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A construction of pseudo-Anosov braids with small normalized entropies

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Abstract. Let b be a pseudo-Anosov braid whose permutation has a fixed point and let M_b be the mapping torus by the pseudo-Anosov homeomorphism defined on the genus 0 fiber F_b associated with b. We prove that there is a 2-dimensional subcone $\mathcal{C}_0$ contained in the fibered cone $\mathcal{C}$ of F_b such that the fiber F_a for each primitive integral class $a \in \mathcal{C}_0$ has genus 0. We also give a constructive description of the monodromy $\phi_a : F_a \to F_a$ of the fibration on M_b over the circle, and consequently provide a construction of many sequences of pseudo-Anosov braids with small normalized entropies. As an application we prove that the smallest entropy among skew-palindromic braids with n strands is comparable to $1/n$, and the smallest entropy among elements of the odd/even spin mapping class groups of genus g is comparable to $1/g$.

Contents

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1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures for $n \geq 0$. We set $\Sigma_g = \Sigma_{g,0}$. By mapping class group $\mathrm{Mod}(\Sigma_{g,n})$, we mean the group of isotopy classes of orientation preserving self-homeomorphisms on $\Sigma_{g,n}$ preserving punctures setwise. By Nielsen-Thurston classification, elements in $\mathrm{Mod}(\Sigma)$ are classified into three types: periodic, reducible, pseudo-Anosov [30, 9]. For $\phi \in \mathrm{Mod}(\Sigma)$ we choose a representative $\Phi \in \phi$ and consider the mapping torus $M_\phi = \Sigma \times \mathbb{R}/\sim$, where $\sim$ identifies $(x, t+1)$ with $(\Phi(x), t)$ for $x \in \Sigma$ and $t \in \mathbb{R}$. Then Σ is a fiber of a fibration on M_ϕ over the circle S^1 and ϕ is called the *monodromy*. A theorem by Thurston [31] asserts that M_ϕ admits a hyperbolic structure of finite volume if and only if ϕ is pseudo-Anosov.

For a pseudo-Anosov element $\phi \in \mathrm{Mod}(\Sigma)$ there is a representative $\Phi : \Sigma \to \Sigma$ of ϕ called a *pseudo-Anosov homeomorphism* with the following property: Φ admits a pair of transverse measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ and a constant $\lambda = \lambda(\phi) > 1$ depending on ϕ such that $\mathcal{F}^u$ and $\mathcal{F}^s$ are invariant under Φ, and μ^u and μ^s are uniformly multiplied by λ and λ^{-1} under Φ. The constant $\lambda(\phi)$ is called the *dilatation* and $\mathcal{F}^u$ and $\mathcal{F}^s$ are called the *unstable* and *stable foliation*. We call the logarithm $\log(\lambda(\phi))$ the *entropy*, and call

$$\mathrm{Ent}(\phi) = |\chi(\Sigma)| \log(\lambda(\phi))$$

the *normalized entropy* of ϕ, where $\chi(\Sigma)$ is the Euler characteristic of Σ. Such normalization of the entropy is suited for the context of 3-manifolds [8, 22].

Penner [27] proved that if $\phi \in \mathrm{Mod}(\Sigma_{g,n})$ is pseudo-Anosov, then

$$\frac{\log 2}{12g - 12 + 4n} \leq \log(\lambda(\phi)). \tag{1.1}$$

See also [22, Corollary 2]. For a fixed surface Σ, the set

$$\{\log \lambda(\phi) \mid \phi \in \mathrm{Mod}(\Sigma) \text{ is pseudo-Anosov}\}$$

is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subgroup or subset $G \subset \mathrm{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\mathrm{Mod}(\Sigma))$. We write $f \asymp h$ if there is a universal constant $P > 0$ such that $1/P \leq f/h \leq P$. It is proved by Penner [27] that the minimal entropy among pseudo-Anosov elements in $\mathrm{Mod}(\Sigma_g)$ on the closed surface of genus g satisfies

$$\log \delta(\mathrm{Mod}(\Sigma_g)) \asymp \frac{1}{g}.$$

See also [16, 32, 33] for other sequences of mapping class groups.

For any $P > 0$, consider the set Ψ_P consisting of all pseudo-Anosov homeomorphisms $\Phi : \Sigma \to \Sigma$ defined on any surface Σ with the normalized entropy $|\chi(\Sigma)| \log \lambda(\Phi) \leq P$. This is an infinite set in general (take $P > 2\log(2 + \sqrt{3})$ for example) and is well-understood in the context of

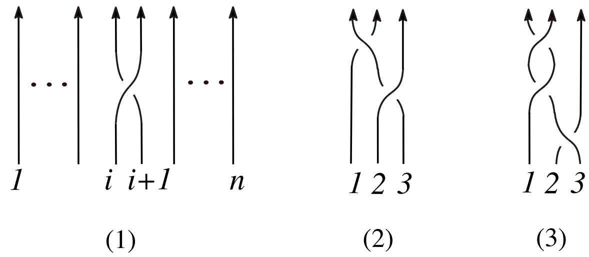

FIGURE 1. (1) σ_i. (2) $\sigma_1^{-1}\sigma_2$ with the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. (3) $\sigma_1^2\sigma_2^{-1}$ whose permutation has a fixed point.

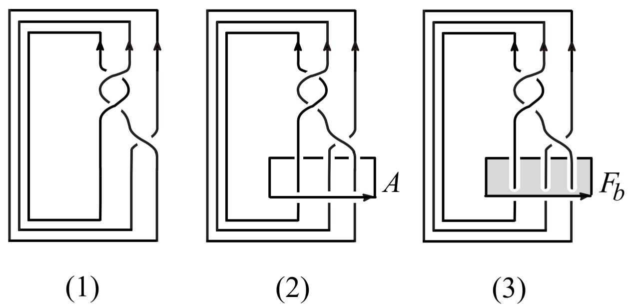

FIGURE 2. $b := \sigma_1^2\sigma_2^{-1}$. (1) $\mathrm{cl}(b)$. (2) $\mathrm{br}(b)$. (3) $F_b \hookrightarrow M_b$.

hyperbolic fibered 3-manifolds. The universal finiteness theorem by Farb-Leininger-Margalit [8] states that the set of homeomorphism classes of mapping tori of pseudo-Anosov homeomprhisms $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is finite, where $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is the fully punctured pseudo-Anosov homeomprhism obtained from $\Phi \in \Psi_P$. (Clearly $\lambda(\Phi^\circ) = \lambda(\Phi)$.) In other words such $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is a monodromy of a fiber in some fibered cone for a hyperbolic fibered 3-manifold in the finite list determined by P. Thus 3-manifolds in the finite list govern all pseudo-Anosov elements in Ψ_P. It is natural to ask the dynamics and a constructive description of elements in Ψ_P. There are some results about this question by several authors [4, 15, 20, 21, 33], but it is not completely understood. In this paper we restrict our attention to the pseudo-Anosov elements in Ψ_P defined on the genus 0 surfaces, and provide an approach for a concrete description of those elements.

Let B_n be the braid group with n strands. The group B_n is generated by the braids $\sigma_1, \cdots, \sigma_{n-1}$ as in Figure 1. Let $\mathcal{S}_n$ be the symmetric group, the group of bijections of $\{1, \ldots, n\}$ to itself. A permutation $\mathcal{P} \in \mathcal{S}_n$ has a *fixed point* if $\mathcal{P}(i) = i$ for some i. We have a surjective homomorphism $\pi : B_n \to \mathcal{S}_n$ which sends each σ_j to the transposition $(j, j+1)$.

The closure $\mathrm{cl}(b)$ of a braid $b \in B_n$ is a knot or link in the 3-sphere S^3. The *braided link*

$$\mathrm{br}(b) = \mathrm{cl}(b) \cup A$$

is a link in S^3 obtained from $\mathrm{cl}(b)$ with its braid axis A (Figure 2). Let M_b denote the exterior of $\mathrm{br}(b)$ which is a 3-manifold with boundary. It is easy to find an $(n+1)$-holed sphere F_b in M_b (Figure 2(3)). Clearly F_b is a fiber of a fibration on $M_b \to S^1$ and its monodromy $\phi_b : F_b \to F_b$ is determined by b. We call F_b the *F-surface* for b.

A braid $b \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if the associated mapping class $f_b \in \mathrm{Mod}(\Sigma_{0,n+1})$ is of the corresponding type (Section 2.3). If b is pseudo-Anosov, then the *dilatation* $\lambda(b)$ is defined by $\lambda(f_b)$ and the *normalized entropy* $\mathrm{Ent}(b)$ is defined by $\mathrm{Ent}(f_b)$. The following theorem is due to Hironaka-Kin [16, Proposition 3.36] together with the observation by Kin-Takasawa [21, Section 4.1].

Theorem 1.1. *There is a sequence of pseudo-Anosov braids $z_n \in B_n$ such that $\mathrm{Ent}(z_n) \neq 2\log(2+\sqrt{3})$, $M_{z_n} \simeq M_{\sigma_1^2\sigma_2^{-1}}$ for each $n \geq 3$ and $\mathrm{Ent}(z_n) \to 2\log(2+\sqrt{3})$ as $n \to \infty$.*

Here $\simeq$ means they are homeomorphic to each other. The limit point $2\log(2+\sqrt{3})$ is equal to $\mathrm{Ent}(\sigma_1^2\sigma_2^{-1})$. By the lower bound (1.1), Theorem 1.1 implies that

$$\log \delta(\mathrm{Mod}(\Sigma_{0,n})) \asymp \frac{1}{n}.$$

In particular, the hyperbolic fibered 3-manifold $M_{\sigma_1^2\sigma_2^{-1}}$ admits an infinitely family of genus 0 fibers of fibrations over S^1.

Let z_n be a pseudo-Anosov braid with d_n strands. We say that a sequence $\{z_n\}$ *has a small normalized entropy* if $d_n \asymp n$ and there is a constant $P > 0$ which does not depend on n such that $\mathrm{Ent}(z_n) \leq P$. By (1.1) a sequence $\{z_n\}$ having a small normalized entropy means $\log(\lambda(z_n)) \asymp 1/n$. One of the aims in this paper is to give a construction of many sequences of pseudo-Anosov braids with small normalized entropies. The following result generalizes Theorem 1.1.

Theorem A. *Suppose that b is a pseudo-Anosov braid whose permutation has a fixed point. There is a sequence of pseudo-Anosov braids $\{z_n\}$ with small normalized entropy such that $\mathrm{Ent}(z_n) \to \mathrm{Ent}(b)$ as $n \to \infty$ and $M_{z_n} \simeq M_b$ for $n \geq 1$.*

The proof of Theorem A is constructive. In fact one can describe braids z_n explicitly. For a more general result see Theorems 5.1, 5.2. Let $\mathcal{C} \subset H_2(M_b, \partial M_b)$ be the fibered cone containing $[F_b]$. A theorem by Thurston [29] states that for each primitive integral class $a \in \mathcal{C}$ there is a connected fiber F_a with the pseudo-Anosov monodromy $\phi_a : F_a \to F_a$ of a fibration on the hyperbolic 3-manifold M_b over S^1. The following theorem states a structure of $\mathcal{C}$.

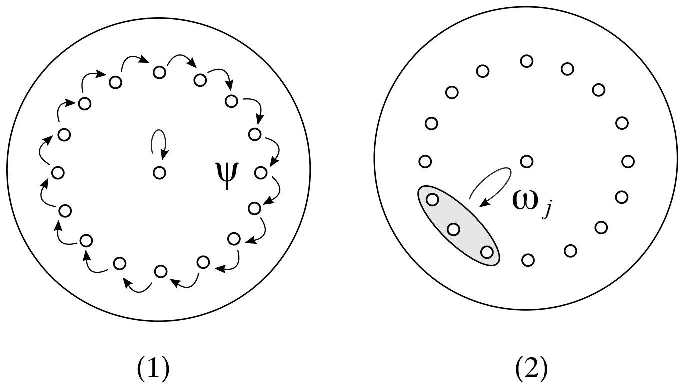

FIGURE 3. Dynamics of ψ and ω_j in Theorem B. (1) Periodic $\psi : F_a \to F_a$. (2) Reducible $\omega_j : F_a \to F_a$. Subsurface $h(S_0)$ is shaded.

Theorem B. *Suppose that b is a pseudo-Anosov braid whose permutation has a fixed point. Then there are a 2-dimensional subcone $\mathcal{C}_0 \subset \mathcal{C}$ and an integer $u \geq 1$ with the following properties.*

(1) *The fiber F_a for each primitive integral class $a \in \mathcal{C}_0$ has genus 0.*
(2) *The monodromy $\phi_a : F_a \to F_a$ for each primitive integral class $a \in \mathcal{C}_0$ is conjugate to*

$$(\omega_1\psi)\cdots(\omega_{u-1}\psi)(\omega_u\psi)\psi^{m-1} : F_a \to F_a,$$

where $m \geq 1$ depends on the class a, ψ is periodic and each ω_j is reducible. Moreover there are homeomorphisms $\widehat{\omega}_j : S_0 \to S_0$ on a surface S_0 for $j = 1, \ldots, u$ determined by b and an embedding $h : S_0 \hookrightarrow F_a$ such that $h(S_0)$ is the support of each w_j and

$$w_j|_{h(S_0)} = h \circ \widehat{\omega}_j \circ h^{-1}.$$

Theorem B gives a constructive description of ϕ_a. Also it states that each $w_j : F_a \to F_a$ is reducible supported on a uniformly bounded subsurface $h(S_0) \subset F_a$. It turns out from the proof that the type of the periodic homeomorphism $\psi : F_a \to F_a$ does not depend on $a \in C_0$ (Remark 3.3), see Figure 3(1). Theorem B reminds us of the symmetry conjecture in [23] by Farb-Leininger-Margalit.

Clearly the permutation of each pure braid has a fixed point. For any pseudo-Anosov braid b, a suitable power b^k becomes a pure braid and one can apply Theorems A, B for b^k.

We have a remark about Theorem A. While the existence of a sequence (F_n, ϕ_n) of fibers and monodromies in $\mathcal{C}$ for which $\mathrm{Ent}(\phi_n) \to \mathrm{Ent}(b)$ is guaranteed by McMullen [25, Theorem 10.2], it does not say anything about the genera of fibers F_n. Theorem B has the extra (constructive) information that each fiber F_n along $\mathcal{C}_0$ is genus 0.

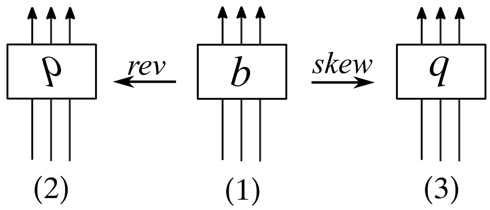

FIGURE 4. Illustration of braids (1) b, (2) $rev(b)$, (3) $skew(b)$.

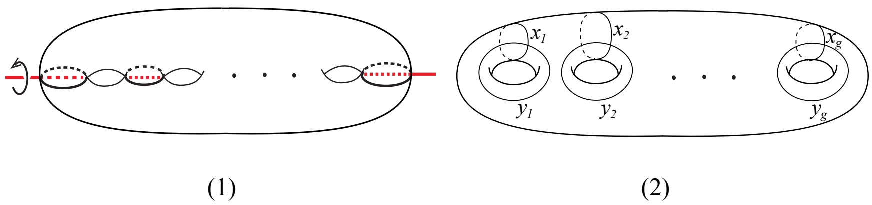

FIGURE 5. (1) $\mathcal{I} : \Sigma_g \to \Sigma_g$. (2) A basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_1(\Sigma_g; \mathbb{Z}_2)$.

