}^{\widetilde{U}}_{A_\star}\big(\widetilde{\mathcal{P}}(\star,\chi)\big)\ ,\ \partial_{U_\star}\circ\mathrm{pr}^{\widetilde{U}}_{A_\star}\big(\widetilde{\mathcal{P}}(\star,\overline{\chi})\big)\right\rangle\!\right\rangle_{U_\star}
\end{aligned}$$

and then applying Lemma 3.2:

$$\begin{aligned}
&\partial_{U_\star}\circ\,\mathrm{pr}^{\widetilde{U}}_{A_\star}\big(\widetilde{\mathcal{P}}(\star,\chi)\big) \\
&= \prod_{\mathfrak{q}|\widetilde{N}}\big(1-C(\mathfrak{q},\star)\chi(\mathfrak{Q})+\chi^2(\mathfrak{Q})\omega(\mathfrak{q})\cdot\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\big)\cdot\partial_{U_\star}\big(\mathcal{P}(\star,\chi)\big), \quad \text{and} \\
&\partial_{U_\star}\circ\,\mathrm{pr}^{\widetilde{U}}_{A_\star}\big(\widetilde{\mathcal{P}}(\star,\overline{\chi})\big) \\
&= \prod_{\mathfrak{q}|\widetilde{N}}\big(1-C(\mathfrak{q},\star)\chi(\mathfrak{Q}^c)+\chi^2(\mathfrak{Q}^c)\omega(\mathfrak{q})\cdot\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})\big)\cdot\partial_{U_\star}\big(\mathcal{P}(\star,\overline{\chi})\big).
\end{aligned}$$

The product of these two sets of Euler factors above yields the (degree four) factor $L_{(\widetilde{N})}\big(\star,\chi,0\big)=\prod_{\mathfrak{q}|\widetilde{N}}L_{\mathfrak{q}}\big(\star\otimes\mathrm{Ind}_E^F(\chi),\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})^{-s}\big)\Big|_{s=0}$, which therefore implies

$$\left\langle\!\left\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\right\rangle\!\right\rangle_{\widetilde{U}} \;=\; L_{(\widetilde{N})}\big(\star,\chi,0\big)\times\left\langle\!\left\langle \mathcal{P}(\star,\chi),\mathcal{P}(\star,\chi^{-1})\right\rangle\!\right\rangle_{A_\star}.$$

Writing out in full the p-adic Gross-Zagier formula from [12, Theorem B],

$$\left\langle\!\left\langle \mathcal{P}(\star,\chi),\mathcal{P}(\star,\chi^{-1})\right\rangle\!\right\rangle_{A_\star} \;=\; \frac{c_E}{2}\cdot Z_p^o(\chi)^{-1}\times\chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\emptyset}^{\mathrm{Dis}}(\star,\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1},$$

where

$$\begin{aligned}
&Z_p^o(\chi) \\
&= \frac{\chi\big(d_F^{(p)}\big)G(\overline{\chi})\sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\mathcal{N}_{E/F}(\mathfrak{f}_\chi)\big)}\;\overline{\chi}(D_{E/F})}{\prod_{\mathfrak{p}|p}\alpha_{\mathfrak{p}}(\star)^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}}\cdot\prod_{\mathfrak{p}|p}\prod_{\mathfrak{P}|\mathfrak{p}}\left(1-\frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\star)}\right).
\end{aligned}$$

As an immediate consequence, one deduces that

$$\begin{aligned}
&\left\langle\!\left\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\right\rangle\!\right\rangle_{\widetilde{U}} \\
&= \frac{c_E\cdot L_{(\widetilde{N})}\big(\star,\chi,0\big)}{2\cdot Z_p^o(\chi)}\cdot\chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\emptyset}^{\mathrm{Dis}}(\star,\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1}.
\end{aligned}$$

If we switch between Σ and the empty set $\emptyset$, the interpolation rule in Equation (2.2) yields the identity

$$\chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\emptyset}^{\mathrm{Dis}}(\star,\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1} \;=\; \chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}^{\mathrm{Dis}}(\star,\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1}\times L_{(\widetilde{N})}\big(\star,\chi,1\big)^{-1}$$

where $L_{(\widetilde{N})}\big(\star,\chi,1\big)=\prod_{\mathfrak{q}|\widetilde{N}}L_{\mathfrak{q}}\big(\star\ \otimes\mathrm{Ind}_E^F(\chi),\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})^{-s}\big)\Big|_{s=1}$.

