On two trigonometric inequalities of Askey and Steinig

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ABSTRACT. We prove that the inequality

$$\frac{5}{8}\cos(x/4) + \sum_{k=1}^{n} \frac{\cos((k+1/4)x)}{k+1} \geq 0$$

as well as its companion, obtained by replacing "cos" by "sin", hold for all $n \geq 1$ and $x \in (0, 2\pi)$. In both cases, the constant factor 5/8 is sharp. This refines a result of Askey and Steinig, who proved the inequalities with the factor 1 instead of 5/8.

CONTENTS

1. Introduction and statement of main results

In this paper, we study the inequalities

$$\sum_{k=0}^{n} \frac{\cos((k+1/4)x)}{k+1} > 0 \quad \text{and} \quad \sum_{k=0}^{n} \frac{\sin((k+1/4)x)}{k+1} > 0 \qquad (1.1)$$

which are valid for all integers $n \geq 0$ and real numbers $x \in (0, 2\pi)$. These elegant inequalities were proved in 1974 by Askey and Steinig [4]. In 2018, Alzer and Kwong [1] published the following counterpart,

$$\sum_{k=0}^{n} \frac{\cos((k+1/4)x) + \sin((k+1/4)x)}{k+1} \geq \frac{1}{\sqrt{2}}.$$

This inequality holds for all $n \geq 0$ and $x \in [0, 2\pi]$. The constant lower bound is best possible.

The inequalities (1.1) are special cases of a more general result. Askey and Steinig showed that if

$$(2k-1)b_{k-1} \geq 2kb_k > 0 \quad (k = 1, ..., n) \tag{1.2}$$

and $x \in (0, 2\pi)$, then

$$\sum_{k=0}^{n} b_k \cos((k+1/4)x) > 0 \quad \text{and} \quad \sum_{k=0}^{n} b_k \sin((k+1/4)x) > 0. \tag{1.3}$$

The assumption (1.2) can be relaxed. Brown and Hewitt [5] proved that (1.3) remains valid if we replace (1.2) by

$$2kb_{k-1} \geq (2k+1)b_k > 0 \quad (k = 1, ..., n). \tag{1.4}$$

Koumandos [6] showed that the cosine inequality in (1.3) holds for $x \in (0, 2\pi)$ if

$$(k-\lambda)b_{k-1} \geq kb_k > 0 \quad (k = 1, ..., n) \quad \text{and} \quad 0.3084438 \leq \lambda < 1. \tag{1.5}$$

We define

$$C_n(a, x) = a\cos(x/4) + \sum_{k=1}^{n} \frac{\cos((k+1/4)x)}{k+1}$$

and

$$S_n(a, x) = a\sin(x/4) + \sum_{k=1}^{n} \frac{\sin((k+1/4)x)}{k+1}.$$

From (1.1) we obtain

$$C_n(1, x) > 0 \quad \text{and} \quad S_n(1, x) > 0 \quad (n \geq 1, 0 < x < 2\pi). \tag{1.6}$$

Is it possible to refine these inequalities? More precisely, we ask for the smallest real numbers α and β such that

$$C_n(\alpha, x) > 0 \quad \text{and} \quad S_n(\beta, x) > 0 \quad (n \geq 1, 0 < x < 2\pi). \tag{1.7}$$

We show that both inequalities in (1.6) can be improved. In fact, (1.7) holds with the best possible constants $\alpha = 5/8$ and $\beta = 5/8$.

Theorem 1.1. *For all natural numbers n and real numbers $x \in (0, 2\pi)$ we have*

$$\frac{5}{8}\cos(x/4) + \sum_{k=1}^{n} \frac{\cos((k+1/4)x)}{k+1} \geq 0. \tag{1.8}$$

The sign of equality holds if and only if $n = 1$ and $x = 4\arccos(\sqrt{5/8})$.

Since $S_n(a, x) = C_n(a, 2\pi - x)$, we conclude from Theorem 1.1 that the following counterpart for sine sums is valid.

