Equilibrium states on the Toeplitz algebras of small higher-rank graphs

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ABSTRACT. We consider a family of operator-algebraic dynamical systems involving the Toeplitz algebras of higher-rank graphs. We explicitly compute the KMS states (equilibrium states) of these systems built from small graphs with up to four connected components.

CONTENTS

Over the past decade there has been a great deal of interest in KMS states (or equilibrium states) on C^*-algebras of directed graphs [16, 12, 4, 3, 20, 25] and their higher-rank analogues [13, 14, 18, 26, 7, 8, 5]. At first this work focused on strongly connected graphs, possibly because these had provided many examples of interesting simple C^*-algebras [24, 22]. A uniform feature of all this work is that the Toeplitz algebras of graphs have much more interesting KMS structure than their Cuntz–Krieger quotients.

More recently, we have been looking at graphs with more than one strongly connected component [15, 11]. This throws up new problems: removing components can create sources, and, as Kajiwara and Watatani demonstrated in [16, Theorem 4.4], sources give rise to extra KMS states (see also [12, Corollary 6.1] for a version phrased with our graph-algebra conventions). For higher-rank graphs, the situation is even more complicated. We saw in [11, §8] that even if the original graph has no sources, removing a component can give sources of several different kinds. So when we tried to test the

general results from [10] and [9, §3–6] by organising them into a coherent program for calculating KMS states, we ruled out the possiblity that there could be non-trivial bridges between different components [9, §7–8]. Here we study graphs like the ones in [11, §8] which led us to rule out non-trivial bridges in [9, §7–8]. We find that, while there are indeed new difficulties at almost every turn, the general results in [9, §3–6] are sharp enough to determine all the KMS states.

Our first problem is that there are different kinds of sources: some are *absolute sources*, which receive no edges, and others may receive edges of some colours but not others. The only Cuntz–Krieger relations which apply to higher-rank graphs like the ones arising in [10] are those proposed in [23]. (There is a lengthy explanation of why these relations are appropriate in [23, Appendix A].) Since these relations are complicated, we have to develop some techniques for finding efficient sets of relations, and these techniques may be useful elsewhere. We discuss this in §1, and introduce the two main families of 2-graphs with sources that we will analyse. The first family contains graphs with two vertices $\{u, v\}$; the second, graphs with three vertices $\{u, v, w\}$, each containing one of the first family as a subgraph. We normalise the dynamics to ensure that the critical inverse temperature is always 1.

In all our classification results, the KMS states at non-critical inverse temperatures are parametrised by a simplex of "subinvariant vectors". Identifying this simplex requires computing the numbers of paths with range a given vertex, and the presence of sources throws up new problems, which we deal with in §2 and §3. The results in §3 for the graphs with three vertices depend on the analogous results for the graphs with two vertices in the previous section.

In §4 we apply the results of [9, §4 and §5] to find the KMS states at the critical inverse temperature for the higher-rank graphs we studied in the preceding sections. This involves proving, first for the example with two vertices $\{u, v\}$, and then for the three-vertex example, that a KMS_1 state cannot see any of the vertices except u. The proofs rely on our understanding of the Cuntz–Krieger relations for graphs with sources.

The examples we have studied arose as subgraphs of graphs with 4 vertices which do not themselves have sources. In §5, we calculate the KMS_1 states of these graphs. We get one by lifting the unique KMS_1 state of $\mathcal{T}C^*(u\Lambda u)$, and we find a second by stepping carefully through the construction of [9, §4]. In the final section, we investigate what happens below the critical inverse temperature for the concrete example discussed in [11, Example 8.4].

Since the article [9] was posted, there have been two very substantial papers which describe the KMS states on larger classes of C^*-algebras: in [6], Christensen uses the groupoid model of Neshveyev [21] to parametrise the KMS states on the Toeplitz algebras of general finite higher-rank graphs, and in [2], Afsar, Larsen and Neshveyev use an approach based on ideas from [19]

to describe the KMS states on a variety of Nica–Pimsner algebras. However, we value the graphical intuition we get from working directly with the graph, and the opportunities this gives to cross-check results. Here we have had to deal with new graphical problems, and we hope that our methods of solving these problems, particularly those involving the Cuntz–Krieger relations for graphs with sources, will be useful elsewhere.

1. Exhaustive sets

Throughout, we consider a finite k-graph Λ, and typically $k = 2$. A vertex $v \in \Lambda^0$ is a *source* if there exists $i \in \{1, \ldots, k\}$ such that $v\Lambda^{e_i}$ is empty. (We believe this is standard: it is the negation of "Λ has no sources" in the sense of the original paper [17].) Our main examples are 2-graphs in which a vertex can receive red edges but not blue edges or vice-versa — for example, the vertex v in the graph in [11, Figure 2] (which is also Example 1.8 below). We call a vertex such that $v\Lambda^{e_i}$ is empty for all i an *absolute source*. For example, the vertex w in Example 1.8 is an absolute source.

For such graphs the only Cuntz–Krieger relations available are those of [23]. As there, if μ, ν is a pair of of paths in Λ with $r(\mu) = r(\nu)$, we set

$$\Lambda^{\min}(\mu, \nu) = \big\{ (\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta \text{ and } d(\mu\alpha) = d(\mu) \vee d(\nu) \big\}$$

for the set of *minimal common extensions*. For a vertex $u \in \Lambda^0$, a finite subset E of $u\Lambda^1 := \bigcup_{i=1}^k u\Lambda^{e_i}$ is *exhaustive* if for every $\mu \in u\Lambda$ there exists $e \in E$ such that $\Lambda^{\min}(\mu, e)$ is nonempty.

We then use the Cuntz–Krieger relations of [23], and in particular the presentation of these relations which uses only edges, as discussed in [23, Appendix C]. As there, we write $\{t_\mu : \mu \in \Lambda\}$ for the universal Toeplitz–Cuntz–Krieger family which generates the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$. The Cuntz–Krieger relations then include the relations (T1), (T2), (T3) and (T5) for Toeplitz–Cuntz–Krieger families, as in [13] and [11], and for every u that is not a source the extra relations

$$\prod_{e \in E} (t_u - t_e t_e^*) = 0 \tag{CK}$$

for all $u \in \Lambda^0$ and finite exhaustive sets $E \subset u\Lambda^1$. Since adding extra edges to an exhaustive set gives another exhaustive set, it is convenient to find small exhaustive sets, so that the resulting Cuntz–Krieger relations for these smallest sets contain minimal redundancy.

Lemma 1.1. *Suppose that Λ is a finite k-graph and $E \subset u\Lambda^1$ is a finite exhaustive set. Consider $i \in \{1, \ldots, k\}$ and $e \in u\Lambda^{e_i}$. If there is a path $e\mu \in u\Lambda^{\mathbb{N}e_i}$ such that $s(\mu)$ is an absolute source, then $e \in E$.*
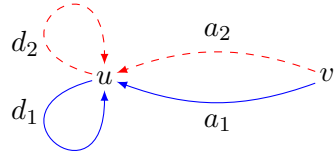
Proof. Since E is exhaustive, there exists $f \in E$ with $\Lambda^{\min}(e\mu, f) \neq \emptyset$. Then there exist paths α, β such that $e\mu\alpha = f\beta$. Since $s(\mu)\Lambda = \{s(\mu)\}$, we

have $\alpha = s(\mu)$. Thus $d(f\beta) = d(e\mu\alpha) = d(e\mu) \in \mathbb{N}e_i$; since $f \in \Lambda^1$ and

$$0 \le d(f) \le d(f) + d(\beta) = d(f\beta) \in \mathbb{N}e_i,$$

we deduce that $d(f) = e_i$. Now uniqueness of factorisations and $e\mu = f\beta$ imply that $f = (e\mu)(0, e_i) = e$. Thus $e \in E$.   □

Example 1.2. We consider a 2-graph Λ with the following skeleton:

in which the label a_1, for example, means that there are a_1 blue edges from v to u. Since each path in $u\Lambda^{e_1+e_2}v$ has unique blue-red and red-blue factorisations, the numbers $a_i, d_i \in \mathbb{N}\backslash\{0\}$ satisfy $d_1 a_2 = d_2 a_1$. Since v is an absolute source, Lemma 1.1 implies that every finite exhaustive set in $u\Lambda^1$ must contain every edge, and hence contains $u\Lambda^1$; since every finite exhaustive set is by definition a subset of $u\Lambda^1$, it is therefore the only finite exhaustive set. Thus the only Cuntz–Krieger relation at u is

$$\prod_{e \in u\Lambda^1} (t_u - t_e t_e^*) = 0.$$

There is no Cuntz–Krieger relation at v.

We now aim to make the Cuntz–Krieger relation (CK) look a little more like the familiar ones involving sums of range projections.

Proposition 1.3. *Suppose that Λ is a finite k-graph, u is a vertex and $E \subset u\Lambda^1$ is a finite exhaustive set. For each nonempty subset J of $\{1, \ldots, k\}$, define $e_J \in \mathbb{N}^k$ by $e_J = \sum_{i \in J} e_i$. Then the Cuntz–Krieger relation (CK) associated to E is equivalent to*

$$t_u + \sum_{\emptyset \ne J \subset \{1,\ldots,k\}} (-1)^{|J|} \sum_{\{\mu \in u\Lambda^{e_J} : \mu(0,e_i) \in E \ for \ i \in J\}} t_\mu t_\mu^* = 0.$$

From the middle of [11, page 120] we have

$$\prod_{e \in E} (t_u - t_e t_e^*) = t_u + \sum_{\emptyset \ne J \subset \{1,\cdots,k\}} (-1)^{|J|} \prod_{i \in J} \Big(\sum_{e \in E \cap u\Lambda^{e_i}} t_e t_e^* \Big). \qquad (1.1)$$

We want to expand the product, and we describe the result in a lemma. Proposition 1.3 then follows from (1.1) and the lemma.