Lastly, observing that $\frac{L_{(\widetilde{N})}(\star,\chi,0)}{L_{(\widetilde{N})}(\star,\chi,1)} = \mathcal{E}_{\widetilde{N}}(\star,\chi)$, the proof of (b) is complete. $\square$

We are now ready to give the demonstration of the main result in this section. We will also indicate how Theorems 2.2 and 3.1 imply (as special cases) the results stated in the Introduction, for the congruent elliptic curves $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathbb{Q}$.

Proof of Theorem 3.1. Statement (i) follows immediately from the simple observation that if the base character χ_0 satisfies Hypothesis **(Odd)**, then so does $\chi = \chi_0 \cdot \chi^\dagger$ for any choice of anticyclotomic (and finite order) character $\chi^\dagger$ on Γ_E.

To show statement (ii), recall from [21, p167] that the p-adic height takes values in $\log_p(\gamma_0) \cdot \prod_{\mathfrak{P}|p} \#\widetilde{A}_\star(\mathcal{O}_E/\mathfrak{P})^{-2} \cdot \mathbb{Z}_p \subset \mathbb{Q}_p$, and is naturally a $\mathbb{Z}_p$-bilinear pairing. Applying Lemma 3.3(a) to $\widetilde{\mathcal{P}}(\star,\chi)$ and $\widetilde{\mathcal{P}}(\star,\chi^{-1})$, one immediately deduces that $\langle\!\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{f},\chi^{-1})\big)\rangle\!\rangle_{\widetilde{U},E}$ is equal to

$$\langle\!\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi^{-1})\big)\rangle\!\rangle_{\widetilde{U},E} + \ \lambda^r \cdot \Big(\langle\!\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi)\big), Q\big(\mathbf{f},\mathbf{g},\chi^{-1}\big)\rangle\!\rangle_{\widetilde{U},E}$$
$$+ \langle\!\langle Q\big(\mathbf{f},\mathbf{g},\chi\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\mathbf{g},\chi^{-1})\big)\rangle\!\rangle_{\widetilde{U},E} + \ \lambda^r \cdot \langle\!\langle Q\big(\mathbf{f},\mathbf{g},\chi\big), Q\big(\mathbf{f},\mathbf{g},\chi^{-1}\big)\rangle\!\rangle_{\widetilde{U},E}\Big)$$

which means that $\langle\!\langle \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi)\big), \partial_{\widetilde{U}}\big(\widetilde{\mathcal{P}}(\star,\chi^{-1})\big)\rangle\!\rangle_{\widetilde{U},E}$ modulo $\log_p(\gamma_0) \cdot \lambda^{r-r_0}$ must be independent of the choice of HMF $\star \in \{\mathbf{f},\mathbf{g}\}$.

Now by applying Lemma 3.3(b), one obtains the following congruence for the period-modified p-adic L-functions:

$$\mathcal{E}_{\widetilde{N}}(\mathbf{f},\chi) \cdot \chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}^{\mathrm{Dis}}(\mathbf{f},\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1} \equiv$$
$$\mathcal{E}_{\widetilde{N}}(\mathbf{g},\chi) \cdot \chi^\dagger\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}^{\mathrm{Dis}}(\mathbf{g},\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1} \quad \bmod \ \frac{2}{c_E} \cdot \log_p(\gamma_0) \cdot \lambda^{r-r_0} \cdot \mathcal{O}_{\mathcal{K},\chi}$$

since for each choice of HMF $\star \in \{\mathbf{f},\mathbf{g}\}$, the p-adic multiplier term

