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Computations of de Rham cohomology rings of classifying stacks at torsion primes

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Abstract. We compute the de Rham cohomology rings of BG_2 and $B\mathrm{Spin}(n)$ for $7 \le n \le 11$ over base fields of characteristic 2.

Contents

Introduction

Let G be a smooth affine algebraic group over a commutative ring R. In [17], Totaro defines the Hodge cohomology group $H^i(BG, \Omega^j)$ for $i, j \ge 0$ to be the ith étale cohomology group of the sheaf of differential forms Ω^j over R on the big étale site of the classifying stack BG. For $n \ge 0$, let $H^n_{\mathrm{H}}(BG/R) := \oplus_j H^j(BG, \Omega^{n-j})$ denote the total Hodge cohomology group of degree n. De Rham cohomology groups $H^n_{\mathrm{dR}}(BG/R)$ are defined to be the étale cohomology groups of the de Rham complex of BG. Let $\mathfrak{g}$ denote the Lie algebra associated to G and let $O(\mathfrak{g}) = S(\mathfrak{g}^*)$ denote the ring of polynomial functions on $\mathfrak{g}$. In [17, Corollary 2.2], Totaro showed that the Hodge cohomology of BG is related to the representation theory of G:

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, S^j(\mathfrak{g}^*)).$$

Let G be a split reductive group defined over $\mathbb{Z}$. From the work of Bhatt-Morrow-Scholze in p-adic Hodge theory [1, Theorem 1.1], one might expect that

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$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i(BG_{\mathbb{C}}, \mathbb{F}_p) \tag{1}$$

for all primes p and $i \geq 0$. The results from [1] do not immediately apply to BG since BG is not proper as a stack over $\mathbb{Z}$. For p a non-torsion prime of a split reductive group G defined over $\mathbb{Z}$, Totaro showed that

$$H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p) \cong H^*(BG_{\mathbb{C}}, \mathbb{F}_p) \tag{2}$$

[17, Theorem 9.2]. It remains to compare $H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p)$ with $H^*(BG_{\mathbb{C}}, \mathbb{F}_p)$ for p a torsion prime of G. For $n \geq 3$, 2 is a torsion prime for the split group $SO(n)$. Totaro showed that

$$H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[w_2, \ldots, w_n]$$

as graded rings where $w_2, \ldots, w_n$ are the Stiefel-Whitney classes [17, Theorem 11.1]. In general, the rings $H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p)$ and $H^*(BG_{\mathbb{C}}, \mathbb{F}_p)$ are different though. For example,

$$\dim_{\mathbb{F}_2} H^{32}_{\mathrm{dR}}(B\mathrm{Spin}(11)_{\mathbb{F}_2}/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2)$$

[17, Theorem 12.1].

In this paper, we verify inequality (1) for more examples. For the torsion prime 2 of the split reductive group G_2 over $\mathbb{Z}$, we show that

$$H^*_{\mathrm{dR}}(B(G_2)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(B(G_2)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[y_4, y_6, y_7]$$

as graded rings where $|y_i| = i$ for $i = 4, 6, 7$. For the spin groups, we show that

$$H^*_{\mathrm{dR}}(B\mathrm{Spin}(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(B\mathrm{Spin}(n)_{\mathbb{C}}, \mathbb{F}_2) \tag{3}$$

for $7 \leq n \leq 10$. Note that 2 is a torsion prime for $\mathrm{Spin}(n)$ for $n \geq 7$. The isomorphism (3) holds for $1 \leq n \leq 6$ by the "accidental" isomorphisms for spin groups along with (2).

For $n = 11$, we make a full computation of the de Rham cohomology ring of $B\mathrm{Spin}(n)_{\mathbb{F}_2}$:

$$H^*_{\mathrm{dR}}(B\mathrm{Spin}(11)_{\mathbb{F}_2}/\mathbb{F}_2) \cong \mathbb{F}_2[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7 y_{10} + y_6 y_{11})$$

where $|y_i| = i$ for all i. We can compare this result with the computation of the singular cohomology of $B\mathrm{Spin}(11)_{\mathbb{C}}$ given by Quillen [14]:

$$H^*(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{64}]/(w_7 w_{10} + w_6 w_{11}, \\ w_{11}^3 + w_{11}^2 w_7 w_4 + w_{11} w_8 w_7^2)$$

where $|w_i| = i$ for all i. Equivalently,

$$H^*(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2) \cong H^*(BSO(11)_{\mathbb{C}}, \mathbb{F}_2)/J \otimes \mathbb{F}_2[w_{64}]$$

where J is the ideal generated by the regular sequence

$$w_2, Sq^1(w_2), Sq^2 Sq^1(w_2), \ldots, Sq^{16} Sq^8 \cdots Sq^1 w_2.$$

Thus, the rings $H^*_{\mathrm{dR}}(B\mathrm{Spin}(n)_{\mathbb{F}_2}/\mathbb{F}_2)$ and $H^*(B\mathrm{Spin}(n)_{\mathbb{C}}, \mathbb{F}_2)$ are not isomorphic in general even though $H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2)$ for all n. Steenrod squares on de Rham cohomology over a base field of characteristic 2 have not yet been constructed. If they exist, our calculation suggests that their action on $H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2)$ would have to be different from the action of the topological Steenrod operations.

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1. Preliminaries

In this section, we recall results from [17] that will be used in our computations. These results were also used by Totaro in [17, Theorem 11.1] to compute the de Rham cohomology of $BSO(n)_k$ for k a field of characteristic 2.

The first result we mention [17, Proposition 9.3] is an analogue of the Leray-Serre spectral sequence from topology.

Proposition 1.1. *Let G be a split reductive group defined over a field F and let P be a parabolic subgroup of G with Levi quotient L (this means that $P \cong R_u(P) \rtimes L$ where $R_u(P)$ is the unipotent radical of P [2, 14.19]). There exists a spectral sequence of algebras*

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/F) \otimes H^j_{\mathrm{H}}((G/P)/F) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/F).$$

Proposition 1.1 is the main tool that we will use to compute Hodge cohomology rings of classifying stacks. To apply Proposition 1.1, we will choose a parabolic subgroup P for which $H^*_{\mathrm{H}}(BL/F)$ is a polynomial ring.

To fill in the 0th column of the E_2 page in Proposition 1.1, we use a result of Srinivas [15].

Proposition 1.2. *Let G be split reductive over a field F and let P be a parabolic subgroup of G. The cycle class map*

$$CH^*(G/P) \otimes_{\mathbb{Z}} F \to H^*_{\mathrm{H}}((G/P)/F)$$

is an isomorphism.

Under the cycle class map, $CH^i(G/P) \otimes_{\mathbb{Z}} F$ maps to $H^i(G/P, \Omega^i)$. From the work of Chevalley [5] and Demazure [6], $CH^*(G/P)$ is independent of the field F and is isomorphic to the singular cohomology ring $H^*(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathbb{Z})$.

The last piece of information we will use to compute $H^*_{\mathrm{H}}(BG/F)$ is the ring of G-invariants $O(\mathfrak{g})^G = \oplus_i H^i(BG, \Omega^i)$. Let T be a maximal torus in G with Lie algebra $\mathfrak{t}$ and Weyl group W. There is a restriction homomorphism

$$O(\mathfrak{g})^G \to O(\mathfrak{t})^W. \tag{4}$$

We will need the following theorem which is due to Chaput and Romagny [4, Theorem 1.1]. For the following theorem, a split algebraic group G over a field F is simple if every proper smooth normal connected subgroup of G is trivial.

Theorem 1.3. *Assume that G is simple over a field F. Then the restriction homomorphism (4) is an isomorphism unless $\mathrm{char}(F) = 2$ and $G_{\overline{F}}$ is a product of copies of $Sp(2n)$ for some $n \in \mathbb{N}$.*

From the rings $O(\mathfrak{g})^G, CH^*(G/P), H^*_{\mathrm{H}}(BL/F)$, we will be able to determine the E_∞ terms of the spectral sequence in Proposition 1.1. This will allow us to determine $H^*_{\mathrm{H}}(BG/F)$ by using the following version of the Zeeman comparison theorem [12, Theorem VII.2.4].