Theorem 1.2. *For all natural numbers n and real numbers $x \in (0, 2\pi)$ we have*

$$\frac{5}{8}\sin(x/4) + \sum_{k=1}^{n} \frac{\sin((k+1/4)x)}{k+1} \geq 0. \tag{1.9}$$

The sign of equality holds if and only if $n = 1$ and $x = 4 \arccos(\sqrt{3/8})$.

Over the years, inequalities for trigonometric sums attracted (and still attract) the attention of numerous researchers. Among the mathematicians, who worked in this field, we find well-known names like Fejér, Szegö, Turán, Vietoris, just to mention a few. Many of these inequalities are exquisitely beautiful and, moreover, have remarkable applications in various branches, like, for example, geometric function theory and approximation theory. For more information on this subject we refer to Askey [2], Askey and Gasper [3], Milovanović et al. [9, chapter 4] and the references cited therein.

In the next section, we collect a few lemmas which we need to prove Theorem 1.1. A proof of Theorem 1.1 is given in Section 2. Finally, in Section 3, we present a few remarks and corollaries.

The numerical and algebraic computations have been carried out using the computer software MAPLE 13.

2. Lemmas

The comparison principle which is stated in our first lemma is a helpful tool to prove inequalities for trigonometric sums; see Koumandos [7] and Kwong [8].

Lemma 2.1. *Let $\alpha_k > 0$, β_k and γ_k $(k = 0, 1, ..., N)$ be real numbers. If*

$$\frac{\beta_0}{\alpha_0} \geq \frac{\beta_1}{\alpha_1} \geq \cdots \geq \frac{\beta_N}{\alpha_N} > 0 \quad and \quad \lambda \leq \sum_{k=0}^{M} \alpha_k \gamma_k \leq \Lambda \quad (M = 0, 1, ..., N),$$

then

$$\frac{\beta_0}{\alpha_0} \lambda \leq \sum_{k=0}^{N} \beta_k \gamma_k \leq \frac{\beta_0}{\alpha_0} \Lambda. \tag{2.1}$$

Proof. Let $\alpha_{N+1} = 1$, $\beta_{N+1} = 0$ and $\delta_k = \sum_{j=0}^{k} \alpha_j \gamma_j$. Then,

$$\sum_{k=0}^{N} \beta_k \gamma_k = \sum_{k=0}^{N} \delta_k \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k+1}}{\alpha_{k+1}} \right)$$

and

$$\lambda \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k+1}}{\alpha_{k+1}} \right) \leq \delta_k \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k+1}}{\alpha_{k+1}} \right) \leq \Lambda \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k+1}}{\alpha_{k+1}} \right) \quad (k = 0, 1, ..., N).$$

By summation we obtain (2.1). $\qquad \square$

With the help of the comparison principle we are able to prove the following two lemmas.

Lemma 2.2. *Let c_k $(k = 0, 1, ..., n)$ be real numbers such that $c_0 \geq c_1 \geq \cdots \geq c_n > 0$. Then, for $x \in (0, \pi/2)$,*

$$\sum_{k=0}^{n} c_k \cos((4k + 1)x) \geq c_0 \frac{\sin(x) - 1}{2 \sin(2x)}. \tag{2.2}$$

Proof. We apply the formulae

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$

$$\sin(a)\cos(b) = \frac{1}{2}\big(\sin(a-b) + \sin(a+b)\big), \tag{2.3}$$

$$\sum_{k=0}^{n} \cos(2ka) = \frac{\cos(na)\sin((n+1)a)}{\sin(a)}$$

and

$$\sum_{k=0}^{n} \sin(2ka) = \frac{\sin(na)\sin((n+1)a)}{\sin(a)}. \tag{2.4}$$

Then, for $x \in (0, \pi/2)$,

$$\sum_{k=0}^{n} \cos((4k+1)x) = \frac{\sin(x) + \sin((4n+3)x)}{2\sin(2x)} \geq \frac{\sin(x) - 1}{2\sin(2x)}.$$