$$Z_p^o(\star,\chi) \ =$$
$$\frac{\chi\big(d_F^{(p)}\big) G(\overline{\chi}) \sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(D_{E/F}\mathcal{N}_{E/F}(\mathfrak{f}_\chi)\big)}\ \overline{\chi}(D_{E/F})}{\prod_{\mathfrak{p}|p} \alpha_{\mathfrak{p}}(\star)^{\mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{E/F}(\mathfrak{f}_\chi))}} \prod_{\mathfrak{p}|p} \prod_{\mathfrak{P}|\mathfrak{p}} \left(1 - \frac{\overline{\chi}(\mathfrak{P})}{\alpha_{\mathfrak{p}}(\star)}\right)$$

is an algebraic number satisfying the congruence $Z_p^o(\mathbf{f},\chi) \equiv Z_p^o(\mathbf{g},\chi)$ modulo λ^r. However, $\mathbf{L}_{p,\Sigma}\big(\star,\rho_0\big) = \dfrac{\Omega_{\infty,K}^{\mathrm{aut},(1)}(\star)}{\Omega_{\infty,K}^{\mathrm{aut},(0)}(\star)} \times \mathbf{L}_{p,\Sigma}^{\mathrm{Dis}}\big(\star,\rho_0\big)$, so defining $\delta_E :=$

$\dfrac{2}{c_E} \cdot \dfrac{\Omega^{\mathrm{aut},(1)}_{\infty,K}(\star)}{\Omega^{\mathrm{aut},(0)}_{\infty,K}(\star)}$ which does not depend on the choice[4] of cusp form $\star$, one thereby concludes

$$\mathcal{E}_{\widetilde{N}}(\mathbf{f},\chi)\cdot\chi^{\dagger}\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\mathbf{f},\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1} \equiv$$
$$\mathcal{E}_{\widetilde{N}}(\mathbf{g},\chi)\cdot\chi^{\dagger}\left(\frac{\mathrm{d}\kappa_{\mathrm{cy}}^{s-1}\mathbf{L}_{p,\Sigma}(\mathbf{g},\rho_0)}{\mathrm{d}s}\right)\Bigg|_{s=1} \quad \bmod \delta_E\cdot\log_p(\gamma_0)\cdot\lambda^{r-r_0}\cdot\mathcal{O}_{\mathcal{K},\chi}$$

thus completing the proof of Theorem 3.1(ii). □

Remarks. (a) We should point out that the special values of the derivatives of the p-adic L-functions $\mathbf{L}_{p,\Sigma}(\mathbf{f},\rho_0)$ and $\mathbf{L}_{p,\Sigma}(\mathbf{g},\rho_0)$ lie inside $\delta_E\cdot\log_p(\gamma_0)\cdot\lambda^{-r_0}$, so we should get a non-trivial congruence. If we want to swap the automorphic periods with motivic periods, we consequently obtain a congruence modulo $\lambda^r\cdot\mathcal{L}_{\mathbf{f},\mathbf{g}}$ where the lattice $\mathcal{L}_{\mathbf{f},\mathbf{g}}\subset\mathbb{C}_p$ is generated by the values of the motivic p-adic L-functions.

(b) Suppose we are in the situation of the Introduction, so that $\mathcal{A}_1$ and $\mathcal{A}_2$ are congruent elliptic curves modulo p^r. In the *odd case*, applying Theorem 3.1 to the base-change $\mathbf{f}$ of $\mathcal{A}_1$ and base-change $\mathbf{g}$ of $\mathcal{A}_2$, yields a congruence mod $p^r\cdot\mathcal{L}^{(1)}_{\mathcal{A}_1,\mathcal{A}_2}$ where $\mathcal{L}^{(1)}_{\mathcal{A}_1,\mathcal{A}_2}$ contains the special values of each $\mathbf{L}_p^{(1)}(\mathcal{A}_i/E,\chi)$ (see Theorem 1.4).