Theorem 1.4. *Fix a field F. Let $\{\bar{E}_r^{i,j}\}, \{E_r^{i,j}\}$ be first quadrant (cohomological) spectral sequences of F-vector spaces such that $\bar{E}_2^{i,j} = \bar{E}_2^{i,0} \otimes_F \bar{E}_2^{0,j}$ and $E_2^{i,j} = E_2^{i,0} \otimes_F E_2^{0,j}$ for all i, j. Let $\{f_r^{i,j} : \bar{E}_r^{i,j} \to E_r^{i,j}\}$ be a morphism of spectral sequences such that $f_2^{i,j} = f_2^{i,0} \otimes f_2^{0,j}$ for all i, j. Fix $N, Q \in \mathbb{N}$. Assume that $f_\infty^{i,j}$ is an isomorphism for all i, j with $i + j < N$ and an injection for $i + j = N$. If $f_2^{0,i}$ is an isomorphism for all $i < Q$ and an injection for $i = Q$, then $f_2^{i,0}$ is an isomorphism for all $i < \min(N, Q + 1)$ and an injection for $i = \min(N, Q + 1)$.*

We recall a result from [17, Section 11] on the degeneration of the Hodge spectral sequence for split reductive groups, under some assumptions. The result in [17, Section 11] was proved for the special orthogonal groups but the proof works more generally.

Proposition 1.5. *Let G be a split reductive group over a field F and assume that the Hodge cohomology ring of BG is generated as an F-algebra by classes in $\oplus_i H^{i+1}(BG, \Omega^i)$ and $\oplus_i H^i(BG, \Omega^i)$. Then the Hodge spectral sequence*

$$E_1^{i,j} = H^j(BG, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(BG/F) \tag{5}$$

for BG degenerates at the E_1 page.

Proof. From [17, Lemma 8.2], there are natural maps

$$H^i(BG, \Omega^i) \to H^{2i}_{\mathrm{dR}}(BG/F)$$

and

$$H^{i+1}(BG, \Omega^i) \to H^{2i+1}_{\mathrm{dR}}(BG/F)$$

for all $i \geq 0$. These maps are compatible with products. Let T denote a maximal torus of G. From the group homomorphism $T \to G$, we have the commuting square

$$\begin{tikzcd} \oplus_i H^i(BG,\Omega^i) \arrow[r] \arrow[d] & \oplus_i H^{2i}_{\mathrm{dR}}(BG/F) \arrow[d] \\ \oplus_i H^i(BT,\Omega^i) \arrow[r, "\cong"] & H^{2i}_{\mathrm{dR}}(BT/F). \end{tikzcd} \tag{6}$$

The restriction homomorphism (4) induces an injection

$$\oplus_i H^i(BG,\Omega^i) \to \oplus_i H^i(BT,\Omega^i)$$

[17, Lemma 8.2]. Hence, from diagram (6), we get that the natural map

$$\oplus_i H^i(BG,\Omega^i) \to \oplus_i H^{2i}_{\mathrm{dR}}(BG/F)$$

is an injection. Hence, any differentials into the diagonal in the spectral sequence (5) must be 0. Then all classes in $\oplus_i H^{i+1}(BT,\Omega^i)$ must be permanent cycles (an element x in the E_2 page of a spectral sequence E_* is called a permanent cycle if $d_i(x) = 0$ for all $i \geq 2$) in (5). Classes in $\oplus_i H^i(BT,\Omega^i)$ must be permanent cycles in the spectral sequence (5) since $H^i(BG,\Omega^j) = 0$ for $i < j$ by [17, Corollary 2.2]. This proves that the Hodge spectral sequence for BG degenerates. $\square$

The following definition will be used later to describe the Hodge cohomology of flag varieties.

Definition 1.6. Let F be a field. For variables $x_1,\dots,x_n$ let $\Delta(x_1,\dots,x_n)$ denote the F-vector space with basis given by the products $x_{i_1}\cdots x_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

2. G_2

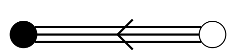
Let k be a field of characteristic 2 and let G denote the split form of G_2 over k.

Theorem 2.1. *The Hodge cohomology ring of BG is freely generated as a commutative k-algebra by generators $y_4 \in H^2(BG,\Omega^2)$, $y_6 \in H^3(BG,\Omega^3)$, and $y_7 \in H^4(BG,\Omega^3)$. The Hodge spectral sequence for BG degenerates at E_1 and we have*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4,y_6,y_7].$$

From the computation [12, Corollary VII.6.3] of the singular cohomology ring of $B(G_2)_{\mathbb{C}}$ with $\mathbb{F}_2$-coefficients, we then have $H^*(B(G_2)_{\mathbb{C}},k) \cong H^*_{\mathrm{dR}}(BG/k)$.

Proof. We first choose a suitable parabolic subgroup of G. Let P be the parabolic subgroup of G corresponding to inclusion of the long root.



From Proposition 1.2, $CH^*(G/P)$ is independent of the field k and the characteristic of k. As discussed in [9, §23.3], if we consider $(G_2)_{\mathbb{C}}$ over $\mathbb{C}$ along with the corresponding parabolic subgroup $P_{\mathbb{C}}$, $(G_2)_{\mathbb{C}}/P_{\mathbb{C}}$ is isomorphic to a smooth quadric Q_5 in $\mathbb{P}^6$. Hence, by [8, Chapter XIII], $H^*_{\mathrm{H}}((G/P)/k)$ is isomorphic to

$$CH^*(Q_5) \otimes_{\mathbb{Z}} k \cong k[v, w]/(v^6, w^2, v^3 - 2w) = k[v, w]/(v^3, w^2)$$

where $|v| = 2$ and $|w| = 6$ in $H^*_{\mathrm{H}}((G/P)/k)$.

We next show that the Levi quotient L of P is isomorphic to $GL(2)_k$. This can be seen by constructing an isomorphism from the root datum of $GL(2)_k$ to the root datum of the Levi quotient. Let $(X_1, R_1, X_1^\vee, R_1^\vee)$ be the usual root datum of $GL(2)_k$ where $X_1 = \mathbb{Z}\chi_1 + \mathbb{Z}\chi_2$, $R_1 = \mathbb{Z}(\chi_1 - \chi_2)$, and we take our torus to be the set of diagonal matrices in $GL(2)_k$. We take $(X_2, R_2, X_2^\vee, R_2^\vee)$ to be the root datum of G as described in [3, Plate IX]. Here, $X_2 = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = 0\}$. The long root α for G is then $(-2, 1, 1)$ and the root datum of $P/R_u(P)$ is $(X_2, \pm\alpha, X_2^\vee, \pm\frac{1}{3}\alpha)$. An isomorphism from the root datum of $GL(2)_k$ to the root datum of G can then be obtained from the isomorphism

$$X_1 \to X_2$$
$$\chi_1 \longmapsto (-1, 1, 0), \chi_2 \longmapsto (1, 0, -1).$$

Thus, $L \cong GL(2)_k$.

We now analyze the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{7}$$

from Proposition 1.1. From [7, Proposition] and [10, II.4.22],

$$H^*_{\mathrm{H}}(BL/k) = S^*(\mathfrak{gl}_2)^{GL(2)_k} \cong S^*(\mathfrak{t})^{S_2} = k[x_1, x_2]$$

where $x_1 \in H^1(BL, \Omega^1)$ and $x_2 \in H^2(BL, \Omega^2)$. Here, $\mathfrak{t}$ is the space of all diagonal matrices in $\mathfrak{gl}_2$ and S_2 acts on $\mathfrak{t}$ by permuting the diagonal entries.

In order to compute $H^*_{\mathrm{H}}(BG/k)$ from the spectral sequence above, we must first compute the ring of invariants of $S^*(\mathfrak{g}_2)^G$. From Theorem 1.3, $S^*(\mathfrak{g}_2)^G \cong S^*(\mathfrak{t}_o)^W$ where $\mathfrak{t}_o$ is the Lie algebra of a maximal torus T in G and W is the corresponding Weyl group of G. By [17, Corollary 2.2],

$$H^i(BG, \Omega^i) \cong S^i(\mathfrak{t}_o)^W$$

for $i \geq 0$.