As an application we will determine asymptotic behaviors of the minimal dilatations of a subset of B_n consisting of braids with a symmetry. A braid $b \in B_n$ is *palindromic* if $rev(b) = b$, where $rev : B_n \to B_n$ is a map such that if w is a word of letters $\sigma_j^{\pm 1}$ representing b, then $rev(b)$ is the braid obtained from b reversing the order of letters in w. A braid $b \in B_n$ is *skew-palindromic* if $skew(b) = b$, where $skew(b) = \Delta rev(b) \Delta^{-1}$ and Δ is a half twist (Section 2.2). See Figure 4. We will prove that dilatations of palindromic braids have the following lower bound.

Theorem C. *If $b \in B_n$ is palindromic and pseudo-Anosov for $n \geq 3$, then*

$$\lambda(b) \geq \sqrt{2 + \sqrt{5}}.$$

In contrast with palindromic braids we have the following result.

Theorem D. *Let PA_n be the set of skew-palindromic elements in B_n. We have*

$$\log \delta(PA_n) \asymp \frac{1}{n}.$$

The *hyperelliptic mapping class group* $\mathcal{H}(\Sigma_g)$ is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution $\mathcal{I} : \Sigma_g \to \Sigma_g$ as in Figure 5(1). It is shown in [16] that $\log \delta(\mathcal{H}(\Sigma_g)) \asymp 1/g$. See also [7, 15, 19] for other subgroups of $\mathrm{Mod}(\Sigma_g)$. As an application we will determine the asymptotic behavior of the minimal dilatations of the odd/even spin mapping class groups of genus g. To define these subgroups let $(\cdot, \cdot)_2$ be the mod-2 intersection form on $H_1(\Sigma_g; \mathbb{Z}_2)$. A map $\mathfrak{q} : H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ is a *quadratic form* if $\mathfrak{q}(v + w) = \mathfrak{q}(v) + \mathfrak{q}(w) + (v, w)_2$ for $v, w \in H_1(\Sigma_g; \mathbb{Z}_2)$. For a quadratic form $\mathfrak{q}$, the *spin mapping class group* $\mathrm{Mod}_g[\mathfrak{q}]$ is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of elements ϕ such that $\mathfrak{q} \circ \phi_* = \mathfrak{q}$. To define the two quadratic forms $\mathfrak{q}_0$ and $\mathfrak{q}_1$ we choose a basis $\{x_1, y_1, \dots, x_g, y_g\}$ of $H_1(\Sigma_g; \mathbb{Z}_2)$ as in Figure 5(2). Let $\mathfrak{q}_0$ be the quadratic form such that $\mathfrak{q}_0(x_i) = \mathfrak{q}_0(y_i) = 0$ for $1 \leq i \leq g$. Let $\mathfrak{q}_1$ be the quadratic form such that $\mathfrak{q}_1(x_1) = \mathfrak{q}_1(y_1) = 1$ and $\mathfrak{q}_1(x_i) = \mathfrak{q}_1(y_i) = 0$ for $2 \leq i \leq g$. A result of Dye [5] tells us that $\mathrm{Mod}_g[\mathfrak{q}]$ for any $\mathfrak{q}$ is conjugate to either $\mathrm{Mod}_g[\mathfrak{q}_0]$ or $\mathrm{Mod}_g[\mathfrak{q}_1]$ in $\mathrm{Mod}(\Sigma_g)$. We call $\mathrm{Mod}_g[\mathfrak{q}_0]$ and $\mathrm{Mod}_g[\mathfrak{q}_1]$ the *even spin* and *odd spin mapping class group* respectively. It is known that $\mathrm{Mod}_g[\mathfrak{q}_1]$ attains the minimum index for a proper subgroup of $\mathrm{Mod}(\Sigma_g)$ and $\mathrm{Mod}_g[\mathfrak{q}_0]$ attains the secondary minimum, see Berrick-Gebhardt-Paris [2].

Theorem E. *We have*

(1) $\log \delta(\mathrm{Mod}_g[\mathfrak{q}_1] \cap \mathcal{H}(\Sigma_g)) \asymp \dfrac{1}{g}$ *and*

(2) $\log \delta(\mathrm{Mod}_g[\mathfrak{q}_0] \cap \mathcal{H}(\Sigma_g)) \asymp \dfrac{1}{g}$.

In particular $\log \delta(\mathrm{Mod}_g[\mathfrak{q}]) \asymp 1/g$ *for each quadratic form* $\mathfrak{q}$.

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2. Preliminaries

2.1. Links. Let L be a link in the 3-sphere S^3. Let $\mathcal{N}(L)$ denote a tubular neighborhood of L and let $\mathcal{E}(L)$ denote the exterior of L, i.e. $\mathcal{E}(L) = S^3 \setminus \mathrm{int}(\mathcal{N}(L))$.

Oriented links L and L' in S^3 are *equivalent*, denoted by $L \sim L'$ if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ such that $f(L) = L'$ with respect to the orientations of the links. Furthermore for components K_i of L and K'_i of L' with $i = 1, \dots, m$ if f satisfies $f(K_i) = K'_i$ for each i, then $(L, K_1, \dots, K_m)$ and $(L', K'_1, \dots, K'_m)$ are *equivalent* and we write

$$(L, K_1, \dots, K_m) \sim (L', K'_1, \dots, K'_m).$$

2.2. Braid groups B_n and spherical braid groups SB_n. Let us set

$$\delta_j = \sigma_1 \sigma_2 \cdots \sigma_{j-1} \quad \text{and} \quad \rho_j = \sigma_1 \sigma_2 \cdots \sigma_{j-2} \sigma_{j-1}^2.$$

The half twist Δ_j is given by

$$\Delta_j = \delta_j \delta_{j-1} \cdots \delta_2.$$

We often omit the subscript n in Δ_n, δ_n and ρ_n when they are precisely n-braids.

We put indices $1, 2, \ldots, n$ from left to right on the bottoms of strands, and give an orientation of strands from the bottom to the top (Figure 1). The closure $\mathrm{cl}(b)$ is oriented by the strands. We think of $\mathrm{br}(b) = \mathrm{cl}(b) \cup A$ as an oriented link in S^3 choosing an orientation of $A = A_b$ arbitrarily. (In Section 3 we assign an orientation of the braid axis for *i-monotonic braids*).

If two braids are conjugate to each other, then their braided links are equivalent. Morton proved that the converse holds if their axises are preserved.

Theorem 2.1 (Morton [26])**.** *If* $(\mathrm{br}(b), A_b)$ *is equivalent to* $(\mathrm{br}(c), A_c)$ *for braids* $b, c \in B_n$*, then* b *and* c *are conjugate in* B_n*.*

Let us turn to the spherical braid group SB_n with n strands. We also denote by σ_i, the element of SB_n as shown in Figure 1(1). The group SB_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$. For a braid $b \in B_n$ represented by a word of letters $\sigma_j^{\pm 1}$, let $S(b)$ denote the element in SB_n represented by the same word as b.

For a braid b in B_n or SB_n the *degree* of b means the number n of the strands, denoted by $d(b)$.

2.3. Mapping classes and mapping tori from braids. Let D_n be the n-punctured disk. Consider the mapping class group $\mathrm{Mod}(D_n)$, the group of isotopy classes of orientation preserving self-homeomorphisms on D_n preserving the boundary ∂D of the disk setwise. We have a surjective homomorphism

$$\Gamma : B_n \to \mathrm{Mod}(D_n)$$

which sends each generator σ_i to the right-handed half twist $\mathfrak{t}_i$ between the ith and $(i+1)$st punctures. The kernel of Γ is an infinite cyclic group generated by the full twist Δ^2.

Collapsing ∂D to a puncture in the sphere we have a homomorphism

$$\mathfrak{c} : \mathrm{Mod}(D_n) \to \mathrm{Mod}(\Sigma_{0,n+1}).$$

We say that $b \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if $f_b := \mathfrak{c}(\Gamma(b))$ is of the corresponding Nielsen-Thurston type. The braids $\delta, \rho \in B_n$ are periodic since some power of each braid is the full twist: $\Delta^2 = \delta^n = \rho^{n-1} \in B_n$.

We also have a surjective homomorphism

$$\widehat{\Gamma} : SB_n \to \mathrm{Mod}(\Sigma_{0,n})$$

sending each generator σ_i to the right-handed half twist $\mathfrak{t}_i$. We say that $\eta \in SB_n$ is *pseudo-Anosov* if $\widehat{\Gamma}(\eta) \in \mathrm{Mod}(\Sigma_{0,n})$ is pseudo-Anosov. In this case $\lambda(\eta)$ is defined by the dilatation of $\widehat{\Gamma}(\eta)$.

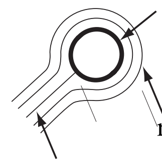


FIGURE 6. Stable foliation which is 1-pronged at a boundary component.

2.4. Stable foliations $\mathcal{F}_b$ for pseudo-Anosov braids b. Recall the surjective homomorphism $\pi : B_n \to \mathcal{S}_n$. We write $\pi_b = \pi(b)$ for $b \in B_n$. Consider a pseudo-Anosov braid $b \in B_n$ with $\pi_b(i) = i$. Removing the ith strand $b(i)$ from b, we get a braid $b - b(i) \in B_{n-1}$. Taking its spherical element, we have $S(b - b(i)) \in SB_{n-1}$. Note that $b - b(i)$ and $S(b - b(i))$ are not necessarily pseudo-Anosov. A well-known criterion uses the stable foliation $\mathcal{F}_b$ for the monodromy $\phi_b : F_b \to F_b$ of a fibration on $M_b \to S^1$ as we recall now. Such a fibration on M_b extends naturally to a fibration on the manifold obtained from M_b by Dehn filling a cusp along the boundary slope of the fiber F_b which lies on the torus $\partial\mathcal{N}(\mathrm{cl}(b(i)))$. Also ϕ_b extends to the monodromy defined on $F_b^\bullet$ of the extended fibration, where $F_b^\bullet$ is obtained from F_b by filling in the boundary component of F_b which lies on $\partial\mathcal{N}(\mathrm{cl}(b(i)))$ with a disk. Then $b - b(i)$ is the corresponding braid for the extended monodromy defined on $F_b^\bullet$. Suppose that $\mathcal{F}_b$ is not 1-pronged at the boundary component in question. (See Figure 6 in the case where F_b is 1-pronged at a boundary component.) Then $\mathcal{F}_b$ extends to the stable foliation for $b - b(i)$, and hence $b - b(i)$ is pseudo-Anosov with the same dilatation as b. Furthermore if $\mathcal{F}_b$ is not 1-pronged at the boundary component of F_b which lies on $\partial\mathcal{N}(A)$, then $S(b - b(i))$ is still pseudo-Anosov with the same dilatation as b.

2.5. Thurston norm. Let M be a 3-manifold with boundary (possibly $\partial M = \emptyset$). If M is hyperbolic, i.e. the interior of M possess a complete hyperbolic structure of finite volume, then there is a norm $\|\cdot\|$ on $H_2(M, \partial M; \mathbb{R})$, now called the Thurston norm [29]. The norm $\|\cdot\|$ has the property such that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$, $\|a\| = \min_S\{-\chi(S)\}$, where the minimum is taken over all oriented surface S embedded in M with $a = [S]$ and with no components of non-negative Euler characteristic. The surface S realizing this minimum is called a *norm-minimizing surface* of a.

Theorem 2.2 (Thurston [29])**.** *The norm $\|\cdot\|$ on $H_2(M, \partial M; \mathbb{R})$ has the following properties.*

(1) *There are a set of maximal open cones $\mathcal{C}_1, \cdots, \mathcal{C}_k$ in $H_2(M, \partial M; \mathbb{R})$ and a bijection between the set of isotopy classes of connected fibers of fibrations $M \to S^1$ and the set of primitive integral classes in the union $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$.*

(2) *The restriction of* $\|\cdot\|$ *to* $\mathcal{C}_j$ *is linear for each* j.
(3) *If we let* F_a *be a fiber of a fibration* $M \to S^1$ *associated with a primitive integral class* a *in each* $\mathcal{C}_j$, *then* $\|a\| = -\chi(F_a)$.

We call the open cones $\mathcal{C}_j$ *fibered cones* and call integral classes in $\mathcal{C}_j$ *fibered classes.*

Theorem 2.3 (Fried [11])**.** *For a fibered cone* $\mathcal{C}$ *of a hyperbolic* 3*-manifold* M, *there is a continuous function* $\mathrm{ent} : \mathcal{C} \to \mathbb{R}$ *with the following properties.*

(1) *For the monodromy* $\phi_a : F_a \to F_a$ *of a fibration* $M \to S^1$ *associated with a primitive integral class* $a \in \mathcal{C}$, *we have* $\mathrm{ent}(a) = \log(\lambda(\phi_a))$.
(2) $\mathrm{Ent} = \|\cdot\|\mathrm{ent} : \mathcal{C} \to \mathbb{R}$ *is a continuous function which becomes constant on each ray through the origin.*
(3) *If a sequence* $\{a_n\} \subset \mathcal{C}$ *tends to a point* $\neq 0$ *in the boundary* $\partial\mathcal{C}$ *as* n *tends to* ∞, *then* $\mathrm{ent}(a_n) \to \infty$. *In particular* $\mathrm{Ent}(a_n) = \|a_n\|\mathrm{ent}(a_n) \to \infty$.

We call $\mathrm{ent}(a)$ and $\mathrm{Ent}(a)$ the *entropy* and *normalized entropy* of the class $a \in \mathcal{C}$.

For a pseudo-Anosov element $\phi \in \mathrm{Mod}(\Sigma)$ we consider the mapping torus M_ϕ. The vector field $\frac{\partial}{\partial t}$ on $\Sigma \times \mathbb{R}$ induces a flow ϕ^t on M_ϕ called the *suspension flow.*

Theorem 2.4 (Fried [10])**.** *Let* ϕ *be a pseudo-Anosov mapping class defined on* Σ *with stable and unstable foliations* $\mathcal{F}^s$ *and* $\mathcal{F}^u$. *Let* $\widehat{\mathcal{F}^s}$ *and* $\widehat{\mathcal{F}^u}$ *denote the suspensions of* $\mathcal{F}^s$ *and* $\mathcal{F}^u$ *by* ϕ. *If* $\mathcal{C}$ *is a fibered cone containing the fibered class* $[\Sigma]$, *then we can modify a norm-minimizing surface* F_a *associated with each primitive integral class* $a \in \mathcal{C}$ *by an isotopy on* M_ϕ *with the following properties.*

(1) F_a *is transverse to the suspension flow* ϕ^t, *and the first return map* $\phi_a : F_a \to F_a$ *is precisely the pseudo-Anosov monodromy of the fibration on* $M_\phi \to S^1$ *associated with* a. *Moreover* F_a *is unique up to isotopy along flow lines.*
(2) *The stable and unstable foliations for* ϕ_a *are given by* $\widehat{\mathcal{F}^s} \cap F_a$ *and* $\widehat{\mathcal{F}^u} \cap F_a$.