(c) Likewise in the *even case*, applying Theorem 2.2 to the base-change cusp forms $\mathbf{f}$ and $\mathbf{g}$ as in (b), this time we obtain a congruence modulo $p^r\cdot\mathcal{L}^{(0)}_{\mathcal{A}_1,\mathcal{A}_2}$ where $\mathcal{L}^{(0)}_{\mathcal{A}_1,\mathcal{A}_2}$ contains the values of $\mathbf{L}_p^{(0)}(\mathcal{A}_i/E,\chi)$ for each $i\in\{1,2\}$ (see Theorem 1.3).

4. Logarithm maps and Coleman integration

In this section, we continue to assume Hypotheses **($\mathbf{f}\equiv\mathbf{g}$ (λ^r))** and **(Odd)** hold. We also assume that $\mathfrak{p}$ splits in E. Generalizing the work of Bertolini-Darmon-Prasanna [1], Liu, Zhang and Zhang have constructed a p-adic L-function on Γ_E^{anti} interpolating the complex Rankin-Selberg L-function of $\star$ twisted by characters on Γ_E^{anti} of positive weight, for each $\star\in\{\mathbf{f},\mathbf{g}\}$ (see in particular [18, Theorem 3.2.10]). At every finite order character χ, the value of this p-adic L-function is related to the logarithm of the corresponding χ-twisted Heegner point $\mathcal{P}(\star,\chi)$ attached to either HMF $\star\in\{\mathbf{f},\mathbf{g}\}$, as given by Theorem 3.3.2 in *op. cit.*

[4]For the record, the explicit form of the factor $\delta_E\in\mathbb{Q}^\times$ can be calculated via the formula
$$\delta_E=\frac{4|D_E|^{1/2}\cdot\zeta_F(2)^{-1}L(1,\eta_{E/F})^2\cdot L\big(1,\mathrm{ad}(\star)\big)\cdot\langle\widetilde{\star},\widetilde{\star}\rangle_{\mathcal{R}^\times}}{|D_F|^{1/2}\cdot(16\pi^3)^{[F:\mathbb{Q}]}\cdot\mathrm{Vol}\big(\mathbf{X}_{U_0(\widetilde{N})}\big)\cdot\langle\phi_\star,\phi_\star\rangle_{\mathrm{Pet}}}.$$

Following the strategy of [17], we shall show that these special values satisfy a congruence relation under $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ via Coleman integration. However, at present, we do not know whether the p-adic L-function of Liu-Zhang-Zhang is an Iwasawa function, so it is unclear to us whether an analogue of Theorem 2.2 holds.

We first recall the notion of Coleman primitives from [7]. Let K be a local field contained in $\mathbb{C}_p$, X a quasiprojective scheme over K and $U \subset X^{\mathrm{rig}}$ an affinoid domain with good reduction. We assume ω is a closed rigid analytic 1-form on U. Suppose that there exists a Frobenius endomorphism ϕ on U (that is, it becomes a power of the Frobenius map on the reduction of U), a locally analytic function F_ω on U, and a polynomial $P(X) \in \mathbb{C}_p[X]$ whose zeroes are not roots of unity, satisfying the twin conditions:

- $\mathrm{d}F_\omega = \omega$;
- $P(\phi^*)F_\omega$ is rigid analytic.

Then F_ω is called a *Coleman primitive* of ω. Furthermore, it is independent of the polynomial $P(X)$, and is uniquely determined up to an additive constant.

We will require the following technical result of Liu-Zhang-Zhang.

Proposition 4.1 ([18], Proposition A.0.1)**.** *Let K, U and X be given as above. Assume A is an abelian variety over K which either has totally degenerate reduction or potentially good reduction. Then for a morphism $f : X \to A$ and a differential form $\omega \in \Omega^1(A/K)$, the restriction to U of the pullback $f^*\omega$ admits $f^* \log_\omega \big|_U$ as a Coleman primitive, where* $\log_\omega$: $A(\mathbb{C}_p) \to \mathbb{C}_p$ *denotes the p-adic logarithmic attached to ω.*

We next briefly review the definition of p-adic HMFs. Let R be a ring which is complete and separated in its p-adic topology, and $\mathfrak{C}$ is a fractional ideal of $\mathcal{O}_F$. Then a p-adic $\mathfrak{C}$-HMF over R is a rule $\mathbf{h}$, which assigns to every isomorphism class of triples (A, C, ϖ) a value in R, and satisfies some standard automorphy conditions (we refer the reader to [16, §1.9] and [13, Chapter 5, §6] for the precise details). Here A is a $\mathfrak{C}$-polarized HBAV over R equipped with real multiplications by F, C denotes a level structure on A, and ϖ is a nowhere vanishing differential on A.

