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On orthogonal systems, two-sided bases and regular subfactors

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Abstract. We prove that a regular subfactor of type II_1 with finite Jones index always admits a two-sided Pimsner-Popa basis. This is preceded by a pragmatic revisit of Popa's notion of orthogonal systems.

Contents

1. Introduction

Let $\mathcal{N} \subset \mathcal{M}$ be a unital inclusion of von Neumann algebras equipped with a faithful normal conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$. Then, a finite set $\mathcal{B} := \{\lambda_1, \ldots, \lambda_n\} \subset \mathcal{M}$ is called a left Pimsner-Popa basis for $\mathcal{M}$ over $\mathcal{N}$ via $\mathcal{E}$ if every $x \in \mathcal{M}$ can be expressed as $x = \sum_{i=1}^{n} \mathcal{E}(x\lambda_i^*)\lambda_i$ - see [14, 17, 16, 9, 20] and the references therein. Similarly, $\mathcal{B}$ is called a right Pimsner-Popa basis for $\mathcal{M}$ over $\mathcal{N}$ via $\mathcal{E}$ if every $x \in \mathcal{M}$ can be expressed as $x = \sum_{j=1}^{n} \lambda_j \mathcal{E}(\lambda_j^* x)$. And, $\mathcal{B}$ is said to be a two-sided basis if it is simultaneously a left and a right Pimsner-Popa basis. It is readily seen that a type II_1 subfactor that admits a two-sided basis is always extremal (Proposition 3.1).

An extensively exploited result of Pimsner and Popa (from [14]) states that if $N \subset M$ is a subfactor of type II_1 with finite Jones index ([7]), then there always exists a left (equivalently, a right) Pimsner-Popa basis for M over N via the unique trace preserving conditional expectation $E_N : M \to N$. As noted above, non-extremal subfactors do not admit two-sided bases. So, it is natural to ask whether there always exists a two-sided basis for every finite index extremal subfactor or not. In fact, it has also been asked publicly

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by Vaughan Jones at various places - see, for instance, the second talk by M. Izumi in the workshop organized in honour of V. S. Sunder's 60th birthday in Chennai during March-April 2012. Given the fact that every irreducible regular subfactor of finite index is a group subfactor, it is not surprising that such a subfactor always admits a two-sided orthonormal basis, as was illustrated in [6] (also see [2]) . However, it seems to be a difficult question to answer in general. In this article, we answer this question in affirmative for all regular subfactors of type II_1 with finite Jones index (without assuming extremality) in:

Theorem 3.10. *Let $N \subset M$ be a regular subfactor of type II_1 with finite Jones index. Then, M admits a two-sided basis over N.*

As a consequence, we deduce that every finite index regular subfactor of type II_1 is extremal.

Recall that an inclusion $\mathcal{Q} \subset \mathcal{P}$ of von Neumann algebras is said to be regular if its group of normalizers $\mathcal{N}_{\mathcal{P}}(\mathcal{Q}) := \{u \in \mathcal{U}(\mathcal{P}) : u\mathcal{Q}u^* = \mathcal{Q}\}$ generates $\mathcal{P}$ as von Neumann algebra, i.e., $\mathcal{N}_{\mathcal{P}}(\mathcal{Q})'' = \mathcal{P}$. Our proof is essentially self contained and does not depend on any structure theorem for regular subfactors.

An effort has been made to keep this article as self-contained as possible. The reader is assumed only to have some basic knowledge of subfactor theory, for instance, as discussed in the first few chapters of [9].

Here is a brief outline of the content of this article.

As mentioned in the abstract, we first revisit, in Section 2, Popa's ([17]) notion of an *orthogonal system* for an inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. This generalizes the notion of an *orthonormal basis* for a subfactor $N \subset M$ of type II_1 introduced by Pimsner and Popa in [14]. Dropping orthogonality, Jones and Sunder, in [9], generalized the notion of *orthonormal basis* and gave another formulation of basis for M over N (as recalled in the first paragraph of Introduction). Very much on the lines of [9], we introduce and discuss the notion of a *Pimsner-Popa system*, which generalizes Popa's notion of an *orthogonal system*.

If $\mathcal{N} \subset \mathcal{M}$ is an inclusion of finite von Neumann algebras with a fixed faithful normal tracial state tr on $\mathcal{M}$, then for any *Pimsner-Popa system* $\{\lambda_1, \cdots, \lambda_k\}$ for $\mathcal{N} \subset \mathcal{M}$ with respect to the unique tr-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, it turns out that the positive operator $f := \sum_{i=1}^{n} \lambda_i e_1 \lambda_i^*$ is a projection in $\mathcal{M}_1$ (Lemma 2.3), which we call the support of the system, where as usual e_1 denotes the Jones projection for the canonical basic construction $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$. An astute reader must have already noticed that, if the support of $\{\lambda_i\}$ equals 1, then it is in fact a *Pimsner-Popa basis* (in the sense of [9]) for $\mathcal{M}$ over $\mathcal{N}$.

On the other hand, for a finite index subfactor $N \subset M$ of type II_1, we observe that for every projection $f \in M_1$ there exists a *Pimsner-Popa system* with support f (Proposition 2.8). An useful consequence of this observation yields:

Theorem 2.10 *Let $N \subset M$ be a subfactor of type II_1 with finite index. Then, any Pimsner-Popa system $\{\lambda_1, \cdots, \lambda_k\}$ for M over N can be extended to a Pimsner-Popa basis for M over N.*

One application being that we deduce in Corollary 2.14 that every subfactor of finite index admits a Pimsner-Popa basis (not necessarily orthonormal) containing at least $|G|$ many unitaries, where G is the *generalized Weyl group* of the subfactor (as defined in the next paragraph).

Given its importance, an important example of an *orthogonal system* for a finite index subfactor $N \subset M$ that we illustrate (in Corollary 2.13) consists of a set containing coset representatives of, what we call, the *generalized Weyl group* of the subfactor $N \subset M$, namely, the quotient group

$$G := \mathcal{N}_M(N)/\mathcal{U}(N)\mathcal{U}(N' \cap M).$$

This group was first considered by Loi in [12]. Clearly, this group agrees with the Weyl group of the subfactor if the subfactor is irreducible, i.e., $N' \cap M = \mathbb{C}$. Such coset representatives were also considered in [4, 8, 14, 15, 11, 6] in the irreducible setup and used effectively.

Our second important class of examples of *Pimsner-Popa systems* comes from unital inclusions of finite dimensional C^*-algebras - see Section 2.2.2. This is done by employing the formalism of *path algebras* introduced independently by Sunder ([19]) and Ocneanu ([13]). Apart from these, Section 2 is also devoted to a detailed discussion of certain other useful properties related to *Pimsner-Popa systems.*

Finally, in Section 3, we settle the question of existence of two-sided basis for any finite index regular subfactor $N \subset M$. This is achieved through a twofold strategy, namely, we first appeal to the formalism of *path algebras* to get hold of a two-sided basis for $N' \cap M$ over $\mathbb{C}$ with respect to the restriction of tr_M (in Proposition 3.3), which also turns out to be a two-sided basis for $\mathcal{R} := N \vee (N' \cap M)$ over N (Lemma 3.4), and then, thanks to the regularity of $N \subset M$, every set of coset representatives of the generalized Weyl group of $N \subset M$ turns out to be a two-sided orthonormal basis consisting of normalizing unitaries for M over $\mathcal{R}$ (Proposition 3.7). Ultimately, with an appropriate patching technique (Proposition 3.9), we deduce (in Theorem 3.10) that the product of these two two-sided bases forms a two-sided Pimsner-Popa basis for M over N. And finally, employing the two-sided bases mentioned above and Watatani's notion of index of a conditional expectation, we derive (in Theorem 3.12) that

$$[M : N] = |G| \dim_{\mathbb{C}}(N' \cap M),$$

where G again denotes the generalized Weyl group of the subfactor $N \subset M$.