2.6. Disk twist. Let L be a link in S^3. Suppose an unknot K is a component of L. Then the exterior $\mathcal{E}(K)$ (resp. $\partial\mathcal{E}(K)$) is a solid torus (resp. torus). We take a disk D bounded by the longitude of a tubular neighborhood $\mathcal{N}(K)$ of K. We define a mapping class T_D defined on $\mathcal{E}(K)$ as follows. We cut $\mathcal{E}(K)$ along D. We have resulting two sides obtained from D, and reglue two sides by twisting either of the sides 360 degrees so that the mapping class defined on $\partial\mathcal{E}(K)$ is the right-handed Dehn twist about ∂D. Such a mapping class on $\mathcal{E}(K)$ is called the *disk twist about* D. For simplicity we also call a self-homeomorphism representing the mapping class T_D the *disk twist* about D, and denote it by the same notation

$$T_D : \mathcal{E}(K) \to \mathcal{E}(K).$$

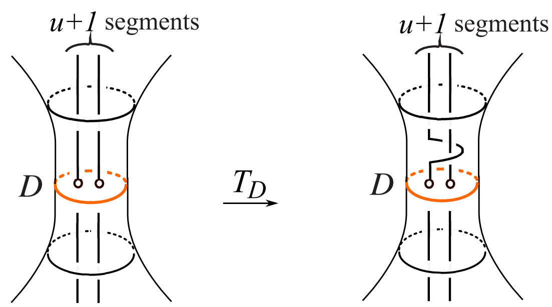


FIGURE 7. Disk twist T_D.

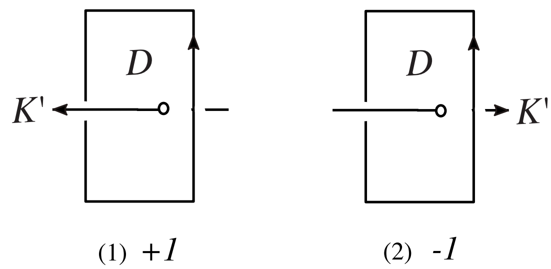


FIGURE 8. Sign of the point of intersection: $+1$ in (1) and -1 in (2).

Clearly T_D equals the identity map outside a neighborhood of D in $\mathcal{E}(K)$. We observe that if $u+1$ segments of $L-K$ pass through D for $u \geq 1$, then $T_D(L-K)$ is obtained from $L-K$ by adding the full twist near D. In the case $u=1$, see Figure 7. We may assume that T_D fixes one of these segments, since any point in D becomes the center of the twisting about D.

For any integer ℓ, consider a homeomorphism

$$T_D^\ell : \mathcal{E}(K) \to \mathcal{E}(K).$$

Observe that T_D^ℓ converts L into a link $K \cup T_D^\ell(L-K)$ such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T_D^\ell(L-K))$. Then T_D^ℓ induces a homeomorphism between the exteriors of links

$$h_{D,\ell} : \mathcal{E}(L) \to \mathcal{E}(K \cup T_D^\ell(L-K)). \tag{2.1}$$

We use the homeomorphism in (2.1) in later section.

3. i-increasing braids and Theorem 3.2

Definitions of i-increasing braids, signs and intersection numbers. Let L be an oriented link in S^3 with a trivial component K. We take an oriented disk D bounded by the longitude of $\mathcal{N}(K)$ so that the orientation of D agrees with the orientation of K. For each component K' of $L-K$ such that D and K' intersect transversally with $D \cap K' \neq \emptyset$, we assign each point of intersection $+1$ or -1 as shown in Figure 8.

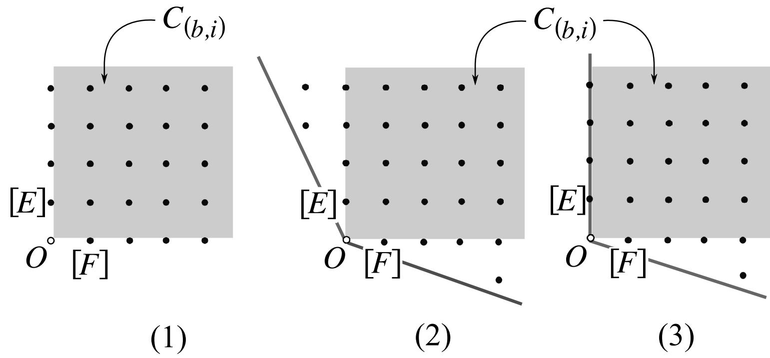


FIGURE 9. $F := F_b$ and $E := E_{(b,i)}$. (1) Subcone $C_{(b,i)}$. (2)(3) Possible shapes of $\mathcal{C} \cap \{x[F]+y[E] \mid x, y \in \mathbb{R}\}$. In case (2), $[E] \in \mathcal{C}$. In case (3), $[E] \notin \mathcal{C}$.

Let b be a braid with $\pi_b(i) = i$. We consider an oriented disk $D = D_{(b,i)}$ bounded by the longitude ℓ_i of $\mathcal{N}(\mathrm{cl}(b(i)))$. Such a disk D is unique up to isotopy on $\mathcal{E}(\mathrm{cl}(b(i)))$. We say that a braid $b \in B_n$ with $\pi_b(i) = i$ is *i-increasing* (resp. *i-decreasing*) if there is a disk $D = D_{(b,i)}$ as above with the following conditions.

(D1) There is at least one component K' of $\mathrm{cl}(b - b(i))$ such that $D \cap K' \neq \emptyset$.

(D2) Each component of $\mathrm{cl}(b - b(i))$ and D intersect with each other transversally, and every point of intersection has the sign $+1$ (resp. -1).

We set $\epsilon(b, i) = 1$ (resp. $\epsilon(b, i) = -1$), and call it the *sign* of the pair (b, i). We also call D the *associated disk* of the pair (b, i). We say that b is *i-monotonic* if b is i-increasing or i-decreasing. Then we set

$$I(b, i) = D \cap \mathrm{cl}(b - b(i))$$

and let $u(b, i) \geq 1$ be the cardinality of $I(b, i)$. We call $u(b, i)$ the *intersection number* of the pair (b, i). If the pair (b, i) is specified, then we simply denote $\epsilon(b, i)$ and $u(b, i)$ by ϵ and u respectively. For example $\sigma_1^2\sigma_2^{-1}$ is 1-increasing with $u(\sigma_1^2\sigma_2^{-1}, 1) = 1$.

A braid b is *positive* if b is represented by a word in letters σ_j, but not σ_j^{-1}. A braid b is irreducible if the Nielsen-Thurston type of b is not reducible.

Lemma 3.1. *Let b be a positive braid with $\pi_b(i) = i$. Then b is i-increasing if b is irreducible.*

Proof. Suppose that a positive braid b with $\pi_b(i) = i$ is irreducible. Since b is positive, there is a disk $D = D_{(b,i)}$ with the condition $(D2)$. Assume that D fails in $(D1)$. Let ∂D_n be the boundary of the disk D_n containing n punctures. Consider a neighborhood of $\partial D_n \cup (D_n \cap D)$ in D_n which is an annulus. One of the boundary components of this annulus is an essential simple closed curve in D_n preserved by $\Gamma(b) \in \mathrm{Mod}(D_n)$. This means that b is reducible, a contradiction. Thus D satisfies $(D1)$, and b is i-increasing. $\square$

Orientation of the axis *A*. Let b be i-monotonic with $\epsilon(b,i) = \epsilon$ and $u(b,i) = u$. Consider the braided link $\mathrm{br}(b) = \mathrm{cl}(b) \cup A$. The associated disk D has a unique point of intersection with A, and the cardinality of $I(b,i) \cup (D \cap A)$ is $u(b,i)+1$. To deal with $\mathrm{br}(b) = \mathrm{cl}(b) \cup A$ as an oriented link, we consider an orientation of $\mathrm{cl}(b)$ as we described before, and assign an orientation of A so that the sign of the intersection between D and A coincides with $\epsilon(b,i)$. See Figure 2(2).

Recall that $M_b = \mathcal{E}(\mathrm{br}(b))$ is the exterior of $\mathrm{br}(b)$ which is a surface bundle over S^1. We consider an orientation of the F-surface F_b which agrees with the orientation of A.

***E*-surface.** We now define an oriented surface $E_{(b,i)}$ of genus 0 embedded in M_b. Consider small $u(b,i)+1$ disks in the oriented disk $D = D_{(b,i)}$ whose centers are points of $I(b,i) \cup (D \cap A)$. Then $E_{(b,i)}$ is a sphere with $u(b,i)+2$ boundary components obtained from D by removing the interiors of those small disks. We choose the orientation of $E_{(b,i)}$ so that it agrees with the orientation of D. We call $E_{(b,i)}$ the *E-surface* for b. For example, the 1-increasing braid $\sigma_1^2\sigma_2^{-1}$ has the E-surface $E_{(\sigma_1^2\sigma_2^{-1},1)}$ homeomorphic to a 3-holed sphere.

Subcone $C_{(b,i)}$. Consider the 2-dimensional subcone of $H_2(M_b, \partial M_b; \mathbb{R})$ spanned by $[F_b]$ and $[E_{(b,i)}]$ (Figure 9):

$$C_{(b,i)} = \{x[F_b] + y[E_{(b,i)}] \mid x > 0,\ y > 0\}.$$

Let $\overline{C_{(b,i)}}$ denote the closure of $C_{(b,i)}$. We write $(x,y) = x[F_b] + y[E_{(b,i)}]$. We prove the following theorem in Section 4.

Theorem 3.2. *For a pseudo-Anosov, i-increasing braid b with $u(b,i) = u$, let $\mathcal{C}$ be the fibered cone containing $[F_b]$. We have the following.*

1. $C_{(b,i)} \subset \mathcal{C}$.
2. *The fiber $F_{(x,y)}$ for each primitive integral class $(x,y) \in C_{(b,i)}$ has genus* 0.
3. *The monodromy $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ for each primitive integral class $(x,y) \in C_{(b,i)}$ is conjugate to*

$$(\omega_1\psi)\cdots(\omega_{u-1}\psi)(\omega_u\psi)\psi^{m-1} : F_{(x,y)} \to F_{(x,y)},$$

where $m \geq 1$ depends on (x,y), ψ is periodic and each ω_j is reducible. Moreover there are homeomorphisms $\widehat{\omega}_j : S_0 \to S_0$ for $j = 1,\ldots,u$ on a surface S_0 determined by b and an embedding $h : S_0 \hookrightarrow F_{(x,y)}$ such that the subsurface $h(S_0)$ of $F_{(x,y)}$ is the support of each w_j and

$$w_j|_{h(S_0)} = h \circ \widehat{\omega}_j \circ h^{-1}.$$

The conclusion of Theorem 3.2 holds for i-decreasing braids as well. We now claim that Theorem 3.2 implies Theorem B.

Proof of Theorem B. Suppose that Theorem 3.2 holds. Let $b \in B_n$ be a pseudo-Anosov braid such that $\pi_b(i) = i$. We consider the braid $b\Delta^{2k} \in B_n$ for $k \geq 1$. The full twist Δ^2 is an element in the center $Z(B_n)$ and $\Delta^2 = \sigma_j P_j$ holds for each $1 \leq j \leq n-1$, where P_j is positive. Such properties imply that $b\Delta^{2k}$ is positive for k large. We fix such large k. Since $\Gamma(b) = \Gamma(b\Delta^{2k})$ in $\mathrm{Mod}(D_n)$, the braid $b\Delta^{2k}$ is certainly pseudo-Anosov. Hence it is i-increasing by Lemma 3.1. One can apply Theorem 3.2 for this braid, and obtains the subcone $C_{(b\Delta^{2k},i)}$. Consider the kth power of the disk twist about the disk D_A bounded by the longitude of $\mathcal{N}(A)$:

$$T_{D_A}^k : \mathcal{E}(A) \to \mathcal{E}(A).$$

Since $A \cup T_{D_A}^k(\mathrm{cl}(b)) = A \cup \mathrm{cl}(b\Delta^{2k}) = \mathrm{br}(b\Delta^{2k})$, we have $S^3 \setminus \mathrm{br}(b) \simeq S^3 \setminus \mathrm{br}(b\Delta^{2k})$. Let us set

$$f_k := h_{D_A,k} : M_b \to M_{b\Delta^{2k}},$$

where $h_{D_A,k}$ is the homeomorphism in (2.1). The isomorphism

$$f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}})$$

sends $[F_b]$ to $[F_{b\Delta^{2k}}]$. (Here we note that the above k is suppose to be large, but the homeomorphism f_k makes sense for all integer k.) The pullback of the subcone $C_{(b\Delta^{2k},i)}$ into $H_2(M_b, \partial M_b)$ is a desired subcone contained in $\mathcal{C}$. $\square$

Remark 3.3. *If $F_{(x,y)}$ is a $(d+1)$-holed sphere, then the periodic homeomorphism $\psi : F_{(x,y)} \to F_{(x,y)}$ in Theorem* 3.2 *is determined by the periodic braid $\rho = \sigma_1\sigma_2 \ldots \sigma_{d-2}\sigma_{d-1}^2 \in B_d$. See the proof of Theorem* 3.2*(3) in Section* 4.3.

4. Proof of Theorem 3.2

We fix integers $n \geq 3$ and $1 \leq i \leq n$. Throughout Section 4, we assume that $b \in B_n$ is pseudo-Anosov and i-increasing with $u(b,i) = u$. We now choose an associated disk about the pair (b,i) suitably. Let $\mathbb{D}$ denote the unit disk with the center $(0,0)$ in the plane $\mathbb{R}^2$. Let $J = (-1,1) \times \{0\} \subset \mathbb{D}$ be the interval and let $A_0 = (-2,0)$ be a point in $\mathbb{R}^2$. We denote by $\mathbb{D}_n$, the disk $\mathbb{D}$ with equally spaced n points in J. Let us denote these n points by $A_1, \ldots, A_n$ from left to right. We take a point $Q_i \neq A_i \in J$ between A_{i-1} and A_i so that the Euclidean distance $d(Q_i, A_i)$ is sufficiently small (e.g. $d(Q_i, A_i) < \frac{1}{n+1}$). Let r_i denote the closed interval in $[-2,1] \times \{0\}$ with endpoints A_0 and Q_i. (Figure 10(1).) We regard b as a braid contained in the cylinder $\mathbb{D} \times [0,1] \subset \mathbb{R}^3$ and b is based at n points $A_1 \times \{0\}, \ldots, A_n \times \{0\}$. Since $\pi_b(i) = i$, one can take a representative of b such that $b(i)$ is an interval in the cylinder:

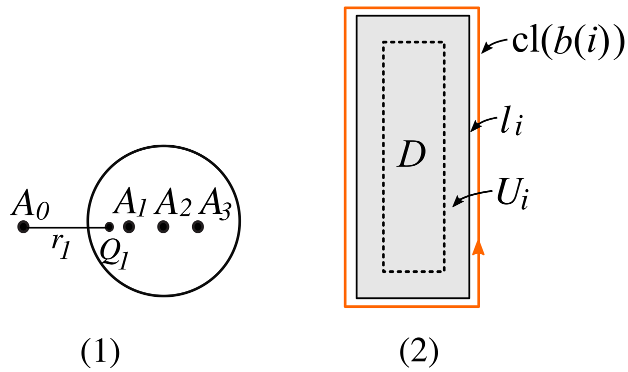


FIGURE 10. (1) $A_0, \dots, A_n$, Q_i, r_i when $n = 3$, $i = 1$. (2) $\partial D = \ell_i$ is a union of four segments. U_i is an annulus in the figure.

$\diamondsuit$1. $b(i) = \displaystyle\bigcup_{0\le t\le 1} A_i \times \{t\}$.

Furthermore we may assume that $\partial D (= \ell_i)$ of an associated disk D of (b, i) is a union of the following four segments as a set (Figure 10):

$\diamondsuit$2. $\big(\displaystyle\bigcup_{-1\le t\le 2} A_0 \times \{t\}\big) \cup \big(r_i \times \{-1\}\big) \cup \big(\displaystyle\bigcup_{-1\le t\le 2} Q_i \times \{t\}\big) \cup \big(r_i \times \{2\}\big)$.