In particular, such p-adic $\mathfrak{C}$-Hilbert modular forms have q-expansions indexed by totally positive elements in $\mathfrak{ab}$ where $\mathfrak{C} = \frac{\mathfrak{a}}{\mathfrak{b}}$. Recall that we are in the odd case, so again $\mathbb{B}_{/F}$ denotes the incoherent quaternion algebra from Section 3, and for each compact open subgroup $U \subset \mathbb{B}^\times_{\mathbb{A}_F}$ the algebraic curve X_U has as its complex points $X_U(\mathbb{C}) = \mathbb{B}\backslash\mathcal{H}^\pm \times \widehat{\mathbb{B}}^\times/U$. The space of p-adic modular forms over X_U is then given by the direct sum of p-adic $\mathfrak{C}$-HMFs, as $\mathfrak{C}$ runs through a complete set of coset representatives for the narrow class group of F.

Let $\mathbf{h}$ be a parallel weight-two p-adic HMF over $\mathcal{O}_{\mathcal{K}}$ on $X_{\widetilde{U}}$ in the sense of [16]. Because it has weight $\underline{2}$, we may identify $\mathbf{h}$ with a differential

$\omega_{\mathbf{h}} \in H^0(X_{\widetilde{U}}, \Omega^1_{X_{\widetilde{U}}})$. Let $\widetilde{\iota} : X_{\widetilde{U}} \to \mathrm{Jac}X_{\widetilde{U}}$ be the Abel-Jacobi map: we shall write $\omega_{\mathbf{h}}^{\#} \in \Omega^1_{\mathrm{Jac}X_{\widetilde{U}}}$ for the differential satisfying $\widetilde{\iota}^*\omega_{\mathbf{h}}^{\#} = \omega_{\mathbf{h}}$.

Let Θ be the Atkin-Serre differential operator of [18, Definition 2.4.7] – this corresponds to the composition of $\theta(\sigma)$ as σ runs through all embeddings $F \hookrightarrow \overline{\mathbb{Q}}$, where $\theta(\sigma)$ is defined as in [16, Corollary 2.6.25]. The Θ-operator shifts the weight of a HMF by exactly $\underline{2}$, i.e. the weight of $\Theta(\mathbf{h})$ equals $(k_\sigma + 2)_{\sigma:F\hookrightarrow\overline{\mathbb{Q}}}$ if the weight of $\mathbf{h}$ is $(k_\sigma)_{\sigma:F\hookrightarrow\overline{\mathbb{Q}}}$. On q-expansions it has the effect $C(\mathfrak{q}, \Theta(\mathbf{h})) = \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})C(\mathfrak{q}, \mathbf{h})$ for all $\mathfrak{q}$ (see [16, (2.6.27)]). If $\mathbf{h}$ is of parallel weight two, let $F_{\mathbf{h}}$ denote the Coleman primitive of $\omega_{\mathbf{h}} \in H^0(X_{\widetilde{U}}, \Omega^1_{X_{\widetilde{U}}})$ as given by Proposition 4.1. In particular, $\mathrm{d}F_{\mathbf{h}} = \omega_{\mathbf{h}}$. On comparing q-expansions, we see that and $\Theta F_{\mathbf{h}} = \mathbf{h}$. Note that $F_{\mathbf{h}}$ is a HMF of parallel weight zero since $\mathbf{h}$ is of parallel weight two. Applying Proposition 4.1 above, we obtain the following important consequence.