2. Pimsner-Popa bases and systems

Recall, from [17], that given a unital inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with a faithful normal conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$, a family $\{m_j\}_j$ in $\mathcal{M}$ is called a right orthogonal system for $\mathcal{M}$ over $\mathcal{N}$ with respect to $\mathcal{E}$ if $\mathcal{E}(m_i^* m_j) = \delta_{ij} f_j$ for some projections $\{f_j\}_j$ in $\mathcal{N}$. In this article, we will be dealing only with finite right orthogonal systems.

2.1. Pimsner-Popa systems. On the lines of [9, § 4.3], Popa's notion of *orthogonal systems* generalizes naturally to the following:

Definition 2.1. Let $\mathcal{N} \subset \mathcal{M}$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$. A finite subset $\{\lambda_j : j \in J\}$ in $\mathcal{M}$ will be called a right Pimsner-Popa system for $\mathcal{M}$ over $\mathcal{N}$ with respect to $\mathcal{E}$ if the matrix $Q = [q_{ij}]$ with entries $q_{ij} := \mathcal{E}(\lambda_i^* \lambda_j)$ is a projection in $M_J(\mathcal{N})$.

Such a Pimsner-Popa system will be called a right orthogonal system if $q_{ij} = \delta_{i,j} q_j$ for some projections $\{q_j : j \in J\} \subset \mathcal{N}$. If each q_j is the identity operator, then such an orthogonal system will be called a right orthonormal system.

Remark 2.2. (1) Similarly, one defines left systems by considering the matrix $\left[\mathcal{E}(\lambda_i \lambda_j^*)\right]$ in $M_J(\mathcal{N})$. A collection which is both a left system and a right system will be called a two-sided system.

(2) Hereafter, by a Pimsner-Popa (resp., an orthogonal) system we will always mean a right Pimsner-Popa (resp., a right orthogonal) system and will henceforth drop the adjective 'right'. And, whenever the conditional expectation is clear from the context, we shall omit the phrase 'with respect to $\mathcal{E}$'.

In this subsection, we systematically study these objects and their generalities in the spirit of Pimsner-Popa basis.

Let $\mathcal{N} \subset \mathcal{M}$ be a unital inclusion of finite von Neumann algebras with a fixed faithful normal tracial state tr on $\mathcal{M}$ and let $E_{\mathcal{N}}$ denote the unique trace preserving normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. As is standard, e_1 will denote the Jones projection that implements the basic construction $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$.

Lemma 2.3. *Let $\mathcal{N} \subset \mathcal{M}$, $E_{\mathcal{N}}$ be as in the preceding paragraph and let $\{\lambda_1, \ldots, \lambda_k\}$ be a Pimsner-Popa system for $\mathcal{M}/\mathcal{N}$. Then, the positive operator $\sum_i \lambda_i e_1 \lambda_i^*$ is a projection in $\mathcal{M}_1$.*

Proof. The idea of the proof is essentially borrowed from [14] and [9]. Consider the projection $Q = [q_{ij}] := [E_{\mathcal{N}}(\lambda_i^* \lambda_j)]$ in $M_k(\mathcal{N})$. Let $v_i := \lambda_i e_1$ for

$1 \leq i \leq k$ and $V \in M_k(\mathcal{M}_1)$ be the matrix given by

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now, since $v_i^* v_j = e_1 \lambda_i^* \lambda_j e_1 = q_{ij} e_1$, we see that $V^*V = QE = EQ$, where E is the diagonal matrix $\operatorname{diag}(e_1, \ldots, e_1)$ in $M_k(\mathcal{M}_1)$. So, V is a partial isometry in $M_k(\mathcal{M}_1)$. In particular, VV^* is a projection in $M_k(\mathcal{M}_1)$, thereby implying that $\sum_i v_i v_i^* = \sum_i \lambda_i e_1 \lambda_i^*$ is a projection in $\mathcal{M}_1$. $\square$

Definition 2.4. Let $\mathcal{N} \subset \mathcal{M}$ and $E_{\mathcal{N}}$ be as in Lemma 2.3. For any Pimsner-Popa system $\{\lambda_i : 1 \leq i \leq n\}$ for $\mathcal{M}$ over $\mathcal{N}$, the projection $\sum_{i=1}^n \lambda_i e_1 \lambda_i^* \in \mathcal{M}_1$ will be called the support of the system $\{\lambda_i : 1 \leq i \leq n\}$.

Remark 2.5. (1) A subcollection of an orthogonal (resp., orthonormal) system is also an orthogonal (resp., orthonormal) system.

(2) A Pimsner-Popa system with support equal to 1 turns out to be a Pimsner-Popa basis for $\mathcal{M}$ over $\mathcal{N}$ (as mentioned in Section 1). For such a basis, the sum $\sum_{i=1}^n \lambda_i \lambda_i^*$ is independent of the basis (see [20]) and is called the Watatani index of $\mathcal{N} \subset \mathcal{M}$. This quantity is denoted by $\operatorname{Index}_w(\mathcal{N} \subset \mathcal{M})$.

If $N \subset M$ is a finite index subfactor of type II_1, then it is known that $\operatorname{Index}_w(N \subset M) = [M : N]$ - see [20]

The following useful equivalence is folklore and will be used on few occasions.

Lemma 2.6. *Let $\mathcal{N} \subset \mathcal{M}$ and $E_{\mathcal{N}}$ be as in Lemma 2.3. Then, for any finite set $\{\lambda_1, \ldots, \lambda_n\}$ in $\mathcal{M}$, $\{\lambda_i : 1 \leq i \leq n\}$ is a Pimsner-Popa basis for $\mathcal{M}/\mathcal{N}$ if and only if $\sum_{i=1}^n \lambda_i e_1 \lambda_i^* = 1$.*

Unlike above characterization of a Pimsner-Popa basis (Lemma 2.6), the converse of Lemma 2.3 may not be true; that is, if for some projection $f \neq 1$ in $\mathcal{M}_1$ there is a finite set $\{\lambda_i\} \subset \mathcal{M}$ satisfying $\sum_i \lambda_i e_1 \lambda_i^* = f$, then there is no obvious reason why $\{\lambda_i\}$ should be a Pimsner-Popa system for $\mathcal{M}/\mathcal{N}$. However, in some specific cases the situation is better.

Proposition 2.7. *Let $N \subset M$ be a subfactor of type II_1 with $[M : N] < \infty$, $\{\lambda_i : 1 \leq i \leq n\}$ be a finite subset of M and f be a projection in M_1 satisfying the following three conditions:*

(1) $f \geq e_1$,
(2) $\sum_i \lambda_i e_1 \lambda_i^* = f$ *and*
(3) $\{\lambda_i : 1 \leq i \leq n\} \subseteq \{f\}' \cap M$.

Then, $\{\lambda_i : 1 \leq i \leq n\}$ is a Pimsner-Popa system for M/N.