Preserving $\diamondsuit 1, 2$ we may further assume the following (Figures 10(2), 11(1)):

$\diamondsuit$3. For a regular neighborhood U_i of ℓ_i in D, we have $I(b, i) \subset U_i$.

This is because every point $x \in D \cap K'$, where K' is a component of $\mathrm{cl}(b - b(i))$, one can slide x along K' so that the resulting point on K' is in U_i. Said differently, preserving ∂D pointwise, we can modify a small neighborhood of D near K' so that the resulting associated disk satisfies $\diamondsuit 3$.

Under the conditions $\diamondsuit 1, 2, 3$ we have the following. For each $x \in D \cap K' \subset U_i$, there is a segment $h' \subset K'$ through x such that h' passes over $b(i)$ since b is i-increasing. See Figure 11(1). Such a local picture of $\mathrm{cl}(b)$ is used in the the next section. Hereafter we assume that associated disks possess conditions $\diamondsuit 1, 2, 3$.

4.1. Proof of Theorem 3.2(1). Let s be the open segment (1-dimensional simplex) in $H_2(M_b, \partial M_b; \mathbb{R})$ with the endpoints $\frac{n-1}{u}[E_{(b,i)}] = (0, \frac{n-1}{u})$ and $[F_b] = (1, 0)$:

$$s = \{(x, y) \in C_{(b,i)} \mid y = -\frac{n-1}{u}x + \frac{n-1}{u},\ 0 < x < 1\}. \tag{4.1}$$

The ray of each point in $C_{(b,i)}$ through the origin intersects with s. Thus for the proof of (1), it suffices to prove that $s \subset \mathcal{C}$.

We now introduce a sequence of braided links $\{\mathrm{br}(b_p)\}_{p=1}^{\infty}$ from an i-increasing braid $b \in B_n$ such that $M_{b_p} \simeq M_b$ for each $p \ge 1$. (We use the

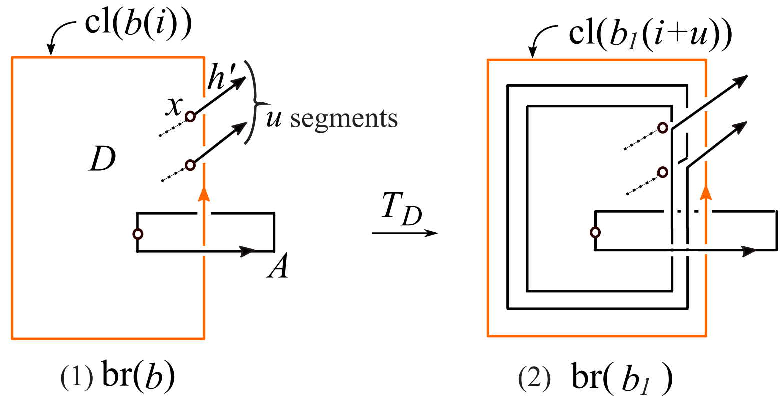


FIGURE 11. Case: b is i-increasing. (1) Associated disk D with conditions $\diamondsuit$ 1,2,3. (2) $\mathrm{br}(b_1)$. Circles $\circ$ indicate points of intersection between D and components of $\mathrm{br}(b - b(i))$. See also Figure 12.

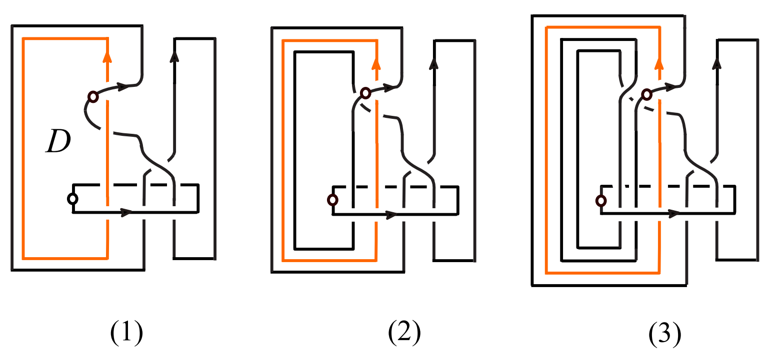


FIGURE 12. Braided links for (1) 1-increasing $\sigma_1^2\sigma_2^{-1}$, (2) 2-increasing $(\sigma_1^2\sigma_2^{-1})_1$ and (3) 3-increasing $(\sigma_1^2\sigma_2^{-1})_2$.

1-increasing braid $\sigma_1^2\sigma_2^{-1} \in B_3$ to illustrate the idea.) Let D be an associated disk of the pair (b, i). We take a disk twist

$$T_D : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$$

so that the point of intersection $D \cap A$ becomes the center of the twisting about D, i.e. $T_D(D \cap A) = D \cap A$. We may assume that $T_D(A) = A$ as a set. Figure 11 illustrates the image of the segment h' under T_D. The condition $\diamondsuit 3$ ensures that T_D equals the identity map outside a neighborhood of U_i in $\mathcal{E}(\mathrm{cl}(b(i)))$. Then by $\diamondsuit 1, 2$, it follows that

$$T_D(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i))$$

is a braided link of some $(i+u)$-increasing braid with $(n+u)$ strands. We define $b_1 \in B_{n+u}$ to be such a braid. The trivial knot $T_D(A)(= A)$ becomes a braid axis of b_1. By definition of the disk twist, we have $M_{b_1} \simeq M_b$. See Figure 12 for $\mathrm{br}((\sigma_1^2\sigma_2^{-1})_1)$.

As discussed below, there is some ambiguity in defining b_1. As we will see, the ambiguity is irrelevant for the study of pseudo-Anosov monodromies defined on fibers of fibrations on the mapping torus. Suppose that both D and D' are the associated disks of the pair (b, i) with conditions $\diamondsuit 1, 2, 3$. We consider the disk twists T_D and $T_{D'}$ with the above condition, i.e. both $D \cap A$ and $D' \cap A$ become the center of the twisting about D and D' respectively. Observe that the resulting two links obtained from D and D' are equivalent:

$$T_D(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i)) \sim T_{D'}(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i)).$$

They are braided links, say $\mathrm{br}(b_1)$ and $\mathrm{br}(b_1')$ of some braids $b_1, b_1' \in B_{n+u}$ respectively with the same axis $T_D(A) = A = T_{D'}(A)$. This means that a more stronger claim holds:

$$(\mathrm{br}(b_1), A) \sim (\mathrm{br}(b_1'), A).$$

Thus b_1 and b_1' are conjugate in B_{n+u} by Theorem 2.1. In particular both b_1 and b_1' are pseudo-Anosov (since the initial braid b is pseudo-Anosov and M_b is hyperbolic) and they have the same dilatation.

To define b_p for $p \geq 1$, we consider the pth power

$$T_D^p : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$$

using the above T_D. As in the case of $p = 1$,

$$T_D^p(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i))$$

is a braided link of some $(i + pu)$-increasing braid with $(n + pu)$ strands. We define $b_p \in B_{n+pu}$ to be such a braid. Then $M_{b_p} \simeq M_b$. As in the case of $p = 1$, such a braid b_p is well-defined up to conjugate. We say that b_p is *obtained from b by the disk twist*. Clearly $u(b_p, i + pu) = u(b, i)$ for $p \geq 1$. See Figure 12.

Let us set

$$g_p := h_{D,p} : M_b \to M_{b_p},$$

where $h_{D,p}$ is the homeomorphism in (2.1). We consider the isomorphism

$$g_{p*} : H_2(M_b, \partial M_b) \to H_2(M_{b_p}, \partial M_{b_p}).$$

Lemma 4.1. *For each integer $p \geq 1$, g_{p*} sends $(0,1) \in \overline{C_{(b,i)}}$ to $(0,1) \in \overline{C_{(b_p,i+pu)}}$, and sends $(1,p) \in \overline{C_{(b,i)}}$ to $(1,0) \in \overline{C_{(b_p,i+pu)}}$. In particular for integers $x, y \geq 1$ with $y = xp + r$ for $0 \leq r < p$, g_{p*} sends $(x,y) \in \overline{C_{(b,i)}}$ to $(x,r) \in \overline{C_{(b_p,i+pu)}}$.*

Proof. We consider the *oriented sum* $F_{(x,y)} := xF_b + yE_{(b,i)}$. This is an oriented surface embedded in M_b, and is obtained from the cut and past construction of parallel x copies of F_b and parallel y copies of $E_{(b,i)}$. The orientation of $F_{(x,y)}$ agrees with those of F_b and $E_{(b,i)}$. We have $[F_{(x,y)}] = (x,y) \in C_{(b,i)}$. Then g_p sends $E_{(b,i)}$ to $E_{(b_p,i+pu)}$, and sends $F_{(1,p)}$ to F_{b_p}.

Thus g_{p_*} sends $(0,1)$ to $(0,1)$, and sends $(1,p)$ to $(1,0)$. This completes the proof. $\square$

By the proof of Lemma 4.1, g_1 sends $F_{(1,1)} = F_b + E_{(b,i)}$ to the fiber F_{b_1} of a fibration on M_b associated with $(1,1) \in C_{(b,i)}$. Since the fibers $F_{(1,1)}$ and F_b are norm-minimizing, $E_{(b,i)}$ is also norm-minimizing.

Proof of Theorem 3.2(1). We have $\|[F_b]\| = n-1$ and $\|[F_{b_p}]\| = n+pu-1$ since F_b and F_{b_p} are fibers, and $\|[E_{(b,i)}]\| = u$ since $E_{(b,i)}$ is norm-minimizing. By Lemma 4.1, $[F_{b_p}] = (1,p) \in C_{(b,i)}$. Consider the rational class

$$c_p := \frac{n-1}{n+pu-1}[F_{b_p}] = \Big(\frac{n-1}{n+pu-1}, \frac{p(n-1)}{n+pu-1}\Big).$$

The classes c_p that are all projectively fibered, and they lie on the 1-dimensional linear simplex s given by (4.1). Note that the closure of s contains $[F_b]$. Moreover, the Thurston norm of all c_p equals that of $[F_b]$ (and it is $n-1$). This is only possible if the simplex s is projectively contained in a single fibered face. The corresponding fibered cone has to contain $[F_b]$ from the above discussion, and hence it is $\mathcal{C}$. Thus $s \subset \mathcal{C}$. This completes the proof. $\square$

Remark 4.2. *From the proof of Theorem* 3.2*(1), one sees the following: If* $[E_{(b,i)}] \in \overline{C_{(b,i)}}$ *is a fibered class, then* $[E_{(b,i)}] \in \mathcal{C}$. *Otherwise* $[E_{(b,i)}] \in \partial\mathcal{C}$. *See Figure* 9*(2)(3).*

4.2. Proof of Theorem 3.2(2). We start with a simple observation: $\Delta^2 \in B_n$ is j-increasing for each $1 \le j \le n$, and $u(\Delta^2, j) = n-1$ holds. The following lemma is immediate.

Lemma 4.3. *If* $b \in B_n$ *is* i*-increasing, then* $b\Delta^2 \in B_n$ *is* i*-increasing with* $u(b\Delta^2, i) = u(b,i) + n - 1$.

We explain the idea of Theorem 3.2(2). Let D be the associated disk of the pair (b,i). We have two types of the disk twist. One is $T^k_{D_A} : \mathcal{E}(A) \to \mathcal{E}(A)$ which appears in the proof of Theorem B in Section 3 and the other is $T^p_D : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$. If k and p are positive, then we obtain the i-increasing $b\Delta^{2k}$ from the former type $T^k_{D_A}$, and another increasing braid b_p from the latter type T^p_D. Since both resulting braids are increasing, we can further apply two types of the disk twist for the resulting braid. This is a key of the proof. Choosing two types of the disk twist alternatively, we get a sequence of increasing and pseudo-Anosov braids (since the initial braid b is pseudo-Anosov). We shall see that the desired monodromies associated with primitive classes in $C_{(b,i)}$ are given by these braids.

Let $p_1, \dots, p_j$ be integers such that $p_1 \ge 0$ and $p_2, \dots, p_j \ge 1$. Given an i-increasing braid $b \in B_n$ with $u(b,i) = u$, we define an integer $i[p_1, \dots, p_j] \ge 1$ and an $i[p_1, \dots, p_j]$-increasing braid $b[p_1, \dots, p_j]$ inductively as follows.

- If $j = 1$ and $p_1 = 0$, then $i[0] = i$ and $b[0] = b$. If $j = 1$ and $p_1 = p \geq 1$, then $i[p] = i + pu$ and $b[p] = b_p$.
- If $j > 1$ is even, then
$$\begin{aligned} i[p_1, \ldots, p_{j-1}, p_j] &= i[p_1, \ldots, p_{j-1}], \\ b[p_1, \ldots, p_{j-1}, p_j] &= \big(b[p_1, \ldots, p_{j-1}]\big)\Delta^{2p_j}. \end{aligned}$$
The right-hand side is $i[p_1, \ldots, p_{j-1}]$-increasing by Lemma 4.3.
- If $j > 1$ is odd, then
$$\begin{aligned} i[p_1, \ldots, p_{j-1}, p_j] &= i[p_1, \ldots, p_{j-1}] + p_j u\big(b[p_1, \ldots, p_{j-1}], i[p_1, \ldots, p_{j-1}]\big), \\ b[p_1, \ldots, p_{j-1}, p_j] &= \big(b[p_1, \ldots, p_{j-1}]\big)_{p_j}. \end{aligned}$$

We say that $b[p_1, \ldots, p_j]$ has *length* j.

Example 4.4.

(1) *$b[p] = b_p$ by definition.*
(2) *Let $\beta = b\Delta^2$. Then $b[0, 1] = \beta$ and $b[0, 1, p] = \beta_p$.*
(3) *We have $b[0, p] = b\Delta^{2p}$ and $b[0, p, 1] = (b\Delta^{2p})_1$, where $(b\Delta^{2p})_1$ is obtained from i-increasing $b\Delta^{2p}$ by the disk twist.*

For each $k \geq 1$, let $f_k : M_b \to M_{b\Delta^{2k}}$ be the homeomorphism which in the proof of Theorem B. Consider the isomorphism $f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}})$. We have the following property.