Proposition 2.2. *The ring of invariants $S^*(\mathfrak{t}_o)^W$ is equal to $k[y_4, y_6]$ where $|y_4| = 2$ and $|y_6| = 3$ in $S^*(\mathfrak{t}_o)^W$.*

Proof. Following the notation in [3, Plate IX], $W \cong Z_2 \times S_3$ acts on the root lattice $X_2 = \{(a, b, c) \in \mathbb{Z}^3 \mid a+b+c = 0\}$ by multiplication by -1 and by permuting the coordinates. Hence, since we are working in characteristic 2, W acts on $S^*(\mathfrak{t}_o) = k[t_1, t_2, t_3]/(t_1 + t_2 + t_3)$ by permuting t_1, t_2, and t_3. We then have $S^*(\mathfrak{t}_o)^W = k[t_1t_2 + t_1t_3 + t_2t_3, t_1t_2t_3] = k[y_4, y_6]$. $\square$

We can now carry out the computation of $H^*_{\mathrm{H}}(BG/k)$. First, we show that the class $v \in E_2^{0,2}$ is a permanent cycle. Consider the filtration on $H^2_{\mathrm{H}}(BL/k) = k{\cdot}v$ given by (7): $H^2_{\mathrm{H}}(BL/k) \hookleftarrow E_\infty^{2,0}$, where $H^2_{\mathrm{H}}(BL/k)/E_\infty^{2,0} \cong E_\infty^{0,2}$. Here, $E_2^{1,1} = 0$ and

$$E_\infty^{2,0} = E_2^{2,0} = H^2_{\mathrm{H}}(BG/k) = H^1(BG, \Omega^1)$$

(we have $H^2(BG, \mathcal{O}) = 0$ since $H^2(BL, \mathcal{O}) = 0$ and there are no differentials entering $E_2^{2,0}$) since $H^*_{\mathrm{H}}((G/P)/k) = \oplus_i H^i(G/P, \Omega^i)$ is concentrated in even degrees. Hence,

$$E_\infty^{2,0} = H^2_{\mathrm{H}}(BG/k) = H^1(BG, \Omega^1) = 0,$$

by Proposition 2.2. It follows that $E_\infty^{0,2} \cong E_2^{0,2} = k \cdot v$ which implies that $d_3(v) = 0$. As (7) is a spectral sequence of algebras, it follows that v and v^2 are permanent cycles. Using that $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees, we then get that $H^3_{\mathrm{H}}(BG/k) = E_2^{3,0} = E_\infty^{3,0} = 0$ and $H^5_{\mathrm{H}}(BG/k) = E_2^{5,0} = E_\infty^{5,0} = 0$.

Next, we show that $w \in H^6_{\mathrm{H}}((G/P)/k) = E_2^{0,6}$ is transgressive with $0 \neq d_7(w) \in E_7^{7,0}$. Note that $\dim_k H^6_{\mathrm{H}}(BL/k) = 2$. As v is a permanent cycle in E_*, we observe that $E_\infty^{4,2} \cong E_2^{4,2} \cong k{\cdot}y_4 \otimes_k k{\cdot}v \cong k$ and $E_\infty^{6,0} \cong E_2^{6,0} \cong k{\cdot}y_6 \cong k$. Hence, $\dim_k H^6_{\mathrm{H}}(BL/k) = 2 = \dim_k E_\infty^{4,2} + \dim_k E_\infty^{6,0}$. From the filtration on $H^6_{\mathrm{H}}(BL/k)$ given by the spectral sequence (7), it follows that $E_\infty^{0,6} = 0$. As $H^3_{\mathrm{H}}(BG/k) = E_2^{3,0} = E_\infty^{3,0} = 0$ and $H^5_{\mathrm{H}}(BG/k) = E_2^{5,0} = E_\infty^{5,0} = 0$, we then get that $0 \neq d_7(w) \in E_7^{7,0}$ and $d_7(w)$ lifts to a non-zero element $y_7 \in H^4(BG, \Omega^3) \subseteq H^7_{\mathrm{H}}(BG/k)$.

$$\begin{tikzcd}
k \cdot w \arrow[ddddddrrrrrrr] & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k \cdot v^2 \arrow[ddddrrrrr] & 0 & 0 & 0 & k \cdot v^2 y_4 & 0 & k \cdot v^2 y_6 & k \cdot v^2 y_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k \cdot v \arrow[ddrrr] & 0 & 0 & 0 & k \cdot v y_4 & 0 & k \cdot v y_6 & k \cdot v y_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 & k \cdot y_4 & 0 & k \cdot y_6 & k \cdot y_7
\end{tikzcd}$$

Now, we can determine the E_∞ terms in (7). For n odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Let $n \in \mathbb{N}$ be even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b = n\}.$$

For $i = 0,1,2$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0,1,2$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b = n - 2i\}$. As v is a permanent cycle in (7), $E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$ for $i = 0,1,2$. As $y_7 \in H^4(BG, \Omega^3)$ and $H^i(BG, \Omega^j) = 0$ for $i < j$,

$$y_7 \cdot x \notin \oplus_j H^j(BG, \Omega^j)$$

for all $x \in H^*_{\mathrm{H}}(BG/k)$. Hence,

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes_k k \cdot v^i \subseteq E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0,1,2$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n}$ by

$$f_n(a,b) = \begin{cases} (b, a/3) \in S_{0,n} \text{ if } a \equiv 0 \mod 3, \\ (b, (a-1)/3) \in S_{1,n} \text{ if } a \equiv 1 \mod 3, \\ (b, (a-2)/3) \in S_{2.n} \text{ if } a \equiv 2 \mod 3. \end{cases}$$

Then

$$\begin{aligned} \dim_k H^n_{\mathrm{H}}(BL/k) &= |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| \\ &\leq \dim_k E_\infty^{n,0} + \dim_k E_\infty^{n-2,2} + \dim_k E_\infty^{n-4,4} \end{aligned}$$

where the inequality follows from the fact proved above that

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0,1,2$. From the filtration on $H^n_{\mathrm{H}}(BL/k)$ defined by the spectral sequence (7), it follows that $H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \cong E_\infty^{n-2i,2i}$ for $i = 0,1,2$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 3$.

We can now finish the computation of the Hodge cohomology of BG by using Zeeman's comparison theorem. Let F_* denote the cohomological spectral sequence of k-vector spaces concentrated on the 0th column with E_2 page given by

$$F_2^{0,i} = \begin{cases} k \text{ if } i = 0, \\ k \cdot v \text{ if } i = 2, \\ k \cdot v^2 \text{ if } i = 4, \\ 0 \text{ if } i \neq 0,\ 2,\ 4. \end{cases}$$

As $v \in E_2^{0,2}$ in the spectral sequence (7) is transgressive with $d_r(v) = 0$ for all $r \geq 2$, there exists a map of of spectral sequences $F_* \to E_*$ that takes $v \in F_2^{0,2}$ to $v \in E_2^{0,2}$ and $v^2 \in F_2^{0,4}$ to $v^2 \in E_2^{0,4}$.

Fixing a variable y, let H_* denote the cohomological spectral sequence with E_2 page given by $H_2 = \Delta(w) \otimes k[y]$ where w is of bidegree $(0,6)$, y is of bidegree $(7,0)$, and w is transgressive with $d_7(wy^i) = y^{i+1}$ for all $i \geq 0$. As $w \in E_2^{0,6}$ is transgressive with $d_7(w) = y_7 \in E_2^{7,0}$, there exists a map of spectral sequence $H_* \to E_*$ such that $w \in H_2^{0,6}$ maps to $w \in E_2^{0,6}$ and $y \in H_2^{7,0}$ maps to $y_7 \in E_2^{7,0}$. Elements of the ring of G-invariants $k[y_4, y_6]$ are permanent cycles in the spectral sequence (7) since they are concentrated on the 0th row. Thus, by tensoring the previous maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6] \to E_*$$

of spectral sequences.

As shown above, the map α induces an isomorphism $I_\infty \cong F_2 \otimes k[y_4, y_6] \to E_\infty$ on E_∞ pages. The 0th columns of the E_2 pages of the spectral sequences I_* and E_* are both isomorphic to $k[v,w]/(v^3, w^2)$ and α induces an isomorphism on the 0th columns of the E_2 pages. Thus, by Theorem 1.4, α induces an isomorphism on the 0th rows of the E_2 pages. Hence,

$$H_{\mathrm{H}}^*(BG/k) = k[y_4, y_6, y_7].$$

From Proposition 1.5, the Hodge spectral sequence for BG degenerates. $\square$

Corollary 2.3. *Let G be a k-form of G_2. Then*

$$H_{\mathrm{H}}^*(BG/k) \cong k[x_4, x_6, x_7]$$

where $|x_i| = i$ for $i = 4, 6, 7$.