Proof. Let $q_{ij} := E_N(\lambda_i^* \lambda_j)$ for $1 \leq i, j \leq n$. Clearly, $q_{ij}^* = q_{ji}$ and we have

$$\begin{aligned}
\Big(\sum_k q_{ik} q_{kj}\Big) e_1 &= \Big(\sum_k E_N(\lambda_i^* \lambda_k) E_N(\lambda_k^* \lambda_j)\Big) e_1 \\
&= \Big(\sum_k E_N\Big(\lambda_i^* \lambda_k E_N(\lambda_k^* \lambda_j)\Big)\Big) e_1 \\
&= \sum_k e_1 \lambda_i^* \lambda_k E_N(\lambda_k^* \lambda_j) e_1 \\
&= \sum_k e_1 \lambda_i^* \lambda_k e_1 \lambda_k^* \lambda_j e_1 \\
&= e_1 \lambda_i^* f \lambda_j e_1 \\
&= e_1 f \lambda_i^* \lambda_j e_1 \\
&= q_{ij} e_1
\end{aligned}$$

for all $1 \leq i, j \leq n$. So, by the uniqueness part of the Pushdown Lemma [14, Lemma 1.2], we deduce that $\sum_k q_{ik} q_{kl} = q_{ij}$ for all $1 \leq i, j \leq n$. Thus, the matrix $Q := [q_{ij}]$ is a projection in $M_n(N)$. This completes the proof. $\square$

The following observation is the crux of this section.

Proposition 2.8. *Let $N \subset M$ be as in Proposition 2.7. Then, for any projection $f \in M_1$, there exists a Pimsner-Popa system $\{\lambda_1, \ldots, \lambda_n\}$ for M/N with support equal to f.*

Proof. The proof that we give is inspired by [9, Proposition 4.3.3(a)]. Fix an $n \geq [M : N]$. Since $0 \leq \mathrm{tr}(f) \leq 1$, we obtain $n \geq \mathrm{tr}(f)[M : N]$. Since $M_n(N)$ is a II_1-factor, we can choose a projection $Q \in M_n(N)$ with $\mathrm{tr}_{M_n(N)}(Q) = \frac{\mathrm{tr}(f)[M:N]}{n}$. Consider the diagonal matrix $P_1 := \mathrm{diag}(f, 0, \ldots, 0)$ in $M_n(M_1)$. Then, P_1 is a projection with $\mathrm{tr}_{M_N(M_1)}(P_1) = \frac{tr(f)}{n}$.

On the other hand, consider the projection $P_0 := QE$ in $M_n(M_1)$, where $E := \mathrm{diag}(e_1, \ldots, e_1)$. Clearly,

$$\mathrm{tr}_{M_n(M_1)}(P_0) = \frac{\sum_i \mathrm{tr}(q_{ii} e_1)}{n} = \frac{\sum_i \mathrm{tr}(q_{ii})}{n\,[M : N]} = \frac{\mathrm{tr}_{M_n(N)}(Q)}{[M : N]} = \frac{\mathrm{tr}(f)}{n};$$

so that, $P_1 \sim P_0$ in $M_n(M_1)$. Hence, there exists a partial isometry $V \in M_n(M_1)$ such that $V^* V = P_0$ and $VV^* = P_1$. Note that, the condition $VV^* = P_1$ forces V to be of the form

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

for some v_i's in M_1. These v_i's then satisfy $\sum_i v_i v_i^* = f$ and $v_i^* v_j = q_{ij} e_1$ for all $1 \leq i, j \leq n$. In particular, $v_i^* v_i = q_{ii} e_1 \leq e_1$ for all $1 \leq i \leq n$.

Thus, $|v_i| \leq e_1 \leq 1$ and this implies that $|v_i| = |v_i|e_1$; so that, by polar decomposition of v_i, we obtain $v_i = w_i|v_i| = w_i|v_i|e_1 = v_ie_1$ for every $1 \leq i \leq n$, where each w_i is an appropriate partial isometry.

Therefore, by the Pushdown Lemma [14, Lemma 1.2], we obtain a set $\{\lambda_1, \ldots, \lambda_n\}$ in M such that $v_i = \lambda_i e_1$ for all $1 \leq i \leq n$. In particular,

$$q_{ij}e_1 = v_i^* v_j = e_1 \lambda_i^* \lambda_j e_1 = E_N(\lambda_i^* \lambda_j)e_1;$$

so that, by the uniqueness component of Pushdown Lemma, $q_{ij} = E_N(\lambda_i^* \lambda_j)$ for all $1 \leq i, j \leq n$. So, $\{\lambda_1, \ldots, \lambda_n\}$ is a Pimsner-Popa system for M/N and its support is given by $\sum_i \lambda_i e_1 \lambda_i^* = \sum_i v_i v_i^* = f$. □

Remark 2.9.
(1) An appropriate customization of above proof actually guarantees the existence of an orthogonal system as well. Indeed, if we choose a projection $q \in N$ such that $\mathrm{tr}(q) = \frac{\mathrm{tr}(f)[M:N]}{n}$ and let $Q := \mathrm{diag}(q, q, \ldots, q) \in M_n(N)$ then clearly Q is a projection with $\mathrm{tr}_{M_n(N)}(Q) = \frac{\mathrm{tr}(f)[M:N]}{n}$. Then, a Pimsner-Popa system $\{\lambda_1, \cdots, \lambda_n\}$ for M/N provided by the proof of Theorem 2.8 is in fact an orthogonal system for M/N with support f.
(2) We could even take a projection $Q = (1, \ldots, 1, q) \in M_n(N)$, where q is a projection in N with $\mathrm{tr}_N(q) = \frac{\mathrm{tr}(f)[M:N]-n+1}{n}$. This choice of Q yields an orthogonal system $\{\lambda_i : 1 \leq i \leq n\}$ with support f such that $E_N(\lambda_i^* \lambda_i) = 1$ for all $1 \leq i \leq n-1$ and $E_N(\lambda_n^* \lambda_n) = q$. In particular, if $f = 1$, then we obtain an orthonormal basis (in the sense of [14]) for M/N.

As mentioned in the Introduction, the following consequence can be used to construct bases with some specific requirements as we shall see, for instance, in Corollary 2.14.

Theorem 2.10. *Let $N \subset M$ be as in Proposition 2.7. Then, any Pimsner-Popa system $\{\lambda_1, \ldots, \lambda_k\}$ for M/N can be extended to a Pimsner-Popa basis for M/N.*

Proof. Let f denote the support of the given system $\{\lambda_i : 1 \leq i \leq k\}$. By Proposition 2.8, there exists a Pimsner-Popa system $\{\lambda_{k+1}, \ldots, \lambda_{k+l}\}$ for M/N with support $1 - f$. Then,

$$\sum_{i=1}^{k+l} \lambda_i e_1 \lambda_i^* = \sum_{i=1}^{k} \lambda_i e_1 \lambda_i^* + \sum_{i=1}^{l} \lambda_{k+i} e_1 \lambda_{k+i}^* = f + (1-f) = 1.$$

Thus, by Lemma 2.6, $\{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+l}\}$ is a Pimsner-Popa basis for M/N. □

2.2. Examples of Pimsner-Popa systems.

2.2.1. Pimsner-Popa bases and intermediate subalgebras. Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras. Let $\mathcal{P}$ be an intermediate von Neumann subalgebra, i.e., $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$. Fix a faithful normal tracial state on $\mathcal{M}$ and let $e_{\mathcal{P}}$ denote the canonical Jones projection for the basic construction $\mathcal{P} \subset \mathcal{M} \subset \mathcal{P}_1$. Let $\{\lambda_i\}$ be a finite set in $\mathcal{P}$. If $\{\lambda_i\}$ is a Pimsner-Popa basis for $\mathcal{P}/\mathcal{N}$, then it is easy to see that $\{\lambda_i\}$ is a Pimsner-Popa system for $\mathcal{M}/\mathcal{N}$ with support $e_{\mathcal{P}}$. Indeed, for any $x \in \mathcal{M}$, we have

$$\begin{aligned}\big(\sum_i \lambda_i e_1 \lambda_i^*\big)x\Omega &= \sum_i \lambda_i E^{\mathcal{M}}_{\mathcal{N}}(\lambda_i^* x)\Omega \\ &= \sum_i \lambda_i E^{\mathcal{P}}_{\mathcal{N}}(\lambda_i^* E^{\mathcal{M}}_{\mathcal{P}}(x))\Omega \\ &= E^{\mathcal{M}}_{\mathcal{P}}(x)\Omega = e_{\mathcal{P}}(x\Omega),\end{aligned}$$

where the second last equality holds because $\{\lambda_i\}$ is a basis for $\mathcal{P}$ over $\mathcal{N}$.