Lemma 1.4. *Suppose that E is a finite exhaustive subset of $u\Lambda^1$ and J is a subset of $\{1,\ldots,k\}$. Write $e_J = \sum_{i \in J} e_i$. Then*

$$\prod_{i \in J}\Big(\sum_{f \in E \cap u\Lambda^{e_i}} t_f t_f^*\Big) = \sum_{\{\mu \in u\Lambda^{e_J} : \mu(0,e_i) \in E \text{ for } i \in J\}} t_\mu t_\mu^*. \qquad (1.2)$$

Proof. For $|J| = 1$, say $J = \{i\}$, we have $e_J = e_i$. Thus

$$\{\mu \in u\Lambda^{e_J} : \mu(0,e_i) \in E \text{ for } i \in J\}$$
$$= \{\mu \in u\Lambda^{e_i} : \mu = \mu(0,e_i) \in E\}$$
$$= E \cap u\Lambda^{e_i}.$$

Now suppose the formula holds for $|J| = n$ and that $K := J \cup \{j\}$ for some $j \in \{1,\ldots,k\} \setminus J$. Then the inductive hypothesis gives

$$\prod_{i \in K}\Big(\sum_{f \in E \cap u\Lambda^{e_i}} t_f t_f^*\Big) = \Big(\prod_{i \in J}\Big(\sum_{f \in E \cap u\Lambda^{e_i}} t_f t_f^*\Big)\Big)\Big(\sum_{e \in E \cap u\Lambda^{e_j}} t_e t_e^*\Big)$$
$$= \Big(\sum_{\{\mu \in u\Lambda^{e_J} : \mu(0,e_i) \in E \text{ for } i \in J\}} t_\mu t_\mu^*\Big)\Big(\sum_{e \in E \cap u\Lambda^{e_j}} t_e t_e^*\Big).$$

Now for each pair of summands $t_\mu t_\mu^*$ and $t_e t_e^*$ the relation (T5) gives

$$(t_\mu t_\mu^*)(t_e t_e^*) = \sum_{(g,\nu) \in \Lambda^{\min}(\mu,e)} t_\mu(t_g t_\nu^*)t_e^* = \sum_{(g,\nu) \in \Lambda^{\min}(\mu,e)} t_{\mu g} t_{\nu e}^*.$$

By definition of $\Lambda^{\min}$ we have $g \in \Lambda^{e_j}$ and $\mu g = e\nu$, so $d(\mu g) = e_K$. Then we have

$$(\mu g)(0,e_j) = (e\nu)(0,e_j) = e \in E, \quad \text{and}$$
$$(\mu g)(0,e_i) = \mu(0,e_i) \in E \text{ for } i \in J.$$

So the paths which arise as μg are precisely those in the set

$$\{\lambda \in u\Lambda^{e_K} : \lambda(0,e_i) \in E \text{ for } i \in J \cup \{j\} = K\}. \qquad \square$$

Remark 1.5. It is possible that the index set on the right-hand side of (1.2) is empty, in which case we are asserting that the product on the left is 0.

In a finite k-graph, the set $u\Lambda^1$ of all edges with range u is always a finite exhaustive subset of Λ. Then Proposition 1.3 applies with $E = u\Lambda^1$. For this choice of E, the condition $\mu(0,e_i) \in E$ is trivially satisfied, and hence we get the following simpler-looking relation.

Corollary 1.6. *Suppose that Λ is a finite k-graph. Then for every $u \in \Lambda^0$ we have*

$$\prod_{f \in u\Lambda^1}(t_u - t_f t_f^*) = t_u + \sum_{\emptyset \neq J \subset \{1,\ldots,k\}} (-1)^{|J|}\Big(\sum_{\mu \in \Lambda^{e_J}} t_\mu t_\mu^*\Big).$$

Example 1.7. We return to a 2-graph Λ with the skeleton desccribed in Example 1.2, and its only finite exhaustive set $u\Lambda^1$. Then (1.1) and Lemma 1.4 imply that

$$\prod_{e\in u\Lambda^1}(t_u - t_e t_e^*) = t_u + \sum_{\emptyset\neq J\subset\{1,2\}}(-1)^{|J|}\Big(\sum_{\{\mu\in\Lambda^{e_J}:\mu(0,e_i)\in u\Lambda^1 \text{ for } i\in J\}} t_\mu t_\mu^*\Big).$$

The nonempty subsets of $\{1,2\}$ are $\{1\}$, $\{2\}$ and $\{1,2\}$. For $J = \{1\}$ the requirement $\mu(0,e_1)\in u\Lambda^1$ just says that μ is a blue edge (that is, $d(\mu) = e_1$), and

$$\sum_{\mu\in\Lambda^{e_J}} t_\mu t_\mu^* = \sum_{e\in u\Lambda^{e_1}} t_e t_e^*.$$

A similar thing happens for $J = \{2\}$. For $J = \{1,2\}$, the condition on $\mu(0,e_i)$ is still trivially satisfied by all $\mu \in u\Lambda^{e_1+e_2}$. Hence the Cuntz–Krieger relation becomes

$$t_u - \sum_{e\in u\Lambda^{e_1}} t_e t_e^* - \sum_{e\in u\Lambda^{e_2}} t_e t_e^* + \sum_{\mu\in u\Lambda^{e_1+e_2}} t_\mu t_\mu^* = 0,$$
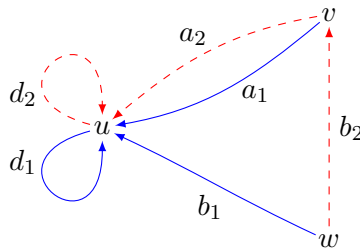
or equivalently

$$t_u = \sum_{e\in u\Lambda^{e_1}} t_e t_e^* + \sum_{e\in u\Lambda^{e_2}} t_e t_e^* - \sum_{\mu\in u\Lambda^{e_1+e_2}} t_\mu t_\mu^*,$$

which does indeed look more like a Cuntz–Krieger relation.

Lemma 1.1 establishes a lower bound for the finite exhaustive sets. In Example 1.2, this lower bound was all of $u\Lambda^1$, and hence this had to be the only finite exhaustive set. However, this was a bit lucky, as the next example shows.

Example 1.8. We consider a 2-graph Λ with skeleton

in which $d_1 a_2 = d_2 a_1$ and $a_1 b_2 = d_2 b_1$. Note that w is an absolute source. Our interest in these graphs arises from [11, §8], where we saw that the Toeplitz algebras of such graphs can arise as quotients of the Toeplitz algebras of graphs with no sources.

The only finite exhaustive subset of $v\Lambda^1 = v\Lambda^{e_2}$ is the whole set, and this yields a Cuntz–Krieger relation

$$\prod_{f\in v\Lambda^{e_2}}(t_v - t_f t_f^*) = 0 \iff t_v = \sum_{f\in v\Lambda^{e_2}} t_f t_f^*. \tag{1.3}$$

For the vertex u the situation is more complicated.

Proposition 1.9. *Suppose that Λ is a 2-graph with the skeleton described in Example* 1.8. *Then*

$$E := (u\Lambda^1 u) \cup (u\Lambda^{e_2} v) \cup (u\Lambda^{e_1} w)$$

is exhaustive, and every other finite exhaustive subset of $u\Lambda^1$ contains E.

Proof. Since w is an absolute source, Lemma 1.1 implies that every finite exhaustive subset of $u\Lambda^1$ contains $u\Lambda^1 u$, $u\Lambda^{e_2} v$ and $u\Lambda^1 w = u\Lambda^{e_1} w$, and hence also the union E. So it suffices for us to prove that E is exhaustive.

To see this, we take $\lambda \in u\Lambda$ and look for $e \in E$ such that $\Lambda^{\min}(\lambda, e) \neq \emptyset$. Unfortunately, this seems to require a case-by-case argument. We begin by eliminating some easy cases.

- If $\lambda = u$, we take $e \in u\Lambda^1 u$; then $e \in \Lambda^{\min}(\lambda, e)$, and we are done. So we suppose that $d(\lambda) \neq 0$.
- If $\lambda \in u\Lambda u \setminus \{u\}$, we choose i such that $e_i \leq d(\lambda)$. Then $\lambda(0, e_i) \in u\Lambda^{e_i} u \subset E$ and $\Lambda^{\min}(\lambda, \lambda(0, e_i)) = \left\{ \left(s(\lambda), \lambda(e_i, d(\lambda)) \right) \right\} \neq \emptyset$.
- We now suppose that $\lambda \in u\Lambda v \cup u\Lambda w$. If $e_2 \leq d(\lambda)$, then $\lambda(0, e_2) \in E$ and $\Lambda^{\min}(\lambda, \lambda(0, e_2)) \neq \emptyset$. Otherwise $d(\lambda) \in \mathbb{N} e_1$.
- If $d(\lambda) \geq 2e_1$, then $\lambda(0, e_1) \in u\Lambda^{e_1} u$ belongs to E, and satisfies $\Lambda^{\min}(\lambda, \lambda(0, e_1)) \neq \emptyset$.
- If $d(\lambda) = e_1$ and $s(\lambda) = w$, then $\lambda \in u\Lambda^{e_1} w$ belongs to E, and we take $e = \lambda$.