Lemma 4.5. *For each integer $k \geq 1$, f_{k*} sends $(1, 0) \in \overline{C_{(b,i)}}$ to $(1, 0) \in \overline{C_{(b\Delta^{2k},i)}}$, and sends $(k, 1) \in \overline{C_{(b,i)}}$ to $(0, 1) \in \overline{C_{(b\Delta^{2k},i)}}$. In particular for integers $x, y \geq 1$ with $x = yk + r$ for $0 \leq r < k$, then f_{k*} sends $(x, y) \in \overline{C_{(b,i)}}$ to $(r, y) \in \overline{C_{(b\Delta^{2k},i)}}$.*

Proof. The homeomorphism f_k sends F_b to $F_{b\Delta^{2k}}$, and sends $F_{(k,1)} = kF_b + E_{(b,i)}$ to $E_{(b\Delta^{2k},i)}$. This implies that the claim holds. □

Proof of Theorem 3.2(2). Let $(x, y) \in C_{(b,i)}$ be a primitive integral class. (Hence x, y are positive integers with $\gcd(x, y) = 1$.) We consider the continued fraction of y/x by the Euclidean algorithm

$$\frac{y}{x} = p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \cdots + \cfrac{1}{p_{j-1} + \cfrac{1}{p_j}}}} := p_1 + \frac{1}{p_2 +}\frac{1}{p_3 +} \cdots + \frac{1}{p_{j-1} +}\frac{1}{p_j}$$

with length j and $p_j \geq 2$ and $p_1 = 0$ if $0 < y < x$. There is another expression

$$\frac{y}{x} = p_1 + \frac{1}{p_2 +}\frac{1}{p_3 +} \cdots + \frac{1}{p_{j-1} +}\frac{1}{(p_j - 1) +}\frac{1}{1}$$

with length $j+1$. We choose one of the two expressions with odd length ℓ:

$$\frac{y}{x} = p_1 + \frac{1}{p_2 +}\frac{1}{p_3 +} \cdots + \frac{1}{p_{\ell-1} +}\frac{1}{p_\ell}.$$

This encodes the fiber $F_{(x,y)}$ and its monodromy $\phi_{(x,y)}$. In fact Lemmas 4.1, 4.5 ensure that

$$(g_{p_\ell} f_{p_{\ell-1}} g_{p_{\ell-2}} \cdots f_{p_2} g_{p_1})_* : H_2(M_b, \partial M_b) \to H_2(M_{b[p_1,\dots,p_\ell]}, \partial M_{b[p_1,\dots,p_\ell]})$$

sends $(x,y) = [xF_b + yE_{(b,i)}]$ to $(1,0)$ which is the integral class of the F-surface of $b[p_1,\dots,p_\ell]$. ($g_{p_1} = id : M_b \to M_b$ if $p_1 = 0$.) Thus $F_{(x,y)}$ has genus 0. Moreover this means that one can take $F_{b[p_1,\dots,p_\ell]}$ as a representative of $(x,y) \in C_{(b,i)}$ and the monodromy $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ is determined by $b[p_1,\dots,p_\ell]$. This completes the proof. $\Box$

We denote by $b_{(x,y)}$ the braid $b[p_1,\dots,p_\ell]$ which determines $\phi_{(x,y)}$. Here is an example: If $(x,y) = (5,14)$, then $\frac{14}{5} = 2 + \frac{1}{1+}\frac{1}{4}$ and $\phi_{(5,14)}$ is determined by $b_{(5,14)} = b[2,1,4]$. If $(x,y) = (14,5)$, then $\frac{5}{14} = 0 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1}$ and $\phi_{(14,5)}$ is determined by $b_{(14,5)} = b[0,2,1,3,1]$.

4.3. Proof of Theorem 3.2(3).

We begin with the following lemma.

Lemma 4.6 (Standard form). *If $b \in B_n$ is i-increasing with $u(b,i) = u$, then b is conjugate to an n-increasing braid b' of the form*

$$b' = (w_1 \sigma_{n-1}^2) \cdots (w_u \sigma_{n-1}^2),$$

where each w_k is a word of $\sigma_1^{\pm 1}, \dots, \sigma_{n-2}^{\pm 1}$, but not $\sigma_{n-1}^{\pm 1}$, possibly $w_k = \emptyset$ for some k.

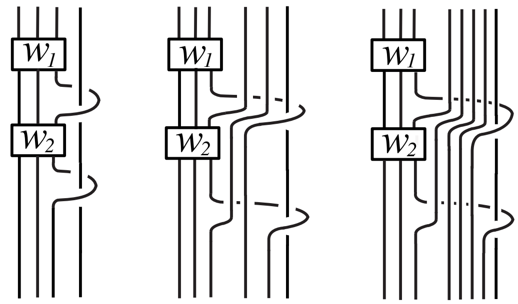
Figure 13(1) shows the form of b' in Lemma 4.6 in case $u = 2$.

Proof. We regard b as a braid in $\mathbb{D} \times [0,1]$. By $\diamondsuit 1$, $b(i)$ is an interval in $\mathbb{D} \times [0,1]$. If $i = n$, then b is n-increasing and it is not hard to see that a representative of b is of the desired form in Lemma 4.6. Suppose that b is i-increasing for $1 \le i < n$. We set $\sigma = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_i$ if $1 \le i < n-1$ and $\sigma = \sigma_{n-1}$ if $i = n-1$. We consider the n-braid $b' = \sigma b \sigma^{-1}$ which is n-increasing with $u(b',n) = u$. We pull $b'(n)$ tight in $\mathbb{D} \times [0,1]$ and make it straight. Then a representative of b' is of the desired form. $\Box$

Proof of Theorem 3.2(3). Since each i-increasing braid is conjugate to an n-increasing braid of a standard form in Lemma 4.6, we may assume that $b \in B_n$ is an n-increasing braid of the form $b = (w_1\sigma_{n-1}^2)\cdots(w_u\sigma_{n-1}^2)$. Since $\rho \in B_n$ is the periodic braid such that $\rho = \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2$ we have $\sigma_{n-1}^2 = (\sigma_1\cdots\sigma_{n-2})^{-1}\rho$. Then b is expressed as follows.

$$b = (\nu_1\rho)\cdots(\nu_u\rho),$$

where $\nu_i = w_i(\sigma_1\cdots\sigma_{n-2})^{-1}$ is written by a word of $\sigma_1^{\pm 1}, \cdots, \sigma_{n-2}^{\pm 1}$, but not $\sigma_{n-1}^{\pm 1}$. Each ν_j in b is a reducible braid and ρ in b is the periodic braid.



(1) (2) (3)

FIGURE 13. The figure illustrates how an initial braid b generates $\{b_p\}$. (1) $b = w_1\sigma_3^2 w_2\sigma_3^2 = (\nu_1\rho)(\nu_2\rho) \in B_4$, where $\nu_j = w_j(\sigma_1\sigma_2)^{-1}$. (2) $b_1 = (\nu_1\rho)(\nu_2\rho) \in B_6$. (3) $b_2 = (\nu_1\rho)(\nu_2\rho) \in B_8$.

Let $\omega_j : F_b \to F_b$ denote a reducible representative whose mapping class is determined by ν_j, and let $\psi : F_b \to F_b$ denote a periodic representative whose mapping class determined by ρ. The monodromy ϕ_b defined on F_b is written by $\phi_b = (\omega_1\psi)\cdots(\omega_u\psi)$.

Recall that $\mathbb{D}_{n-1}$ is the disk $\mathbb{D}$ with marked points $A_1, \cdots, A_{n-1}$. Let S_0 be an n-holed sphere obtained from $\mathbb{D}_{n-1}$ by removing the interiors of small $(n-1)$ disks with centers $A_1, \cdots, A_{n-1}$. Each ν_j as an $(n-1)$-braid determines a homeomorphism $\widehat{\omega}_j : S_0 \to S_0$. We may assume that $\widehat{\omega}_j$ fixes one of the boundary components corresponding to $\partial\mathbb{D}$ pointwise. It is clear that we have an embedding $h : S_0 \hookrightarrow F_b$ such that each ω_j in ϕ_b is reducible supported on the subsurface $h(S_0)$ and the restriction of ω_j to $h(S_0)$ is given by $h \circ \widehat{\omega}_j \circ h^{-1}$.

By the proof of Theorem 3.2(2), $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ associated with each primitive class $(x,y) \in C_{(b,i)}$ is determined by the braid of the form $b[p_1, \ldots, p_\ell]$. We now prove by the induction on length ℓ that

$$b[p_1, \ldots, p_\ell] = (\nu_1\rho)\cdots(\nu_{u-1}\rho)(\nu_u\rho)\rho^{m-1} = (\nu_1\rho)\cdots(\nu_{u-1}\rho)(\nu_u\rho^m)$$

for some $m \geq 1$ depending on (x,y). Here each ν_j in $b[p_1, \ldots, p_\ell]$ is a reducible braid which is an extension of ν_j in b and ρ is the periodic braid with the degree of $b[p_1, \ldots, p_j]$. If this holds, then $\phi_{(x,y)}$ has a desired property as in Theorem 3.2(3). Suppose that $\ell = 1$. If $p_1 = 0$, then $b[0] = b$ and we are done. If $p_1 \geq 1$, then $b[p_1] = b_{p_1}$. Using the above expression of b we observe that b_{p_1} is written by

$$b_{p_1} = (\nu_1\rho)\cdots(\nu_u\rho) \in B_{n+p_1u}$$

(see Figure 13). We are done.

For $\ell \geq 2$, suppose that $b[p_1, \ldots, p_{\ell-1}] = (\nu_1\rho_d)\cdots(\nu_{u-1}\rho_d)(\nu_u\rho_d^m)$ for some m, where d is the degree of $b[p_1, \ldots, p_{\ell-1}]$. Consider $b[p_1, \ldots, p_\ell]$ with length ℓ. If ℓ is even, then by induction hypothesis

$$b[p_1,\dots,p_\ell] = \big(b[p_1,\dots,p_{\ell-1}]\big)\Delta_d^{2p_\ell} = (\nu_1\rho_d)\cdots(\nu_{u-1}\rho_d)(\nu_u\rho_d^m)\Delta_d^{2p_\ell}.$$

Since $\Delta_d^2 = \rho_d^{d-1}$ we have $(\nu_u\rho_d^m)\Delta_d^{2p_\ell} = \nu_u\rho_d^{m+p_\ell(d-1)}$. Thus $b[p_1,\dots,p_\ell]$ has a desired expression and we are done. If ℓ is odd, then by induction hypothesis again

$$b[p_1,\dots,p_\ell] = \big(b[p_1,\dots,p_{\ell-1}]\big)_{p_\ell} = \big((\nu_1\rho_d)\cdots(\nu_{u-1}\rho_d)(\nu_u\rho_d^m)\big)_{p_\ell}.$$

As in the case of $\ell = 1$, the braid in the right-hand side is expressed as

$$\big((\nu_1\rho_d)\cdots(\nu_{u-1}\rho_d)(\nu_u\rho_d^m)\big)_{p_\ell} = (\nu_1\rho_\dagger)\cdots(\nu_{u-1}\rho_\dagger)(\nu_u\rho_\dagger^m),$$

where $\dagger$ is the degree of $b[p_1,\dots,p_\ell]$. This completes the proof. □

5. Sequences of pseudo-Anosov braids with small normalized entropies

In this section we prove Theorem A. We begin with an observation. Let $\Omega \subset \{a \in \mathcal{C} \mid \|a\| = 1\}$ be a compact set in $H_2(M_b, \partial M_b; \mathbb{R})$ and let $\mathcal{C}_\Omega \subset \mathcal{C}$ denote the cone over Ω through the origin. By Theroem 2.3(2) there is a constant $P = P(\Omega) > 0$ depending on Ω such that $\mathrm{Ent}(a) < P$ for any $a \in C_\Omega$. This observation provides us many sequences of pseudo-Anosov braids with small normalized entropies from a single pseudo-Anosov braid b.

Theorem 5.1. *Suppose that b is a pseudo-Anosov braid whose permutation has a fixed point. We fix any $0 < \ell < \infty$. Let $\{(x_p, y_p)\}$ be a sequence of primitive integral classes in $C_{(b,i)}$ such that $y_p/x_p < \ell$ and $\|(x_p,y_p)\| \asymp p$. Then the sequence of pseudo-Anosov braids $\{b_{(x_p,y_p)}\}$ has a small normalized entropy.*

Proof. If $\{(x_p, y_p)\}$ is the sequence under the assumption, then we have $d(b_{(x_p,y_p)}) \asymp \|(x_p,y_p)\| \asymp p$. (Recall that $d(\cdot)$ denotes the degree of the braid, i.e., the number of the strands.) Since $(1,0) \in C_{(b,i)} \subset \mathcal{C}$ and the slope of y_p/x_p is bounded by ℓ from above, the set of projective classes (x_p, y_p) is contained in some compact set in $\{a \in \mathcal{C} \mid \|a\| = 1\}$ (Figure 9). Thus there is a constant $P = P(\ell) > 1$ such that $\mathrm{Ent}(b_{(x_p,y_p)}) < P$ for any p. This completes the proof. □

Let us discuss three sequences coming from Example 4.4. They are $\{b_p\}$, $\{\beta_p\}$ and $\{(b\Delta^{2p})_1\}$ varying p. It is not hard to see that $d(b_p)$, $d(\beta_p)$, $d((b\Delta^{2p})_1) \asymp p$.

Theorem 5.2. *For an i-increasing and pseudo-Anosov $b \in B_n$, we have the following on the sequences of pseudo-Anosov braids.*

(1) *$\{b_p\}$ has a small normalized entropy if and only if $[E_{(b,i)}]$ is a fibered class.*

(2) *For* $\beta = b\Delta^2 \in B_n$, $\{\beta_p\}$ *has a small normalized entropy and* $\mathrm{Ent}(\beta_p) \to \mathrm{Ent}((1,1))$ *as* $p \to \infty$.
(3) $\{(b\Delta^{2p})_1\}$ *has a small normalized entropy and* $\mathrm{Ent}((b\Delta^{2p})_1) \to \mathrm{Ent}(b)$ *as* $p \to \infty$.

Proof of Theorem 5.2. For $a = (x, y) \in \overline{C_{(b,i)}}$, let $\underline{a} = \underline{(x, y)}$ denote its projective class. We have $\underline{[F_{b_p}]} = \underline{(1, p)} \to \underline{[E_{(b,i)}]} = \underline{(0, 1)}$ as $p \to \infty$. If $[E_{(b,i)}]$ is a fibered class, then $[E_{(b,i)}] \in \mathcal{C}$ by Remark 4.2 and $\mathrm{Ent}(b_p) \to \mathrm{Ent}([E_{(b,i)}])$ as $p \to \infty$ by Theorem 2.3(2). If $[E_{(b,i)}]$ is a non-fibered class, then $[E_{(b,i)}] \in \partial\mathcal{C}$ by Remark 4.2, and $\mathrm{Ent}(b_p) \to \infty$ as $p \to \infty$ by Theorem 2.3(3). We finish the proof of (1). We turn to (2). Since $[F_{\beta_p}] = (p+1, p) \in C_{(b,i)}$, its projective class goes to $\underline{(1, 1)}$ as $p \to \infty$. Since $(1, 1) \in C_{(b,i)} \subset \mathcal{C}$ by Theorem 3.2(1), $\mathrm{Ent}(\beta_p) \to \mathrm{Ent}((1, 1))$ as $p \to \infty$ by Theorem 2.3(2). This completes the proof of (2). Finally we prove (3). The fibered class of F-surface of $(b\Delta^{2p})_1$ is given by $(p + 1, 1) \in C_{(b,i)}$. Its projective class goes to $\underline{[F_b]} = \underline{(1, 0)}$ as $p \to \infty$. Thus $\mathrm{Ent}((b\Delta^{2p})_1) \to \mathrm{Ent}(b)$ as $p \to \infty$. This completes the proof. □

We use Theorem 5.2(1)(2) in Section 8. For an application using (3), see [19].