2.2.2. Inclusion of finite dimensional C^*-algebras. Let $A_0 \subset A_1$ be a unital inclusion of finite dimensional C^*-algebras with dimension vectors $\overrightarrow{m} = [m_1, \cdots, m_k]$ and $\overrightarrow{n} = [n_1, \cdots, n_l]$, respectively; so that

$$A_0 \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_k}(\mathbb{C}) \text{ and } A_1 \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C}).$$

We briefly recall the formalism of *path algebras* associated to such an inclusion, introduced independently by Ocneanu ([13]) and Sunder ([19]). For details, we refer the reader to [9, §5.4].

Let $\widehat{C}$ denote the set of minimal central projections of a finite dimensional C^*-algebra C. With this notation, let $\widehat{A_0} = \{p^{(0)}_1, \ldots, p^{(0)}_k\}$ and $\widehat{A_1} = \{p^{(1)}_1, \ldots, p^{(1)}_l\}$. Let $A_{-1} := \mathbb{C}$ and put $\widehat{\mathbb{C}} = \{\star\}$. Consider the Bratteli diagram for $\mathbb{C} \subset A_0$ and let $\Omega_{0]}$ denote the set of all directed edges starting from $\star$ and ending at $p^{(0)}_i$ for some $1 \leq i \leq k$. Similarly, let $\Omega_{[0,1]}$ denote the set of edges in the Bratelli diagram of $A_0 \subset A_1$, and $\Omega_{1]}$ denote the set of all paths starting from $\star$ and ending at $p^{(1)}_j$ for some $1 \leq j \leq l$. For any edge or path β, $s(\beta)$ and $r(\beta)$ denotes the source vertex and range vertex of β. Let $\mathcal{H}_{0]}, \mathcal{H}_{[0,1]}$ and $\mathcal{H}_{1]}$ denote the corresponding Hilbert spaces with orthonormal bases indexed by $\Omega_{0]}, \Omega_{[0,1]}$ and $\Omega_{1]}$, respectively. Then, from [19] (also see [9]), there exist C^*-subalgebras $B_0 \subset B_1 \subseteq \mathcal{L}(\mathcal{H}_{1]})$ such that the inclusion $A_0 \subset A_1$ is isomorphic to the inclusion $B_0 \subset B_1$ - see [9, Proposition 5.4.1(v)]. The pair $B_0 \subset B_1$ is called the path algebra model of the pair $A_0 \subset A_1$.

Fix $\lambda, \mu \in \Omega_{1]}$ with same end points. Define $e_{\lambda,\mu} \in B_1$ by

$$e_{\lambda,\mu}(\alpha, \beta) = \delta_{\lambda,\alpha}\delta_{\mu,\beta} \text{ for all } \alpha, \beta \in \Omega_{1]}.$$

Then, the set $\{e_{\lambda,\mu} : \lambda, \mu \in \Omega_{1]} \text{ with } r(\lambda) = r(\mu)\}$ forms a system of matrix units for B_1 - see [9, Proposition 5.4.1 (iv)].

Now, let us assume that $A_0 \subset A_1$ has a faithful tracial state tr on A_1. Let $E^{A_1}_{A_0} : A_1 \to A_0$ denote the unique tr-preserving conditional expectation. Let $\bar{t}^{(1)}$ be the trace vector corresponding to tr and $\bar{t}^{(0)}$ be the one corresponding to $\mathrm{tr}_{|A_0}$. Then, by [19] (also see [9]), we have

$$E_{B_0}(e_{\lambda,\mu}) = \delta_{\lambda_{[0},\mu_{[0}} \frac{\bar{t}^{(1)}_{r(\lambda)}}{\bar{t}^{(0)}_{r(\lambda_{0]})}} e_{\lambda_{0]},\mu_{0]}}. \tag{2.1}$$

Now, consider $I := \{(\kappa, \beta) : \kappa \in \Omega_{[0,1]}, \beta \in \Omega_{1]}, r(\kappa) = r(\beta)\}$ and, for each $(\kappa, \beta) \in I$, let

$$a_{\kappa,\beta} := \sum_{\{\theta \in \Omega_{0]} : r(\theta) = s(\kappa)\}} e_{\theta \circ \kappa, \beta}.$$

Then, by [9, Proposition 5.4.3], we have

$$E_{B_0}\Big(a_{\kappa,\beta}(a_{\kappa',\beta'})^*\Big) = \delta_{(\kappa,\beta),(\kappa',\beta')} \frac{\bar{t}^{(1)}_{r(\kappa)}}{\bar{t}^{(0)}_{s(\kappa)}} \sum_{\substack{\theta,\theta' \in \Omega_{0]} \\ r(\theta) = r(\theta') = s(\kappa)}} e_{\theta,\theta'}. \tag{2.2}$$

Further, for each $p \in \widehat{A_0}$, consider a projection $j_p \in B_0$ (as in [9, Lemma 5.7.3]) given by

$$j_p = \frac{1}{\bar{n}^{(0)}_p} \sum_{\substack{\alpha,\alpha' \in \Omega_{0]} \\ r(\alpha) = r(\alpha') = p}} e_{\alpha,\alpha'},$$

where $\left(\bar{n}^{(0)}_p\right)^2 = \dim pA_0$, and let $\lambda_{\kappa,\beta} := \left(\bar{n}^{(0)}_{s(\kappa)} \frac{\bar{t}^{(1)}_{r(\kappa)}}{\bar{t}^{(0)}_{s(\kappa)}}\right)^{-1/2} a_{\kappa,\beta}$. Then, by Equation 2.2, we obtain

$$E_{B_0}\Big(\lambda_{\kappa,\beta}(\lambda_{\kappa',\beta'})^*\Big) = \delta_{(\kappa,\beta),(\kappa',\beta')} \quad j_{s(\kappa)}.$$

Therefore, $\{\lambda_{\kappa,\beta} : (\kappa, \beta) \in I\}$ is a left orthogonal system for A_1/A_0. This example will have a significant role to play in Section 3.

We will discuss some further useful properties of Pimsner-Popa systems in Section 2.4. Before that, let us digress to an important class of examples of orthonormal systems consisting of unitaries.

2.3. Generalized Weyl group and orthonormal systems.

In this subsection, we illustrate an important example of an orthonormal system consisting of unitaries, which will attract a good share of limelight of this article. Let $N \subset M$ be a subfactor of type II_1 (which is not necessarily irreducible), let $\mathcal{U}(N)$ (resp., $\mathcal{U}(M)$) denote the group of unitaries of N (resp., M) and $\mathcal{N}_M(N) := \{u \in \mathcal{U}(M) : uNu^* = N\}$ denote the group of unitary normalizers of N in M. It is straightforward to see that $\mathcal{U}(N)\mathcal{U}(N' \cap M)\big(= \mathcal{U}(N' \cap M)\mathcal{U}(N)\big)$ is a normal subgroup of $\mathcal{N}_M(N)$.

Definition 2.11. [12] The generalized Weyl group of a subfactor $N \subset M$ is defined as the quotient group

$$G := \mathcal{N}_M(N)/\mathcal{U}(N)\mathcal{U}(N' \cap M).$$

This group first appeared in [12, Proposition 5.2]. Note that the generalized Weyl group of an irreducible subfactor agrees with its Weyl group, namely, the quotient group $\mathcal{N}_M(N)/\mathcal{U}(N)$.