We are left to deal with paths $\lambda \in u\Lambda^{e_1} v$. Choose $f \in v\Lambda^{e_2} w$, and consider λf. Since $d(\lambda f) = d(\lambda) + d(f) = e_1 + e_2$, λf has a red-blue factorisation

$$\lambda f = (\lambda f)(0, e_2)(\lambda f)(e_2, e_1 + e_2).$$

But now $(\lambda f)(0, e_2) \in u\Lambda^{e_2} u \subset E$, and we have

$$(f, (\lambda f)(e_2, e_1 + e_2)) \in \Lambda^{\min}\big(\lambda, (\lambda f)(0, e_2)\big).$$

Thus in all cases λ has a common extension with some edge in E, and E is exhaustive. $\qquad\square$

So for the graphs Λ with skeleton described in Example 1.8, there is a single Cuntz–Krieger relation at the vertex u, namely $\prod_{e \in E}(t_u - t_e t_e^*) = 0$. Now we rewrite this relation as a more familiar-looking sum.

Lemma 1.10. *Suppose that Λ is a 2-graph with skeleton described in Example* 1.8, *and E is the finite exhaustive set of Proposition* 1.9. *Then we have*

$$\prod_{e \in E}(t_u - t_e t_e^*) = t_u - \sum_{e \in u\Lambda^{e_1}\{u,w\}} t_e t_e^* - \sum_{f \in u\Lambda^{e_2}\{u,v\}} t_f t_f^* + \sum_{\mu \in u\Lambda^{e_1+e_2}\{u,v\}} t_\mu t_\mu^*.$$

$$(1.4)$$

Proof. From (1.1) and Lemma 1.4 we deduce that

$$\prod_{e\in E}(t_u - t_e t_e^*)$$

$$= t_u - \sum_{e\in u\Lambda^{e_1}\cap E} t_e t_e^* - \sum_{f\in u\Lambda^{e_2}\cap E} t_f t_f^* + \sum_{\{\mu\in\Lambda^{e_1+e_2}:\mu(0,e_i)\in E \text{ for } i=1,2\}} t_\mu t_\mu^*$$

$$= t_u - \sum_{e\in u\Lambda^{e_1}\{u,w\}} t_e t_e^* - \sum_{f\in u\Lambda^{e_2}\{u,v\}} t_f t_f^*$$

$$+ \sum_{\{\mu\in\Lambda^{e_1+e_2}:\mu(0,e_i)\in E \text{ for } i=1,2\}} t_\mu t_\mu^*.$$

To understand the last term, we claim that $\mu \in u\Lambda^{e_1+e_2}$ has $\mu(0,e_1) \in E$ and $\mu(0,e_2) \in E$ if and only if $s(\mu) = u$ or $s(\mu) = v$. The point is that if $s(\mu) = u$ or $s(\mu) = v$ then $s(\mu(0,e_i)) = u$ for $i = 1,2$, and $u\Lambda^1 u \subset E$. The alternative is that $s(\mu) = w$, and then $\mu(0,e_1)$ belongs to $u\Lambda^{e_1}v$, which is not in E. Thus

$$\{\mu \in \Lambda^{e_1+e_2} : \mu(0,e_i) \in E \text{ for } i = 1,2\} = u\Lambda^{e_1+e_2}\{u,v\},$$

and this completes the proof. □

Corollary 1.11. *Suppose that Λ is a 2-graph with skeleton described in Example 1.8. Then the Cuntz–Krieger algebra is the quotient of $\mathcal{T}C^*(\Lambda)$ by the Cuntz–Krieger relations (1.3) and*

$$t_u = \sum_{e\in u\Lambda^{e_1}\{u,w\}} t_e t_e^* + \sum_{f\in u\Lambda^{e_2}\{u,v\}} t_f t_f^* - \sum_{\mu\in u\Lambda^{e_1+e_2}\{u,v\}} t_\mu t_\mu^*.$$

2. KMS states for the graphs of Example 1.2

We wish to compute the KMS_β states for a 2-graph Λ with skeleton described in Example 1.2. Such graphs have one absolute source v. We list the vertex set as $\{u,v\}$, and write A_i for the vertex matrices, so that

$$A_i = \begin{pmatrix} d_i & a_i \\ 0 & 0 \end{pmatrix} \quad \text{for } i = 1,2.$$

We then fix $r \in (0,\infty)^2$, and consider the associated dynamics $\alpha^r : \mathbb{R} \to \text{Aut}\,\mathcal{T}C^*(\Lambda)$ such that

$$\alpha_t(t_\mu t_\nu^*) = e^{itr\cdot(d(\mu)-d(\nu))} t_\mu t_\nu^*.$$

We then consider $\beta \in (0,\infty)$ such that

$$\beta r_i > \ln\rho(A_i) \quad \text{for } i = 1 \text{ and } i = 2. \tag{2.1}$$

As observed at the start of [11, §8], even though Λ has a source, we can still apply Theorem 6.1 of [13] to find the KMS_β states.

First we need to compute the vector $y = (y_u, y_v) \in [0,\infty)^{\Lambda^0}$ appearing in that theorem. We find:

Lemma 2.1. *We have*

$$y_u = \sum_{\mu \in \Lambda u} e^{-\beta r \cdot d(\mu)} = (1 - d_1 e^{-\beta r_1})^{-1}(1 - d_2 e^{-\beta r_2})^{-1}, \quad and \qquad (2.2)$$

$$y_v = 1 + a_1 e^{-\beta r_1} y_u + a_2 e^{-\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1}. \qquad (2.3)$$

Proof. We first evaluate

$$y_u := \sum_{\mu \in \Lambda u} e^{-\beta r \cdot d(\mu)} = \sum_{n \in \mathbb{N}^2} \sum_{\mu \in \Lambda^n u} e^{-\beta r \cdot n}.$$

Each path of degree n is uniquely determined by (say) its blue-red factorisation. Then we have $d_1^{n_1}$ choices for the blue path and $d_2^{n_2}$ choices for the red path. Thus

$$y_u = \sum_{n \in \mathbb{N}^2} d_1^{n_1} d_2^{n_2} e^{-\beta(n_1 r_1 + n_2 r_2)} = \sum_{n \in \mathbb{N}^2} (d_1 e^{-\beta r_1})^{n_1}(d_2 e^{-\beta r_2})^{n_2}$$

$$= \Big(\sum_{n_1=0}^{\infty} (d_1 e^{-\beta r_1})^{n_1} \Big) \Big(\sum_{n_2=0}^{\infty} (d_2 e^{-\beta r_2})^{n_2} \Big),$$

and summing the geometric series gives (2.2).

To compute y_v, we need to list the distinct paths μ in Λv. First, if $d(\mu)_1 > 0$, then μ has a factorisation $\mu = \nu f$ with $d(f) = e_1$. Note that $s(f) = s(\mu) = v$, and hence $s(\nu) = r(f) = u$, so $\nu \in \Lambda u$. Otherwise we have $d(\mu) \in \mathbb{N}e_2$, and Λv is the disjoint union of the singleton $\{v\}$, $\bigcup_{e \in \Lambda^{e_1} v}(\Lambda u)e$, and $\bigcup_{l=0}^{\infty} \Lambda^{(l+1)e_2} v$. Counting the three sets gives

$$y_v = 1 + a_1 e^{-\beta r_1} y_u + \sum_{l=0}^{\infty} a_2 d_2^l e^{-\beta(l+1)r_2}$$

$$= 1 + a_1 e^{-\beta r_1} y_u + a_2 e^{-\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1},$$

and hence we have (2.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.2. We made a choice when we computed y_v: we considered the complementary cases $d(\mu)_1 > 0$ and $d(\mu)_1 = 0$. We could equally well have chosen to use the cases $d(\mu)_2 > 0$ and $d(\mu)_2 = 0$, and we would have found

$$y_v = 1 + a_2 e^{-\beta r_2}(1 - d_1 e^{-\beta r_1})^{-1}(1 - d_2 e^{-\beta r_2})^{-1} + a_1 e^{-\beta r_1}(1 - d_1 e^{-\beta r_1})^{-1}, \tag{2.4}$$

which looks different. To see that they are in fact equal, we look at the difference. To avoid messy formulas, we write

$$\Delta := (1 - d_1 e^{-\beta r_1})(1 - d_2 e^{-\beta r_2}),$$

and observe that, for example, $(1 - d_1 e^{-\beta r_1})^{-1} = (1 - d_2 e^{-\beta r_2})\Delta^{-1}$. Then the difference $(2.3) - (2.4)$ is

$$a_1 e^{-\beta r_1} \Delta^{-1} + a_2 e^{-\beta r_2}(1 - d_1 e^{-\beta r_1})\Delta^{-1} - a_2 e^{-\beta r_2}\Delta^{-1}$$

$$- a_1 e^{-\beta r_1}(1 - d_2 e^{-\beta r_2})\Delta^{-1}.$$

When we expand the brackets we find that the terms $a_1 e^{-\beta r_1}\Delta^{-1}$ and $a_2 e^{-\beta r_2}\Delta^{-1}$ cancel out, leaving

$$-a_2 e^{-\beta r_2}d_1 e^{-\beta r_1}\Delta^{-1}+a_1 e^{-\beta r_1}d_2 e^{-\beta r_2}\Delta^{-1}$$
$$= (-a_2 d_1 + a_1 d_2)e^{-\beta(r_1+r_2)}\Delta^{-1},$$

which vanishes because the factorisation property forces $a_1 d_2 = a_2 d_1$.