Corollary 4.2. *If $\mathcal{P} \in X_{\widetilde{U}}(\mathbb{C}_p)$, then*

$$F_{\mathbf{h}}(\mathcal{P}) = \log_{\omega_{\mathbf{h}}^{\#}}(\mathcal{P}).$$

Proof. We simply take f, X, A and ω in Proposition 4.1 to be $\widetilde{\iota}$, $X_{\widetilde{U}}$, $\mathrm{Jac}X_{\widetilde{U}}$ and $\omega_{\mathbf{h}}^{\#}$ respectively, and the rest follows immediately. $\square$

We shall regard $\mathbf{f}$ and $\mathbf{g}$ as p-adic HMFs on $X_{U_{\mathbf{f}}}$ and $X_{U_{\mathbf{g}}}$ respectively, as well as on $X_{\widetilde{U}}$ of course. If one makes a choice of HMF $\star \in \{\mathbf{f}, \mathbf{g}\}$, then recall from Definition 1.6 the notation $\widetilde{\star}$ refers to the depleted form on $X_{U_\star}$ obtained from $\star$. For a p-adic HMF $\mathbf{h}$ and an $\mathcal{O}_F$-ideal $\mathcal{I}$, we denote the $\mathcal{I}$-depletion of $\mathbf{h}$ by $\mathbf{h}^{(\mathcal{I})}$.

Lemma 4.3. *The Hypothesis* $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ *implies that*

$$F_{\widetilde{\mathbf{f}}^{(p)}} \;=\; F_{\widetilde{\mathbf{g}}^{(p)}} \;+\; \lambda^r \cdot \sum_j c_j \cdot F_{\mathbf{h}_j^{(p)}}.$$

Proof. We follow [17, proof of Theorem 3.9]. Since the operator Θ is $\mathcal{O}_{\mathcal{K}}$-linear, one immediately deduces that

$$\Theta^n \widetilde{\mathbf{f}}^{(p)} \;=\; \Theta^n \widetilde{\mathbf{g}}^{(p)} \;+\; \lambda^r \cdot \sum_j c_j \cdot \Theta^n \mathbf{h}_j^{(p)} \tag{4.1}$$

for all integers $n \geq 1$. Note that $\Theta^n : q^m \mapsto m^n q^m$ within the q-expansion of $\mathbf{h}^{(p)}$, and recall from [13, Corollary 5.1] that the q-expansion map over $\mathbf{C}$ is injective. Because we have p-depleted our HMFs and the map $n \mapsto m^n$ is continuous in the p-adic topology whenever $p \nmid m$, the HMFs $\Theta^n \mathbf{h}^{(p)}$ varies p-adically continuously in n. If we define

$$\Theta^{-1}\mathbf{h}^{(p)} := \lim_{n \to -1} \Theta^n \mathbf{h}^{(p)}$$

where the limit is taken under the p-adic topology, then $\Theta^{-1}\mathbf{h}^{(p)} = F_{\mathbf{h}^{(p)}}$ on comparing q-expansions. Thus, our result follows on letting $n \to -1$ in (4.1). $\square$

Theorem 4.4. *Under the Hypothesis* $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$, *we have the congruence*

$$\prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q}, \mathbf{f})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \cdot \log_{A_{\mathbf{f}}}(\mathcal{P}(\mathbf{f}, \chi)) \equiv$$
$$\prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q}, \mathbf{g})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \cdot \log_{A_{\mathbf{g}}}(\mathcal{P}(\mathbf{g}, \chi)) \mod \lambda^r \cdot \mathcal{O}_{\mathcal{K},\chi}.$$

Proof. The Hypothesis $(\mathbf{f} \equiv \mathbf{g}\ (\lambda^r))$ together with Lemma 4.3 tell us that $F_{\mathbf{f}}^{(p\widetilde{N})}$ and $F_{\mathbf{g}}^{(p\widetilde{N})}$ must be congruent mod λ^r, as weight-zero p-adic HMFs on $X_{\widetilde{U}}$. In particular,

$$F_{\mathbf{f}}^{(p\widetilde{N})}(\mathcal{P}) \equiv F_{\mathbf{g}}^{(p\widetilde{N})}(\mathcal{P}) \mod \lambda^r \cdot \mathcal{O}_{\mathbb{C}_p} \tag{4.2}$$

for every $\mathcal{P} \in X_{\widetilde{U}}(\mathbb{C}_p)$.