Proof. Letting k_s denote the separable closure of k, we have $BG \times_k \mathrm{Spec}(k_s) \cong B(G_2)_{k_s}$. From Theorem 2.1, $H_{\mathrm{H}}^*(B(G_2)_{k_s})/k_s) \cong k_s[x_4', x_6', x_7']$ for some $x_4', x_6', x_7' \in H_{\mathrm{H}}^*(B(G_2)_{k_s}/k_s)$ with $|x_i'| = i$ for all i. As Hodge cohomology commutes with extensions of the base field,

$$H_{\mathrm{H}}^*((BG \times_k \mathrm{Spec}(k_s))/k_s) \cong H_{\mathrm{H}}^*(BG/k) \otimes_k k_s.$$

It follows that $H_{\mathrm{H}}^*(BG/k) \cong k[x_4, x_6, x_7]$ for some $x_4, x_6, x_7 \in H_{\mathrm{H}}^*(BG/k)$. $\square$

3. Spin groups

Let k be a field of characteristic 2 and let G denote the split group $\mathrm{Spin}(n)_k$ over k for $n \geq 7$.

Let $P_0 \subset SO(n)_k$ denote a parabolic subgroup that stabilizes a maximal isotropic subspace. Let $P \subset G$ denote the inverse image of P_0 under the double cover map $G \to SO(n)_k$. The Hodge cohomology of G/P is given by Proposition 1.2 and [12, Theorem III.6.11].

Proposition 3.1. *There is an isomorphism*

$$H_{\mathrm{H}}^*((G/P)/k) \cong k[e_1, \ldots, e_s]/(e_i^2 = e_{2i}),$$

where $s = \lfloor (n-1)/2 \rfloor$, $e_m = 0$ for $m > s$, and $|e_i| = 2i$ for all i.

The Levi quotient of P_0 is isomorphic to $GL(r)_k$ where $r = \lfloor n/2 \rfloor$. Hence, the Levi quotient L of P is a double cover of $GL(r)_k$.

Proposition 3.2. *The torsion index of L is equal to* 1.

Proof. We show that the torsion index of the corresponding compact connected Lie group M is equal to 1. As M is a double cover of $U(r)$, M is isomorphic to $(S^1 \times SU(r))/2\mathbb{Z}$ where $k \in \mathbb{Z}$ acts on $S^1 \times SU(r)$ by

$$(z, A) \mapsto (ze^{2\pi ik/r}, e^{-2\pi ik/r}A).$$

Hence, the derived subgroup $[M, M]$ of M is isomorphic to $SU(r)$. As $SU(r)$ has torsion index 1, M has torsion index 1 by [16, Lemma 2.1]. Thus, L has torsion index equal to 1. □

Corollary 3.3. *We have*

$$H^*_{\mathrm{H}}(BL/k) = O(\mathfrak{l})^L = k[A, c_2, \ldots, c_r]$$

where $|c_i| = 2i$ in $H^*_{\mathrm{H}}(BL/k)$ for all i and $|A| = 2$.

Proof. From Proposition 3.2 and [17, Theorem 9.1],

$$H^*_{\mathrm{H}}(BL/k) = O(\mathfrak{l})^L.$$

Let T be a maximal torus in L with Lie algebra $\mathfrak{t}$ and Weyl group W. From Theorem 1.3, $O(\mathfrak{l})^L \cong O(\mathfrak{t})^W$. To compute $O(\mathfrak{t})^W$, we use that L is a double cover of $GL(r)_k$. We have

$$\begin{aligned} S(X^*(T) \otimes k) &\cong \mathbb{Z}[x_1, \ldots, x_r, A]/(2A = x_1 + \cdots + x_r) \otimes k \\ &\cong k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r). \end{aligned}$$

The Weyl group W of L is isomorphic to the symmetric group S_r and acts on $S(X^*(T) \otimes k)$ by permuting $x_1, \ldots, x_r$. From [13, Proposition 4.1],

$$(k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r))^{S_r} = k[A, c_2, \ldots, c_r]$$

where $c_1, \ldots, c_r$ are the elementary symmetric polynomials in the variables

$$x_1, \ldots, x_r.$$

□

For our calculations, we will need to know the Hodge cohomology of $BSO(n)_k$ [17, Theorem 11.1].

Theorem 3.4. *The Hodge spectral sequence for $BSO(n)_k$ degenerates and*

$$H^*_{\mathrm{H}}(BSO(n)_k/k) = k[u_2, \ldots, u_n]$$

where $u_{2i} \in H^i(BSO(n)_k, \Omega^i)$ and $u_{2i+1} \in H^{i+1}(BSO(n)_k, \Omega^i)$ for all relevant i.

We'll also need to know the ring of invariants of $G = \mathrm{Spin}(n)_k$ for all $n \geq 6$. This can be found in [17, Section 12].

Lemma 3.5. *For $n \geq 6$,*

$$O(\mathfrak{g})^G = \begin{cases} k[c_2, \dots, c_r, \eta_{r-1}] \text{ if } n = 2r+1 \\ k[c_2, \dots, c_r, \mu_{r-1}] \text{ if } n = 2r \text{ and } r \text{ is even} \\ k[c_2, \dots, c_r, \mu_r] \text{ if } n = 2r \text{ and } r \text{ is odd} \end{cases}$$

where $|c_i| = i$, $|\eta_j| = 2^j$, and $|\mu_j| = 2^{j-1}$ in $O(\mathfrak{g})^G$ for all i and j.

Note that under the inclusion $O(\mathfrak{g})^G \subset H^*_{\mathrm{H}}(BG/k)$, the degree of an invariant function in $H^*_{\mathrm{H}}(BG/k)$ is twice its degree in $O(\mathfrak{g})^G$.

Theorem 3.6. *Let $n = 7$. The Hodge spectral sequence for BG degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$.

Proof. From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{8}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^4, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3].$$

First, we show that $e_1 \in E_2^{0,2}$ is a permanent cycle. From the filtration on $H^2_{\mathrm{H}}(BL/k) = k \cdot A$ given by (8), we have

$$1 = \dim_k E_\infty^{0,2} + \dim_k E_\infty^{2,0} = \dim_k E_\infty^{0,2} + \dim_k E_2^{2,0}.$$

As $H^*_{\mathrm{H}}(BL/k) = \oplus_i H^i(BL, \Omega^i)$, $E_2^{2,0} = H^1(BG, \Omega^1) = 0$. Hence, $E_\infty^{0,2} = E_2^{0,2} = k \cdot e_1$ which implies that e_1 is a permanent cycle. As $e_2 = e_1^2$, it follows that e_2 is a permanent cycle. Hence, $E_\infty^{4,2} \cong E_2^{4,2} \cong k \cdot (y_4 \otimes e_1)$ and $E_\infty^{6,0} \cong E_2^{6,0} \cong k \cdot y_6$.

We next show that $e_3 \in E_2^{0,6}$ is transgressive with $d_7(e_3) \neq 0$. As e_1 is a permanent cycle and $H^i_{\mathrm{H}}(BL/k) = 0$ for i odd, the spectral sequence (8) implies that $E_2^{3,0} = E_2^{5,0} = 0$. Consider the filtration of (8) on $H^6_{\mathrm{H}}(BL/k)$. We have

$$\dim_k H^6_{\mathrm{H}}(BL/k) = 3 = \dim_k E_\infty^{6,0} + \dim_k E_\infty^{4,2} + \dim_k E_\infty^{0,6} = 2 + \dim_k E_\infty^{0,6}$$

which implies that $E_\infty^{0,6} \cong k \cdot e_1 e_2$. As $E_2^{3,0} = E_2^{5,0} = 0$, we must then have $e_3 \in E_7^{0,6}$ and $0 \neq d_7(e_3) \in E_7^{7,0}$. The class $d_7(e_3)$ lifts to a non-zero class $y_7 \in H^4(BG, \Omega^3) \subseteq E_2^{7,0} = H^7_{\mathrm{H}}(BG/k)$.

$$\begin{tikzcd}
k\cdot e_1e_2\oplus k\cdot e_3 \arrow[dddddrrrrrrr] & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k\cdot e_2 \arrow[ddddrrrrr] & 0 & 0 & 0 & k\cdot e_2y_4 & 0 & k\cdot e_2y_6 & k\cdot e_2y_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k\cdot e_1 \arrow[ddrrr] & 0 & 0 & 0 & k\cdot e_1y_4 & 0 & k\cdot e_1y_6 & k\cdot e_1y_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 & k\cdot y_4 & 0 & k\cdot y_6 & k\cdot y_7
\end{tikzcd}$$

We can now determine the E_∞ page of (8). For n odd, $E_\infty^{i,n-i}=0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n\in\mathbb{N}$ is even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n=\{(a,b,c)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}:2a+4b+6c=n\}.$$

For $i=0,1,2,3$, set $V_{i,n}\coloneqq H^{(n-2i)/2}(BG,\Omega^{(n-2i)/2})$. For $i=0,1,2,3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n}=\{(a,b,c)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}:4a+6b+8c=n-2i\}$. As e_1 is a permanent cycle in (8),

$$V_{i,n}\cong V_{i,n}\otimes k\cdot e_1^i\subseteq E_\infty^{n-2i,2i}$$

for $i=0,1,2,3$.