We recall that we are considering β satisfying (2.1). The first step in the procedure of [9, §8] for such β is to apply [13, Theorem 6.1]. Then the KMS states of $(\mathcal{T}C^*(\Lambda),\alpha^r)$ have the form ϕ_ϵ for $\epsilon \in [0,\infty)^{\{u,v\}}$ satisfying $\epsilon \cdot y = 1$. This is a 1-dimensional simplex with extreme points $(y_u^{-1},0)$ and $(0,y_v^{-1})$. The values of the state ϕ_ϵ on the vertex projections t_u and t_v are the coordinates of the vector

$$m = \Big(\prod_{i=1}^{2}(1 - e^{-\beta r_i}A_i)^{-1}\Big)\epsilon.$$

To find m, we compute

$$\prod_{i=1}^{2}(1 - e^{-\beta r_i}A_i)^{-1} = \Delta^{-1}\begin{pmatrix} 1 & a_2 e^{-\beta r_2} + a_1 e^{-\beta r_1}(1 - d_2 e^{-\beta r_2}) \\ 0 & (1 - d_1 e^{-\beta r_1})(1 - d_2 e^{-\beta r_2}) \end{pmatrix}.$$

For the first extreme point $\epsilon = (y_u^{-1},0)$, we get $m = (1,0)$ and the corresponding KMS_β state ϕ_1 satisfies

$$\begin{pmatrix} \phi_1(t_u) \\ \phi_1(t_v) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.5}$$

Lemma 6.2 of [1] (for example) implies that ϕ_1 factors through a state of the quotient by the ideal of $\mathcal{T}C^*(\Lambda)$ generated by t_v, which is the ideal denoted $I_{\{v\}}$ in [11, §2.4]. Thus the quotient is $\mathcal{T}C^*(\Lambda\backslash\{v\}) = \mathcal{T}C^*(u\Lambda u)$. The general theory of [13] says that $(\mathcal{T}C^*(u\Lambda u),\alpha^r)$ has a unique KMS_β state ψ, and we therefore have $\phi_1 = \psi \circ q_{\{v\}}$, where $q_{\{v\}}$ is the quotient map of $\mathcal{T}C^*(\Lambda)$ onto $\mathcal{T}C^*(\Lambda\backslash\{v\})$ for the hereditary subset $\{v\}$ of Λ^0 from [11, Proposition 2.2].

Now we consider the other extreme point $\epsilon = (0, y_v^{-1})$. This yields a KMS_β state ϕ_2 such that

$$\begin{pmatrix} \phi_2(t_u) \\ \phi_2(t_v) \end{pmatrix} = \begin{pmatrix} y_v^{-1}\Delta^{-1}\big(a_2 e^{-\beta r_2} + a_1 e^{-\beta r_1} - a_1 d_2 e^{-\beta(r_1+r_2)}\big) \\ y_v^{-1} \end{pmatrix}. \tag{2.6}$$

Because this vector ϵ is supported on the absolute source v, Proposition 8.2 of [11] implies that ϕ_2 factors through a state of $(C^*(\Lambda),\alpha^r)$ (and we can also verify this directly — see the remark below).

We summarise our findings as follows.

Proposition 2.3. *Suppose that Λ, r and β are as described at the start of the section. Then $(\mathcal{T}C^*(\Lambda),\alpha^r)$ has a 1-dimensional simplex of KMS_β states with extreme points ϕ_1 and ϕ_2 satisfying (2.5) and (2.6). The KMS state*

ϕ_1 *factors through a state* ψ *of* $\mathcal{T}C^*(u\Lambda u)$, *and the KMS state* ϕ_2 *factors through a state of* $C^*(\Lambda)$.

Remark 2.4. At this stage we can do some reassuring reality checks. First, we check that $\phi_2(t_u) + \phi_2(t_v) = 1$. We multiply through by y_v to take the y_v^{-1} out. Then we compute using that $a_1 d_2 = a_2 d_1$:

$$
\begin{aligned}
y_v \phi_2(t_u) + y_v \phi_2(t_v) &= y_v \phi_2(t_u) + 1 \\
&= \Delta^{-1}\big(a_2 e^{-\beta r_2} + a_1 e^{-\beta r_1} - a_1 d_2 e^{-\beta(r_1 + r_2)}\big) + 1 \\
&= \Delta^{-1}\big(a_2 e^{-\beta r_2} + a_1 e^{-\beta r_1} - a_2 e^{-\beta r_2} d_1 e^{-\beta r_1}\big) + 1 \\
&= \Delta^{-1}\big(a_2 e^{-\beta r_2}(1 - d_1 e^{-\beta r_1}) + a_1 e^{-\beta r_1}\big) + 1 \\
&= a_2 e^{-\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1} + a_1 e^{-\beta r_1} \Delta^{-1} + 1,
\end{aligned}
$$

which is the formula for y_v reshuffled.

Next, we verify directly that ϕ_2 factors through a state of $C^*(\Lambda)$. We saw in Example 1.2 that the only finite exhaustive subset of $u\Lambda^1$ is $u\Lambda^1$, and then Corollary 1.6 implies that

$$
\phi_2\Big(\prod_{e \in u\Lambda^1} (t_u - t_e t_e^*)\Big) \tag{2.7}
$$

$$
= \phi_2\Big(t_u - \sum_{e \in u\Lambda^{e_1}} t_e t_e^* - \sum_{f \in u\Lambda^{e_2}} t_f t_f^* + \sum_{\mu \in u\Lambda^{e_1 + e_2}} t_\mu t_\mu^*\Big).
$$

Now we break each sum into two sums over subsets of $u\Lambda u$ and $u\Lambda v$, and apply the KMS condition to each $\phi_2(t_\mu t_\mu^*) = e^{-\beta r \cdot d(\mu)} \phi_2(t_{s(\mu)})$. We find that (2.7) is

$$
\Delta \phi_2(t_u) - \big(a_1 e^{-\beta r_1} + a_2 e^{-\beta r_2} - a_1 d_2 e^{-\beta(r_1 + r_2)}\big)\phi_2(t_v),
$$

which vanishes by (2.6). Now the standard argument (using, for example, [1, Lemma 6.2]) shows that ϕ_2 factors though a state of the Cuntz–Krieger algebra $C^*(\Lambda)$, which by Example 1.2 is the quotient of $\mathcal{T}C^*(\Lambda)$ by the single Cuntz–Krieger relation $\prod_{e \in u\Lambda^1}(t_u - t_e t_e^*) = 0$.

3. KMS states for the graphs of Example 1.8

We now consider a 2-graph Λ with the skeleton described in Example 1.8. Such graphs have one absolute source w, and $\Lambda \backslash \{w\}$ is the graph discussed in the previous section. As usual, we consider a dynamics determined by $r \in (0, \infty)^2$, and we want to use Theorem 6.1 of [13] to find the KMS_β states for β satisfying $\beta r_i > \ln \rho(A_i)$. Our first task is to find the vector $y = (y_u, y_v, y_w)$.

Since the sets Λu and Λv lie entirely in the subgraph with vertices $\{u, v\}$, the numbers $y_u := \sum_{\mu \in \Lambda u} e^{-\beta r \cdot d(\mu)}$ and y_v are given by Lemma 2.1. So it remains to compute y_w. We find:

Lemma 3.1. *We define $\Delta := (1 - d_1 e^{-\beta r_1})(1 - d_2 e^{-\beta r_2})$. Then we have*

$$y_u = \Delta^{-1},$$

$$y_v = 1 + a_1 e^{-\beta r_1} \Delta^{-1} + a_2 e^{-\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1}$$

$$= 1 + a_1 e^{-\beta r_1} \Delta^{-1} + a_2 e^{-\beta r_2}(1 - d_1 e^{-\beta r_1})\Delta^{-1}, \quad and \qquad (3.1)$$

$$y_w = 1 + b_2 e^{-\beta r_2} + b_1 e^{-\beta r_1} \Delta^{-1} + a_2 b_2 e^{-2\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1}. \qquad (3.2)$$

Proof. As foreshadowed above, the formula for y_u and the first formula for y_v follow from Lemma 2.1. The formula (3.1) is just a rewriting of the previous one which will be handy in computations (and this trick will be used a lot later).