The following two useful observations are well known for irreducible subfactors - see, for instance, [6, 8, 14, 15, 11, 12]. For the non-irreducible case, their proofs can be extracted readily from [12, Proposition 5.2].

Lemma 2.12. [12] *Let $w \in \mathcal{N}_M(N) \setminus \mathcal{U}(N)\mathcal{U}(N' \cap M)$. Then, $E_N(w) = 0$.*

In particular, for any two elements $v, u \in \mathcal{N}_M(N)$, $E_N(vu^*) = 0 = E_N(v^*u)$ if $[u] \neq [v]$ in the generalized Weyl group G.

Corollary 2.13. [12] *Suppose $[M : N] < \infty$ and G denotes the generalized Weyl group of the subfactor $N \subset M$. Then, any set of coset representatives $\{u_g : g = [u_g] \in G\}$ of G in $\mathcal{N}_M(N)$ forms a two-sided orthonormal system for M/N. Also, G is a finite group with order $\leq [M : N]$.*

Corollary 2.14. *Every finite index subfactor of type II_1 admits a Pimsner-Popa basis containing at least $|G|$ many unitaries.*

Proof. By Corollary 2.13, there exists an orthonormal system for M/N consisting of unitaries. Then, by Theorem 2.10, this orthonormal system can be extended to a Pimsner-Popa basis for M/N. This completes the proof. □

Remark 2.15. Corollary 2.14 could be related somewhat to a recent question asked by Popa in [18] about the maximum number of unitaries possible in an orthonormal basis (in the sense of [14]) of a given subfactor. It, at least, tells us that every finite index subfactor $N \subset M$ of type II_1 always admits a Pimsner-Popa basis (not necessarily orthonormal) containing at least $|G|$ many unitaries.

In view of Corollary 2.14, calculating cardinality of G becomes quite relevant. However, in practice, we are yet to find a suitable way to calculate the cardinality of G. Since the generalized Weyl group is the same as the Weyl group of an irreducible subfactor, it is always non-trivial for such a subfactor.

2.4. Some useful properties related to Pimsner-Popa systems.

Let (N, P, Q, M) be a quadruple of II_1-factors, i.e., $N \subset P, Q \subset M$, with $[M : N] < \infty$. Let $\{\lambda_i : i \in I\}$ and $\{\mu_j : j \in J\}$ be (right) Pimsner-Popa bases for P/N and Q/N, respectively. Consider two auxiliary operators $p(P, Q)$ and $p(Q, P)$ (as in [1]) given by

$$p(P,Q) = \sum_{i,j} \lambda_i \mu_j e_1 \mu_j^* \lambda_i^* \quad \text{and} \quad p(Q,P) = \sum_{i,j} \mu_j \lambda_i e_1 \lambda_i^* \mu_j^*.$$

By [1, Lemma 2.18], $p(P,Q)$ and $p(Q,P)$ are both independent of choice of bases. And, by [1, Proposition 2.22], $Jp(P,Q)J = p(Q,P)$, where J is the usual modular conjugation operator on $L^2(M)$; so that, $\|p(P,Q)\| = \|p(Q,P)\|$. Let us denote this common value by λ.

Proposition 2.16. *Let (N,P,Q,M) be a quadruple of type II_1 factors such that $N' \cap M = \mathbb{C}$ and $[M:N] < \infty$, and let $\{\lambda_i : i \in I\}$ be a Pimsner-Popa basis for P/N. Then, the following hold:*

(1) $\left\{\frac{1}{\sqrt{\lambda}}\lambda_i : i \in I\right\}$ is a Pimsner-Popa system for M/Q with support $\frac{1}{\lambda}p(P,Q)$.

(2) If (N,P,Q,M) is a commuting square, then $\{\lambda_i\}$ can be extended to a Pimsner-Popa basis for M/Q.

Proof. (1) From [1, Lemma 3.2], we know that $\frac{1}{\lambda}p(P,Q)\left(= \frac{1}{\lambda}\sum_i \lambda_i e_Q \lambda_i^*\right)$ is a projection and, by [1, Lemma 3.4], e_Q is a subprojection of $\frac{1}{\lambda}p(P,Q)$. Further, by [1, Proposition 2.25], we know that $p(P,Q) \in P' \cap Q_1$; so, it follows that $\left\{\lambda_i : i \in I\right\} \subseteq \left\{\frac{1}{\lambda}p(P,Q)\right\}' \cap M$. Also, we have

$$\sum_i \frac{1}{\sqrt{\lambda}}\lambda_i e_Q \frac{1}{\sqrt{\lambda}}\lambda_i^* = \frac{1}{\lambda}p(P,Q).$$

Thus, in view of Proposition 2.7, $\left\{\frac{1}{\sqrt{\lambda}}\lambda_i : i \in I\right\}$ is a Pimsner-Popa system for M/Q with support $\frac{1}{\lambda}p(P,Q)$

(2) Suppose that (N,P,Q,M) is a commuting square. Then, by [1, Propositions 2.14 & 2.20], we know that $p(P,Q)$ is a projection. Thus, $\lambda = \|p(P,Q)\| = 1$ and the conclusion follows from (1) and Theorem 2.10. $\square$

Proposition 2.17. *Let $N \subset M$ be an irreducible subfactor of type II_1 with finite index and $\{\lambda_i\}$ be a Pimsner-Popa system for M/N with support lying in $N' \cap M_1$. Then, $1 \leq \sum_i \lambda_i {\lambda_i}^* \leq [M:N]$.*

Proof. Let f denote the support of $\{\lambda_i\}$, i.e., $f = \sum_i \lambda_i e_1 \lambda_i^*$. Then, we obtain $\sum_i \lambda_i \lambda_i^* = [M:N]E_M(f)$. Since $N' \cap M = \mathbb{C}$, we have $E_M(f) = \mathrm{tr}(f) \in [0,1]$. Therefore, $\sum_i \lambda_i \lambda_i^* \leq [M:N]$.

On the other hand, since $f \in N' \cap M_1$ and $N' \cap M = \mathbb{C}$, by [14, Proposition 1.9], we have $\mathrm{tr}(f) \geq \tau$. Then, by irreducibility of $N \subset M$ again, we have $\mathrm{tr}(f) = E_M(f) = \tau \sum_i \lambda_i \lambda_i^*$. Hence, $\sum_i \lambda_i \lambda_i^* \geq 1$. $\square$

We conclude this section with a small observation on a kind of local behaviour of orthogonal systems. Recall, from [7], that for a subfactor $N \subset M$ and a projection $f \in N' \cap M$, the index of N at f is given by $[M_f : N_f] = [M:N]_f$. Also, a finite index subfactor $N \subset M$ is said to be extremal, if $\mathrm{tr}_{N'}$ and tr_M agree on $N' \cap M$. Clearly, if $N \subset M$ is irreducible, then it is extremal.

Proposition 2.18. *Let $N \subset M$ be an irreducible subfactor of type II_1 with $[M:N] < \infty$ and $f \in N' \cap M_1$ be a projection. Then, for any orthogonal system $\{\lambda_i\}$ with support f, we have $\sum_i \lambda_i \lambda_i^* = \sqrt{[M_1 : N]_f}$.*

Proof. Since $N \subset M$ is extremal, the following local index formula holds (see [7]):

$$[fM_1f : Nf] = [M_1 : N]\big(\mathrm{tr}_{M_1}(f)\big)^2 = \Big([M : N]\mathrm{tr}_{M_1}(f)\Big)^2.$$

On the other hand, since $\{\lambda_i\}$ is an orthogonal system, we obtain $\sum_i \lambda_i \lambda_i^* = [M : N]\mathrm{tr}_{M_1}(f)$. This completes the proof. $\square$

3. Regular subfactor and two-sided basis

Before we pursue our hunt for a two-sided basis in a regular subfactor, as asserted in the Introduction, we first show that every finite index subfactor with a two-sided basis is extremal, which, most likely, is folklore.