Define a bijection $f_n:S_n\to S_{0,n}\cup S_{1,n}\cup S_{2,n}\cup S_{3,n}$ by

$$f_n(a,b,c)=\begin{cases}(b,c,a/4)\in S_{0,n} \text{ if } a\equiv 0 \mod 4,\\ (b,c,(a-1)/4)\in S_{1,n} \text{ if } a\equiv 1 \mod 4,\\ (b,c,(a-2)/4)\in S_{2,n} \text{ if } a\equiv 2 \mod 4,\\ (b,c,(a-3)/4)\in S_{3,n} \text{ if } a\equiv 3 \mod 4.\end{cases}$$

Then

$$\dim_k H^n_{\mathrm{H}}(BL/k)=|S_n|=|S_{0,n}|+|S_{1,n}|+|S_{2,n}|++|S_{3,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k)\geq E_\infty^{n,0}+E_\infty^{n-2,2}+E_\infty^{n-4,4}+E_\infty^{n-6,6}$$

and $V_{i,n}\subseteq E_\infty^{n-2i,2i}$ for $i=0,1,2,3$, it follows that $V_{i,n}\cong E_\infty^{n-2i,2i}$ for $i=0,1,2,3$ and $E_\infty^{n-2i,2i}=0$ for $i\geq 4$.

We now use Theorem 1.4 to finish the computation of the Hodge cohomology of BG. Let F_* denote the cohomological spectral sequence of k-vector spaces concentrated on the 0th column given by $F_2=\Delta(e_1,e_2)$ where e_i is of bidegree $(0,2i)$ for $i=1,2$. As e_1 is a permanent cycle in (8) , there is a map of spectral sequences $F_* \to E_*$ taking $e_i \in F_2^{0,2i}$ to $e_i \in E_2^{0,2i}$ for $i = 1, 2$. Fix a variable y. Let H_* be the spectral sequence with E_2 page given by $H_2 = \Delta(e_3) \otimes k[y]$ where e_3 is of bidegree $(0, 6)$, y is of bidegree $(7, 0)$, and e_3 is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all i. As $e_3 \in E_2^{0,6}$ is transgressive with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3 \in H_2^{0,6}$ to $e_3 \in E_2^{0,6}$ and $y \in H_2^{7,0}$ to $y_7 \in E_2^{7,0}$.

Elements in the ring of G-invariants $k[y_4, y_6, y_8]$ are permanent cycles in the spectral sequence (8). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8] \to E_*$$

of spectral sequences. As $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8]$, α induces isomorphisms on E_∞ terms and on the 0th columns of the E_2 pages. Hence, by Theorem 1.4, α induces an isomorphism on the 0th rows of the E_2 pages. Thus,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8].$$

The Hodge spectral sequence for BG degenerates by Proposition 1.5. □

As Hodge cohomology commutes with extensions of the base field, we have the following result.

Corollary 3.7. *Let k be a field of characteristic 2 and let G be a k-form of* $\mathrm{Spin}(7)$. *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[x_4, x_6, x_7, x_8]$$

where $|x_i| = i$ for all i.

Theorem 3.8. *Let $n = 8$. The Hodge spectral sequence for BG degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_8']$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y_8'| = 8$.

Proof. From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_8']$$

where $|y_i| = i$ and $|y_8'| = 8$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{9}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^4, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that e_1 is a permanent cycle in (9) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(7)_k/k)$ for $m < 8$ and $H^8_{\mathrm{H}}(BG/k) = k \cdot y_8 \oplus k \cdot y_8'$.

We can now determine the E_∞ terms for (9). For n odd, $E_\infty^{i,n-i}=0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n\in\mathbb{N}$ is even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n=\{(a,b,c,d)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}:2a+4b+6c+8d=n\}.$$

For $i=0,1,2,3$, set $V_{i,n}\coloneqq H^{(n-2i)/2}(BG,\Omega^{(n-2i)/2})$. For $i=0,1,2,3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n}=\{(a,b,c,d)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}:4a+6b+8c+8d=n-2i\}$. As e_1 is a permanent cycle in (9),

$$V_{i,n}\cong V_{i,n}\otimes k\cdot e_1^i\subseteq E_\infty^{n-2i,2i}$$

for $i=0,1,2,3$.

Define a bijection $f_n:S_n\to S_{0,n}\cup S_{1,n}\cup S_{2,n}\cup S_{3,n}$ by

$$f_n(a,b,c,d)=\begin{cases}(b,c,d,a/4)\in S_{0,n} \text{ if } a\equiv 0 \mod 4,\\ (b,c,d,(a-1)/4)\in S_{1,n} \text{ if } a\equiv 1 \mod 4,\\ (b,c,d,(a-2)/4)\in S_{2,n} \text{ if } a\equiv 2 \mod 4,\\ (b,c,d,(a-3)/4)\in S_{3,n} \text{ if } a\equiv 3 \mod 4.\end{cases}$$

Then

$$\dim_k H^n_{\mathrm{H}}(BL/k)=|S_n|=|S_{0,n}|+|S_{1,n}|+|S_{2,n}|++|S_{3,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k)\geq E_\infty^{n,0}+E_\infty^{n-2,2}+E_\infty^{n-4,4}+E_\infty^{n-6,6}$$

and $V_{i,n}\subseteq E_\infty^{n-2i,2i}$ for $i=0,1,2,3$, it follows that $V_{i,n}\cong E_\infty^{n-2i,2i}$ for $i=0,1,2,3$ and $E_\infty^{n-2i,2i}=0$ for $i\geq 4$.

Let F_* denote the spectral sequence concentrated on the 0th column with $F_2=\Delta(e_1,e_2,e_4)$ where e_i is of bidegree $(0,2i)$. There is a map of spectral sequences $F_*\to E_*$ taking e_i to e_i for $i=1,2,4$. Fix a variable y. Let H_* denote the spectral sequence with E_2 page $H_2=\Delta(e_3)\otimes k[y]$ where e_3 is of bidegree $(0,6)$, y is of bidegree $(7,0)$, and e_3 is transgressive with $d_7(e_3y^i)=y^{i+1}$ for all i. There is an obvious map of spectral sequences $H_*\to E_*$. Classes in the ring of G-invariants are permanent cycles in the spectral sequence (9). Tensoring these maps, we get a map of spectral sequences

$$\alpha:I_*\coloneqq F_*\otimes H_*\otimes k[y_4,y_6,y_8,y_8']\to E_*.$$

The map α induces an isomorphism on E_∞ terms and on the 0th columns of the E_2 pages. Theorem 1.4 then implies that α induces an isomorphism on the 0th rows of the E_2 pages. Thus,

$$H^*_{\mathrm{H}}(BG/k)=k[y_4,y_6,y_7,y_8,y_8'].$$

Proposition 1.5 implies that the Hodge spectral sequence for BG degenerates.

□

Corollary 3.9. *Let k be a field of characteristic 2 and let G be a k-form for* Spin(8). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_8']$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y_8'| = 8$.

Theorem 3.10. *Let $n = 9$. The Hodge spectral sequence for BG degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{16}]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 16$.

Proof. From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_{16}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{10}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that e_1 is a permanent cycle in (10) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(7)_k/k)$ for $m \leq 10$.

We now determine the E_∞ terms for (10). For n odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c + 8d = n\}.$$

For $0 \leq i \leq 7$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 7$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c + 16d = n - 2i\}$. As e_1 is a permanent cycle in (10),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{7} S_{i,n}$ by $f_n(a, b, c, d) = (b, c, d, (a - i)/8) \in S_{i,n}$ for $a \equiv i \mod (8)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{7} |S_{i,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$.

Let F_* denote the cohomological spectral sequence concentrated on the 0th column with E_2 page given by $F_2 = \Delta(e_1, e_2, e_4)$ where e_i has bidegree $(0, 2i)$ for $i = 1, 2, 4$. As e_1 is a permanent cycle in the spectral sequence (10), there exists a map $F_* \to E_*$ of spectral sequences taking e_i to e_i for $i = 1, 2, 4$. Let y be a free variable and let H_* denote the spectral sequence with E_2 page $H_2 = \Delta(e_3) \otimes k[y]$ where e_3 is of bidegree $(0, 6)$, y is of bidegree $(7, 0)$, and e_3 is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all i. As e_3 is transgressive in the spectral sequence (10) with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking e_3 to e_3 and y to y_7.