Next, we use Lemma 2.1 with

$$\alpha_k = 1, \quad \beta_k = c_k, \quad \gamma_k = \cos((4k+1)x), \quad \lambda = \frac{\sin(x) - 1}{2\sin(2x)}.$$

This leads to (2.2). □

Lemma 2.3. *Let m, n be integers with $0 \leq m \leq n$ and let c_k ($k = m, m+1, ..., n$) be real numbers such that $c_m \geq c_{m+1} \geq \cdots \geq c_n > 0$. Then, for $x \in (0, \pi/2)$,*

$$\sum_{k=m}^{n} c_k \sin(4kx) \leq \frac{c_m}{\sin(2x)}. \tag{2.5}$$

Proof. Using

$$\sin(a)\sin(b) = \frac{1}{2}\big(\cos(a-b) - \cos(a+b)\big)$$

and (2.4) gives for $N \geq m$ and $x \in (0, \pi/2)$,

$$\sum_{k=m}^{N} \sin(4kx) = \frac{\cos((4m-2)x) - \cos((4N+2)x)}{2\sin(2x)} \leq \frac{1}{\sin(2x)}.$$

From Lemma 2.1 with

$$N = n - m, \quad \alpha_k = 1, \quad \beta_k = c_{k+m},$$

$$\gamma_k = \sin(4(k+m)x), \quad \Lambda = \frac{1}{\sin(2x)}$$

we conclude that (2.5) holds. □

The following inequality for cosine polynomials is due to Rogosinski and Szegö [10].

Lemma 2.4. *For all natural numbers n and real numbers x we have*

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \geq 0.$$

3. Proof of Theorem 1.1

Since

$$C_1(5/8, x) = \frac{5}{8}\cos(x/4) + \frac{1}{2}\cos(5x/4) = \frac{1}{8}\cos(x/4)\left(8\cos^2(x/4) - 5\right)^2,$$

we obtain $C_1(5/8, x) \geq 0$ for $x \in (0, 2\pi)$ with equality if and only if $x = 4\arccos(\sqrt{5/8})$.

We define

$$\Theta_n(x) = C_n(5/8, 4x) = \frac{5}{8}\cos(x) + \sum_{k=1}^{n} \frac{\cos((4k+1)x)}{k+1}.$$

It remains to show that for $n \geq 2$ and $x \in (0, \pi/2)$ we get $\Theta_n(x) > 0$.

First, we consider the cases $n = 2, 3, 4$. We have

$$\Theta_2(x) = \frac{1}{24}\cos(x)F_2\left(\cos^2(x)\right), \quad \Theta_3(x) = \frac{1}{24}\cos(x)F_3\left(\cos^2(x)\right),$$
$$\Theta_4(x) = \frac{1}{120}\cos(x)F_4\left(\cos^2(x)\right), \tag{3.1}$$

where

$$F_2(z) = 2048z^4 - 4608z^3 + 3648z^2 - 1200z + 147,$$
$$F_3(z) = 24576z^6 - 79872z^5 + 101888z^4 - 64512z^3 + 21120z^2 - 3384z + 225,$$
$$F_4(z) = 1572864z^8 - 6684672z^7 + 11821056z^6 - 11261952z^5 + 6254080z^4$$
$$- 2045952z^3 + 379776z^2 - 36504z + 1533.$$

Next, we apply Sturm's theorem to determine the number of distinct real roots of a polynomial in an interval; see van der Waerden [11, section 79]. We obtain that each of the functions F_2, F_3, F_4 has no zero on $[0, 1]$. Since $F_n(0) > 0$ ($n = 2, 3, 4$), we get $F_n(z) > 0$ for $z \in [0, 1]$. From (3.1) we conclude that for $x \in (0, \pi/2)$ we have $\Theta_n(x) > 0$ ($n = 2, 3, 4$).

Let $n \geq 5$. We consider two cases.

Case 1. $0 < x \leq 0.055$.