Proposition 3.1. *Let $N \subset M$ be a type II_1 subfactor with finite index. If there exists a two-sided basis for M over N, then $N \subset M$ is extremal.*

Proof. Given any right basis $\{\lambda_i : 1 \leq i \leq n\}$ for M/N, it is known (see, for instance, [1, Lemma 2.23]) that the $\mathrm{tr}_{N'}$ preserving conditional expectation $E_{M'} : N' \to M'$ is given by

$$E_{M'}(x) = [M : N]^{-1} \sum_i \lambda_i x \lambda_i^*, \ x \in N'.$$

Thus, if $x \in N' \cap M$, then $\mathrm{tr}_{N'}(x) = E_{M' \cap M}(x) = [M : N]^{-1} \sum_i \lambda_i x \lambda_i^*$.

Now, let $\{\lambda_i : 1 \leq i \leq n\}$ be any two-sided basis for M/N. Then, we have $\sum_i \lambda_i^* e_1 \lambda_i = 1 = \sum_i \lambda_i e_1 \lambda_i^*$ so that $\sum_i \lambda_i^* \lambda_i = [M : N]\, 1_M$ (after applying $E_M^{M_1}$ on both sides of first equality). Thus, for any $x \in N' \cap M$, we have

$$\begin{aligned} \mathrm{tr}_M(x) &= [M : N]^{-1}\mathrm{tr}_M\big(x \sum_i \lambda_i^* \lambda_i\big) \\ &= [M : N]^{-1}\mathrm{tr}_M\big(\sum_i \lambda_i x \lambda_i^*\big) \\ &= \mathrm{tr}_M\big(\mathrm{tr}_{N'}(x)\, 1_M\big) \\ &= \mathrm{tr}_{N'}(x). \end{aligned}$$

Hence, $N \subset M$ is extremal. $\square$

As the header suggests, this section is devoted to proving the existence of two-sided basis for a finite index regular subfactor. Keeping this in mind, from now onward, throughout this section, $N \subset M$ will denote a finite index subfactor of type II_1, which is not necessarily irreducible, and $\mathcal{R}$ will denote the intermediate von Neumann subalgebra generated by N and $N' \cap M$, i.e., $\mathcal{R} = N \vee (N' \cap M)$. We first present some preparatory results that we require to deduce the main theorem.

Lemma 3.2. *With notations as in the preceding paragraph, we have*

$$\mathcal{N}_M(N) \subseteq \mathcal{N}_M(\mathcal{R}).$$

Proof. Let $u \in \mathcal{N}_M(N)$. Then, $uNu^* = N$, and for $x \in N' \cap M$, we have

$$(uxu^*)n = uxu^*nuu^* = uu^*nuxu^* = n(uxu^*) \text{ for all } n \in N,$$

i.e., $u(N' \cap M)u^* = N' \cap M$. So, $u(nx)u^* = (unu^*)(uxu^*) \in N \vee (N' \cap M)$ for all $n \in N$ and $x \in N' \cap M$. Thus, we readily deduce that $u\mathcal{R}u^* = \mathcal{R}$. $\square$

The following crucial ingredient is an adaptation of [9, Lemma 5.7.3].

Proposition 3.3. *Let* tr *denote the restriction of* tr_M *on* $N' \cap M$. *Then,* $N' \cap M$ *has a two-sided Pimsner-Popa basis over* $\mathbb{C}$ *with respect to* tr.

Proof. Let $\overrightarrow{n} = [n_1, n_2, \cdots, n_k]$ denote the dimension vector of $N' \cap M$ and $\bar{t}$ denote the trace vector of tr. Consider the path algebra model $B_{-1} \subseteq B_0 \subseteq B_1$ for the inclusion $\mathbb{C} \subseteq N' \cap M$ as recalled in Section 2.2.2. Since $(\mathbb{C} \subseteq N' \cap M) \cong (B_0 \subseteq B_1)$, it is enough to show that $B_0 \subseteq B_1$ admits a two-sided basis with respect to the tracial state (on B_1) determined by the trace vector $\bar{t}$. Let

$$J := \{(\kappa, \beta) : \kappa, \beta \in \Omega_{1]} \text{ such that } r(\kappa) = r(\beta)\}.$$

Then, by [9, Proposition 5.4.1(iv)] (or see Section 2.2.2), $\{e_{\kappa,\beta} : (\kappa, \beta) \in J\}$ is a system of matrix units for B_1. So, by [9, Proposition 5.4.3 (iii)], we easily deduce that

$$E_{B_0}\big(e_{\kappa,\beta}(e_{\kappa',\beta'})^*\big) = \delta_{(\kappa,\beta),(\kappa',\beta')}\bar{t}_{r(\kappa)} \text{ for all } (\kappa, \beta), (\kappa', \beta') \in J.$$

Then, defining

$$\lambda_{\kappa,\beta} = \frac{1}{\sqrt{\bar{t}_{r(\kappa)}}} e_{\kappa,\beta} \text{ for } (\kappa, \beta) \in J,$$

we obtain

$$\sum_{(\kappa',\beta') \in I} E_{B_0}\big(e_{\kappa,\beta}(\lambda_{\kappa',\beta'})^*\big)\lambda_{\kappa',\beta'} = e_{\kappa,\beta} \quad \text{for all } (\kappa, \beta) \in J.$$

In particular, since $\{e_{\kappa,\beta} : (\kappa, \beta) \in J\}$ is a system of matrix units for B_1, we have

$$\sum_{(\kappa',\beta') \in J} E_{B_0}\big(x(\lambda_{\kappa',\beta'})^*\big)\lambda_{\kappa',\beta'} = x \quad \text{for all } x \in B_1,$$

that is, $\mathcal{B} := \{\lambda_{\kappa',\beta'} : (\kappa', \beta') \in J\}$ is a left Pimsner-Popa basis for $(N' \cap M)/\mathbb{C}$. Hence, being a self-adjoint set, $\mathcal{B}$ is in fact a two-sided Pimsner-Popa basis for B_1 over $\mathbb{C}$. $\square$

Lemma 3.4. $\mathcal{R}$ *has a two-sided basis over* N *contained in* $N' \cap M$.

Proof. First observe that $(\mathbb{C}, N'\cap M, N, M)$ is a commuting square (see, for instance, [5, Lemma 4.6.2]). Now the quadruple $\big(\mathbb{C}, N'\cap M, N, N\vee(N'\cap M)\big)$ is non-degenerate because $N\vee(N'\cap M)$ is the SOT closure of the algebra $N(N'\cap M)$ $(= (N'\cap M)N)$ (see [16, Proposition 1.1.5]). Therefore, the conclusion follows once we apply Lemma 3.3 and [16, Proposition 1.1.5] again. $\square$

The following useful result is implicit in [10], and was also observed in [3, Lemma 4.2]. For the sake of completeness, we include a proof using Pimsner-Popa basis.