Let $\mathbf{x} = (A, C, \varpi) \in X_{\widetilde{U}}$ be any CM-point, and consider $t \in \mathrm{Pic}(\mathcal{O}_\chi)$ as in §3. If $\mathbf{h}$ is a weight-zero p-adic HMF on $X_{\widetilde{U}}$ with central character ω, recall that

$$\mathbf{h}^{(\mathfrak{q})} = \left(1 - \mathcal{T}(\mathfrak{q})\mathcal{V}(\mathfrak{q}) + \frac{\langle q\rangle \mathcal{V}(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right)\mathbf{h} \quad \text{if } \mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F) \text{ with } \mathfrak{q} + N_{\mathbf{h}} = \mathcal{O}_F;$$

otherwise, it is given by $\big(1 - \mathcal{U}(\mathfrak{q}) \circ \mathcal{V}(\mathfrak{q})\big)\mathbf{h}$ if $\mathfrak{q} + N_{\mathbf{h}} \neq \mathcal{O}_F$. Our calculations on the images of $\mathbf{x}$ under these operators, which are described in the proof of Lemma 3.2, directly imply that

$$\begin{aligned} &\sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)\mathbf{h}^{(p\widetilde{N})}(t * \mathbf{x}) \\ &= \prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q}, \mathbf{h})\chi(\mathfrak{Q}) + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \times \sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)\mathbf{h}(t * \mathbf{x}) \end{aligned} \tag{4.3}$$

(see also [17, Lemma 3.6] for the same result for p-adic elliptic modular forms).

Recall once more that $\Theta : q^m \mapsto mq^m$ on q-expansions, so for either $\star \in \{\mathbf{f}, \mathbf{g}\}$ we have $C(\mathfrak{q}, F_\star) = \frac{C(\mathfrak{q},\star)}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}$ at each $\mathfrak{q}|p\widetilde{N}$. Hence, we may rewrite Equation (4.3) as:

$$\begin{aligned} &\sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)F_\star^{(p\widetilde{N})}(t * \mathbf{x}) \\ &= \prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q}, \star)\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \times \sum_{t \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)F_\star(t * \mathbf{x}) \end{aligned}$$

and upon combining this with (4.2), one therefore deduces

$$\prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{f})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)F_{\mathbf{f}}\,(t*\mathbf{x})$$
$$\equiv \prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{g})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right) \sum_{t\in\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}_\chi})} \chi(t)F_{\mathbf{g}}\,(t*\mathbf{x})$$
$$\mod \lambda^r\cdot\mathcal{O}_{\mathcal{K},\chi}.$$

Finally, Corollary 4.2 informs us that

$$\prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{f})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right)\cdot\log_{\omega_{\mathbf{f}}^{\#}}(\mathcal{P}(\mathbf{f},\chi))$$
$$\equiv \prod_{\mathfrak{q}|p\widetilde{N}} \left(1 - C(\mathfrak{q},\mathbf{g})\frac{\chi(\mathfrak{Q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})} + \frac{\chi^2(\mathfrak{Q})\omega(\mathfrak{q})}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})}\right)\cdot\log_{\omega_{\mathbf{g}}^{\#}}(\mathcal{P}(\mathbf{g},\chi))$$
$$\mod \lambda^r\cdot\mathcal{O}_{\mathcal{K},\chi}.$$

However, $\log_{\omega_\star^{\#}} = \log_{A_\star}$ by their definition, so the proof is now complete. □