Elements in the ring of G-invariants $k[y_4, y_6, y_8, y_{16}]$ are permanent cycles in the spectral sequence (10). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{16}] \to E_*.$$

The map α induces an isomorphism on E_∞ terms and on the 0th columns of the E_2 pages. Hence, Theorem 1.4 implies that α induces an isomorphism on the 0th rows of the E_2 pages. Thus,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{16}].$$

Proposition 5 implies that the Hodge spectral sequence for BG degenerates. $\square$

Corollary 3.11. *Let k be a field of characteristic 2 and let G be a k-form for* Spin(9). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{16}]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 16$.

Remark 3.12. Assume that k is perfect. Let μ_2 denote the group scheme of the 2nd roots of unity over k. For $n \geq 10$, the Hodge cohomology of BG is no longer a polynomial ring. To determine the relations that hold in $H^*_{\mathrm{H}}(BG/k)$, we will restrict cohomology classes to the classifying stack of a certain subgroup of G considered in [17, Section 12]. Let $r = \lfloor n/2 \rfloor$ and let $T \cong \mathbb{G}_m^r$ denote a split maximal torus of G. Assume that $n \not\equiv 2 \mod 4$ so that the Weyl group W of G contains -1, acting by inversion on T. Then -1 acts by the identity on $T[2] \cong \mu_2^r$ (for $n \in \mathbb{N}$, $T[n] \subset T$ is the kernel of the nth power map $T \to T$) and G contains a subgroup $Q \cong \mu_2^r \times \mathbb{Z}/2$. Under the double cover $G \to SO(n)_k$, the image of Q is isomorphic to $K \cong \mu_2^{r-1} \times \mathbb{Z}/2$ and $Q \to K$ is a split surjection. We will need to know the Hodge cohomology rings of the classifying stacks of these groups. For a commutative ring R, we let $\mathrm{rad} \subset R$ denote the ideal of nilpotent elements. From [17, Proposition 10.1],

$$H^*_{\mathrm{H}}(B\mu_2/k)/\mathrm{rad} \cong k[t]$$

where $t \in H^1(B\mu_2, \Omega^1)$. From [17, Lemma 10.2],

$$H^*_{\mathrm{H}}((B\mathbb{Z}/2)/k) = k[s]$$

where $s \in H^1(B\mathbb{Z}/2, \Omega^0)$. The Künneth formula [17, Proposition 5.1] then lets us calculate the Hodge cohomology ring of $B\mu_2^i \times B(\mathbb{Z}/2)^j$ for any $i, j \geq 0$. Fix $i, j > 0$. Then

$$H^*_{\mathrm{H}}((B\mu_2^i \times B(\mathbb{Z}/2)^j)/k)/\mathrm{rad} \cong k[t_1, \ldots, t_i, s_1, \ldots, s_j]$$

where $t_l \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^1)$ for all l and $s_l \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^0)$ for all l.

Theorem 3.13. *Let $n = 10$. The Hodge spectral sequence for BG degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10})$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 10, 32$.

Proof. We may assume that $k = \mathbb{F}_2$ so that Remark 3.12 applies. From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_{10}, y_{32}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{11}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4, c_5].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that e_1 is a permanent cycle in (11) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(9)_k/k)$ for $m < 10$.

Let F_* be the spectral sequence concentrated on the 0th column with E_2 page given by $F_2 = \Delta(e_1, e_2, e_4)$ where e_i has bidegree $(0, 2i)$ for all i. As e_1 is a permanent cycle in (11), there exists a map of spectral sequence $F_* \to E_*$ taking e_i to e_i for $i = 1, 2, 4$. Fix a variable y. Let H_* denote the spectral sequence with E_2 page $H_2 = \Delta(e_3) \otimes k[y]$ where e_3 has bidegree $(0, 6)$, y has bidegree $(7, 0)$, and e_3 is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all i. As e_3 is transgessive in (11) with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking e_3 to e_3 and y to y_7. Elements in the ring of G-invariants $k[y_4, y_6, y_8, y_{10}, y_{32}]$ are permanent cycles in (11). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_* \tag{12}$$

which induces an isomorphism on the 0th columns of the E_2 pages.

Let n be even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 2a + 4b + 6c + 8d + 10e = n\}.$$

For $0 \leq i \leq 15$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^2_{\mathrm{H}}((G/P)/k)$ is a permanent cycle in (11),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$. Hence, the map α in (12) induces injections on all E_∞ terms. For n odd, α induces isomorphisms $0 = I_\infty^{n-i,i} \cong E_\infty^{n-i,i} = 0$ for all i since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}. \tag{13}$$

Now assume that $n \leq 14$. Then f_n gives a bijection

$$S_n \to \bigcup_{i=0}^{7} S_{i,n}.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$. As α induces injections on all E_∞ terms, Theorem 1.4 implies that α in (12) induces an isomorphism $I_2^{n,0} \to E_2^{n,0}$ for $n < 16$.

Now we consider the filtration on $H^{16}_{\mathrm{H}}(BL/k)$ given by (11). From the bijection f_{16} defined in the previous paragraph, we have

$$\dim_k H^{16}_{\mathrm{H}}(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,n}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,n} \otimes k \cdot e_1^i.$$

As e_1 is a permanent cycle and α induces isomorphisms on 0th row terms of the E_2 pages in degrees less than 16, we must then have

$$E_\infty^{10,6} \cong (H^{10}_{\mathrm{H}}(BG/k) \otimes k \cdot e_1^3) \oplus (k \cdot z \otimes k \cdot e_3)$$

for some $0 \neq z \in H^{10}_{\mathrm{H}}(BG/k)$. Hence, $y_7 z = 0$ in $H^*_{\mathrm{H}}(BG/k)$. Write $z = ay_4y_6 + by_{10}$ for some $a, b \in k$.

We now show that $a = 0$ by restricting $y_7 z = 0$ to the Hodge cohomology of the classifying stack of the subgroup $\mathrm{Spin}(8)_k$ of G. Under the isomorphism

$$H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k) \cong k[y_4, y_6, y_7, y_8, y_{16}]$$

of Theorem 3.10, the pullback from $H^*_{\mathrm{H}}(BG/k)$ to $H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$ maps $y_4, y_6, y_{10} \in H^*_{\mathrm{H}}(BG/k)$ to y_4, y_6, and 0 respectively in $H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. Hence, to show that $a = 0$, it suffices to show that $y_7 \in H^*_{\mathrm{H}}(BG/k)$ restricts to $y_7 \in H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. From the isomorphism

$$H^*_{\mathrm{H}}(BSO(m)_k/k) \cong k[u_2, \ldots, u_m]$$

of Theorem 3.4 for $m \geq 0$, the class $u_7 \in H^7_{\mathrm{H}}(BSO(10)_k/k)$ restricts to $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$. Thus, we are reduced to showing that $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$ pulls back to a non-zero multiple of $y_7 \in H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$.

Consider the subgroups $\mu_2^4 \times \mathbb{Z}/2 \cong Q \subseteq \mathrm{Spin}(8)_k$ and $\mu_2^3 \times \mathbb{Z}/2 \cong K \subseteq SO(8)_k$ defined in Remark 3.12. As the morphism $Q \to K$ is split surjective, if we can show that u_7 restricts to a nonzero class in $H^*_{\mathrm{H}}(BK/k)$, then u_7 would restrict to a nonzero class in $H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. From the inclusion $O(2)_k^4 \subset O(8)_k$, $O(8)_k$ contains a subgroup of the form $\mu_2^4 \times (\mathbb{Z}/2)^4$. As $SO(8)_k$ is the kernel of the Dickson determinant (also called the Dickson invariant in some sources [11, §23]) $O(8)_k \to \mathbb{Z}/2$, it follows that $SO(8)_k$ contains a subgroup $H \cong \mu_2^4 \times (\mathbb{Z}/2)^3$. Write

$$H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} \cong k[t_1, \ldots, t_4, s_1, \ldots, s_4]/(s_1 + s_2 + s_3 + s_4)$$

using Remark 3.12. From the proof of [17, Lemma 11.4], the pullback of u_7 to $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$ followed by pullback to

$$H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_4, s]/(t_1 + \cdots + t_4)$$

is given by

$$u_7 \mapsto \sum_{j=1}^{3} s_j(t_j + t_4) \sum_{\substack{1 \leq i_1 < i_2 \leq 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2} \mapsto \sum_{j=1}^{3} s(t_j + t_4) \sum_{\substack{1 \leq i_1 < i_2 \leq 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2}$$

$$= s \sum_{1 \leq i_1 < i_2 \leq 3} (t_{i_1} + t_{i_2}) t_{i_1} t_{i_2} \neq 0.$$