We have

$$\Theta_n(x) = \cos(x)U_n(x) - \sin(x)V_n(x) \tag{3.2}$$

with

$$U_n(x) = \frac{5}{8} + \sum_{k=1}^{n} \frac{\cos(4kx)}{k+1} \quad \text{and} \quad V_n(x) = \sum_{k=1}^{n} \frac{\sin(4kx)}{k+1}.$$

Using Lemma 2.4 gives

$$\cos(x)U_n(x) \geq \cos(0.055) \cdot \frac{1}{8}. \tag{3.3}$$

We apply Lemma 2.3 with $m = 6$ and $c_k = 1/(k+1)$. Then,

$$
\begin{aligned}
\sin(x)V_n(x) &= \sin(x)\left(V_5(x) + \sum_{k=6}^{n} \frac{\sin(4kx)}{k+1}\right) \\
&\leq \sin(x)\left(V_5(x) + \frac{1}{7\sin(2x)}\right) \qquad (3.4) \\
&= \sin(x)\sum_{k=1}^{5} \frac{\sin(4kx)}{k+1} + \frac{1}{14\cos(x)} = G(x), \quad \text{say.}
\end{aligned}
$$

Since each of the six terms of G is increasing on $[0, 0.055]$, we obtain

$$
G(x) \leq G(0.055). \qquad (3.5)
$$

Combining (3.2) - (3.5) yields

$$
\Theta_n(x) \geq \frac{1}{8}\cos(0.055) - G(0.055) = 0.0144....
$$

Case 2. $0.055 \leq x < \pi/2$.
Let

$$
c_0 = c_1 = c_2 = c_3 = \frac{1}{5}, \quad c_k = \frac{1}{k+1} \quad (k \geq 4)
$$

and

$$
B(x) = \frac{17}{40}\cos(x) + \frac{3}{10}\cos(5x) + \frac{2}{15}\cos(9x) + \frac{1}{20}\cos(13x).
$$

Then,

$$
\Theta_n(x) = B(x) + \sum_{k=0}^{n} c_k \cos((4k+1)x).
$$

An application of Lemma 2.2 gives

$$
\sum_{k=0}^{n} c_k \cos((4k+1)x) \geq \frac{\sin(x) - 1}{10\sin(2x)}.
$$

It follows that

$$
\Theta_n(x) \geq B(x) + \frac{\sin(x) - 1}{10\sin(2x)}. \qquad (3.6)
$$

Let

$$
W(x) = 10\sin(2x)B(x) + \sin(x) - 1.
$$

We obtain

$$
W(x) = \sin(x)P(\cos^2(x)) - 1
$$

with

$$
P(z) = 4096z^7 - 13312z^6 + \frac{51968}{3}z^5 - 11520z^4 + 4160z^3 - 804z^2 + \frac{151}{2}z + 1.
$$

Let $T = T(x) = \tan(x/2)$. Since T is strictly increasing on $[0, \pi)$, we have $0.027... = T(0.055) \le T < T(\pi/2) = 1$. Using

$$\sin(x) = \frac{2T}{1 + T^2} \quad \text{and} \quad \cos(x) = \frac{1 - T^2}{1 + T^2}$$

leads to the representation

$$W(x) = \frac{2T}{1 + T^2} P\left(\left(\frac{1 - T^2}{1 + T^2}\right)^2\right) - 1 = \frac{(1 - T)^2}{3(1 + T^2)^{15}} Q(T) \tag{3.7}$$

with

$$Q(T) = \sum_{k=0}^{28} \mu_k T^k$$

and

$$\mu_0 = \mu_{28} = -3, \quad \mu_1 = \mu_{27} = 109, \quad \mu_2 = \mu_{26} = 176,$$
$$\mu_3 = \mu_{25} = -4887, \quad \mu_4 = \mu_{24} = -10265, \quad \mu_5 = \mu_{23} = 117334,$$
$$\mu_6 = \mu_{22} = 243568, \quad \mu_7 = \mu_{21} = -1235098, \quad \mu_8 = \mu_{20} = -2717859,$$
$$\mu_9 = \mu_{19} = 6533455, \quad \mu_{10} = \mu_{18} = 15775760,$$
$$\mu_{11} = \mu_{17} = -15448645, \quad \mu_{12} = \mu_{16} = -46688065,$$
$$\mu_{13} = \mu_{15} = 11893220, \quad \mu_{14} = 70455200.$$