Remarks. (a) For those readers familiar with the notation of Liu-Zhang-Zhang in [18, Theorem 3.3.2], their p-adic Waldspurger formula states that

$$\log_{A_\star^+}(\mathcal{P}(\star,\chi))\cdot\log_{A_\star^-}\left(\mathcal{P}(\star,\chi^{-1})\right)$$
$$= (\text{Euler factor at } \mathfrak{p})\cdot\chi\big(\mathfrak{L}(A_\star)\big)\cdot\alpha_\chi\big(f_{\star,+},f_{\star,-}\big)$$

where $\mathfrak{L}(A_\star)$ denotes the p-adic L-function attached to $\star$ in [18, Theorem 3.2.10], and $\alpha_\chi\big(f_{\star,+},f_{\star,-}\big)$ is a distinguished generator for the $\mathcal{K}$-line

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\left(\Pi_\star^+\otimes\chi,\mathcal{K}\right)\otimes_{\mathcal{K}}\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\left(\Pi_\star^-\otimes\chi^{-1},\mathcal{K}\right).$$

(b) Applying Theorem 4.4 directly to $\log_{A_\star}\left(\mathcal{P}(\star,\chi^{\pm1})\right)$, a simple calculation reveals that

$$\mathcal{E}_{1,p\widetilde{N}}(\mathbf{f})\times\log_{A_{\mathbf{f}}}(\mathcal{P}(\mathbf{f},\chi))\cdot\log_{A_{\mathbf{f}}}\left(\mathcal{P}(\mathbf{f},\chi^{-1})\right)$$
$$\equiv\ \mathcal{E}_{1,p\widetilde{N}}(\mathbf{g})\times\log_{A_{\mathbf{g}}}(\mathcal{P}(\mathbf{g},\chi))\cdot\log_{A_{\mathbf{g}}}\left(\mathcal{P}(\mathbf{g},\chi^{-1})\right)\quad\mod\lambda^r\cdot\mathcal{O}_{\mathcal{K}}.$$

(c) Under the strong assumption that $\mathfrak{L}(A_{\mathbf{f}})$ *and* $\mathfrak{L}(A_{\mathbf{g}})$ *correspond to bounded Iwasawa functions* (which is so far only known over $F=\mathbb{Q}$), as a corollary (b) yields a congruence modulo λ^r linking together the Σ-imprimitive p-adic L-functions $\mathfrak{L}_\Sigma(A_{\mathbf{f}})$ and $\mathfrak{L}_\Sigma(A_{\mathbf{g}})$, for suitably chosen isomorphisms $\phi_\star$ between the local field $\mathcal{K}$ and the lines

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\left(\Pi_\star^+\otimes\chi,\mathcal{K}\right)\otimes_{\mathcal{K}}\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\left(\Pi_\star^-\otimes\chi^{-1},\mathcal{K}\right)$$
$$\otimes_{F^M}\left(\mathrm{Lie}(A_\star^+)\otimes_{F^M}\mathrm{Lie}(A_\star^-)\right).$$

(d) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are congruent elliptic curves mod p^r as in §1, one thereby obtains a congruence between $\mathfrak{L}_\Sigma(\mathcal{A}_1)$ and $\mathfrak{L}_\Sigma(\mathcal{A}_2)$ modulo $p^r\cdot$

$\mathcal{L}^{\natural}_{\mathcal{A}_1,\mathcal{A}_2}[[\Gamma_E^{\mathrm{anti}}]]$, again assuming that $\mathfrak{L}(\mathcal{A}_i)$ for $i=1,2$ correspond to bounded Iwasawa functions, and where $\mathcal{L}^{\natural}_{\mathcal{A}_1,\mathcal{A}_2}$ is the $\mathcal{O}_{\mathbb{C}_p}$-submodule generated by the values $\chi\big(\mathfrak{L}(\mathcal{A}_1)\big)$ and $\chi\big(\mathfrak{L}(\mathcal{A}_2)\big)$ as $\chi^{\dagger}$ ranges over $\mathrm{Hom}\big(\Gamma_E^{\mathrm{anti}},\overline{\mathbb{Q}}_p^{\times}\big)$ – we refer the reader to Theorem 1.5 for the precise statement.

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