To find y_w, we consider the paths $\mu = \nu e$ with $e \in \Lambda^{e_1} w$ and $\nu \in \Lambda u$ (these are the ones with $d(\mu) \geq e_1$). There are b_1 such e, and hence we have a contribution $b_1 e^{-\beta r_1} y_u = b_1 e^{-\beta r_1} \Delta^{-1}$ to y_w. The remaining paths are in $\Lambda^{\mathbb{N}e_2} w$, and give a contribution of

$$1 + b_2 e^{-\beta r_2} + b_2 e^{-\beta r_2} a_2 e^{-\beta r_2} \sum_{l=0}^{\infty} d_2^l e^{-\beta r_2 l}$$

$$= 1 + b_2 e^{-\beta r_2} + b_2 e^{-\beta r_2} a_2 e^{-\beta r_2}(1 - d_2 e^{-\beta r_2})^{-1}.$$

Adding the two contributions gives (3.2). $\qquad \square$

Remark 3.2. We could also have computed y_w by counting the paths with $d(\mu) \geq e_2$ and those in $\Lambda^{\mathbb{N}e_1} w$. This gives

$$y_w = 1 + b_1 e^{-\beta r_1}(1 - d_1 e^{-\beta r_1})^{-1} + b_2 e^{-\beta r_2} y_v. \qquad (3.3)$$

We found the check that this is the same as the right-hand side of (3.2) instructive. First, we use the alternative formula (2.4) for y_v (whose proof in Remark 2.2 used the crucial identity $a_1 d_2 = a_2 d_1$). Then the right-hand side of (3.3) becomes

$$1 + b_1 e^{-\beta r_1}(1 - d_1 e^{-\beta r_1})^{-1}$$
$$+ b_2 e^{-\beta r_2}\left(1 + a_2 e^{-\beta r_2} \Delta^{-1} + a_1 e^{-\beta r_1}(1 - d_1 e^{-\beta r_1})^{-1}\right).$$

Now we write $(1 - d_1 e^{-\beta r_1})^{-1} = (1 - d_2 e^{-\beta r_2})\Delta^{-1}$, similarly for the term $(1 - d_2 e^{-\beta r_2})^{-1}$, and expand the brackets: we get

$$1 + b_1 e^{-\beta r_1} \Delta^{-1} - b_1 d_2 e^{-\beta(r_1+r_2)} \Delta^{-1} + b_2 e^{-\beta r_2} + b_2 a_2 e^{-2\beta r_2} \Delta^{-1}$$
$$+ b_2 a_1 e^{-\beta(r_1+r_2)} \Delta^{-1} - b_2 a_1 d_2 e^{-\beta(r_1+2r_2)} \Delta^{-1}.$$

Now we recall from Example 1.8 that $b_1 d_2 = b_2 a_1$, and hence the third and sixth terms cancel. Next we use the identity $a_1 d_2 = a_2 d_1$ in the last term. We arrive at

$$1 + b_1 e^{-\beta r_1} \Delta^{-1} + b_2 e^{-\beta r_2} + b_2 a_2 e^{-2\beta r_2} \Delta^{-1} - b_2 a_2 d_1 e^{-\beta(r_1+2r_2)} \Delta^{-1}$$
$$= 1 + b_1 e^{-\beta r_1} \Delta^{-1} + b_2 e^{-\beta r_2} + b_2 a_2 e^{-2\beta r_2}(1 - d_1 e^{-\beta r_1})\Delta^{-1}$$

$$= 1 + b_1 e^{-\beta r_1} \Delta^{-1} + b_2 e^{-\beta r_2} + b_2 a_2 e^{-2\beta r_2} (1 - d_2 e^{-\beta r_2})^{-1},$$

which is the the formula for y_w in (3.2). We find it reassuring that we had to explicitly use both relations $b_1 d_2 = b_2 a_1$ and $a_1 d_2 = a_2 d_1$ that are imposed on us by the assumption that our coloured graph is the skeleton of a 2-graph.

Theorem 6.1 of [13] says that for each β satisfying $\beta r_i > \ln \rho(A_i)$ for $i = 1, 2$, there is a simplex of KMS_β states ϕ_ϵ on $(\mathcal{T}C^*(\Lambda), \alpha^r)$ parametrised by the set

$$\Delta_\beta = \{ \epsilon \in [0, \infty)^{\{u,v,w\}} : \epsilon \cdot y = 1 \}.$$

Here, the set Δ_β is a 2-dimensional simplex with extreme points $e_u := (y_u^{-1}, 0, 0)$, $e_v := (0, y_v^{-1}, 0)$, and $e_w = (0, 0, y_w^{-1})$. The values of ϕ_ϵ on the vertex projections are the entries in the vector $m(\epsilon) = \prod_{i=1}^{2} (1 - e^{-\beta r_i} A_i)^{-1} \epsilon$. Since the matrices $1 - e^{-\beta r_i} A_i$ are upper-triangular, so are their inverses, and we deduce that both $m(e_u)$ and $m(e_v)$ have final entry $m(e_u)_w = 0 = m(e_v)_w$. So the corresponding KMS states are the compositions of the states of $(\mathcal{T}C^*(\Lambda\backslash\{w\}), \alpha^{(r_1, r_2)})$ with the quotient map $q_{\{w\}} : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda\backslash\{w\})$. Thus the extreme points of the simplex of KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ are $\phi_1 \circ q_{\{w\}} = (\psi \circ q_{\{v\}}) \circ q_{\{w\}}$, $\phi_2 \circ q_{\{w\}}$ and $\psi_3 := \phi_{e_w}$.

Remark 3.3. We recall from the end of the previous section that the state ϕ_2 of $\mathcal{T}C^*(\Lambda\backslash\{w\})$ factors through a state of the Cuntz–Krieger algebra $C^*(\Lambda\backslash\{w\})$. So it is tempting to ask whether $\phi_2 \circ q_{\{w\}}$ factors through a state of $C^*(\Lambda)$. Hoewever, this is not the case. The point is that in the graph $\Lambda\backslash\{w\}$, the vertex v is an absolute source, and hence there is no Cuntz–Krieger relation involving t_v. However, in the larger graph Λ, v is not an absolute source: the set $v\Lambda^{e_2}$ is a nontrivial finite exhaustive subset of $v\Lambda^1$, and hence the Cuntz–Krieger family generating $C^*(\Lambda)$ must satisfy the relation

$$\prod_{e \in v\Lambda^{e_2}} (t_v - t_e t_e^*) = 0 \iff t_v - \sum_{e \in v\Lambda^{e_2}} t_e t_e^* = 0.$$

The KMS condition implies that the state $\phi := \phi_2 \circ q_{\{w\}}$ satisfies

$$\phi(t_e t_e^*) = e^{-\beta r_2} \phi(t_{s(e)}) = e^{-\beta r_2} \phi(t_w)$$
$$= e^{-\beta r_2} \phi_2 \circ q_{\{w\}}(t_w) = e^{-\beta r_2} \phi_2(0) = 0$$

for all $e \in v\Lambda^{e_2}$. Since we know from (2.6) that $\phi(t_v) = \phi_2(t_v) = y_v^{-1}$ is not zero, we deduce that

$$\phi\left(t_v - \sum_{e \in v\Lambda^{e_2}} t_e t_e^*\right) = \phi(t_v) \neq 0.$$

Thus $\phi = \phi_2 \circ q_{\{w\}}$ does not factor through a state of $C^*(\Lambda)$.

We now focus on the new extreme point is ϕ_{e_w}. To compute it, we need to calculate $\prod_{i=1}^{2} (1 - e^{-\beta r_i} A_i)^{-1}$. Since the matrices A_1 and A_2 commute,

so do the matrices $1 - e^{-\beta r_i} A_i$, and it suffices to compute the inverse of

$$\prod_{i=1}^{2}(1 - e^{-\beta r_i} A_i) =$$

$$\begin{pmatrix} \Delta & -(1 - d_1 e^{-\beta r_1})a_2 e^{-\beta r_2} - a_1 e^{-\beta r_1} & e^{-\beta(r_1+r_2)}a_1 b_2 - b_1 e^{-\beta r_1} \\ 0 & 1 & -b_2 e^{-\beta r_2} \\ 0 & 0 & 1 \end{pmatrix},$$

where as before we write $\Delta = \prod_{i=1}^{2}(1 - d_i e^{-\beta r_i})$. We find that the inverse is Δ^{-1} times

$$\begin{pmatrix} 1 & (1 - d_1 e^{-\beta r_1})a_2 e^{-\beta r_2} + a_1 e^{-\beta r_1} & (1 - d_1 e^{-\beta r_1})a_2 b_2 e^{-2\beta r_2} + b_1 e^{-\beta r_1} \\ 0 & \Delta & \Delta b_2 e^{-\beta r_2} \\ 0 & 0 & \Delta \end{pmatrix}.$$

Thus the corresponding KMS_β state ϕ_{e_w} satisfies

$$\begin{pmatrix} \phi_{e_w}(t_u) \\ \phi_{e_w}(t_v) \\ \phi_{e_w}(t_w) \end{pmatrix} = \begin{pmatrix} \Delta^{-1}\big((1 - d_1 e^{-\beta r_1})a_2 b_2 e^{-2\beta r_2} + b_1 e^{-\beta r_1}\big)y_w^{-1} \\ b_2 e^{-\beta r_2} y_w^{-1} \\ y_w^{-1} \end{pmatrix}. \qquad (3.4)$$

Remark 3.4. As usual, we take the opportunity for a reality check: since $t_u + t_v + t_w$ is the identity of $\mathcal{T}C^*(\Lambda)$ and ϕ_{e_w} is a state, we must have $\phi_{e_w}(t_u) + \phi_{e_w}(t_v) + \phi_{e_w}(t_w) = 1$. But since

$$\Delta^{-1}(1 - d_1 e^{-\beta r_1}) = (1 - d_2 e^{-\beta r_2})^{-1},$$

the formula (3.2) says that this sum is precisely $y_w y_w^{-1} = 1$.