Proof of Theorem A. Suppose that $b \in B_n$ is pseudo-Anosov with $\pi_b(i) = i$. Let $\beta(k)$ denote $b\Delta^{2k} \in B_n$ for $k \geq 1$. Clearly $\beta(k)$ is pseudo-Anosov with the same dilatation as b (for any k) and $\beta(k)$ is positive for k large. We fix such large k. By Lemma 3.1 $\beta(k)$ is i-increasing. If we let $z_p = (\beta(k)\Delta^{2p})_1$, then $M_{z_p} \simeq M_{\beta(k)} \simeq M_b$ holds for $p \geq 1$. By Theorem 5.2(3), $\{z_p\}$ has a small normalized entropy and $\mathrm{Ent}(z_p) \to \mathrm{Ent}(\beta(k)) = \mathrm{Ent}(b)$ as $p \to \infty$. □

Let $b_p^\bullet$ denote the braid obtained from $(i + pu)$-increasing b_p by removing the strand of the index $i + pu$. Taking its spherical element we have $S(b_p^\bullet)$. A mild generalization of the sequence $\{b_p\}$ is the ones $\{b_p^\bullet\}$ and $\{S(b_p^\bullet)\}$ varing p. Although $b_p^\bullet$, $S(b_p^\bullet)$ may not be pseudo-Anosov, they are frequently pseudo-Anosov. To be more precise, we need to consider the number of prongs of singularities in the stable foliation $\mathcal{F}_{b_p}$ for b_p as we explained in Section 2.3. This is the motivation of the study in Section 6.

6. Stable foliation for the monodromy

Let b be pseudo-Anosov and i-monotonic with the sign $\epsilon(b, i) = \epsilon$. For any primitive integral class $(x, y) \in C_{(b,i)}$, the oriented sum $F_{(x,y)} = xF_b + yE_{(b,i)}$ is connected. Let $\mathcal{T}_{(b,A)}$ and $\mathcal{T}_{(b,i)}$ denote the tori $\partial\mathcal{N}(A)$ and $\partial\mathcal{N}(\mathrm{cl}(b(i)))$ respectively. Let us set

$$\partial_{(b,A)}F_{(x,y)} = \partial F_{(x,y)} \cap \mathcal{T}_{(b,A)} \quad \text{and} \quad \partial_{(b,i)}F_{(x,y)} = \partial F_{(x,y)} \cap \mathcal{T}_{(b,i)},$$

each of which is a single simple closed curve on the torus (since $\gcd(x, y) = 1$). Recall that we chose the orientation of the axis for the i-monotonic b

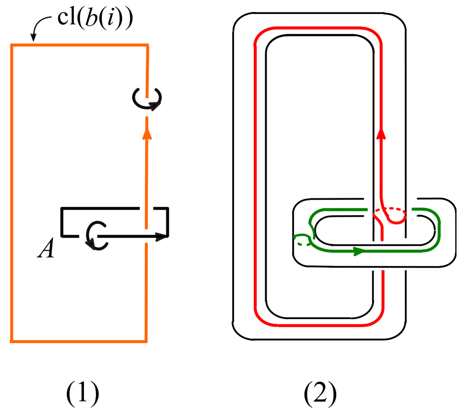


FIGURE 14. Case: b is i-increasing. (1) Meridian and longitude basis. (2) Two boundary slopes $\partial_{(b,A)}F_{(1,1)}$ (in green) on $\mathcal{T}_{(b,A)}$ and $\partial_{(b,i)}F_{(1,1)}$ (in red) on $\mathcal{T}_{(b,i)}$ when $(x,y)=(1,1)$.

in Section 3. We use the meridian and longitude basis $\{m_A, \ell_A\}$ for $\mathcal{T}_{(b,A)}$ to represent a homology class of a disjoint union of simple closed curves on $\mathcal{T}_{(b,A)}$. We also use the meridian and the longitude basis $\{m_i, \ell_i\}$ for $\mathcal{T}_{(b,i)}$. Observe that the homology classes $[\partial_{(b,A)}F_{(x,y)}]$ and $[\partial_{(b,i)}F_{(x,y)}]$ are given by the pairs of integers

$$[\partial_{(b,A)}F_{(x,y)}] = (-\epsilon y, x) \quad \text{and} \quad [\partial_{(b,i)}F_{(x,y)}] = (-\epsilon x, y). \tag{6.1}$$

They are called *boundary slopes* of $F_{(x,y)}$. See Figure 14.

Let $\phi_b : F_b \to F_b$ be the pseudo-Anosov monodromy of a fiber F_b of the fibration on $M_b \to S^1$. The stable foliation $\mathcal{F}_b$ of ϕ_b has singularities on each boundary component of F_b. Now we consider the suspension flow ϕ_b^t $(t \in \mathbb{R})$ on the mapping torus M_b. We obtain a disjoint union of simple closed curves $c_A = c_{(b,A)}$ on $\mathcal{T}_{(b,A)}$ (possibly a single simple closed curve) which is a union of closed orbits for singularities in $\partial_{(b,A)}F_b$ under the flow. Similarly we have a disjoint union of simple closed curves $c_i = c_{(b,i)}$ on $\mathcal{T}_{(b,i)}$ (possibly a single simple closed curve again) which is a union of closed orbits for singularities in $\partial_{(b,i)}F_b$. (Figure 17 depicts these closed curves for some pseudo-Anosov 3-braid.) A useful tool is *train track maps* which encode those data ϕ_b, $\mathcal{F}_b$. They also enable us to compute homology classes $[c_A]$ and $[c_i]$.

The following lemma is a consequence of Theorem 2.4(2) by Fried.

Lemma 6.1. *Let $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ be the monodromy of a fibration on $M_b \to S^1$ associated with a primitive integral class $(x,y) \in C_{(b,i)}$. Then the stable foliation $\mathcal{F}_{(x,y)}$ for $\phi_{(x,y)}$ is $\mathfrak{i}([c_A], [\partial_{(b,A)}F_{(x,y)}])$-pronged at $\partial_{(b,A)}F_{(x,y)}$, and is $\mathfrak{i}([c_i], [\partial_{(b,i)}F_{(x,y)}])$-pronged at $\partial_{(b,i)}F_{(x,y)}$, where $\mathfrak{i}(\cdot,\cdot)$ means the geometric intersection number between homology classes of closed curves.*

Remark 6.2. *Every closed orbit of the suspension flow ϕ_b^t on the mapping torus M_b travels around S^1 direction at least once. This implies that*

$[c_A]$ has a non-zero first coordinate of the meridian and longitude basis for $\mathcal{T}_{(b,A)}$, i.e., we have $[c_A] = (k, \ell) \in \mathbb{Z}^2$ with $k \neq 0$, since the meridian for $\mathcal{T}_{(b,A)}$ corresponds to the flow direction. Similarly, $[c_i]$ has a non-zero second coordinate of the meridian and longitude basis for $\mathcal{T}_{(b,i)}$, that is we have $[c_i] = (k', \ell') \in \mathbb{Z}^2$ with $\ell' \neq 0$, since the longitude for $\mathcal{T}_{(b,i)}$ corresponds to the flow direction in this case.

Recall that given a braid $b \in B_n$, we denote by $S(b) \in SB_n$, the spherical n-braid with the same word as b. For an i-increasing braid b of pseudo-Anosov type, consider the braid $(b\Delta^{2p})_1 = b[0,p,1]$ in Example 4.4(3). This is an $i[0,p,1]$-increasing braid. Then we have its spherical braid $S((b\Delta^{2p})_1)$. We now define other braids obtained from $(b\Delta^{2p})_1$. Let $(b\Delta^{2p})_1^\bullet$ denote the braid obtained from $(b\Delta^{2p})_1$ by removing the strand of the index $i[0,p,1]$. Let $S((b\Delta^{2p})_1)$ and $S((b\Delta^{2p})_1^\bullet)$ be the spherical braids corresponding to $(b\Delta^{2p})_1$ and $(b\Delta^{2p})_1^\bullet$ respectively. Then we have the following result.

Lemma 6.3. *Suppose that b is an i-increasing braid of pseudo-Anosov type. For p large, the braid $(b\Delta^{2p})_1^\bullet$ and the spherical braids $S((b\Delta^{2p})_1)$, $S((b\Delta^{2p})_1^\bullet)$ are all pseudo-Anosov with the same dilatation as $(b\Delta^{2p})_1$.*

Before proving Lemma 6.3, we recall a formula of the geometric intersection number $\mathfrak{i}([c],[c'])$ between two homology classes of simple closed curves c, c' on a torus. Let (p,q) and (p',q') be primitive elements of $\mathbb{Z}^2$ which represent $[c]$ and $[c']$ respectively. Then

$$\mathfrak{i}([c],[c']) = |pq' - p'q|.$$

Proof of Lemma 6.3. The fibered class of F-surface of $(b\Delta^{2p})_1$ is $(p+1,1) \in C_{(b,i)}$. We have $[\partial_{(b,A)} F_{(p+1,1)}] = (-1, p+1)$ and $[\partial_{(b,i)} F_{(p+1,1)}] = (-(p+1), 1)$, see (6.1). By Remark 6.2, one can write $[c_A] = (k,\ell)$ with $k \neq 0$ and $[c_i] = (k',\ell')$ with $\ell' \neq 0$. Then $\mathfrak{i}([c_A],[\partial_{(b,A)} F_{(p+1,1)}]) = |k(p+1)+\ell|$ and $\mathfrak{i}([c_i],[\partial_{(b,i)} F_{(p+1,1)}]) = |k' + \ell'(p+1)|$. Since $k \neq 0$ and $\ell' \neq 0$, these intersection numbers are increasing with respective to p and they are clearly greater than 1 when p is large. Then Lemma 6.1 says that when p is large, the stable foliation $\mathcal{F}_{(p+1,1)}$ for the monodromy $\phi_{(p+1,1)}$ is not 1-pronged at each component of $\partial_{(b,A)} F_{(p+1,1)} \cup \partial_{(b,i)} F_{(p+1,1)}$. By the discussion in Section 2.4, we are done. □

7. Properties of *F*-surfaces and *E*-surfaces

The aim of this section is to study properties of E-, F-surfaces and to present the technique used in the last section.

Lemma 7.1. *For an i-increasing braid $b \in B_n$ with $u(b,i) = u$, we set $\beta = b\Delta^2 \in B_n$. Then there is an n-increasing braid $\gamma \in B_{n+u}$ such that*

$$(\mathrm{br}(\beta), \mathrm{cl}(\beta(i)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(n))).$$

In particular $M_b \simeq M_\beta \simeq M_\gamma$ and $E_{(\beta,i)} = F_\gamma$, $F_\beta = E_{(\gamma,n)}$ up to isotopy in M_β. Moreover if b is pseudo-Anosov, then γ is also pseudo-Anosov.

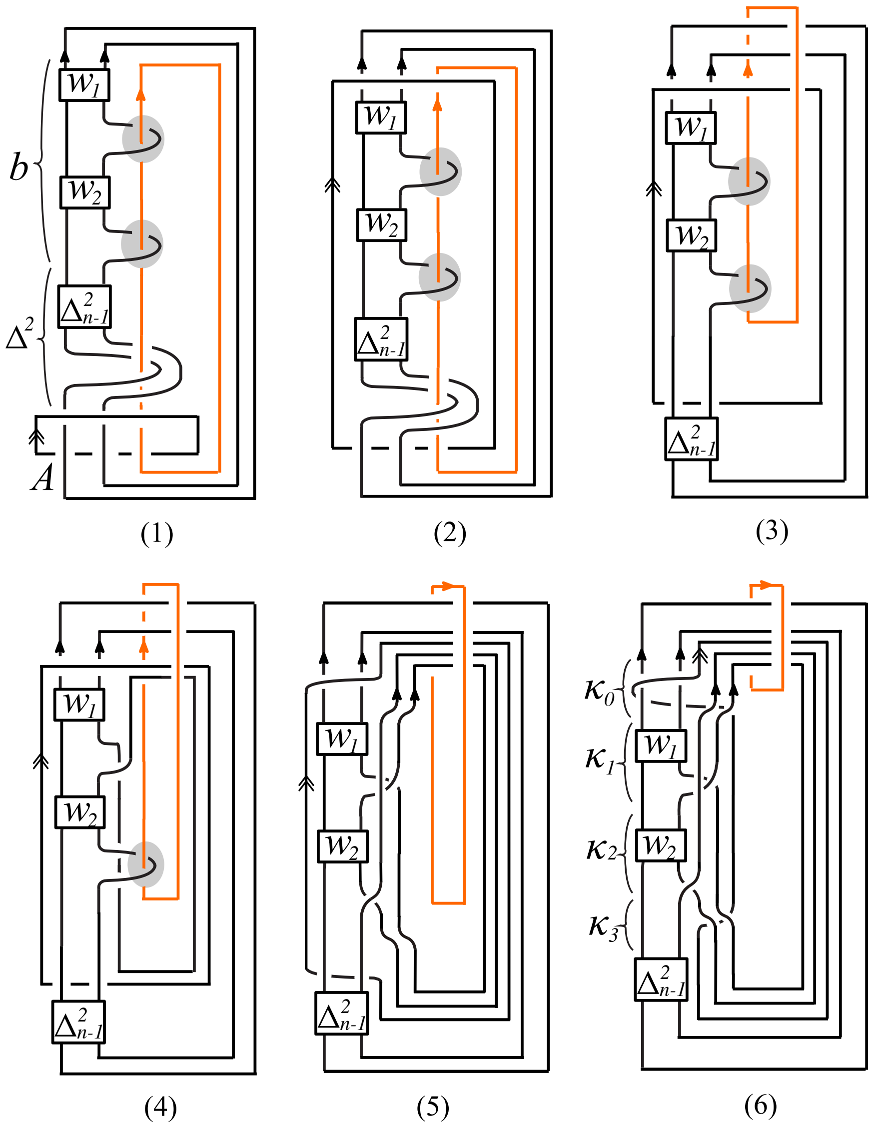


FIGURE 15. Demonstration of Lemma 7.1 when b is n-increasing with $u(b,n) = 2$. (1) br(β) of $\beta = w_1\sigma_{n-1}^2 w_2\sigma_{n-1}^2\Delta^2$. (5)(6) br$(\gamma)$ of $\gamma = \kappa_0\kappa_1\kappa_2\kappa_3\Delta_{n-1}^2$.

A similar claim holds for i-decreasing braids.