Lemma 3.5. *Let θ be an automorphism of $\mathcal{R}$ such that its restriction to N is an outer automorphism of N. Then, θ is a free automorphism of $\mathcal{R}$.*

Proof. Suppose θ is not a free automorphism of $\mathcal{R}$. Then, by definition, there exists a non-zero $r\in\mathcal{R}$ such that

$$rx = \theta(x)r \text{ for all } x\in\mathcal{R}. \tag{3.1}$$

By Lemma 3.4, there exists a basis $\{\lambda_1,\ldots,\lambda_n\}$ for $\mathcal{R}/N$ contained in $N'\cap M$. Since $\sum_{i=1}^k \lambda_i E_N(\lambda_i^* r) = r\neq 0$, we must have $E_N(\lambda_j^* r)\neq 0$ for at least one λ_j. Thus, multiplying both sides of Equation (3.1) by λ_j^* on the left, we obtain

$$\lambda_j^* r x = \lambda_j^*\theta(x) r = \theta(x)\lambda_j^* r \;\text{ for all } x\in N. \tag{3.2}$$

Then, taking conditional expectation E_N on both sideds of Equation (3.2), we get

$$E_N(\lambda_j^* r)x = \theta(x)E_N(\lambda_j^* r) \;\text{ for all } x\in N.$$

This shows that $\theta|_N$ is not free. But a free automorphism of a factor is outer ([10], [9, §$A.4$]). Hence, we have a contradiction as $\theta_{|_N}$ is given to be outer. $\square$

Proposition 3.6. *Let G denote the generalized Weyl group of $N\subset M$. Then, any set of coset representatives $\{u_g : g=[u_g]\in G\}$ of G in $\mathcal{N}_M(N)$ forms a two-sided orthonormal system for $M/\mathcal{R}$.*

Proof. Let $w\in\mathcal{N}_M(N)$. We first assert that

$$E_{\mathcal{R}}(w) = 0 \text{ if and only if } w\in\mathcal{N}_M(N)\setminus\mathcal{U}(N)\mathcal{U}(N'\cap M).$$

Necessity is obvious. Conversely, suppose $w\notin\mathcal{U}(N)\mathcal{U}(N'\cap M)$. Note that, by Lemma 3.2, $wxw^*\in\mathcal{R}$ for all $x\in\mathcal{R}$. So, $\beta:\mathcal{R}\to\mathcal{R}$ defined by $\beta(x)=wxw^*$ is an automorphism of $\mathcal{R}$, which restricts to an outer automorphism on N (since $w\notin\mathcal{U}(N)\mathcal{U}(N'\cap M)$). Thus, by Proposition 3.5, β is a free automorphism of $\mathcal{R}$. Then, applying $E_{\mathcal{R}}$ on both sides of the equation $wx=\beta(x)w$, we obtain $E_{\mathcal{R}}(w)x=\beta(x)E_N(w)$ for all $x\in\mathcal{R}$. Since β is free, we must have $E_{\mathcal{R}}(w)=0$. This proves the assertion.

Now, fix a set of coset representatives $\{u_g : g=[u_g]\in G\}$ of G in $\mathcal{N}_M(N)$. Then, by above assertion, we have

$$E_{\mathcal{R}}(u_g u_h^*) = 0 = E_{\mathcal{R}}(u_g^* u_h) \text{ if and only if } g\neq h. \tag{3.3}$$

Hence, $\{u_g : g \in G\}$ forms a two-sided orthonormal system for M over $\mathcal{R}$. $\square$

Proposition 3.7. *Let $\mathcal{P} := \mathcal{N}_M(N)''$ and $\{u_g : g \in G\}$ be an orthonormal system for $M/\mathcal{R}$ as in Proposition 3.6. If p denotes the support of $\{u_g : g \in G\}$, then $p = e_{\mathcal{P}}$.*

In particular, if $N \subset M$ is regular, then $\{u_g : g \in G\}$ forms a two-sided orthonormal basis for M over $\mathcal{R}$.

Proof. We have $p = \sum_g u_g e_{\mathcal{R}} u_g^* \in \langle \mathcal{M}, e_{\mathcal{R}} \rangle \in B(L^2(\mathcal{M}))$ (see Definition 2.4). We first assert that $p|_{L^2(\mathcal{P})} = id$.

Let $A = \operatorname{span}\big(\mathcal{N}_M(N)\big)$. Then, $\mathcal{P} = A''$ and since A is a unital $*$-subalgebra of $\mathcal{P}$, by Double Commutant Theorem, we have $A'' = \overline{A}^{\text{SOT}}$. Let $x \in \mathcal{P}$. Then, there exists a net $(x_i) \subset A$ such that x_i converges to x in SOT. Thus, $x_i\Omega$ converges to $x\Omega$ in $L^2(M)$. So, it suffices to show that $p(u\Omega) = u\Omega$ for every $u \in \mathcal{N}_M(N)$ for then we will have

$$p(x\Omega) = \lim_i p(x_i\Omega) = \lim_i x_i\Omega = x\Omega.$$

Let $u \in \mathcal{N}_M(N)$. Then, $[u] = [u_g]$ for a unique $g \in G$. So, $u = u_g v$ for some $v \in \mathcal{U}(N)\mathcal{U}(N' \cap M)$. Thus,

$$\begin{aligned} p(u\Omega) &= \sum_{t \in G} u_t e_{\mathcal{R}} u_t^* u\Omega = \sum_{t\in G} u_t E_{\mathcal{R}}(u_t^* u)\Omega \\ &= \sum_{t\in G} u_t E_{\mathcal{R}}(u_t^* u_g) v\Omega = u_g v\Omega = u\Omega, \end{aligned}$$

where the second last equality holds because of Equation 3.3.

Now, it just remains to show that

$$p_{|_{\left(L^2(P)\right)^{\perp}}} = 0.$$

For this, it suffices to show that, for all $y \in M$ satisfying $\operatorname{tr}_M(x^*y) = 0$ for all $x \in \mathcal{P}$, we must have $p(y\Omega) = 0$, that is, we just need to show that $\sum_{g\in G} u_g E_{\mathcal{R}}(u_g^* y)\Omega = 0$ for any such y. In fact, we assert that $E_{\mathcal{R}}(u_g^* y) = 0$ for all $g \in G$.

For $z \in \mathcal{U}(N)\mathcal{U}(N' \cap M)$, $u_g z^* \in \mathcal{P}$ so that $\operatorname{tr}_M(z u_g^* y) = 0$ for all $g \in G$. Further, since

$$\mathcal{R} = \overline{\operatorname{span}\{\mathcal{U}(N)\mathcal{U}(N' \cap M)\}}^{\text{SOT}}$$

and tr_M is SOT-continuous on bounded sets, it follows that $\operatorname{tr}_M(r u_g^* y) = 0$ for all $r \in \mathcal{R}$ and $g \in G$. Hence, by the trace preserving property of the conditional expectation, we deduce that $E_{\mathcal{R}}(u_g^* y) = 0$ for all $g \in G$. This completes the proof. $\square$

The following two elementary observations turn out to be catalytic in proving the existence of two-sided basis for an arbitrary regular subfactor of type II_1 with finite index.

Lemma 3.8. *Let $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras with a faithful tracial state* tr *on $\mathcal{M}$ and $\{\lambda_i : 1 \leq i \leq m\}$ be a basis for $\mathcal{P}/\mathcal{N}$. Then, for any $u \in \mathcal{N}_{\mathcal{M}}(\mathcal{P}) \cap \mathcal{N}_{\mathcal{M}}(\mathcal{N})$, $\{u\lambda_i u^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$.*

Proof. Note that the map $\theta : \mathcal{P} \to \mathcal{P}$ given by $\theta(x) = uxu^*$ is a $\mathrm{tr}_{\mathcal{M}}$ (and hence $\mathrm{tr}_{\mathcal{P}}$) preserving automorphism of $\mathcal{P}$ that keeps $\mathcal{N}$ invariant. Then, a routine verification shows that $\{u\lambda_i u^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$, which we leave to the reader. □