Thus, $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$ pulls back to a nonzero multiple of

$$y_7 \in H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$$

which implies that $y_7 y_{10} = 0$ in $H^*_{\mathrm{H}}(BG/k)$.

$$\begin{tikzcd}
u_7 \in H^7_{\mathrm{H}}(BSO(10)_k/k) \arrow[r] \arrow[d] & y_7 \in H^7_{\mathrm{H}}(BG/k) \arrow[d] \\
u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k) \arrow[r] \arrow[d] & y_7 \in H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k) \arrow[d] \\
\sum_{j=1}^{3} s(t_j + t_4) \sum_{\substack{1 \le i_1 < i_2 \le 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2} \in H^7_{\mathrm{H}}(BK/k) \arrow[r, "\neq 0"] & H^7_{\mathrm{H}}(BQ/k)
\end{tikzcd}$$

Using the relation $y_7 y_{10} = 0$, we now modify the spectral sequence I_* defined above to define a new spectral sequence J_* that better approximates (and will actually be isomorphic to) the spectral sequence (11). Let

$$(yy_{10}) \coloneqq F_2 \otimes (\Delta(e_3) \otimes yk[y]) \otimes y_{10} k[y_4, y_6, y_8, y_{10}, y_{32}].$$

Define the E_2 page of J_* by $J_2 = I_2/(yy_{10})$. Define the differentials d'_m of J_* so that $I_2 \to J_2$ induces a map $I_* \to J_*$ of cohomological spectral sequences of k-vector spaces and $d'_m = 0$ for $m > 7$. This means that $d'_7(f \otimes e_3 \otimes y_{10} g) = f \otimes y \otimes y_{10} g = 0$ and $d'_m(f \otimes e_3 \otimes y_{10} g) = 0$ for $m > 7$, $f \in F_2$, and $g \in k[y_4, y_6, y_8, y_{10}, y_{32}]$. The E_∞ page of J_* is given by

$$J_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes e_3 \otimes y_{10} k[y_4, y_6, y_8, y_{10}, y_{32}]).$$

As $y_7 y_{10} = 0$ in $H^*_{\mathrm{H}}(BG/k)$, α induces a map $\alpha' : J_* \to E_*$ of spectral sequences. To finish the calculation, we will show that α' induces an isomorphism on E_∞ terms so that Theorem 1.4 will apply. For n odd, $E_\infty^{n-i,i} = 0$ for all i since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Now assume that n is even. For $0 \le i \le 7$,

$$V_{i,n} \cong H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^i \subseteq E_\infty^{n-2i,2i}.$$

For $8 \le i \le 15$,

$$V_{i,n} \cong y_{10} H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^{i-8} e_3 \subseteq E_\infty^{n-2i+10,2i-10}.$$

Hence, from the description of the E_∞ terms of J_* given above, it follows that α' induces an injection $J_\infty^{n-2i,2i} \to E_\infty^{n-2i,2i}$ for all i. Equation (13) then implies that $J_\infty^{n-2i,2i} \cong E_\infty^{n-2i,2i}$ for all i.

Thus, α' induces an isomorphism on E_∞ pages and an isomorphism on the 0th columns of the E_2 pages of the 2 spectral sequences. Theorem 1.4 then implies that

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10}).$$

From Proposition 5, the Hodge spectral sequence for BG degenerates. $\square$

Corollary 3.14. *Let G be a k-form of* $\mathrm{Spin}(10)$*. Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10})$$

where $|y_i| = i$ *for all* i.

Theorem 3.15. *Let* $n = 11$. *The Hodge spectral sequence for* BG *degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11})$$

where $|y_i| = i$ *for* $i = 4, 6, 7, 8, 10, 11, 32$.

Proof. We may assume that $k = \mathbb{F}_2$ so that Remark 3.12 applies. From Lemma 3.5,

$$O(\mathfrak{g})^G \cong k[y_4, y_6, y_8, y_{10}, y_{32}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{14}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4, e_5]/(e_i^2 = e_{2i}) = k[e_1, e_3, e_5]/(e_1^8, e_3^2, e_5^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4, c_5].$$

Using Theorem 3.4, write $H^*_{\mathrm{H}}(BSO(11)_k/k) = k[u_2, \ldots, u_{11}]$. From the inclusions $O(2)^5_k \subset O(10)_k \subset SO(11)_k$, $SO(11)_k$ contains a subgroup $H \cong \mu_2^5 \times (\mathbb{Z}/2)^5$. Write $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s_1, \ldots, s_5]$ as described in Remark 3.12. Under the pullback map $H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$, u_{2m} pulls back to the mth elementary symmetric polynomial

$$\sum_{1 \le i_1 < \cdots < i_m \le 5} t_{i_1} \cdots t_{i_m} \tag{15}$$

and u_{2m+1} pulls back to

$$\sum_{j=1}^{5} s_j \sum_{\substack{1 \le i_1 < \cdots < i_m \le 5 \\ \text{one equal to j}}} t_{i_1} \cdots t_{i_m}$$

for $1 \le m \le 5$ [17, Lemma 11.4]. To be concise, from now on we will write u_{2m} to denote the image of u_{2m} under pullback maps to $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$ or $H^*_{\mathrm{H}}(BK/k)/\mathrm{rad}$ whenever we are dealing with these two rings.

Let $Q \cong (\mu_2^5 \times \mathbb{Z}/2) \subset G$ and $K \cong (\mu_2^4 \times \mathbb{Z}/2) \subset SO(11)_k$ be the subgroups described in Remark 3.12. Write $H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s]/(t_1 + \cdots + t_5)$. Under the pullback map $H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BK/k)/\mathrm{rad}$, u_7 maps to $su_6 \neq 0$ and u_{11} maps to $su_{10} \neq 0$. As $Q \to K$ is split, it follows that u_7, u_{11} restrict to nonzero classes $y_7 \in H^7_{\mathrm{H}}(BG/k)$ and $y_{11} \in H^{11}_{\mathrm{H}}(BG/k)$. Also, y_4y_7 and y_{11} are linearly independent in $H^{11}_{\mathrm{H}}(BG/k)$.

Returning to the spectral sequence (14), calculations similar to those performed in the proof of Proposition 3.6 show that e_1 is a permanent cycle in (14) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(10)_k/k)$ for $m \le 10$.

Let F_* be the spectral sequence concentrated on the 0th column with E_2 page given by $\Delta(e_1, e_2, e_4)$ with e_i of bidegree $(0, 2i)$ for $i = 1, 2, 4$. Fix a variable y and let H_* be the spectral sequence with $H_2 = \Delta(e_3) \otimes k[y]$ where e_3 is of bidegree $(0, 6)$, y is of bidegree $(7, 0)$, and e_3 is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all i. There exists a map of spectral sequence

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_*$$

taking e_i to e_i for $i = 1, 2, 3, 4$ and taking y to y_7. The E_∞ page of I_* is given by $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]$ and α induces an injection $I_\infty^{i,j} \to E_\infty^{i,j}$ for all i, j with $i + j \leq 17$. For n odd, α induces an isomorphism $0 = I_\infty^{n-i,i} \cong E_\infty^{n-i,i} = 0$ for all i since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees.

Let n be even. The k-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 2a + 4b + 6c + 8d + 10e = n\}.$$

For $0 \leq i \leq 15$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^2_{\mathrm{H}}((G/P)/k)$ is a permanent cycle in (14),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$ and $n \leq 16$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}. \tag{16}$$

Now assume that $n \leq 14$. Then f_n gives a bijection

$$S_n \to \bigcup_{i=0}^{7} S_{i,n}.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$. In particular, $E_\infty^{0,10} \cong k \cdot e_1^5$. As mentioned above, we have $H^m_{\mathrm{H}}(BG/k) = 0$ for $m = 3, 5, 9$. After adding a k-multiple of $e_3 e_1^2$ to e_5, we can assume that $d_7(e_5) = 0$. Then the isomorphism $E_\infty^{0,10} \cong k \cdot e_1^5$ implies that $d_{11}(e_5) \neq 0$. Hence, e_5 is transgressive in (14) and $y_{11} \in H^6(BG, \Omega^5)$ is a lifting of $d_{11}(e_5)$ to $E_2^{11,0}$.