Proof. By Lemma 4.6 we may assume that $b \in B_n$ is an n-increasing braid of a standard form $b = (w_1\sigma_{n-1}^2)\cdots(w_u\sigma_{n-1}^2)$ containing u subwords σ_{n-1}^2. Using the identity

$$\Delta^2 = \Delta_{n-1}^2\sigma_{n-1}\cdots\sigma_2\sigma_1\sigma_1\sigma_2\cdots\sigma_{n-1} \in B_n,$$

we have (Figure 15(1))

$$\mathrm{br}(\beta) = \mathrm{br}(b\Delta^2) = \mathrm{br}(w_1\sigma_{n-1}^2 \cdots w_u\sigma_{n-1}^2\Delta_{n-1}^2\sigma_{n-1}\cdots\sigma_2\sigma_1\sigma_1\sigma_2\cdots\sigma_{n-1}).$$

We first deform $\mathrm{br}(\beta)$ into a link as in Figure 15(3). The same figure(1)(2)(3) tells us the process to get the desired link in (3). Then we perform the local moves in the shaded regions containing u subwords σ_{n-1}^2 in b so that the link in question is a union of the closure of some n-increasing braid $\gamma \in B_{n+u}$ and its braided axis, namely a braided link, see Figure 15(3)(4)(5). As a result,

$$(\mathrm{br}(\beta), \mathrm{cl}(\beta(n)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(n))).$$

This expression says that $M_\beta \simeq M_\gamma$ and the E-, F-surfaces for β are equal to the F-, E-surfaces for γ. Since $M_b \simeq M_\beta$ we are done. □

Here we introduce a simple representative of $\gamma \in B_{n+u}$ in Lemma 7.1. By the deformation as in (5)(6) of Figure 15, we can take the following representative of γ.

$$\begin{aligned}
\gamma &= \kappa_0\kappa_1\cdots\kappa_{u+1}\Delta_{n-1}^2, \text{ where}\\
\kappa_0 &= \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\sigma_1\sigma_2\cdots\sigma_{n+u-1},\\
\kappa_j &= w_j\sigma_{n-1}\sigma_n\cdots\sigma_{n+u-j-1}\sigma_{n+u-j-2}^{-1}\cdots\sigma_{n-1}^{-1} \quad \text{if } 1 \le j \le u-1,\\
\kappa_u &= w_u\sigma_{n-1},\\
\kappa_{u+1} &= \sigma_n^{-1} \quad \text{if } u = 1,\\
\kappa_{u+1} &= \sigma_{n+u-1}^{-1}\sigma_{n+u-2}^{-1}\cdots\sigma_n^{-1} \quad \text{if } u \ge 2.
\end{aligned}$$

For example if $(n, u) = (3, 2)$, then

$$\gamma = \kappa_0\kappa_1\kappa_2\kappa_3\Delta_2^2 = \sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4w_1\sigma_2\sigma_3\sigma_2^{-1}w_2\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_1^2. \tag{7.1}$$

If $(n, u) = (3, 3)$, then $\gamma = \kappa_0\kappa_1\kappa_2\kappa_3\kappa_4\Delta_2^2$, that is

$$\gamma = \sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5w_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_2^{-1}w_2\sigma_2\sigma_3\sigma_2^{-1}w_3\sigma_2\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_1^2. \tag{7.2}$$

Lemma 7.1 is used in the following situation. Suppose that $\alpha \in B_{n+u}$ is a j-increasing braid and our task is to prove that α is pseudo-Anosov and its E-surface $E_{(\alpha,j)}$ is a fiber of a fibration on $M_\alpha \to S^1$. (The conditions are needed to apply Theorem 5.2(1) for α.) To do this, we need to find an i-increasing and pseudo-Anosov braid $b \in B_n$ with $u = u(b, i)$ and need to check the resulting n-increasing braid $\gamma \in B_{n+u}$ in Lemma 7.1 satisfies the property

$$(\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(n))) \sim (\mathrm{br}(\alpha), A_\alpha, \mathrm{cl}(\alpha(j))),$$

i.e. γ is conjugate to α preserving the corresponding strand. If this equivalence holds, then by Lemma 7.1 together with the above equivalence $\sim$, our task is done. As a result $\{\alpha_p\}$ has a small normalized entropy by Theorem 5.2(1).

8. Application

In the last section we prove Theorems C, D and E. We first recall a study of pseudo-Anosov 3-braids [14, 24]. Let w be a word in σ_1^{-1} and σ_2. If both σ_1^{-1} and σ_2 occur at least once in w, then we say that w is a *pA word*. It is known that the 3-braid represented by a pA word is pseudo-Anosov. Conversely a 3-braid b is pseudo-Anosov, then there is a pA word w such that the braid represented by w is conjugate to b up to a power of the full twist.

The stable foliation $\mathcal{F}_b$ is 1-pronged at each boundary component of F_b for each pseudo-Anosov 3-braid b. Figure 17(3) exhibits a train track automaton. A train track map for the 3-braid represented by a pA word w is obtained from the closed loop corresponding to w in the automaton. For more details, see Ham-Song [13].

8.1. Palindromic/Skew-palindromic braids. We define a map

$$\begin{array}{rcl} rev : B_n & \to & B_n \\ \sigma_{i_1}^{\mu_1}\sigma_{i_2}^{\mu_2}\cdots\sigma_{i_k}^{\mu_k} & \mapsto & \sigma_{i_k}^{\mu_k}\cdots\sigma_{i_2}^{\mu_2}\sigma_{i_1}^{\mu_1}, \quad \mu_j = \pm 1, \end{array}$$

which is an anti-homomorphism. A braid $b \in B_n$ is palindromic if $rev(b) = b$. Clearly $b \cdot rev(b)$ is palindromic for any $b \in B_n$. Let us consider another anti-homomorphism

$$\begin{array}{rcl} skew : B_n & \to & B_n \\ \sigma_{i_1}^{\mu_1}\sigma_{i_2}^{\mu_2}\cdots\sigma_{i_k}^{\mu_k} & \mapsto & \sigma_{n-i_k}^{\mu_k}\cdots\sigma_{n-i_2}^{\mu_2}\sigma_{n-i_1}^{\mu_1}, \quad \mu_j = \pm 1. \end{array}$$

A braid $b \in B_n$ is skew-palindromic if $skew(b) = b$. Clearly $b \cdot skew(b)$ is skew-palindromic for any $b \in B_n$.

We now prove Theroems C and D which indicate the asymptotic behaviors of minimal entropies among these subsets are quite distinct.

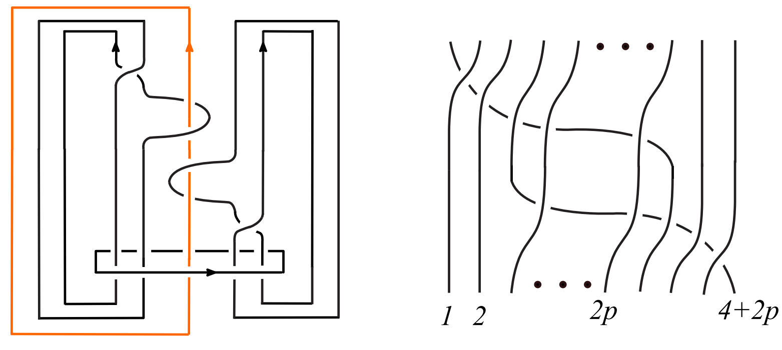
Proof of Theorem C. For the surjective homomorphism $\pi : B_n \to \mathcal{S}_n$ we write $\pi_j = \pi(\sigma_j)$. Suppose that an n-braid $b = \sigma_{i_1}^{\mu_1}\sigma_{i_2}^{\mu_2}\cdots\sigma_{i_k}^{\mu_k}$ is palindromic. Since $rev(b) = b$ we have

$$(\pi_{rev(b)} =)\pi_{i_k}\cdots\pi_{i_2}\pi_{i_1} = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}(= \pi_b).$$

Multiply the both side by $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ from the left:

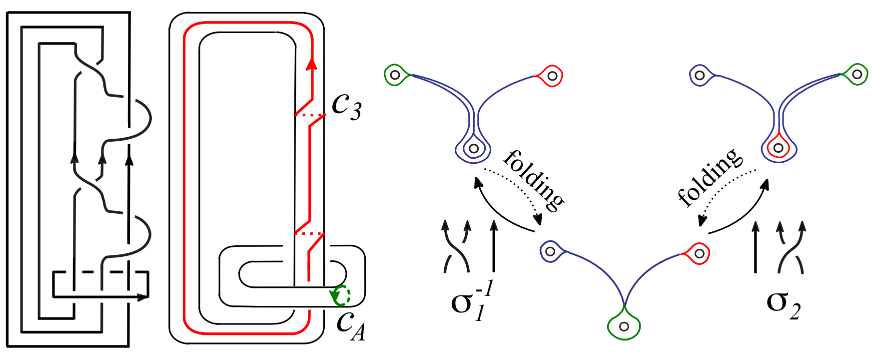
$$(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k})\cdot(\pi_{i_k}\cdots\pi_{i_2}\pi_{i_1}) = (\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k})\cdot(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \pi_b^2.$$

Since $\pi_j^2 = id$ the left-hand side equals id. Hence $id = \pi_b^2$ which means that the square b^2 is pure. A theorem by Song [28] states that for a pseudo-Anosov pure element $b' \in B_n$, its dilatation has a uniform lower bound $2+\sqrt{5} \le \lambda(b')$. In particular if $b' = b^2$, then $2+\sqrt{5} \le \lambda(b^2) = (\lambda(b))^2$. This completes the proof. $\square$



(1) (2)

FIGURE 16. (1) br(ξ). (2) Skew-palindromic $\xi_p^\bullet \in B_{4+2p}$.



(1) (2) (3)

FIGURE 17. (1) br(b) for $b = \sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2$. (2) $c_A \subset \mathcal{T}_{(b,A)}$ and $c_3 \subset \mathcal{T}_{(b,3)}$. (3) Train track automaton.

Proof of Theorem D. We separate the proof into two cases, depending on the parity of the braid degree. We first prove $\log \delta(PA_{2n}) \asymp 1/n$. Let us take $\xi = \sigma_1\sigma_2^2\sigma_3^2\sigma_4 \in B_5$ (Figure 16). The braid ξ is 3-increasing with $u(\xi, 3) = 2$. We consider the disk twist about $D_{(\xi,3)}$. We obtain the braid ξ_p which is $(3+2p)$-increasing for each $p \geq 1$. Observe that $\xi_p^\bullet$ is a skew-palindromic braid with even degree for each $p \geq 1$:

$$\xi_p^\bullet = (\sigma_1 \cdots \sigma_{1+2p})(\sigma_3 \cdots \sigma_{3+2p}) \in B_{4+2p}.$$

(For the definition of $\xi_p^\bullet$, see Section 5.) By the lower bound of dilatations by Penner, it is enough to prove that the sequence $\{\xi_p^\bullet\}$ has a small normalized entropy. We prove this in the following two steps. In Step 1 we prove that $\{\xi_p\}$ has a small normalized entropy. In Step 2 we prove that the stable foliation $\mathcal{F}_{\xi_p}$ is not 1-pronged at $\partial_{(\xi_p,3+2p)}F_{\xi_p}$ for $p \geq 1$. This tells us that $\xi_p^\bullet$

is pseudo-Anosov with the same dilatation as ξ_p. By Step 1 it follows that $\{\xi_p^\bullet\}$ has a small normalized entropy.

Step 1. The sequence $\{\xi_p\}$ has a small normalized entropy.

By Theorem 5.2(1) it suffices to prove that ξ is pseudo-Anosov and $[E_{(\xi,3)}]$ is a fibered class. Consider a pseudo-Anosov braid $b = \sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2 \in B_3$. It is 3-increasing with $u(b, 3) = 2$. For $\beta = b\Delta^2$ we have $M_b \simeq M_\beta$. By Lemma 7.1 $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(3)))$, where $\gamma \in B_5$ is the braid in (7.1) substituting σ_1^{-1} for w_1 and σ_1^{-1} for w_2. It is not hard to check that[1] γ is conjugate to ξ in B_5 and their permutations have a common fixed point 3. Hence

$$(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\xi), A_\xi, \mathrm{cl}(\xi(3))). \tag{8.1}$$

In particular $E_{(\xi,3)} = F_\beta$ which means that $E_{(\xi,3)}$ is a fiber of a fibration on the hyperbolic mapping torus $M_b \simeq M_\xi$ over S^1. Thus ξ is pseudo-Anosov.

Step 2. $\mathcal{F}_{\xi_p}$ is $(p+1)$-pronged at $\partial_{(\xi_p,3+2p)}F_{\xi_p}$ for $p \geq 1$.

We read the singularity data of $\mathcal{F}_{\xi_p}$ from the monodromy $\phi_\beta : F_\beta \to F_\beta$ of the fibration on $M_\beta \to S^1$. First consider the suspension flow ϕ_b^t on the mapping torus M_b. Since $\mathcal{F}_b$ is 1-pronged at each component of F_b, we have simple closed curves $c_A \subset \mathcal{T}_{(b,A)}$ and $c_3 \subset \mathcal{T}_{(b,3)}$ such that $[c_A] = (1, 0)$, $[c_3] = (2, 1) \in \mathbb{Z}^2$ (Figure 17(1)(2)).

Next we turn to $\beta = b\Delta^2 \in B_3$ and the suspension flow ϕ_β^t on $M_\beta \simeq M_b$. We have simple closed curves $c_{(\beta,A)} \subset \mathcal{T}_{(\beta,A)}$ and $c_{(\beta,3)} \subset \mathcal{T}_{(\beta,3)}$. Since β is the product of b and Δ^2, we get $[c_{(\beta,A)}] = (1, 0) + (0, 1) = (1, 1)$. The first term $(1, 0)$ comes from $[c_A]$ and the second one $(0, 1)$ comes from Δ^2. Similarly we have $[c_{(\beta,3)}] = (2, 1) + (1, 0) = (3, 1)$. By (8.1) we have $F_\beta = E_{(\xi,3)}$ and $E_{(\beta,3)} = F_\xi$. We also have $\mathcal{T}_{(\beta,A)} = \mathcal{T}_{(\xi,3)}$ and $\mathcal{T}_{(\beta,3)} = \mathcal{T}_{(\xi,A)}$. Since

$$p[F_\beta] + [E_{(\beta,3)}] = [F_\xi] + p[E_{(\xi,3)}] = [F_\xi + pE_{(\xi,3)}] = (1, p) \in C_{(\xi,3)},$$

the stable foliation $\mathcal{F}_{(1,p)}$ associated with an integral class $(1, p) \in C_{(\xi,3)}$ is the stable foliation associated with $(p, 1) \in C_{(\beta,3)}$. By (6.1) for $(x, y) = (p, 1)$

$$[\partial_{(\beta,A)}(F_\xi + pE_{(\xi,3)})] = (-1, p), \quad [\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})] = (-p, 1) \in \mathbb{Z}^2.$$

From $\mathfrak{i}([c_{(\beta,A)}], [\partial_{(\beta,A)}(F_\xi+pE_{(\xi,3)})]) = p+1$ and $\mathfrak{i}([c_{(\beta,3)}], [\partial_{(\beta,3)}(F_\xi+pE_{(\xi,3)})]) = p+3$ together with Lemma 6.1, one sees that $\mathcal{F}_{(1,p)}$ associated with $(1, p) \in C_{(\xi,3)}$ is $(p+1)$-pronged at $\partial_{(\beta,A)}F_{(1,p)}(= \partial_{(\xi,3)}F_{(1,p)})$, and is $(p+3)$-pronged at $\partial_{(\beta,3)}F_{(1,p)}(= \partial_{(\xi,A)}F_{(1,p)})$.

[1]There is a solution for the conjugacy problem on B_n [6]. The software *Braiding* [12] can be used to determine whether two braids are conjugate.

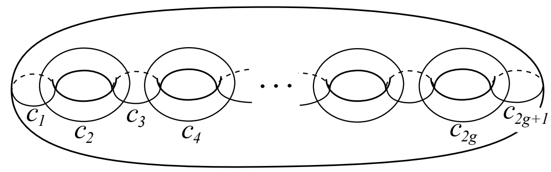


FIGURE 18. Simple closed curve C_j on Σ_g.

Since $g_p : M_\xi \to M_{\xi_p}$ sends $F_{(1,p)}$ to F_{ξ_p} the stable foliation $\mathcal{F}_{(1,p)}$ associated with $(1,p) \in C_{(\xi,3)}$ is identified with $\mathcal{F}_{\xi_p}$ via g_p. The boundary components $\partial_{(\xi,A)} F_{(1,p)}$ and $\partial_{(\xi,3)} F_{(1,p)}$ correspond to $\partial_{(\xi_p,A)} F_{\xi_p}$ and $\partial_{(\xi_p,3+2p)} F_{\xi_p}$ respectively via g_p. Thus $\mathcal{F}_{\xi_p}$ is $(p+1)$-pronged at $\partial_{(\xi_p,3+2p)} F_{\xi_p}$. This completes the proof of Step 2.