Proposition 3.9. *Let $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ be as in Lemma 3.8. Suppose $\mathcal{P}/\mathcal{N}$ has a two-sided basis $\{\lambda_i : 1 \leq i \leq m\}$ and $\mathcal{M}/\mathcal{P}$ has a two-sided basis $\{\mu_j : 1 \leq j \leq n\}$ contained in $\mathcal{N}_{\mathcal{M}}(\mathcal{P}) \cap \mathcal{N}_{\mathcal{M}}(\mathcal{N})$. Then, $\{\mu_j \lambda_i : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a two-sided basis for $\mathcal{M}/\mathcal{N}$.*

Proof. Let $\lambda'_{i,j} := \mu_j \lambda_i \mu_j^*$, $1 \leq i \leq m, 1 \leq j \leq n$. Then, by Lemma 3.8, $\{\lambda'_{i,j} : 1 \leq i \leq m\}$ is a basis for $\mathcal{P}/\mathcal{N}$ for each j. Similarly, $\{(\lambda'_{i,j})^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$. Since $\{\lambda_i\}$ is a basis for $\mathcal{P}/\mathcal{N}$, we have $\sum_i \lambda_i e_1 \lambda_i^* = e_{\mathcal{P}}$ (see Section 2.2.1). So, by Lemma 2.6, we obtain $\sum_{i,j} \mu_j \lambda_i e_1 \lambda_i^* \mu_j^* = \sum_j \mu_j e_{\mathcal{P}} \mu_j^* = 1$. Therefore, by Lemma 2.6 again, $\{\mu_j \lambda_i\}$ is a basis for $\mathcal{M}/\mathcal{N}$. On the other hand, we have

$$\sum_{i,j} \lambda_i^* \mu_j^* e_1 \mu_j \lambda_i = \sum_{i,j} \mu_j^* (\lambda'_{i,j})^* e_1 \lambda'_{i,j} \mu_j = \sum_j \mu_j^* e_{\mathcal{P}} \mu_j = 1,$$

where the second last equality holds because $\{\lambda'_{i,j} : i \leq i \leq m\}$ is a basis for $\mathcal{P}/\mathcal{N}$ and the last equality follows because $\{\mu_j^* : 1 \leq j \leq n\}$ is a basis for $\mathcal{M}/\mathcal{P}$. Thus, we conclude that $\{(\mu_j \lambda_i)^*\}$ is also a basis for $\mathcal{M}/\mathcal{N}$. This completes the proof. □

We are now all set to deduce the main theorem of this article.

Theorem 3.10. Let $N \subset M$ be a regular subfactor of type II_1 with finite index. Then, M admits a two-sided Pimsner-Popa basis over N.

Proof. We observed in Lemma 3.4 that $\mathcal{R} := N \vee \big(N' \cap M\big)$ admits a two-sided basis, say, $\{\lambda_i\}$, over N. Further, we readily deduce, from Proposition 3.7, that M also admits a two-sided basis, say, $\{\mu_j\}$, over $\mathcal{R}$, which is contained in $\mathcal{N}_M(\mathcal{N})$. By Lemma 3.2, we know that $\mathcal{N}_M(\mathcal{N}) \subseteq \mathcal{N}_M(\mathcal{R})$. Hence, by Proposition 3.9, we conclude that $\{\mu_j \lambda_i\}$ is a two-sided Pimsner-Popa basis for M over N. □

In view of Proposition 3.1, we obtain the following:

Corollary 3.11. *Every regular subfactor of type II_1 with finite index is extremal.*

It is well known to the experts that every regular subfactor has integer index; for instance, there is a mention of this fact in [5, Page 150] (without a proof). As final application of some of the results proved above, we deduce this fact along with a precise expression for the index of such a subfactor. We will use Watatani's notion of index of a conditional expectation to do so.

Recall, from [20], that, given an inclusion $B \subset A$ of unital C^*-algebras, a conditional expectation $E : A \to B$ is said to have finite index if there exists a right Pimsner-Popa basis $\{\lambda_i : 1 \leq i \leq n\}$ for A over B via E and the Watatani index of E is defined as

$$\mathrm{Ind}(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*,$$

which is independent of the basis $\{\lambda_i\}$ and is an element of $\mathcal{Z}(A)$.

Theorem 3.12. *Every regular subfactor $N \subset M$ of type II_1 with finite Jones index has integer valued index and the index is given by*

$$[M : N] = |G| \dim_{\mathbb{C}}(N' \cap M),$$

where G denotes the generalized Weyl group of the inclusion $N \subset M$.

Proof. Consider the inclusion $\mathbb{C} \subseteq N' \cap M$. Let Λ denote its inclusion matrix. Let $\{\lambda_i\} \subset N' \cap M$ be a two-sided basis for $N' \cap M$ over $\mathbb{C}$ with respect to tr as in Proposition 3.3. We observed in Lemma 3.4 that $\{\lambda_i\}$ is a two-sided basis for $\mathcal{R} := N \vee \big(N' \cap M\big)$ as well over N with respect to $E_{N|_{\mathcal{R}}}$.

Further, from Proposition 3.7, M admits a two-sided basis consisting of unitaries, say, $\{\mu_j : 1 \leq j \leq |G|\}$, over $\mathcal{R}$, which is contained in $\mathcal{N}_M(\mathcal{N})$. As seen in Theorem 3.10, $\{\mu_j \lambda_i\}$ is a two-sided Pimsner-Popa basis for M over N with respect to E_N. Thus, $\{\lambda_i^* \mu_j^*\}$ is also a basis for M over N, and we obtain

$$[M : N] = \sum_{i,j} \lambda_i^* \mu_j^* \mu_j \lambda_i = |G| \sum_i \lambda_i^* \lambda_i = |G| \sum_i \lambda_i \lambda_i^* = |G| \, \mathrm{Ind}(\mathrm{tr}),$$

where the second last equality holds because $\{\lambda_i\}$ is a two-sided basis for tr. In particular, Ind(tr) is scalar-valued. So, if Λ denotes the matrix of the inclusion $\mathbb{C} \subseteq N' \cap M$ and $\bar{s} = (s_1, \ldots, s_k)$ denotes the trace vector of tr, then by [20, Corollary 2.4.3], there exists a $\beta > 0$ such that $\bar{s}\, \Lambda \Lambda^t = \beta \bar{s}$ and $\mathrm{Ind}(\mathrm{tr}) = \beta$.

Now, if $[n_1, \ldots, n_k]$ is the dimension vector of $N' \cap M$, then by Watatani's convention, we have $\Lambda = [n_1, \ldots, n_k]^t$. Since $\sum_{i=1}^{k} s_i n_i = 1$, we obtain

$$\begin{aligned} \bar{s}\, \Lambda \Lambda^t &= \left(\Big(\sum_{i=1}^{k} s_i n_i \Big) n_1, \Big(\sum_{i=1}^{k} s_i n_i \Big) n_2, \ldots, \Big(\sum_{i=1}^{k} s_i n_i \Big) n_k \right) \\ &= (n_1, n_2, \ldots, n_k), \end{aligned}$$

which yields $\beta = \frac{n_i}{s_i}$ for all $1 \leq i \leq k$. Thus, if p_i denotes a minimal projection in the i-th summand of $N' \cap M$ and $\tilde{p}_i$ denotes the i-th minimal central projection, then $\mathrm{tr}(p_i) = s_i = n_i/\beta$ for all $1 \leq i \leq k$; so, $\mathrm{tr}(\tilde{p_i}) = n_i^2/\beta = s_i n_i$ for all $1 \leq i \leq k$. This gives

$$1 = \mathrm{tr}(1) = \sum_{i=1}^{k} \mathrm{tr}(\tilde{p}_i) = \sum_{i=1}^{k} n_i^2/\beta;$$

so that $\beta = \sum_{i=1}^{k} n_i^2 = \dim_{\mathbb{C}}(N' \cap M)$. Hence,

$$[M : N] = |G| \dim_{\mathbb{C}}(N' \cap M).$$

This completes the proof. □

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