We summarise our findings in the following theorem.

Theorem 3.5. *Suppose that Λ is a 2-graph with skeleton described in Example 1.8 and vertex matrices A_1, A_2. We suppose that $r \in (0,\infty)^2$, and consider the dynamics α^r on $\mathcal{T}C^*(\Lambda)$. We suppose that $\beta > 0$ satisfies $\beta r_i > \ln \rho(A_i)$ for $i = 1, 2$. We write ϕ_1 and ϕ_2 for the KMS_β states of $(\mathcal{T}C^*(\Lambda\backslash\{w\}), \alpha^r)$ described before Remark 2.4. Then $\phi_1 \circ q_{\{w\}}$ and $\phi_2 \circ q_{\{w\}}$ are KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. There is another KMS_β state $\phi_3 = \phi_{e_w}$ satisfying (3.4). Every KMS_β state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ is a convex combination of the three states $\phi_1 \circ q_{\{w\}}$, $\phi_2 \circ q_{\{w\}}$ and ϕ_3. None of these KMS_β states factors through a state of $(C^*(\Lambda), \alpha^r)$.*

The only thing we haven't proved is the assertion that every KMS state is a convex combination of the states that we have described. But this follows from the general results in [13, Theorem 6.1], because the vectors $(y_u^{-1}, 0, 0)$, $(0, y_v^{-1}, 0)$ and $(0, 0, y_w^{-1})$ are the extreme points of the simplex Δ_β.

4. KMS states at the critical inverse temperature

We begin with the graphs of Example 1.2. We observe that the hypothesis of rational independence in the two main results of this section is in practice

easy to verify using Proposition A.1 of [11]: loosely, $\ln d_1$ and $\ln d_2$ are rationally independent unless d_1 and d_2 are different powers of the same integer.

Proposition 4.1. *Suppose that Λ is a 2-graph with the skeleton described in Example 1.2 and that $r \in (0, \infty)^2$ has $r_i \geq \ln d_i$ for both i, $r_i = \ln d_i$ for at least one i, and $\{r_1, r_2\}$ are rationally independent. Consider the quotient map $q_{\{v\}} : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda \backslash \{v\})$ from [11, Proposition 2.2]. Then $(\mathcal{T}C^*(\Lambda \backslash \{v\}), \alpha^r)$ has a unique KMS_1 state ϕ, and $\phi \circ q_{\{v\}}$ is the only KMS_1 state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.*

Lemma 4.2. *Suppose that Λ and r are as in Proposition 4.1, and that ϕ is a KMS_1 state ϕ of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then $\phi(t_v) = 0$.*

Proof. We recall from Example 1.2 that the only finite exhaustive subset of $u\Lambda^1$ is $u\Lambda^1$ itself, and from Example 1.7 we then have

$$\prod_{e \in u\Lambda^1} (t_u - t_e t_e^*) = t_u - \sum_{e \in u\Lambda^1} t_e t_e^* + \sum_{\mu \in u\Lambda^{e_1+e_2}} t_\mu t_\mu^*.$$

Thus positivity of $\phi\big(\prod_{e \in u\Lambda^1}(t_u - t_e t_e^*)\big)$ implies that

$$0 \leq \phi(t_u) - \sum_{e \in u\Lambda^{e_1}} \phi(t_e t_e^*) - \sum_{e \in u\Lambda^{e_2}} \phi(t_e t_e^*) + \sum_{\mu \in u\Lambda^{e_1+e_2}} \phi(t_\mu t_\mu^*).$$

Now we use the KMS relation and count paths of various degrees to get

$$0 \leq \phi(t_u) - \sum_{e \in u\Lambda^{e_1}} e^{-r_1} \phi(t_{s(e)}) - \sum_{e \in u\Lambda^{e_2}} e^{-r_2} \phi(t_{s(e)})$$
$$+ \sum_{\mu \in u\Lambda^{e_1+e_2}} e^{-(r_1+r_2)} \phi(t_{s(\mu)})$$
$$= \phi(t_u) - e^{-r_1}\big(d_1\phi(t_u) + a_1\phi(t_v)\big) - e^{-r_2}\big(d_2\phi(t_u) + a_2\phi(t_v)\big)$$
$$+ e^{-(r_1+r_2)}\big(d_1 d_2 \phi(t_u) + d_1 a_2 \phi(t_v)\big)$$
$$= \big(1 - d_1 e^{-r_1} - d_2 e^{-r_2} + d_1 d_2 e^{-(r_1+r_2)}\big)\phi(t_u)$$
$$- \big(a_1 e^{-r_1} + a_2 e^{-r_2} - d_1 a_2 e^{-(r_1+r_2)}\big)\phi(t_v)$$
$$= \prod_{i=1}^{2}(1 - d_i e^{-r_i})\phi(t_u) - \big(a_1 e^{-r_1} + a_2 e^{-r_2} - d_1 a_2 e^{-(r_1+r_2)}\big)\phi(t_v)$$
$$= -\big(a_1 e^{-r_1} + a_2 e^{-r_2} - d_1 a_2 e^{-(r_1+r_2)}\big)\phi(t_v),$$

where the coefficient of $\phi(t_u)$ vanished because for at least one of $i = 1, 2$ we have $1 - d_i e^{-r_i} = 1 - d_i d_i^{-1} = 0$ by the hypotheses on r_i. If $r_1 = \ln d_1$, then we write this as

$$0 \leq -\big(a_1 e^{-r_1} + (1 - d_1 e^{-r_1})a_2 e^{-r_2}\big)\phi(t_v) = -a_1 e^{-r_1} \phi(t_v),$$

and positivity of t_v implies that $\phi(t_v)$. If $r_2 = \ln d_2$, then we use the identity $d_1 a_2 = d_2 a_1$ to rewrite it as

$$0 \le -\big(a_2 e^{-r_2} + (1 - d_2 e^{-r_2})a_1 e^{-r_1}\big)\phi(t_v) = -a_2 e^{-r_2}\phi(t_v),$$

which also implies that $\phi(t_v) = 0$. □

Proof of Proposition 4.1. Proposition 4.2 of [10] implies that the system $(\mathcal{T}C^*(\Lambda\backslash\{v\}), \alpha^r)$ has a unique KMS$_1$ state ϕ. Since $q_{\{v\}}$ interwines the two actions α^r, the composition $\phi \circ q_{\{v\}}$ is a KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. On the other hand, if ψ is a KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$, then Lemma 4.2 implies that $\psi(t_v) = 0$. The standard argument using [1, Lemma 6.2] shows that ψ factors through the quotient by the ideal generated by t_v, which is precisely the kernel of $q_{\{v\}}$. Thus there is a KMS$_1$ state θ of $(\mathcal{T}C^*(\Lambda\backslash\{v\}), \alpha^r)$ such that $\psi = \theta \circ q_{\{v\}}$, and uniqueness of the KMS$_1$ state implies that $\theta = \phi$. Hence $\psi = \phi \circ q_{\{v\}}$, as required. □

Theorem 4.3. *Suppose that Λ is a 2-graph with the skeleton described in Example 1.8 and that $r \in (0, \infty)^2$ has $r_i \ge \ln d_i$ for both i, $r_i = \ln d_i$ for at least one i, and $\{r_1, r_2\}$ are rationally independent. Consider the quotient map $q_{\{v,w\}}$ of $\mathcal{T}C^*(\Lambda)$ onto $\mathcal{T}C^*(\Lambda\backslash\{v,w\})$ discussed in [11, Proposition 2.2]. Then $(\mathcal{T}C^*(\Lambda\backslash\{v,w\}), \alpha^r)$ has a unique KMS$_1$ state ϕ, and $\phi \circ q_{\{v,w\}}$ is the only KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.*

We need an analogue of Lemma 4.2 for the present situation.