Next we prove $\log \delta(PA_{2n+1}) \asymp 1/n$ following the above arguments in Steps 1,2. Take an initial braid

$$\eta = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_3\sigma_4\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7 \in B_8.$$

It is 4-increasing with $u(\eta,4) = 2$. Consider $\eta_p \in B_{8+2p}$ obtained from η by the disk twist. Then $\eta_p^\bullet$ is a skew-palindromic braid with odd degree for each $p \geq 1$:

$$\eta_p^\bullet = (\sigma_1\sigma_2 \cdots \sigma_{4+2p})(\sigma_3\sigma_4 \cdots \sigma_{6+2p}) \in B_{7+2p}.$$

For our purpose it suffices to prove that $\{\eta_p^\bullet\}$ has a small normalized entropy. Following Step 1 we first prove that η is pseudo-Anosov and $[E_{(\eta,4)}]$ is a fibered class. Consider a pseudo-Anosov braid $b = \sigma^{-1}\sigma_2^6\Delta^2 \in B_3$ which is 3-increasing with $u(b,3) = 5$. For $\beta = b\Delta^2$ Lemma 7.1 tells us that $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(3)))$, where $\gamma = \kappa_0\kappa_1 \cdots \kappa_6\Delta_2^2 \in B_8$. One sees that γ is conjugate to η in B_8. Since the permutation π_η has a unique fixed point it follows that $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\eta), A_\eta, \mathrm{cl}(\eta(4)))$. This expression says that $E_{(\eta,4)} = F_\beta$ is a fiber of a fibration on the hyperbolic $M_b \simeq M_\eta$ over S^1. Hence η is pseudo-Anosov. We conclude that $\{\eta_p\}$ has a small normalized entropy.

Following Step 2 one sees that $\mathcal{F}_{\eta_p}$ is $(p+2)$-pronged at $\partial_{(\eta_p,4+2p)} F_{\eta_p}$ for $p \geq 1$. Thus $\eta_p^\bullet$ is pseudo-Anosov with the same dilatation as η_p. This completes the proof. □

8.2. Spin mapping class groups. In this section we prove Theorem E. We first recall a connection between $\mathcal{H}(\Sigma_g)$ and $\mathrm{Mod}(\Sigma_{0,2g+2})$. Let $t_j \in \mathrm{Mod}(\Sigma_g)$ for $1 \leq j \leq 2g+1$ be the right-handed Dehn twist about the simple closed curve C_j as in Figure 18. Birman-Hilden [3] proved that $\mathcal{H}(\Sigma_g)$ is generated by $t_1, t_2, \ldots, t_{2g+1}$. In fact they prove that

$$\begin{array}{rcl} Q : \mathcal{H}(\Sigma_g) & \to & \mathrm{Mod}(\Sigma_{0,2g+2}) \\ t_j & \mapsto & \mathfrak{t}_j \end{array}$$

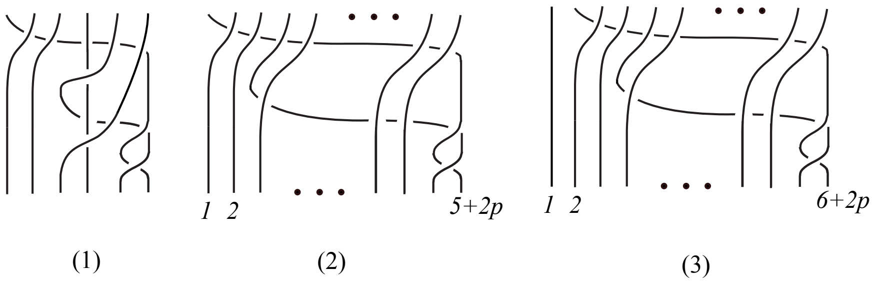


FIGURE 19. (1) $o \in B_6$. (2) $o_p^\bullet \in B_{5+2p}$. (3) $sh(o_p^\bullet) \in B_{6+2p}$.

sending t_j to the right-handed half twist $\mathfrak{t}_j$ (see Section 2.3) is well-defined and it is a surjective homomorphism whose kernel is generated by the involution $\iota = [\mathcal{I}]$ as in Figure 5. Using the relation between $\mathrm{Mod}(\Sigma_{0,2g+2})$ and SB_{2g+2} we have

$$\mathcal{H}(\Sigma_g)/\langle\iota\rangle \simeq \mathrm{Mod}(\Sigma_{0,2g+2}) \simeq SB_{2g+2}/\langle\Delta^2\rangle.$$

It is well-known that $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov if and only if $Q(\phi)$ is pseudo-Anosov and in this case $\lambda(\phi) = \lambda(Q(\phi))$ holds. The following lemma is useful to find elements of the odd/even spin mapping class groups.

Lemma 8.1 (Theorem 6.1 in [18] for (1), Theorem 3.1 in [17] for (2))**.** *Suppose that* $g \geq 3$.

(1) $t_2,\ t_3,\ t_{j+1}t_jt_{j+1}^{-1},\ t_k^2 \in \mathrm{Mod}_g[\mathfrak{q}_1]$ *for* $4 \leq j \leq 2g$ *and* $1 \leq k \leq 2g+1$.
(2) $t_{j+1}t_jt_{j+1}^{-1},\ t_k^2 \in \mathrm{Mod}_g[\mathfrak{q}_0]$ *for* $1 \leq j \leq 2g$ *and* $1 \leq k \leq 2g+1$.

By the above result of Birman-Hilden, all mapping classes in Lemma 8.1 are elements of $\mathcal{H}(\Sigma_g)$. Using the braid relations: $t_it_j = t_jt_i$ if $|i-j| \geq 2$ and $t_jt_{j+1}t_j = t_{j+1}t_jt_{j+1}$ for $1 \leq j \leq 2g$, we have

$$t_jt_{j+1}t_j^{-1} = t_{j+1}^{-1}t_jt_{j+1} = t_{j+1}^{-2}(t_{j+1}t_jt_{j+1}^{-1})t_{j+1}^2.$$

Thus Lemma 8.1 tells us that $t_jt_{j+1}t_j^{-1} \in \mathrm{Mod}_g[\mathfrak{q}_1]$ for $4 \leq j \leq 2g$ and $t_jt_{j+1}t_j^{-1} \in \mathrm{Mod}_g[\mathfrak{q}_0]$ for $1 \leq j \leq 2g$.

The following spin mapping classes are used in the proof of Theorem E.

Lemma 8.2. *Let* $p \geq 1$ *be an integer.*

(1) $t_2t_3(t_4t_5\cdots t_{5+2p})^2t_{5+2p} \in \mathrm{Mod}_g[\mathfrak{q}_1]$ *for any* $g \geq p+2$.
(2) $(t_2t_3\cdots t_{5+2p})^2t_{5+2p}^3 \in \mathrm{Mod}_g[\mathfrak{q}_0]$ *for any* $g \geq p+2$.

Proof. We prove the lemma by the induction on p. We first prove (1). When $p=1$

$$t_2t_3(t_4t_5t_6t_7)^2t_7 = t_2 \cdot t_3 \cdot t_4t_5t_4^{-1} \cdot t_4^2 \cdot t_6t_7t_6^{-1} \cdot t_6t_5t_6^{-1} \cdot t_6^2 \cdot t_7^2$$

which is an element of $\mathrm{Mod}_g[\mathfrak{q}_1]$ for $g \geq 3$ by Lemma 8.1(1).

Assume that $t_2t_3(t_4t_5\cdots t_{5+2(p-1)})^2t_{5+2(p-1)} \in \mathrm{Mod}_g[\mathfrak{q}_1]$ for $g \geq p-1+2$. By the braid relations, $t_2t_3(t_4t_5\cdots t_{4+2(p-1)}t_{5+2(p-1)}t_{4+2p}t_{5+2p})^2t_{5+2p}$ is equal to

$$t_2t_3(t_4t_5\cdots t_{5+2(p-1)})^2t_{5+2(p-1)}\cdot t_{5+2(p-1)}^{-2}\cdot t_{4+2p}t_{5+2p}t_{5+2(p-1)}t_{4+2p}\cdot t_{5+2p}^2.$$

Note that $t_jt_{j+1}t_{j-1}t_j = (t_jt_{j+1}t_j^{-1})(t_jt_{j-1}t_j^{-1})t_j^2$. Then the assumption together with Lemma 8.1(1) implies that $t_2t_3(t_4t_5\cdots t_{5+2p})^2t_{5+2p} \in \mathrm{Mod}_g[\mathfrak{q}_1]$ for $g \geq p+2$.

Let us turn to (2). When $p=1$

$$(t_2t_3t_4t_5t_6t_7)^2t_7^3 = t_2t_3t_2^{-1}\cdot t_2^2\cdot t_4t_3t_4^{-1}\cdot t_4t_5t_4^{-1}\cdot t_4^2\cdot t_6t_7t_6^{-1}\cdot t_6t_5t_6^{-1}\cdot t_6^2\cdot t_7^2\cdot t_7^2$$

which is an element of $\mathrm{Mod}_g[\mathfrak{q}_0]$ for $g \geq 3$.

Assume that $(t_2t_3\cdots t_{5+2(p-1)})^2t_{5+2(p-1)}^3 \in \mathrm{Mod}_g[\mathfrak{q}_0]$ for any $g \geq p-1+2$. By the braid relations again, we have

$$\begin{aligned} & (t_2t_3\cdots t_{4+2(p-1)}t_{5+2(p-1)}t_{4+2p}t_{5+2p})^2t_{5+2p}^3 \\ = \ & (t_2t_3\cdots t_{5+2(p-1)})^2t_{5+2(p-1)}^3\cdot t_{5+2(p-1)}^{-4}\cdot t_{4+2p}t_{5+2p}t_{5+2(p-1)}t_{4+2p}\cdot t_{5+2p}^4. \end{aligned}$$

The assumption together with Lemma 8.1(2) says that $(t_2t_3\cdots t_{5+2p})^2t_{5+2p}^3 \in \mathrm{Mod}_g[\mathfrak{q}_0]$ for $g \geq p+2$. This completes the proof. □

The *shift map* $sh: B_n \to B_{n+1}$ is an injective homomorphism sending σ_j to σ_{j+1} for $1 \leq j \leq n-1$. Suppose that $b \in B_n$ is pseudo-Anosov. Then $S(sh(b)) \in SB_{n+1}$ is pseudo-Anosov with the same dilatation as b since $\widehat{\Gamma}(S(sh(b)))$ is conjugate to $f_b = \mathfrak{c}(\Gamma(b))$ in $\mathrm{Mod}(\Sigma_{0,n+1})$. (See Section 2.3 for definitions Γ, $\widehat{\Gamma}$.) We finally prove Theorem E.

Proof of Theorem E(1). Consider $o = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_3^2\sigma_4\sigma_5\sigma_3\sigma_5 \in B_6$. It is a 4-increasing braid with $u(o,4)=2$ (Figure 19). The braid o_p is obtained from o by disk twist for each $p \geq 1$. Then

$$\begin{aligned} o_p^{\bullet} &= \sigma_1\sigma_2(\sigma_3\sigma_4\cdots\sigma_{4+2p})^2\sigma_{4+2p} \in B_{5+2p}, \\ S(sh(o_p^{\bullet})) &= \sigma_2\sigma_3(\sigma_4\sigma_5\cdots\sigma_{5+2p})^2\sigma_{5+2p} \in SB_{6+2p}. \end{aligned}$$

By Lemma 8.2(1) $t_2t_3(t_4t_5\cdots t_{5+2p})^2t_{5+2p} \in \mathrm{Mod}_{p+2}[\mathfrak{q}_1]$ for $p \geq 1$, and it is pseudo-Anosov if $S(sh(o_p^{\bullet}))$ is pseudo-Anosov. In this case they have the same dilatation. Thus by the relation between $o_p^{\bullet}$ and $S(sh(o_p^{\bullet}))$ it is enough to prove that $\{o_p^{\bullet}\}$ has a small normalized entropy. We first claim that $\{o_p\}$ has a small normalized entropy. By Theorem 5.2(1) it suffices to prove that o is a pseudo-Anosov and $[E_{(o,4)}]$ is a fibered class. Consider a 3-braid $b = \sigma_1^2\sigma_2^2\cdot\sigma_2^2\cdot\sigma_2^2$ which is 3-increasing with $u(b,3)=3$. Let β denote $b\Delta^2$. By Lemma 7.1 $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(3)))$, where $\gamma \in B_6$ is the braid in (7.2) substituting σ_1^2, $\emptyset$, $\emptyset$ for w_1, w_2, w_3 respectively. In this case γ is conjugate to o in B_6. Since the permutation π_o has a unique fixed point 4, it follows that $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(o), A_o, \mathrm{cl}(o(4)))$. This tells us that $M_\beta \simeq M_o$ and $[E_{(o,4)}] = [F_\beta]$ is a fibered class. On the other hand β

is conjugate to $\sigma_1^4\sigma_2^{-2}\Delta^4$ in B_3 which means that β is pseudo-Anosov. Thus $M_\beta \simeq M_o$ is hyperbolic and o is pseudo-Anosov.

Next we prove that $o_p^\bullet$ is pseudo-Anosov with the same dilatation as o_p for $p \geq 1$. By the same argument as in the proof of Theorem D one sees that $\mathcal{F}_{o_p}$ is $(p+2)$-pronged at $\partial_{(o_p,4+2p)}F_{o_p}$. Thus $o_p^\bullet$ has the desired property for $p \geq 1$. We finish the proof of (1).

We turn to (2). Let us consider $v = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^2\sigma_1\sigma_2\sigma_5^3 \in B_6$ which is 3-increasing with $u(v,3) = 2$. Let $v_p \in B_{6+2p}$ be the braid obtained from v by the disk twist. Then v_p is $(3+2p)$-increasing and

$$\begin{aligned} v_p^\bullet &= (\sigma_1\sigma_2\cdots\sigma_{4+2p})^2\sigma_{4+2p}^3 \in B_{5+2p}, \\ S(sh(v_p^\bullet)) &= (\sigma_2\sigma_3\cdots\sigma_{5+2p})^2\sigma_{5+2p}^3 \in SB_{6+2p}. \end{aligned}$$

By Lemma 8.2(2) it is enough to prove that $\{v_p^\bullet\}$ has a small normalized entropy. To do this we first prove that $\{v_p\}$ has a small normalized entropy. Consider a pseudo-Anosov 3-braid

$$b = \sigma_1^2\sigma_2^{-2}\Delta^4 = \sigma_1^3\sigma_2^2\sigma_1\Delta^2 = \sigma_1^3\sigma_2^2 \cdot \sigma_1^2\sigma_2^2 \cdot \sigma_1\sigma_2^2$$

which is 3-increasing with $u(b,3) = 3$. Lemma 7.1 tells us that for $\beta = b\Delta^2$ we have $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(\gamma), A_\gamma, \mathrm{cl}(\gamma(3)))$, where $\gamma \in B_6$ is the braid in (7.2) substituting σ_1^3 for w_1, σ_1^2 for w_2 and σ_1 for w_3. One sees that γ is conjugate to v in B_6. Thus $(\mathrm{br}(\beta), \mathrm{cl}(\beta(3)), A_\beta) \sim (\mathrm{br}(v), A_v, \mathrm{cl}(v(3)))$. This implies that $[E_{(v,3)}] = [F_\beta]$ is a fibered class of the hyperbolic $M_\beta \simeq M_v$, and hence v is pseudo-Anosov. By Theorem 5.2(1), $\{v_p\}$ has a small normalized entropy.

One sees that $\mathcal{F}_{v_p}$ is $(p+3)$-pronged at $\partial_{(v_p,3+2p)}F_{v_p}$. Thus $v_p^\bullet$ is pseudo-Anosov with the same dilatation as v_p for $p \geq 1$. This completes the proof. □

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