An application of Sturm's theorem shows that Q has no zero on $[T(0.055), 1]$, so that $Q(1) > 0$ reveals that $Q(T) > 0$ for $T \in [T(0.055), 1]$. From (3.7) we conclude that W is positive on $[0.055, \pi/2)$. Using (3.6) gives $\Theta_n(x) > 0$. This completes the proof of Theorem 1.1.

4. Remarks and corollaries

We set $b_0 = 5/8$ and $b_k = 1/(k+1)$ $(k \ge 1)$. A short calculation reveals that none of the conditions (1.2), (1.4), (1.5) is fulfilled for $k = 1$. Thus, neither (1.8) nor (1.9) is included in (1.3).

Since

$$C_n(a, 0) = S_n(a, 2\pi) = a + \sum_{k=1}^{n} \frac{1}{k+1},$$

we conclude that there are no constant upper bounds for $C_n(a, x)$ and $S_n(a, x)$ which are valid for all $n \ge 1$ and $x \in (0, 2\pi)$.

We use

$$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a - b) + \cos(a + b)\right)$$

with

$$a = (k + 1/4)x \quad \text{and} \quad b = (k + 1/4)y$$

and apply Theorem 1.1. Then we obtain an extension of (1.8).

Corollary 4.1. *For all natural numbers n and real numbers x, y with $0 < x - y < 2\pi$, $0 < x + y < 2\pi$ we have*

$$\frac{5}{8}\cos(x/4)\cos(y/4) + \sum_{k=1}^{n} \frac{\cos((k+1/4)x)\cos((k+1/4)y)}{k+1} \geq 0. \qquad (4.1)$$

The sign of equality holds if and only if $n = 1$ and $x = 4\arccos(\sqrt{5/8})$, $y = 0$.

An application of (2.3) and Theorem 1.2 leads to a companion of (4.1).

Corollary 4.2. *For all natural numbers n and real numbers x, y with $0 < x - y < 2\pi$, $0 < x + y < 2\pi$ we have*

$$\frac{5}{8}\sin(x/4)\cos(y/4) + \sum_{k=1}^{n} \frac{\sin((k+1/4)x)\cos((k+1/4)y)}{k+1} \geq 0.$$

The sign of equality holds if and only if $n = 1$ and $x = 4\arccos(\sqrt{3/8})$, $y = 0$.

Let

$$C_n^*(x) = \sum_{k=1}^{n}(-1)^k \frac{\cos((k+1/4)x)}{k+1} \quad \text{and} \quad S_n^*(x) = \sum_{k=1}^{n}(-1)^k \frac{\sin((k+1/4)x)}{k+1}.$$

Alzer and Kwong [1] proved that for $n \geq 1$ and $x \in [0, 2\pi]$ we have

$$\cos(x/4) + \sin(x/4) + C_n^*(x) + S_n^*(x) \geq \frac{13 - \sqrt{85}}{200}\sqrt{300 + 20\sqrt{85}} = 0.41601.... \qquad (4.2)$$

The constant lower bound is sharp.

We replace in (1.9) x by $\pi - x$ and make use of

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b).$$

Then we multiply both sides of the inequality by $\sqrt{2}$. This yields the following counterpart of (4.2).

Corollary 4.3. *For all natural numbers n and real numbers $x \in (-\pi, \pi)$ we have*

$$\frac{5}{8}\big(\cos(x/4) - \sin(x/4)\big) + C_n^*(x) - S_n^*(x) \geq 0.$$

The sign of equality holds if and only if $n = 1$ and $x = \pi - 4\arccos(\sqrt{3/8})$.

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