Lemma 4.4. *Under the hypotheses of the preceding theorem, suppose that ϕ is a KMS$_1$ state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then $\phi(t_w) = 0$.*

Proof. We use an argument like that in the proof of Lemma 4.2 for the exhaustive subset $E = (u\Lambda^1 u) \cup (u\Lambda^{e_2} v) \cup (u\Lambda^{e_1} w)$ of $u\Lambda^1$ discussed in Example 1.8. Then the state satisfies

$$\phi\big(\textstyle\prod_{e\in E}(t_u - t_e t_e^*)\big) \ge 0.$$

Now using Lemma 1.10 to write $\prod_{e\in E}(t_u - t_e t_e^*)$ as a sum gives

$$\phi\Big(t_u - \sum_{e\in u\Lambda^{e_1}\{u,w\}} t_e t_e^* - \sum_{f\in u\Lambda^{e_2}\{u,v\}} t_f t_f^* + \sum_{\mu\in u\Lambda^{e_1+e_2}\{u,v\}} t_\mu t_\mu^*\Big) \ge 0.$$

Now we use linearity of ϕ and the KMS condition to get

$$0 \le \phi(t_u) - \sum_{e\in u\Lambda^{e_1}\{u,w\}} \phi(t_e t_e^*) - \sum_{f\in u\Lambda^{e_2}\{u,v\}} \phi(t_f t_f^*)$$

$$+ \sum_{\mu\in u\Lambda^{e_1+e_2}\{u,v\}} \phi(t_\mu t_\mu^*)$$

$$= \phi(t_u) - \big(d_1 e^{-r_1}\phi(t_u) + b_1 e^{-r_1}\phi(t_w)\big) - \big(d_2 e^{-r_2}\phi(t_u) + a_2 e^{-r_2}\phi(t_v)\big)$$

$$+ \big(d_1 d_2 e^{-(r_1+r_2)}\phi(t_u) + d_1 a_2 e^{-(r_1+r_2)}\phi(t_v)\big)$$

$$= \prod_{i=1}^{2}(1 - d_i e^{-r_i})\phi(t_u) - (1 - d_1 e^{-r_1})a_2 e^{-r_2}\phi(t_v) - b_1 e^{-r_1}\phi(t_w).$$

Since at least one r_i is $\ln d_i$, we have $(1 - d_1 e^{-r_1})(1 - d_2 e^{-r_2}) = 0$, and we deduce that

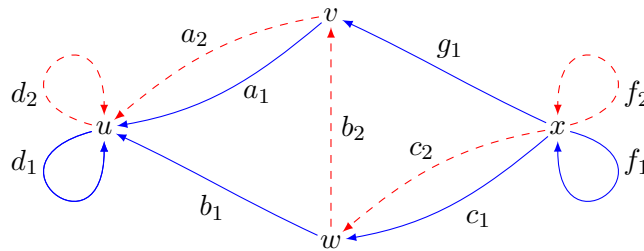$$0 \leq -(1 - d_1 e^{-r_1})a_2 e^{-r_2}\phi(t_v) - b_1 e^{-r_1}\phi(t_w).$$

Since $(1 - d_1 e^{-r_1})a_2 e^{-r_2}$, $\phi(t_v)$, and $b_1 e^{-r_1}$ are all nonnegative, we must have both $(1 - d_1 e^{-r_1})a_2 e^{-r_2}\phi(t_v) = 0$ and $b_1 e^{-r_1}\phi(t_w) = 0$. In particular, we deduce that $\phi(t_w) = 0$. $\qquad\square$

Proof of Theorem 4.3. We suppose that ψ is a KMS$_1$ state of the system $(\mathcal{T}C^*(\Lambda), \alpha^r)$. Then Lemma 4.4 implies that $\psi(t_w) = 0$. The formula in [11, Proposition 2.1(1)] implies that ψ vanishes on the ideal $I_{\{w\}}$ generated by t_w, and by [1, Lemma 6.2] ψ factors through a KMS$_1$ state θ of the system $(\mathcal{T}C^*(\Lambda\backslash\{w\}), \alpha^r)$. The 2-graph $\Lambda\backslash\{w\}$ is the graph in Proposition 4.1, and hence that Proposition implies that $\theta = \phi \circ q_{\{v\}}$. The kernel of the composition $q_{\{v\}} \circ q_{\{w\}}$ is the ideal generated by $\{t_v, t_w\}$, and a glance at the definition of the homomorphism in [11, Proposition 2.2(2)] shows that $q_{\{v\}} \circ q_{\{w\}} = q_{\{v,w\}}$. Thus

$$\psi = \theta \circ q_{\{w\}} = (\phi \circ q_{\{v\}}) \circ q_{\{w\}} = \phi \circ q_{\{v,w\}}. \qquad\square$$

5. Where our examples came from

We consider a family of 2-graphs Λ with skeleton of the form

In these graphs there are two nontrivial strongly connected components $\{u\}$ and $\{x\}$, and the bridges $\mu \in u\Lambda x$ all have $|d(\mu)| > 1$. The graphs in Examples 1.2 and 1.8 are then the graphs $\Lambda\backslash\{w, x\}$ and $\Lambda\backslash\{x\}$, respectively. We assume that all the integers $d_i, a_i, b_i, c_i, g_i, f_i$, are nonzero.

The vertex matrices of the graph Λ are then

$$A_1 = \begin{pmatrix} d_1 & a_1 & b_1 & 0 \\ 0 & 0 & 0 & g_1 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & f_1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} d_2 & a_2 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & f_2 \end{pmatrix}. \quad (5.1)$$

Thus we have $\rho(A_i) = \max\{d_i, f_i\}$ for $i = 1, 2$.

As in the last section, we consider a dynamics $\alpha^r : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ given by $r \in (0, \infty)^2$ such that $r_i \geq \ln \rho(A_i)$ for $i = 1, 2$, $r_i = \ln \rho(A_i)$ for at least one i and $\{r_1, r_2\}$ are rationally independent. We are interested in the KMS_1 states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

First, we suppose that $r_i = \ln \rho(A_i) = \ln d_i$ for one of $i = 1$ or $i = 2$. Then the strongly connected component $\{u\}$ is i-critical in the sense of [9, §3], and Proposition 3.1 and Corollary 3.2 in [9] imply that all the KMS_1 states factor through states of $\Lambda \backslash \{v, w, x\} = u\Lambda u$. Proposition 4.2 of [10] then implies that $(\mathcal{T}C^*(\Lambda), \alpha^r)$ has a unique KMS_1 state.

Second, we suppose that there exists $j \in \{1, 2\}$ such that $r_j = \ln \rho(A_j) = \ln f_j > \ln d_j$. We aim to apply [9, Proposition 4.1] to the hereditary set $D = \{x\}$, and our hypothesis on j says that j belongs to the set L_D described in that Proposition. We will deal separately with the cases $j = 2$ and $j = 1$.

We do $j = 2$ first. Then in [9, Proposition 4.1] we have $H = \emptyset$ and $D = \{u, v, w\}$, and the block decomposition of A_2 in [9, (4.2)] is

$$A_2 = \begin{pmatrix} E_2 & B_2 \\ 0 & f_2 \end{pmatrix}$$

where E_2 is 3×3 and $B_2 = \begin{pmatrix} 0 & 0 & c_2 \end{pmatrix}^T$. The unimodular Perron–Frobenius eigenvector of the matrix (f_2) is the scalar 1, and the condition $\rho(A_2) = f_2 > d_2$ means that $\rho(A_{\{x\},2})1 - E_2 = f_2 1 - E_2$ is invertible. So the vector y in [9, Proposition 4.1] is

$$
\begin{aligned}
y &= \left(\rho(A_{\{x\},2})1 - E_2 \right)^{-1} B_2 \\
&= \begin{pmatrix} f_2 - d_2 & -a_2 & 0 \\ 0 & f_2 & -b_2 \\ 0 & 0 & f_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix} \\
&= \frac{1}{(f_2 - d_2)f_2^2} \begin{pmatrix} f_2^2 & a_2 f_2 & a_2 b_2 \\ 0 & (f_2 - d_2)f_2 & (f_2 - d_2)b_2 \\ 0 & 0 & (f_2 - d_2)f_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix} \\
&= \frac{1}{(f_2 - d_2)f_2^2} \begin{pmatrix} a_2 b_2 c_2 \\ (f_2 - d_2)b_2 c_2 \\ (f_2 - d_2)f_2 c_2 \end{pmatrix} = \begin{pmatrix} (f_2 - d_2)^{-1} f_2^{-2} a_2 b_2 c_2 \\ f_2^{-2} b_2 c_2 \\ f_2^{-1} c_2 \end{pmatrix}. \quad (5.2)
\end{aligned}
$$

When we repeat this for $j = 1$, we find that $y =$

$$
\frac{1}{(f_1 - d_1)f_1^2} \begin{pmatrix} a_1 f_1 g_1 + f_1 b_1 c_1 \\ (f_1 - d_1)f_1 g_1 \\ (f_1 - d_1)f_1 c_1 \end{pmatrix} = \begin{pmatrix} (f_1 - d_1)^{-1} f_1^{-1}(a_1 g_1 + b_1 c_1) \\ f_1^{-1} g_1 \\ f_1^{-1} c_1 \end{pmatrix}. \quad (5.3)
$$

Remark 5.1. It is a little disconcerting that we get different eigenvectors for the same matrix. Indeed, it is quite possible that we have $r_i = \ln \rho(A_i) = \ln f_i$ for both $i = 1$ and $i = 2$, in which case both 1 and 2 belong the set $L_{\{x\}}$, and Proposition 4.1(ii) of [9] implies that the corresponding vectors y

are equal. So we had a look at the two formulas, and found the results to be instructive.

Equality of the third entries in (5.2) and (5.3) is equivalent to $f_1 c_2 = f_2 c_1$, which is one of the relations imposed by the requirement that Λ is a 2-graph. Similar reasoning works for the second entries. But when we removed the inverses by cross-multiplying, the top entry in the second calculation became

$$(f_2 - d_2) f_2^2 (a_1 g_1 + b_1 c_1) = f_2^3 a_1 g_1 + f_2^3 b_1 c_1 - d_2 f_2^2 a_1 g_1 - d_2 f_2^2 b_1 c_1. \quad (5.4)$$

In the other calculation, the top entry has just two summands, which are again products of 5 factors. After staring at them for a bit, we realised that these products have meaning: for example, $f_2^3 a_1 g_1$ is the number $a_1 g_1 f_2^3$ of paths in $u \Lambda^{2e_1 + 3e_2} x$ counted using their BBRRR factorisations. Similarly, $d_2 f_2^2 b_1 c_1 = d_2 b_1 c_1 f_2^2$ counts the same set using the RBBRR factorisations. Now looking at the skeleton confirms that

$$a_1 g_1 f_2^3 = a_1 (g_1 f_2) f_2^2 = a_1 (b_2 c_1) f_2^2 = (a_1 b_2) c_1 f_2^2 = d_2 b_1 c_1 f_2^2,$$

and the first and last terms on the right of (5.4) cancel. Similar considerations using the 1-skeleton of the graph Λ match up the remaining terms in the top entries in (5.2) and (5.3).