Fix a variable x. Let J_* denote the spectral sequence with E_2 page $J_2 = \Delta(e_5) \otimes k[x]$ where e_5 has bidegree $(0, 10)$, x has bidegree $(11, 0)$, and e_5 is transgressive with $d_{11}(e_5 x^i) = x^{i+1}$ for all i.

$$\begin{tikzcd}
k \cdot e_5 \arrow[dr, "d_{11}"] & k \cdot e_5 x \arrow[dr, "d_{11}"] & \cdots & \\
0 & k \cdot x & k \cdot x^2 & \cdots
\end{tikzcd}$$

As e_5 is transgressive in (14), there exists a map of spectral sequences $J_* \to E_*$ taking e_5 to e_5 and x to y_{11}. Tensoring with the map α defined above, we get a map

$$\alpha' : K_* := I_* \otimes J_* \to E_*$$

which induces an isomorphism on the 0th columns of the E_2 pages. The E_∞ page of K_* is given by

$$K_\infty \cong I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}].$$

As mentioned above, α and hence α' induce isomorphisms on $E_\infty^{n-i,i}$ terms for $n < 16$ and injections on all E_∞ terms on or below the line $i + j = 17$. Theorem 1.4 implies that α' induces an isomorphism $K_2^{n,0} \to E_2^{n,0}$ for $n < 16$.

Next, we consider the filtration on $H^{16}_{\mathrm{H}}(BL/k)$ given by (14). From (16),

$$\dim_k H^{16}_{\mathrm{H}}(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,16}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,16} \otimes k \cdot e_1^i.$$

We must then have either $d_7(e_3 f) = y_7 f = 0 \in H^{17}_{\mathrm{H}}(BG/k)$ for some $0 \neq f \in H^{10}_{\mathrm{H}}(BG/k)$ or $d_{11}(e_5 g) = y_{11} g = d_7(e_3) h = y_7 h \in H^{17}_{\mathrm{H}}(BG/k)$ for some $0 \neq g \in H^6_{\mathrm{H}}(BG/k)$ and $h \in H^{10}_{\mathrm{H}}(BG/k)$. Let $a, b, c \in k$, not all zero, such that $a y_{11} y_6 + b y_7 y_{10} + c y_7 y_4 y_6 = 0 \in H^{17}_{\mathrm{H}}(BG/k)$.

The class $a u_{11} u_6 + b u_7 u_{10} + c u_7 u_4 u_6 \in H^{17}_{\mathrm{H}}(BSO(11)_k/k)$ pulls back to $a y_{11} y_6 + b y_7 y_{10} + c y_7 y_4 y_6 = 0 \in H^{17}_{\mathrm{H}}(BG/k)$. Under the pullback map

$$H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s]/(t_1 + \cdots + t_5),$$

$a u_{11} u_6 + b u_7 u_{10} + c u_7 u_4 u_6$ maps to $a s u_{10} u_6 + b s u_6 u_{10} + c s u_6 u_4 u_6$, which equals 0 since $Q \to K$ is split. Then $c = 0$ and $a = b$ since the elementary symmetric polynomials (15) in

$$k[t_1, \ldots, t_5]/(t_1 + \cdots + t_5)$$

generate a polynomial subring.

$$\begin{tikzcd}
a u_{11} u_6 + b u_7 u_{10} + c u_7 u_4 u_6 \in H^{17}_{\mathrm{H}}(BSO(11)_k/k) \arrow[r] \arrow[d] & 0 \in H^{17}_{\mathrm{H}}(BG/k) \arrow[d] \\
a s u_{10} u_6 + b s u_6 u_{10} + c s u_6 u_4 u_6 \in H^{17}_{\mathrm{H}}(BK/k)/\mathrm{rad} \arrow[r] & 0 \in H^{17}_{\mathrm{H}}(BQ/k)/\mathrm{rad}
\end{tikzcd}$$

Thus, the relation $y_7y_{10} + y_6y_{11} = 0$ holds in $H^*_{\mathrm{H}}(BG/k)$ and $E^{6,10}_\infty \cong (k \cdot y_6 \otimes e_5) \oplus (k \cdot y_6 \otimes e_1^5)$. We now use the relation $y_7y_{10} + y_6y_{11} = 0$ to define a new spectral sequence L_* from K_*. Let $(y_6x + yy_{10}) \subset K_2$ denote the ideal generated by $y_6x + yy_{10}$ and let $L_2 \coloneqq K_2/(y_6x + yy_{10})$. Define the differentials d'_m of L_* so that $K_2 \to L_2$ induces a map of spectral sequences $K_* \to L_*$ and $d'_m = 0$ for $m > 11$. Then $\alpha' : K_* \to E_*$ induces a map of spectral sequences $\alpha'' : L_* \to E_*$. The E_∞ page of L_* is given by

$$L_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes y_6k[y_4, y_6, y_8, y_{10}, y_{32}] \otimes e_5).$$

We now show by induction that α'' induces an isomorphism $L^{n,0}_2 \to E^{n,0}_2$ for all n. For $n < 16$, we have shown that $L^{n,0}_2 \cong E^{n,0}_2$. Now let $n \geq 16$ and assume that α'' induces an isomorphism $L^{m,0}_2 \to E^{m,0}_2$ for all $m < n$. First, suppose that n is even. As $L^{m,0}_2 \cong E^{m,0}_2$ for $m < n$, $y_7g \neq 0 \in H^*_{\mathrm{H}}(BG/k)$ for all $0 \neq g \in H^*_{\mathrm{H}}(BG/k)$ with $|g| < n - 7$. Hence, for any $0 \neq g \in H^m_{\mathrm{H}}(BG/k)$ with $|g| = m < n - 7$, $g \otimes e_3e_5 \in E^{m,16}_2 \cong E^{m,16}_7$ is not in the kernel of the differential $d_7 : E^{m,16}_7 \to E^{m+7,10}_7 \cong E^{m+7,10}_2$. As $y_7 \in H^4(BG, \Omega^3)$ and $y_{11} \in H^6(BG, \Omega^5)$, $y_7z, y_{11}z \notin \oplus_i H^i(BG, \Omega^i)$ for all $z \in H^*_{\mathrm{H}}(BG/k)$. It follows that α'' induces an injection $L^{i,j}_\infty \to E^{i,j}_\infty$ for all i, j with $m = i + j \leq n$:

$$L^{m-2i,2i}_\infty \cong V_{i,m} \otimes e_1^i \subseteq E^{m-2i,2i}_\infty$$

for $0 \leq i \leq 4$,

$$L^{m-2i,2i}_\infty \cong (V_{i,m} \otimes e_1^i) \oplus (y_6V_{i+3,m} \otimes e_1^{i-5}e_5) \subseteq E^{m-2i,2i}_\infty$$

for $5 \leq i \leq 7$, and

$$L^{m-2i,2i}_\infty \cong y_6V_{i+3,m} \otimes e_1^{i-5}e_5 \subseteq E^{m-2i,2i}_\infty$$

for $8 \leq i \leq 12$. The equality in (16) then implies that α'' induces isomorphisms $L^{i,j}_\infty \to E^{i,j}_\infty$ for all i, j with $i + j \leq n$. As mentioned above, α'' induces isomorphisms $0 = L^{n+1-i,i}_\infty \to E^{n+1-i,i}_\infty = 0$ for all i since $n + 1$ is odd. Theorem 1.4 then implies that α'' induces an isomorphism $L^{n,0}_2 \cong E^{n,0}_2 = H^n_{\mathrm{H}}(BG/k)$.

Now assume that n is odd. We have $0 = L^{i,j}_\infty \cong E^{i,j}_\infty = 0$ for all i, j with $i + j = n$. An argument similar to the one used above for when n is even shows that α'' induces injections $L^{i,j}_\infty \to E^{i,j}_\infty$ for all i, j with $i + j \leq n + 1$. Equation (16) then implies that α'' induces isomorphisms $L^{i,j}_\infty \to E^{i,j}_\infty$ for all i, j with $i + j \leq n + 1$. It follows that α'' induces an isomorphism $L^{n,0}_2 \cong E^{n,0}_2 = H^n_{\mathrm{H}}(BG/k)$ by an application of Theorem 1.4. Thus, by induction, we have obtained that the 0th row of L_2 is isomorphic to the 0th row of E_2 :

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11}).$$

The Hodge spectral sequence for BG degenerates by Proposition 5.

□

Corollary 3.16. *Let G be a k-form of* $\mathrm{Spin}(11)$. *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11})$$

where $|y_i| = i$ *for* $i = 4, 6, 7, 8, 10, 11, 32$.

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