We now use the results of [9, §5] to describe all the KMS$_1$ states on $(\mathcal{T}C^*(\Lambda), \alpha^r)$ when Λ has skeleton described at the start of the section. Notice that the set $\mathcal{C}$ of *nontrivial* components of our graph is $\mathcal{C} = \{\{u\}, \{x\}\}$; both components are coordinatewise irreducible, and the existence of j such that $r_j = \ln \rho(A_j) = \ln f_j > \ln d_j$ means that [9, (5.1)] holds. Thus we can apply [9, Proposition 5.1] with $z = \binom{y}{1}$ to get a KMS$_1$ state ψ of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ such that

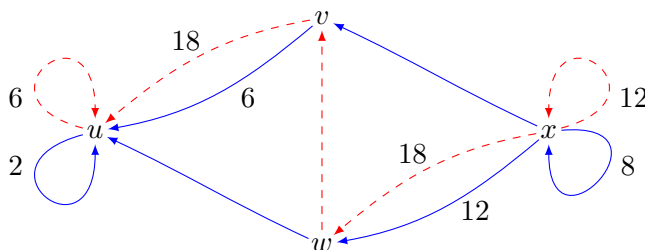$$\psi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-r \cdot d(\mu)} \|z\|_1^{-1} z_{s(\mu)}.$$

It factors through a KMS$_1$ state of $(C^*(\Lambda), \alpha^r)$ if and only if $r_i = \ln \rho(A_i) = \ln f_i$ for both $i = 1$ and $i = 2$.

Now Theorem 5.2 of [9] implies that every KMS$_1$ state of the system $(\mathcal{T}C^*(\Lambda), \alpha^r)$ is a convex combination of ψ and a state $\phi \circ q_{\{x\}}$ lifted from a KMS$_1$ state of $\mathcal{T}C^*(\Lambda \backslash \{x\})$. Since $\Lambda \backslash \{x\}$ is the graph considered in §3 and we are assuming that $\ln d_i < \ln f_i = \ln \rho(A_i)$, we are in the range where Theorem 3.5 gives an explicit description of these KMS$_1$ states.

6. Computing the KMS states on a specific graph

In [11, Example 8.4], we tested the general results in [11] about the preferred dynamics by computing all the KMS states of the system $(\mathcal{T}C^*(\Lambda), \alpha^r)$

when Λ has the following skeleton:

Here we do the same for a non-preferred dynamics on the same graph algebra using the stronger versions in [9]. We consider a dynamics α^r in which

$$r_1 = \ln 8 \quad \text{and} \quad r_2 > \ln 12, \tag{6.1}$$

and $\{r_1, r_2\}$ are rationally independent. For $\beta > 1$, [13, Theorem 6.1] describes a 3-dimensional simplex of KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

Now we consider the KMS_1 states, and aim to apply the results of [9]. The common unimodular Perron–Frobenius eigenvector of $A_{\{x\},i}$ is the 1-vector 1, and as in of [9, Proposition 4.1], this extends to a common eigenvector $z := (y, 1)$ of the matrices A_i with eigenvalue $\rho(A_{\{x\},1}) = 8$ and $\rho(A_{\{x\},2}) = 12$. (Since [9, Proposition 4.1] is linear-algebraic, it applies verbatim here.) Now Proposition 5.1 of [9] gives a KMS_1 state ψ of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ with

$$\begin{pmatrix} \psi(q_u) \\ \psi(q_v) \\ \psi(q_w) \\ \psi(q_x) \end{pmatrix} = \|z\|_1^{-1} z = \frac{1}{24} \begin{pmatrix} 3 \\ 1 \\ 12 \\ 8 \end{pmatrix}.$$

Since the only critical component of Λ for the dynamics α^r is $\{x\}$, Theorem 6.1 of [9] implies that every KMS_1 state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$ is a convex combination of ψ and a state $\phi \circ q_{\{x\}}$ lifted from a KMS_1 state ϕ of $(\mathcal{T}C^*(\Lambda \backslash \{x\}), \alpha^r)$.

The graph $\Lambda \backslash \{x\} = \Lambda_{\{u,v,w\}}$ is one of those we studied in §3. Since

$$r_1 = \ln 8 > \rho(A_{\{u,v,w\},1}) = \ln 2 \quad \text{and} \quad r_2 > \ln 12 > \rho(A_{\{u,v,w\},2}) = \ln 6,$$

$\beta = 1$ is in the range for which Theorem 3.5 gives a concrete description of the KMS_1 states of $(\mathcal{T}C^*(\Lambda \backslash \{x\}), \alpha^r)$. So the original system has a 3-dimensional simplex of KMS_1 states with extreme points ψ, $\phi_1 \circ q_{\{w,x\}}$, $\phi_2 \circ q_{\{w,x\}}$ and $\phi_3 \circ q_{\{x\}}$.

With the next lemma, we can continue below the inverse temperature $\beta = 1$.

Lemma 6.1. *Suppose that $\beta < 1$ and ϕ is a KMS_β state of the system $(\mathcal{T}C^*(\Lambda), \alpha^r)$ considered above. Then ϕ factors through the quotient map $q_{\{x\}}$.*

Proof. We aim to prove that $\phi(t_x) = 0$. We certainly have $\phi(t_x) \geq 0$. The relation (T4) with $n = e_1$ implies that

$$\phi(t_x) \geq \sum_{e \in x\Lambda^{e_1}} \phi(t_e t_e^*) = \sum_{e \in x\Lambda^{e_1}} e^{-\beta r_1} \phi(t_e^* t_e)$$

$$= e^{-\beta(\ln 8)}|x\Lambda^{e_1}|\phi(t_x) = 8^{-\beta} 8 \phi(t_x) = 8^{1-\beta}\phi(t_x),$$

which we can rewrite as $(1 - 8^{1-\beta})\phi(t_x) \geq 0$. But $\beta < 1$ implies that $8^{1-\beta} > 1$, and this is only compatible with $\phi(t_x) \geq 0$ if $\phi(t_x) = 0$. Now it follows from [1, Lemma 6.2] that ϕ factors through $q_{\{x\}}$. $\square$

So we are interested in the KMS_β states of $(\mathcal{T}C^*(\Lambda_{\{u,v,w\}}), \alpha^r)$ for $\beta < 1$. Recall from the start of the section that we are assuming that $r_1 = \ln 8$ and $r_2 > \ln 12$. The next critical level is

$$\beta_c := \max\left\{(\ln 8)^{-1} \ln 2, r_2^{-1} \ln 6\right\} = \max\left\{3^{-1}, r_2^{-1} \ln 6\right\}. \qquad (6.2)$$

For β satisfying $\beta_c < \beta < 1$, we deduce from Theorem 3.5 that the KMS_β states of $(\mathcal{T}C^*(\Lambda_{\{u,v,w\}}), \alpha^r)$ form a 2-dimensional simplex; Theorem 3.5 also provides explicit formulas for the extreme points. Composing with $q_{\{w\}}$ gives a two-dimensional simplex of KMS_β states of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.

Remark 6.2. Strictly speaking, to apply Theorem 3.5 we need to scale the dynamics to ensure that the critical inverse temperature is 1 rather than β_c. Lemma 2.1 of [10] gives the formulas which achieve this. We will assume that this can be done mentally (or at least "in principle").

For $\beta = \beta_c$, at least one of the inequalities $r_i\beta \geq \ln \rho(A_{\{u\},i})$ becomes an equality. Provided $\{r_1, r_2\}$ are rationally independent, Theorem 4.3 implies that $(\mathcal{T}C^*(\Lambda_{\{u,v,w\}}), \alpha^r)$ has a unique KMS_{β_c} state which factors through a state of $(\mathcal{T}C^*(\Lambda_{\{u\}}), \alpha^r)$. It follows from [10, Proposition 6.1] that this state factors through a state of $(C^*(\Lambda_{\{u\}}), \alpha^r)$ if and only if we have $r_i\beta = \ln \rho(A_{\{u\},i})$ for both i. For $\beta < \beta_c$, at least one of the inequalities $r_i\beta \geq \ln \rho(A_i)$ fails, and it follows from [13, Corollary 4.3] that there are no KMS_β states on any of these algebras.

Remark 6.3. For a dynamics satisfying (6.1), the constraint $r_2 > \ln 12$ implies that $r_2^{-1} \ln 6 \leq (\ln 12)^{-1} \ln 6$, and which is the bigger in (6.2) will depend on r_2. A calculator tells us that

$$(\ln 12)^{-1} \ln 6 = \frac{\ln 6}{\ln 2 + \ln 6} \sim 0.72.$$

Thus for small r_2, we have $3^{-1} < r_2^{-1} \ln 6$, but for big r_2, we can have $3^{-1} > r_2^{-1} \ln 6$. So β_c could be either value.

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